Short-time asymptotics for marginal distributions of semimartingales

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Abstract

We study the short-time aymptotics of conditional expectations of smooth and non-smooth functions of a (discontinuous) Ito semimartingale; we compute the leading term in the asymptotics in terms of the local characteristics of the semimartingale. We derive in particular the asymptotic behavior of call options with short maturity in a semimartingale model: whereas the behavior of *out-of-the-money* options is found to be linear in time, the short time asymptotics of *at-the-money* options is shown to depend on the fine structure of the semimartingale.

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We study the short-time asymptotics of expectations of the form $E[f(\xi_t)]$ where ξ_t a discontinuous Itô semimartingale. We study two different cases: first, the case where f is a smooth (C^2) function, then the case where $f(x) = (x-K)_+$. We compute the leading term in the asymptotics in terms of the local characteristics of the semimartingale.

As an application, we derive the asymptotic behavior of call option prices close to maturity in a semimartingale model: whereas the behavior of *out-of-the-money* options is found to be linear in time, the short time asymptotics of *at-the-money* options is shown to depend on the fine structure of the semimartingale.

These results generalize and extend various asymptotic results previously derived for diffusion models [7, 9], Lévy processes [18, 14, 26], Markov jump-diffusion models [4] and one-dimensional martingales [21] to the more general case of a discontinuous semimartingale. In particular, the independence of increments or the Markov property do not play any role in our derivation.

1   Introduction

In applications such as stochastic control, statistics of processes and mathematical finance, one is often interested in computing or approximating conditional expectations of the type

$$\mathbb{E}\left[f(\xi_t)|\mathcal{F}_{t_0}\right], \tag{1}$$

where ξ is a stochastic process. Whereas for Markov process various well-known tools –partial differential equations, Monte Carlo simulation, semigroup methods– are available for the computation and approximation of conditional expectations, such tools do not carry over to the more general setting of semimartingales. Even in the Markov case, if the state space is high dimensional exact computations may be computationally prohibitive and there has been a lot of interest in obtaining approximations of (1) as $t \to t_0$. Knowledge of such *short-time asymptotics* is very useful not only for computation of conditional expectations but also for the estimation and calibration of such models. Accordingly, short-time asymptotics for (1) (which, in the Markov case, amounts to studying transition densities of the process ξ) has been previously studied for diffusion models [7, 9, 12], Lévy processes [18, 19, 25, 2, 14, 13, 26], Markov jump-diffusion models [1, 4] and scalar martingales driven by a one-dimensional Poisson random measure [21], using a variety of techniques. The proofs of these results in the case of Lévy processes makes heavy use of the independence of increments; proofs in other case rely on the Markov property, estimates for heat kernels for second-order differential operators or Malliavin calculus. What is striking, however, is the similarity of the results obtained in these different settings.

We reconsider here the short-time asymptotics of conditional expectations in a more general framework which contains existing models but allows to go beyond the Markovian setting and to incorporate path-dependent features. Such a framework is provided by the class of *Itô semimartingales*, which contains all the examples cited above but allows to use the tools of stochastic analysis.

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An *Itô semimartingale* on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a stochastic process ξ with the representation

$$\xi_t = \xi_0 + \int_0^t \beta_s \, ds + \int_0^t \delta_s \, dW_s + \int_0^t \int_{\mathbb{R}^d} \kappa(y) \, \tilde{M}(dsdy) + \int_0^t \int_{\mathbb{R}^d} (y - \kappa(y)) \, M(dsdy), \tag{2}$$

where ξ_0 is in $\mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, M is an integer-valued random measure on $[0, \infty] \times \mathbb{R}^d$ with compensator $\mu(\omega, dt, dy) = m(\omega, t, dy)dt$ and $\tilde{M} = M - \mu$ its compensated random measure, β (resp. δ) is an adapted process with values in $\mathbb{R}^d$ (resp. $M_{d \times n}(\mathbb{R})$) and

$$\kappa(y) = \frac{y}{1 + \|y\|^2}$$

is a truncation function.

We study the short-time asymptotics of conditional expectations of the form (1) where ξ is an Ito semimartingale of the form (2), for various classes of functions $f : \mathbb{R}^d \to \mathbb{R}$. First, we prove a general result for the case of $f \in C_b^2(\mathbb{R}^d, \mathbb{R})$. Then we will treat, when $d = 1$, the case of

$$\mathbb{E}\left[(\xi_t - K)^+ | \mathcal{F}_{t_0} \right], \tag{3}$$

which corresponds to the value at t_0 of a call option with strike K and maturity t in a model described by equation (2). We show that whereas the behavior of (3) in the case $K > \xi_{t_0}$ (*out-of-the-money* options) is linear in $t - t_0$, the asymptotics in the case $K = \xi_{t_0}$ (which corresponds to *at-the-money* options) depends on the fine structure of the semimartingale ξ at t_0. In particular, we show that for continuous semimartingales the short-maturity asymptotics of at-the-money options is determined by the local time of ξ at t_0. In each case we identify the leading term in the asymptotics and express this term in terms of the local characteristics of the semimartingale at t_0.

Our results unify various asymptotic results previously derived for particular examples of stochastic models and extend them to the more general case of a discontinuous semimartingale. In particular, we show that the independence of increments or the Markov property do not play any role in the derivation of such results.

Short-time asymptotics for expectations of the form (1) have been studied in the context of statistics of processes [18] and option pricing [1, 7, 9, 4, 14, 26, 21]. Berestycki, Busca and Florent [7, 9] derive short maturity asymptotics for call options when ξ_t is a diffusion, using analytical methods. Durrleman [11] studied the asymptotics of implied volatility in a general, non-Markovian stochastic volatility model. Jacod [18] derived asymptotics for (1) for various classes of functions f, when ξ_t is a Lévy process. Lopez [14] and Tankov [26] study the asymptotics of (3) when ξ_t is the exponential of a Lévy process. Lopez [14] also studies short-time asymptotic expansions for (1), by iterating the infinitesimal generator of the Lévy process ξ_t. Alos et al [1] derive short-maturity expansions for call options and implied volatility in a Heston model using Malliavin calculus.

Benhamou et al. [4] derive short-maturity expansions for call options in a model where ξ is the solution of a Markovian SDE whose jumps are described by a compound Poisson process. These results apply to processes with independence of increments or solutions of a "Markovian" stochastic differential equation.

Durrleman studied the convergence of implied volatility to spot volatility in a stochastic volatility model with finite-variation jumps [10]. More recently, Nutz and Muhle-Karbe [21] study short-maturity asymptotics for call options in the case where ξ_t is a one-dimensional Itô semimartingale driven by a (one-dimensional) Poisson random measure whose Lévy measure is absolutely continuous. Their approach consists in "freezing" the characteristic triplet of ξ at t_0, approximating ξ_t by the corresponding Lévy process and using the results cited above [18, 14] to derive asymptotics for call option prices.

Our contribution is to extend these results to the more general case when ξ is a d-dimensional semimartingale with jumps. In contrast to previous derivations, our approach is purely based on Itô calculus and makes no use of the Markov property or independence of increments. Also, our multidimensional setting allows to treat examples which are not accessible using previous results. For instance, when studying index options in jump-diffusion model, one considers an index $I_t = \sum w_i S_t^i$ where $(S^1, ..., S^d)$ are Itô semimartingales. In this framework, I is indeed an Itô semimartingale whose stochastic integral representation is implied by those of S^i but it is naturally represented in terms of a d-dimensional integer-valued random measure, not a one-dimensional Poisson random measure, so the setting of [21] does not apply. Our setting provides a natural framework for treating such examples.

Note that these 'short-time' asymptotics are different from the 'extreme-strike' asymptotics studied by Lee [20] and extended by Friz, Gulisashvili and others [3, 15, 16, 17]. But, in specific models, the two asymptotic regimes may be related using scaling arguments.

2 Short time asymptotics for conditional expectations

2.1 Main result

We make the following assumptions on the characteristics of the semimartingale ξ:

Assumption 2.1 (Right-continuity of characteristics at t_0)**.**

$$\lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[\|\beta_t - \beta_{t_0}\| \,|\, \mathcal{F}_{t_0}\right] = 0, \qquad \lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[\|\delta_t - \delta_{t_0}\|^2 \,|\, \mathcal{F}_{t_0}\right] = 0,$$

where $\|.\|$ *denotes the Euclidean norm on* $\mathbb{R}^d$ *and for* $\varphi \in \mathcal{C}_0^b(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$,

$$\lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[\int_{\mathbb{R}^d} \|y\|^2 \, \varphi(\xi_t, y) \, m(t, dy) | \mathcal{F}_{t_0}\right] = \int_{\mathbb{R}^d} \|y\|^2 \, \varphi(\xi_{t_0}, y) \, m(t_0, dy).$$

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The second requirement, which may be viewed as a weak (right) continuity of $m(t, dy)$ along the paths of ξ, is satisfied for instance if $m(t, dy)$ is absolutely continuous with a density which is right-continuous in t at t_0.

Assumption 2.2 (Integrability condition). $\exists T > t_0$,

$$\mathbb{E}\left[\int_{t_0}^T \|\beta_s\| \, ds \Big| \mathcal{F}_{t_0}\right] < \infty, \qquad \mathbb{E}\left[\int_{t_0}^T \|\delta_s\|^2 \, ds \Big| \mathcal{F}_{t_0}\right] < \infty,$$

$$\mathbb{E}\left[\int_{t_0}^T \int_{\mathbb{R}^d} \|y\|^2 \, m(s, dy) \, ds \Big| \mathcal{F}_{t_0}\right] < \infty.$$

Under these assumptions, the following result describes the asymptotic behavior of $\mathbb{E}\left[f(\xi_t)|\mathcal{F}_{t_0}\right]$ when $t \to t_0$:

Theorem 2.1. *Under Assumptions 2.1 and 2.2, for all $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$,*

$$\lim_{t \downarrow t_0} \frac{1}{t - t_0} \left(\mathbb{E}\left[f(\xi_t)|\mathcal{F}_{t_0}\right] - f(\xi_{t_0})\right) = \mathcal{L}_{t_0} f(\xi_{t_0}). \tag{4}$$

where $\mathcal{L}_{t_0}$ is the (random) integro-differential operator given by

$$\forall f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{L}_{t_0} f(x) = \beta_{t_0} . \nabla f(x) + \frac{1}{2} \text{tr} \left[{}^t\delta_{t_0} \delta_{t_0} \nabla^2 f \right](x) \tag{5}$$

$$+ \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1 + \|y\|^2} \, y . \nabla f(x)] m(t_0, dy).$$

Before proving Theorem 2.1, we recall a useful lemma:

Lemma 2.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be right-continuous at 0, then*

$$\lim_{t \to 0} \frac{1}{t} \int_0^t f(s) \, ds = f(0). \tag{6}$$

Proof. Let F denote the primitive of f, then

$$\frac{1}{t} \int_0^t f(s) \, ds = \frac{1}{t} \left(F(t) - F(0)\right).$$

Letting $t \to 0^+$, this is nothing but the right derivative at 0 of F, which is $f(0)$ by right continuity of f. $\qquad \square$

We can now prove Theorem 2.1.

Proof. of Theorem 2.1
We first note that, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_{t_0}}$ given $\mathcal{F}_{t_0}$, we may replace the conditional expectation in (4) by an expectation with respect to the marginal distribution of ξ_t under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality,

we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. Let $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$. Itô's formula yields

$$
\begin{aligned}
f(\xi_t) &= f(\xi_0) + \int_0^t \nabla f(\xi_{s-}) d\xi_s^i + \frac{1}{2} \int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s-})\, {}^t\delta_s \delta_s\right] ds \\
&\quad + \sum_{s \le t} \left[f(\xi_{s-} + \Delta\xi_s) - f(\xi_{s-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\xi_{s-}) \Delta\xi_s^i \right] \\
&= f(\xi_0) + \int_0^t \nabla f(\xi_{s-}).\beta_s\, ds + \int_0^t \nabla f(\xi_{s-}).\delta_s dW_s \\
&\quad + \frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s-})\, {}^t\delta_s \delta_s\right]\, ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \nabla f(\xi_{s-}).\kappa(y)\, \tilde{M}(ds\, dy) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s-} + y) - f(\xi_{s-}) - \kappa(y).\nabla f(\xi_{s-}) \right) M(ds\, dy).
\end{aligned}
$$

We note that

- since ∇f is bounded and given Assumption 2.2, $\int_0^t \int_{\mathbb{R}^d} \nabla f(\xi_{s-}).\kappa(y)\, \tilde{M}(ds\, dy)$ is a square-integrable martingale.

- since ∇f is bounded and given Assumption 2.2, $\int_0^t \nabla f(\xi_{s-}).\delta_s dW_s$ is a martingale.

Hence, taking expectations, we obtain

$$
\begin{aligned}
\mathbb{E}\left[f(\xi_t)\right] &= \mathbb{E}\left[f(\xi_0)\right] + \mathbb{E}\left[\int_0^t \nabla f(\xi_{s-}).\beta_s\, ds\right] + \mathbb{E}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s-})\, {}^t\delta_s \delta_s\right]\, ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s-} + y) - f(\xi_{s-}) - \kappa(y).\nabla f(\xi_{s-}) \right) M(ds\, dy)\right] \\
&= \mathbb{E}\left[f(\xi_0)\right] + \mathbb{E}\left[\int_0^t \nabla f(\xi_{s-}).\beta_s\, ds\right] + \mathbb{E}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s-})\, {}^t\delta_s \delta_s\right]\, ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s-} + y) - f(\xi_{s-}) - \kappa(y).\nabla f(\xi_{s-}) \right) m(s, dy)\, ds\right],
\end{aligned}
$$

that is

$$
\mathbb{E}\left[f(\xi_t)\right] = \mathbb{E}\left[f(\xi_0)\right] + \mathbb{E}\left[\int_0^t \mathcal{L}_s f(\xi_s)\, ds\right]. \tag{7}
$$

where $\mathcal{L}$ denote the integro-differential operator given, for all $t \in [0, T]$ and for

all $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$, by

$$
\mathcal{L}_t f(x) = \beta_t . \nabla f(x) + \frac{1}{2} \text{tr} \left[{}^t\delta_t \delta_t \nabla^2 f \right] (x)
$$
$$
+ \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1 + \|y\|^2} \, y . \nabla f(x)] m(t, dy), \tag{8}
$$

Equation (7) yields

$$
\frac{1}{t} \, \mathbb{E} \left[f(\xi_t) \right] - \frac{1}{t} f(\xi_0) - \mathcal{L}_0 f(\xi_0)
$$
$$
= \quad \mathbb{E} \left[\frac{1}{t} \int_0^t ds \ (\nabla f(\xi_s) . \beta_s - \nabla f(\xi_0) . \beta_0) \right]
$$
$$
+ \quad \frac{1}{2} \, \mathbb{E} \left[\frac{1}{t} \int_0^t ds \, \text{tr} \left[\nabla^2 f(\xi_s) \, {}^t\delta_s \delta_s - \nabla^2 f(\xi_0) \, {}^t\delta_0 \delta_0 \right] \right]
$$
$$
+ \quad \mathbb{E} \Big[\int_{\mathbb{R}^d} \frac{1}{t} \int_0^t ds \, \big[m(s, dy) \ (f(\xi_s + y) - f(\xi_s) - \kappa(y) . \nabla f(\xi_s))
$$
$$
- m(0, dy), (f(\xi_0 + y) - f(\xi_0) - \kappa(y) . \nabla f(\xi_0)) \big] \Big].
$$

Define

$$
\Delta_1(t) \quad = \quad \mathbb{E} \left[\frac{1}{t} \int_0^t ds \ (\nabla f(\xi_s) . \beta_s - \nabla f(\xi_0) . \beta_0) \right],
$$
$$
\Delta_2(t) \quad = \quad \frac{1}{2} \, \mathbb{E} \left[\frac{1}{t} \int_0^t ds \, \text{tr} \left[\nabla^2 f(\xi_{s-}) \, {}^t\delta_s \delta_s - \nabla^2 f(\xi_0) \, {}^t\delta_0 \delta_0 \right] \right],
$$
$$
\Delta_3(t) \quad = \quad \mathbb{E} \Big[\int_{\mathbb{R}^d} \frac{1}{t} \int_0^t ds \, \big[m(s, dy) \ (f(\xi_s + y) - f(\xi_s) - \kappa(y) . \nabla f(\xi_{s-}))
$$
$$
- m(0, dy) \ (f(\xi_0 + y) - f(\xi_0) - \kappa(y) . \nabla . f(\xi_0)) \big] \Big].
$$

Thanks to Assumptions 2.1 and 2.2,

$$
\mathbb{E} \left[\int_0^t ds \ |\nabla f(\xi_s) . \beta_s - \nabla f(\xi_0) . \beta_0| \right] \leq \mathbb{E} \left[\int_0^t ds \ \|\nabla f\| \, (\|\beta_s\| + \|\beta_0\|) \right] < \infty.
$$

Fubini's theorem then applies:

$$
\Delta_1(t) = \frac{1}{t} \int_0^t ds \, \mathbb{E} \left[\nabla f(\xi_s) . \beta_s - \nabla f(\xi_0) . \beta_0 \right].
$$

Let us prove that

$$
g_1 : [0, T[\to \mathbb{R}
$$
$$
t \to \mathbb{E} \left[\nabla f(\xi_t) . \beta_t - \nabla f(\xi_0) . \beta_0 \right],
$$

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is right-continuous at 0 with $g_1(0) = 0$, yielding $\Delta_1(t) \to 0$ when $t \to 0^+$ if one applies Lemma 2.1.

$$
\begin{aligned}
|g_1(t)| &= |\mathbb{E}\left[\nabla f(\xi_t).\beta_t - \nabla f(\xi_0).\beta_0\right]| \\
&= |\mathbb{E}\left[(\nabla f(\xi_t) - \nabla f(\xi_0)).\beta_0 + \nabla f(\xi_t).(\beta_t - \beta_0)\right]| \qquad (9) \\
&\leq \|\nabla f\|_\infty \mathbb{E}\left[\|\beta_t - \beta_0\|\right] + \|\beta_0\| \|\nabla^2 f\|_\infty \mathbb{E}\left[\|\xi_t - \xi_0\|\right],
\end{aligned}
$$

where $\|\|_\infty$ denotes the supremum norm on $\mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$. Assumption 2.1 implies that:
$$
\lim_{t \to 0^+} \mathbb{E}\left[\|\beta_t - \beta_0\|\right] = 0.
$$
Thanks to Assumption 2.2, one may decompose ξ_t as

$$
\begin{aligned}
\xi_t &= \xi_0 + A_t + M_t, \\
A_t &= \int_0^t \left(\beta_s \, ds + \int_{\mathbb{R}^d} (y - \kappa(y)) \, m(s, dy)\right) ds, \qquad (10) \\
M_t &= \int_0^t \delta_s \, dW_s + \int_0^t \int_{\mathbb{R}^d} y \, \tilde{M}(ds \, dy),
\end{aligned}
$$

where A_t is of finite variation and M_t is a local martingale. First, applying Fubini's theorem (using Assumption 2.2),

$$
\begin{aligned}
\mathbb{E}\left[\|A_t\|\right] &\leq \mathbb{E}\left[\int_0^t \|\beta_s\| \, ds\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \|y - \kappa(y)\| \, m(s, dy) \, ds\right] \\
&= \int_0^t ds \, \mathbb{E}\left[\|\beta_s\|\right] + \int_0^t ds \, \mathbb{E}\left[\int_{\mathbb{R}^d} \|y - \kappa(y)\| \, m(s, dy)\right].
\end{aligned}
$$

Thanks to Assumption 2.1, one observes that if $s \in [0, T[\to \mathbb{E}\left[\|\beta_s - \beta_0\|\right]$ is right-continuous at 0 so is $s \in [0, T[\to \mathbb{E}\left[\|\beta_s\|\right]$. Furthermore, Assumption 2.1 yields that
$$
s \in [0, T[\to \mathbb{E}\left[\int_{\mathbb{R}^d} \|y - \kappa(y)\| \, m(s, dy)\right]
$$
is right-continuous at 0 and Lemma 2.1 implies that
$$
\lim_{t \to 0^+} \mathbb{E}\left[\|A_t\|\right] = 0.
$$
Furthermore, writing $M_t = (M_t^1, \cdots, M_t^d)$,
$$
\mathbb{E}\left[\|M_t\|^2\right] = \sum_{1 \leq i \leq d} \mathbb{E}\left[|M_t^i|^2\right].
$$

Burkholder's inequality [23, Theorem IV.73] implies that there exists $C > 0$ such that

$$
\begin{aligned}
\sup_{s \in [0,t]} \mathbb{E}\left[|M_s^i|^2\right] &\leq C \, \mathbb{E}\left[[M^i, M^i]_t\right] \\
&= C \, \mathbb{E}\left[\int_0^t ds \, |\delta_s^i|^2 + \int_0^t ds \int_{\mathbb{R}^d} |y_i|^2 m(s, dy)\right].
\end{aligned}
$$

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Using Assumption 2.2 we may apply Fubini's theorem to obtain

$$
\begin{aligned}
\sup_{s \in [0,t]} \mathbb{E}\left[\|M_s\|^2\right] & \leq C \sum_{1 \leq i \leq d} \mathbb{E}\left[\int_0^t ds\, |\delta_s^i|^2\right] + \mathbb{E}\left[\int_0^t ds \int_{\mathbb{R}^d} |y_i|^2 m(s,dy)\right] \\
& = C\left(\mathbb{E}\left[\int_0^t ds\, \|\delta_s\|^2\right] + \mathbb{E}\left[\int_0^t ds \int_{\mathbb{R}^d} \|y\|^2 m(s,dy)\right]\right) \\
& = C\left(\int_0^t ds\, \mathbb{E}\left[\|\delta_s\|^2\right] + \int_0^t ds\, \mathbb{E}\left[\int_{\mathbb{R}^d} \|y\|^2 m(s,dy)\right]\right).
\end{aligned}
$$

Thanks to Assumption 2.1, Lemma 2.1 yields

$$
\lim_{t \to 0^+} \mathbb{E}\left[\|M_t\|^2\right] = 0.
$$

Using the Jensen inequality, one obtains

$$
\mathbb{E}\left[\|M_t\|\right] = \mathbb{E}\left[\sqrt{\sum_{1 \leq i \leq d} |M_t^i|^2}\right] \leq \sqrt{\mathbb{E}\left[\sum_{1 \leq i \leq d} |M_t^i|^2\right]} = \mathbb{E}\left[\|M_t\|^2\right].
$$

Hence,

$$
\lim_{t \to 0^+} \mathbb{E}\left[\|M_t\|\right] = 0,
$$

and

$$
\lim_{t \to 0^+} \mathbb{E}\left[\|\xi_t - \xi_0\|\right] \leq \lim_{t \to 0^+} \mathbb{E}\left[\|A_t\|\right] + \lim_{t \to 0^+} \mathbb{E}\left[\|M_t\|\right] = 0.
$$

Going back to the inequalities (9), one obtains

$$
\lim_{t \to 0^+} g_1(t) = 0.
$$

Similarly, $\Delta_2(t) \to 0$ and $\Delta_3(t) \to 0$ as $t \to 0^+$. This ends the proof. $\quad\square$

Remark 2.1. *In applications where a process is constructed as the solution to a stochastic differential equation driven by a Brownian motion and a Poisson random measure, one usually starts from a representation of the form*

$$
\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int \psi_s(y)\, \tilde{N}(ds\, dy), \tag{11}
$$

where $\xi_0 \in \mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, β and δ are non-anticipative càdlàg processes, N is a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with intensity $\nu(dy)\, dt$ where ν is a Lévy measure

$$
\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu(dy) < \infty, \qquad \tilde{N} = N - \nu(dy)dt,
$$

and $\psi : [0,T] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is a predictable random function representing jump amplitude. This representation is different from (2), but as shown in [6, Lemma 1] one can switch from the representation (11) to the representation (2) in an explicit manner.

In particular, if one rewrites Assumption 2.1 in the framework of equation (11), one recovers the Assumptions of [21] as a special case.

2.2 Examples

If one has further information on the behavior of f in the neighborhood of ξ_0, then the quantity $L_0 f(\xi_0)$ may be computed more explicitly. We summarize some commonly encountered situations in the following Proposition.

Proposition 2.1. *Under Assumptions 2.1 and 2.2,*

1. *If $f(\xi_0) = 0$ and $\nabla f(\xi_0) = 0$, then*

$$\lim_{t \to 0^+} \frac{1}{t} \, \mathbb{E}\left[f(\xi_t)\right] = \frac{1}{2}\mathrm{tr}\left[{}^t\delta_0\delta_0 \, \nabla^2 f(\xi_0)\right] + \int_{\mathbb{R}^d} f(\xi_0 + y) \, m(0, dy). \quad (12)$$

2. *If furthermore $\nabla^2 f(\xi_0) = 0$, then*

$$\lim_{t \to 0^+} \frac{1}{t} \, \mathbb{E}\left[f(\xi_t)\right] = \int_{\mathbb{R}^d} f(\xi_0 + y) \, m(0, dy). \quad (13)$$

Proof. Applying Theorem 2.1, $\mathcal{L}_0 f(\xi_0)$ writes

$$\mathcal{L}_0 f(\xi_0) = \beta_0.\nabla f(\xi_0) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(\xi_0) \, {}^t\delta_0\delta_0\right](\xi_0)$$

$$+ \int_{\mathbb{R}^d} [f(\xi_0 + y) - f(\xi_0) - \frac{1}{1 + \|y\|^2} \, y.\nabla f(\xi_0)]m(0, dy).$$

The proposition follows immediately. $\qquad\qquad\square$

Remark 2.2. *As observed by Jacod [18, Section 5.8] in the setting of Lévy processes, if $f(\xi_0) = 0$ and $\nabla f(\xi_0) = 0$, then $f(x) = O(\|x - \xi_0\|^2)$. If furthermore $\nabla^2 f(\xi_0) = 0$, then $f(x) = o(\|x - \xi_0\|^2)$.*

Let us now compute in a more explicit manner the asymptotics of (1) for specific semimartingales.

2.2.1 Functions of a Markov process

An important situations which often arises in applications is when a stochastic processe ξ is driven by an underlying Markov process, i.e.

$$\xi_t = f(Z_t) \qquad f \in C^2(\mathbb{R}^d, \mathbb{R}), \quad (14)$$

where Z_t is a Markov process, defined as the weak solution on $[0, T]$ of a stochastic differential equation

$$Z_t = Z_0 + \int_0^t b(u, Z_{u-}) \, du + \int_0^t \Sigma(u, Z_{u-}) \, dW_u$$

$$+ \int_0^t \int \psi(u, Z_{u-}, y) \, \tilde{N}(du \ dy), \quad (15)$$

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where (W_t) is an n-dimensional Brownian motion, N is a Poisson random measure on $[0, T] \times \mathbb{R}^d$ with Lévy measure $\nu(y)\,dy$, $\tilde{N}$ the associated compensated random measure, $\Sigma : [0, T] \times \mathbb{R}^d \mapsto M_{d \times d}(\mathbb{R})$, $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\psi : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ are measurable functions such that

$$\psi(.,.,0) = 0 \qquad \psi(t, z, .) \text{ is a } \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d) - \text{diffeomorphism}$$

$$\forall t \in [0, T], \quad \mathbb{E}\left[\int_0^t \int_{\{\|y\| \geq 1\}} \sup_{z \in \mathbb{R}^d} \left(1 \wedge \|\psi(s, z, y)\|^2\right) \nu(y)\,dy\,ds \right] < \infty. \tag{16}$$

In this setting, as shown in Proposition [6, Proposition 3], one may verify the regularity of Assumption 2.1 and Assumption 2.2 by requiring easy-to-check assumptions on the coefficients:

Assumption 2.3. $b(.,.)$, $\Sigma(.,.)$ and $\psi(.,.,y)$ are continuous in a neighborhood of $(0, Z_0)$

Assumption 2.4. There exist $T > 0, R > 0$ such that

$$\text{Either} \qquad \forall t \in [0, T] \quad \inf_{\|z - Z_0\| \leq R} \inf_{x \in \mathbb{R}^d, \|x\| = 1} {}^t x . \Sigma(t, z) . x > 0$$

$$\text{or} \qquad \Sigma \equiv 0.$$

We then obtain the following result:

Proposition 2.2. Let $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$ such that

$$\forall (z_1, \cdots, z_{d-1}) \in \mathbb{R}^{d-1}, \quad u \mapsto f(z_1, \ldots, z_{d-1}, u) \text{ is a } \mathcal{C}^1(\mathbb{R}, \mathbb{R}) - \text{diffeomorphism}.$$

Define

$$\begin{cases} \beta_0 &= \nabla f(Z_0).b(0, Z_0) + \frac{1}{2}\text{tr}\left[\nabla^2 f(Z_0)^t \Sigma(0, Z_0)\Sigma(0, Z_0)\right] \\ &\quad + \int_{\mathbb{R}^d} \left(f(Z_0 + \psi(0, Z_0, y)) - f(Z_0) - \psi(0, Z_0, y).\nabla f(Z_0)\right) \nu(y)\,dy, \\ \delta_0 &= \|\nabla f(Z_0)\Sigma(0, Z_0)\|, \end{cases}$$

and the measure $m(0, .)$ via

$$m(0, [u, \infty[) = \int_{\mathbb{R}^d} 1_{\{f(Z_0 + \psi(0, Z_0, y)) - f(Z_0) \geq u\}} \nu(y)\,dy \quad u > 0,$$

$$m(0, [-\infty, u]) = \int_{\mathbb{R}^d} 1_{\{f(Z_0 + \psi(0, Z_0, y)) - f(Z_0) \leq u\}} \nu(y)\,dy \quad u < 0. \tag{17}$$

Under Assumptions 2.3 and 2.4, $\forall g \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$,

$$\lim_{t \to 0^+} \frac{\mathbb{E}\left[g(\xi_t)\right] - g(\xi_0)}{t} = \beta_0\,g'(\xi_0) + \frac{\delta_0^2}{2}\,g''(\xi_0) + \int_{\mathbb{R}^d} [g(\xi_0 + u) - g(\xi_0) - u g'(\xi_0)]\,m(0, du). \tag{18}$$

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Proof. Under the conditions (16) and the Assumption 2.4, [6, Proposition 3] shows that ξ_t admits the semimartingale decomposition

$$\xi_t = \xi_0 + \int_0^t \beta_s \, ds + \int_0^t \delta_s \, dB_s + \int_0^t \int u \, \tilde{K}(ds \, du),$$

where

$$\begin{cases} \beta_t &= \nabla f(Z_{t-}).b(t, Z_{t-}) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_{t-})^t \Sigma(t, Z_{t-})\Sigma(t, Z_{t-})\right] \\ &\quad + \int_{\mathbb{R}^d} \left(f(Z_{t-} + \psi(t, Z_{t-}, y)) - f(Z_{t-}) - \psi(t, Z_{t-}, y).\nabla f(Z_{t-})\right) \nu(y) \, dy, \\ \delta_t &= \|\nabla f(Z_{t-})\Sigma(t, Z_{t-})\|, \end{cases}$$

and K is an integer-valued random measure on $[0, T] \times \mathbb{R}$ with compensator $k(t, Z_{t-}, u) \, du \, dt$ defined via

$$\begin{aligned} k(t, Z_{t-}, u) &= \int_{\mathbb{R}^{d-1}} |\det \nabla_y \Phi(t, Z_{t-}, (y_1, \cdots, y_{d-1}, u))| \\ &\qquad\qquad \nu(\Phi(t, Z_{t-}, (y_1, \cdots, y_{d-1}, u))) \, dy_1 \cdots dy_{d-1}, \end{aligned}$$

with

$$\begin{cases} \Phi(t, z, y) = \phi(t, z, \kappa_z^{-1}(y)) & \kappa_z^{-1}(y) = (y_1, \cdots, y_{d-1}, F_z(y)), \\ F_z(y) : \mathbb{R}^d \to \mathbb{R} & f(z + (y_1, \cdots, y_{d-1}, F_z(y))) - f(z) = y_d. \end{cases}$$

From Assumption 2.3 it follows that Assumptions 2.1 and 2.2 hold for β_t, δ_t and $k(t, Z_t-, .)$ on $[0, T]$. Applying Theorem 2.1, the result follows immediately. $\square$

Remark 2.3. *Benhamou et al. [4] studied the case where Z_t is the solution of a 'Markovian' SDE whose jumps are given by a compound Poisson Process. The above results generalizes their result to the (general) case where the jumps are driven by an arbitrary integer-valued random measure.*

2.2.2 Time-changed Lévy processes

Models based on time–changed Lévy processes provide another class of examples of non-Markovian models which have generated recent interest in mathematical finance. Let L_t be a real-valued Lévy process, (b, σ^2, ν) be its characteristic triplet, N its jump measure. Define

$$\xi_t = L_{\Theta_t} \qquad \Theta_t = \int_0^t \theta_s ds, \tag{19}$$

where (θ_t) is a locally bounded $\mathcal{F}_t$-adapted positive càdlàg process, interpreted as the rate of time change.

Proposition 2.3. *If*

$$\int_{\mathbb{R}} |y|^2 \, \nu(dy) < \infty \quad \text{and} \quad \lim_{t \to 0, \, t > 0} \mathbb{E}\left[|\theta_t - \theta_0|\right] = 0 \tag{20}$$

then

$$\forall f \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R}), \qquad \lim_{t \to 0^+} \frac{\mathbb{E}\left[f(\xi_t)\right] - f(\xi_0)}{t} = \theta_0 \, \mathcal{L}_0 f(\xi_0) \tag{21}$$

where $\mathcal{L}_0$ is the infinitesimal generator of the L:

$$\mathcal{L}_0 f(x) = b \, f'(x) + \frac{\sigma^2}{2} \, f''(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1+|y|^2} \, y f'(x)] \nu(dy). \tag{22}$$

Proof. Considering the Lévy-Itô decomposition of L:

$$
\begin{aligned}
L_t &= \left(b - \int_{\{|y| \leq 1\}} (y - \kappa(y)) \, \nu(dy) \right) t + \sigma W_t \\
&\quad + \int_0^t \int_{\mathbb{R}} \kappa(z) \tilde{N}(ds\,dz) + \int_0^t \int_{\mathbb{R}} (z - \kappa(z)) \, N(ds\,dz),
\end{aligned}
$$

then ξ has the representation

$$
\begin{aligned}
\xi_t &= \xi_0 + \int_0^t \sigma \sqrt{\theta_s} \, dZ_s + \int_0^t \left(b - \int_{\{|y| \leq 1\}} (y - \kappa(y)) \, \nu(dy) \right) \theta_s \, ds \\
&\quad + \int_0^t \int_{\mathbb{R}} \kappa(z) \theta_s \, \tilde{N}(ds\,dz) + \int_0^t \int_{\mathbb{R}} (z - \kappa(z)) \, \theta_s \, N(ds\,dz).
\end{aligned}
$$

where Z is a Brownian motion. With the notation of equation (2), one identifies

$$\beta_t = \left(b - \int_{\{|y| \leq 1\}} (y - \kappa(y)) \, \nu(dy) \right) \theta_t, \quad \delta_t = \sigma \sqrt{\theta_t}, \quad m(t, dy) = \theta_t \, \nu(dy).$$

If (20) holds, then Assumptions 2.1 and 2.2 hold for (β, δ, m) and Theorem 2.1 may be applied to obtain the result. $\qquad \square$

3   Short-maturity asymptotics for call options

Consider a (strictly positive) price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by a stochastic volatility model with jumps:

$$S_t = S_0 + \int_0^t r(s) S_{s^-} \, ds + \int_0^t S_{s^-} \delta_s dW_s + \int_0^t \int_{-\infty}^{+\infty} S_{s^-} (e^y - 1) \tilde{M}(ds \, dy), \tag{23}$$

where $r(t) > 0$ represents a (deterministic) bounded discount rate. For convenience, we shall assume that $r \in \mathcal{C}_0^b(\mathbb{R}^+, \mathbb{R}^+)$. δ_t represents the volatility process and M is an integer-valued random measure with compensator $\mu(\omega; dt \, dy) = m(\omega; t, dy) \, dt$, representing jumps in the log-price, and $\tilde{M} = M - \mu$ its compensated random measure. We make the following assumptions on the characteristics of S:

Assumption 3.1 (Right-continuity at t_0).

$$\lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[|\delta_t - \delta_{t_0}|^2 | \mathcal{F}_{t_0}\right] = 0.$$

For all $\varphi \in \mathcal{C}_0^b(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$,

$$\lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[\int_{\mathbb{R}} \left(e^{2y} \wedge |y|^2\right) \varphi(S_t, y)\, m(t, dy) | \mathcal{F}_{t_0}\right] = \int_{\mathbb{R}} \left(e^{2y} \wedge |y|^2\right) \varphi(S_{t_0}, y)\, m(t_0, dy).$$

Assumption 3.2 (Integrability condition).

$$\exists T > t_0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_0}^{T} \delta_s^2\, ds + \int_{t_0}^{T} ds \int_{\mathbb{R}} (e^y - 1)^2 m(s, dy)\right) | \mathcal{F}_{t_0}\right] < \infty \quad .$$

We recall that the value $C_{t_0}(t, K)$ at time t_0 of a call option with expiry $t > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(t, K) = e^{-\int_{t_0}^{t} r(s)\, ds} \mathbb{E}[\max(S_t - K, 0) | \mathcal{F}_{t_0}]. \tag{24}$$

The discounted asset price

$$\hat{S}_t = e^{-\int_{t_0}^{t} r(u)\, du} S_t,$$

is the stochastic exponential of the martingale ξ defined by

$$\xi_t = \int_0^t \delta_s\, dW_s + \int_0^t \int (e^y - 1)\tilde{M}(ds\, dy).$$

Under Assumption 3.2, we have

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle \xi, \xi \rangle_T^d + \langle \xi, \xi \rangle_T^c\right)\right] < \infty,$$

where $\langle \xi, \xi \rangle^c$ and $\langle \xi, \xi \rangle^d$ denote the continuous and purely discontinuous parts of $[\xi, \xi]$ and [22, Theorem 9] implies that $(\hat{S}_t)_{t \in [t_0, T]}$ is a $\mathbb{P}$-martingale. In particular the expectation in (24) is finite.

3.1 Out-of-the money call options

We first study the asymptotics of out-of-the money call options i.e. the case where $K > S_{t_0}$.

Theorem 3.1 (Short-maturity behavior of out-of-the money options). *Under Assumption 3.1 and Assumption 3.2, if $S_{t_0} < K$ then*

$$\frac{1}{t - t_0} C_{t_0}(t, K) \xrightarrow[t \to t_0^+]{} \int_0^{\infty} (S_{t_0} e^y - K)_+ m(t_0, dy). \tag{25}$$

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This limit can also be expressed using the exponential double tail ψ_{t_0} of the compensator, defined as

$$\psi_{t_0}(z) = \int_z^{+\infty} dx \; e^x \int_x^\infty m(t_0, du) \quad z > 0. \tag{26}$$

Then, as shown in [5, Lemma 1],

$$\int_0^\infty (S_{t_0} e^y - K)_+ m(t_0, dy) = S_{t_0} \psi_{t_0}\left(\ln\left(\frac{K}{S_{t_0}}\right)\right).$$

Proof. The idea is to apply Theorem 2.1 to smooth approximations f_n of the function $x \to (x - K)^+$ and conclude using a dominated convergence argument.

First, as argued in the proof of Theorem 2.1, we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. Applying the Itô formula to $X_t \equiv \ln(S_t)$, we obtain

$$
\begin{aligned}
X_t &= \ln(S_0) + \int_0^t \frac{1}{S_{s-}} dS_s + \frac{1}{2} \int_0^t \frac{-1}{S_{s-}^2} (S_{s-}\delta_s)^2 \, ds \\
&\quad + \sum_{s \le t} \left[\ln(S_{s-} + \Delta S_s) - \ln(S_{s-}) - \frac{1}{S_{s-}} \Delta S_s \right] \\
&= \ln(S_0) + \int_0^t \left(r(s) - \frac{1}{2} \delta_s^2 \right) ds + \int_0^t \delta_s \, dW_s \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} (e^y - 1) \tilde{M}(ds \, dy) - \int_0^t \int_{-\infty}^{+\infty} (e^y - 1 - y) \, M(ds \, dy)
\end{aligned}
$$

Note that there exists $C > 0$ such that

$$\left| e^y - 1 - y \frac{1}{1 + |y|^2} \right| \le C \left(e^y - 1 \right)^2.$$

Thanks to Jensen's inequality, Assumption 3.2 implies that this quantity is

finite, allowing us to write

$$\int_0^t \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} (e^y - 1 - y)\,M(ds\,dy)$$

$$= \int_0^t \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\frac{1}{1+|y|^2}\right)M(ds\,dy)$$

$$+ \int_0^t \int_{-\infty}^{+\infty} \left(y - y\frac{1}{1+|y|^2}\right)M(ds\,dy)$$

$$= \int_0^t \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\frac{1}{1+|y|^2}\right)\tilde{M}(ds\,dy)$$

$$- \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\frac{1}{1+|y|^2}\right)m(s,y)\,ds\,dy + \int_0^t \int_{-\infty}^{+\infty} \left(y - y\frac{1}{1+|y|^2}\right)M(ds\,dy)$$

$$= \int_0^t \int_{-\infty}^{+\infty} y\frac{1}{1+|y|^2}\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\frac{1}{1+|y|^2}\right)m(s,y)\,ds\,dy$$

$$+ \int_0^t \int_{-\infty}^{+\infty} \left(y - y\frac{1}{1+|y|^2}\right)M(ds\,dy).$$

We can thus represent X_t as in (2)):

$$X_t = X_0 + \int_0^t \beta_s\,dt + \int_0^t \delta_s\,dW_s$$

$$+ \int_0^t \int_{-\infty}^{+\infty} y\frac{1}{1+|y|^2}\tilde{M}(ds\,dy) + \int_0^t \int_{-\infty}^{+\infty} \left(y - y\frac{1}{1+|y|^2}\right)M(ds\,dy),$$

$$(27)$$

with

$$\beta_t = r(t) - \frac{1}{2}\delta_t^2 - \int_{-\infty}^{\infty} \left(e^y - 1 - y\frac{1}{1+|y|^2}\right)m(t,y)\,dt\,dy.$$

Hence, if δ and $m(.,dy)$ satisfy Assumption 3.1 then β, δ and $m(.,dy)$ satisfy Assumption 2.1. Thanks to Jensen's inequality, Assumption 3.2 implies that β, δ and m satisfy Assumption 2.2. One may apply Theorem 2.1 to X_t for any function of the form $f \circ exp$, $f \in \mathcal{C}_b^2(\mathbb{R},\mathbb{R})$. Let us introduce a family $f_n \in \mathcal{C}_b^2(\mathbb{R},\mathbb{R})$ such that

$$\begin{cases} f_n(x) = (x-K)^+ & |x-K| > \frac{1}{n} \\ (x-K)^+ \le f_n(x) \le \frac{1}{n} & |x-K| \le \frac{1}{n}. \end{cases}$$

Then for $x \ne K$, $f_n(x) \underset{n\to\infty}{\longrightarrow} (x-K)^+$. Define, for $f \in C_0^\infty(\mathbb{R}^+,\mathbb{R})$,

$$\mathcal{L}_0 f(x) = r(0)xf'(x) + \frac{x^2\delta_0^2}{2}f''(x)$$

$$+ \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1).f'(x)]m(0,dy).$$

$$(28)$$

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First, observe that if $N_1 \geq 1/|S_0 - K|$,

$$\forall n \geq N_1, f_n(S_0) = (S_0 - K)^+ = 0, \quad \text{so}$$

$$\frac{1}{t}\mathbb{E}\left[(S_t - K)^+\right] \leq \frac{1}{t}\mathbb{E}\left[f_n(S_t)\right] = \frac{1}{t}\left(\mathbb{E}\left[f_n(S_t)\right] - f_n(S_0)\right).$$

Letting $t \to 0^+$ yields

$$\limsup_{t \to 0^+} \frac{1}{t} e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[(S_t - K)^+\right] \leq \mathcal{L}_0 f_n(S_0). \tag{29}$$

Furthermore,

$$
\begin{aligned}
\mathbb{E}\left[(S_t - K)^+\right] &\geq \mathbb{E}\left[f_n(S_t)\mathbf{1}_{\{|S_t - K| > \frac{1}{n}\}}\right] \\
&= \mathbb{E}\left[f_n(S_t)\right] - \mathbb{E}\left[f_n(S_t)\mathbf{1}_{\{|S_t - K| \leq \frac{1}{n}\}}\right] \\
&\geq \mathbb{E}\left[f_n(S_t)\right] - f_n(S_0) - \frac{1}{n}\mathbb{E}\left[\mathbf{1}_{\{|S_t - K| \leq \frac{1}{n}\}}\right].
\end{aligned}
$$

But

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{|S_t - K| \leq \frac{1}{n}\}}\right] &\leq \mathbb{P}\left(S_t - K \geq -\frac{1}{n}\right) \\
&\leq \mathbb{P}\left(S_t - S_0 \geq K - S_0 - \frac{1}{n}\right).
\end{aligned}
$$

There exists $N_2 \geq 0$ such that for all $n \geq N_2$,

$$
\begin{aligned}
\mathbb{P}\left(S_t - S_0 \geq K - S_0 - \frac{1}{n}\right) &\leq \mathbb{P}\left(S_t - S_0 \geq \frac{K - S_0}{2}\right) \\
&\leq \left(\frac{2}{K - S_0}\right)^2 \mathbb{E}\left[(S_t - S_0)^2\right],
\end{aligned}
$$

by the Bienaymé-Chebyshev inequality. Hence,

$$\frac{1}{t}\mathbb{E}\left[(S_t - K)^+\right] \geq \frac{1}{t}\left(\mathbb{E}\left[f_n(S_t)\right] - f_n(S_0)\right) - \frac{1}{n}\left(\frac{2}{K - S_0}\right)^2 \frac{1}{t}\mathbb{E}\left[\phi(S_t) - \phi(S_0)\right],$$

with $\phi(x) = (x - S_0)^2$. Applying Theorem 2.1 yields

$$\liminf_{t \to 0^+} \frac{1}{t} e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[(S_t - K)^+\right] \geq \mathcal{L}_0 f_n(S_0) - \frac{1}{n}\left(\frac{2}{K - S_0}\right)^2 \mathcal{L}_0 \phi(S_0).$$

Letting $n \to +\infty$,

$$\lim_{t \to 0^+} \frac{1}{t} e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[(S_t - K)^+\right] = \lim_{n \to \infty} \mathcal{L}_0 f_n(S_0).$$

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Since $S_0 < K$, $f_n = 0$ in a neighborhood of S_0 for $n \geq N_1$ so $f_n(S_0) = f_n''(S_0) = f_n'(S_0) = 0$ and $\mathcal{L}_0 f_n(S_0)$ reduces to

$$\mathcal{L}_0 f_n(S_0) = \int_{\mathbb{R}} [f_n(S_0 e^y) - f_n(S_0)] m(0, dy).$$

A dominated convergence argument then yields

$$\lim_{n \to \infty} \mathcal{L}_0 f_n(S_0) = \int_{\mathbb{R}} [(S_0 e^y - K)_+ - (S_0 - K)_+] m(0, dy).$$

Using integration by parts, this last expression may be rewritten [5, Lemma 1] as

$$S_0 \psi_0 \left(\ln \left(\frac{K}{S_0} \right) \right)$$

where ψ_0 is given by (26). This ends the proof.

$\square$

Remark 3.1. *Theorem 3.1 also applies to in-the-money options, with a slight modification: for $K < S_{t_0}$,*

$$\frac{1}{t - t_0} \left(C_{t_0}(t, K) - (S_{t_0} - K) \right) \xrightarrow[t \to t_0^+]{} r(t_0) S_{t_0} + S_{t_0} \psi_{t_0} \left(\ln \left(\frac{K}{S_{t_0}} \right) \right), \qquad (30)$$

where

$$\psi_{t_0}(z) = \int_{-\infty}^{z} dx \ e^x \int_{-\infty}^{x} m(t_0, du), \qquad \text{for} \quad z < 0 \qquad (31)$$

denotes the exponential double tail of $m(0, .)$.

3.2   At-the-money call options

When $S_{t_0} = K$, Theorem 3.1 does not apply. Indeed, as already noted in the case of Lévy processes by Tankov [26] and Figueroa-Lopez and Forde [13], the short maturity behavior of at-the-money options depends on whether a continuous martingale component is present and, in absence of such a component, on the degree of activity of small jumps, measured by the Blumenthal-Getoor index of the Lévy measure which measures its singularity at zero [18]. We will show here that similar results hold in the semimartingale case. We distinguish three cases:

1. S is a pure jump process of finite variation: in this case at-the-money call options behave linearly in $t - t_0$ (Proposition 3.1).

2. S is a pure jump process of infinite variation and its small jumps resemble those of an α-stable process: in this case at-the-money call options have an asymptotic behavior of order $|t - t_0|^{1/\alpha}$ as $t - t_0 \to 0^+$ (Proposition 3.2).

3. S has a continuous martingale component which is non-degenerate in the neighborhood of t_0: in this case at-the-money call options are of order $\sqrt{t - t_0}$ as $t \to t_0^+$, whether or not jumps are present (Theorem 3.2).

These statements are made precise in the sequel. We observe that, contrarily to the case of out-of-the money options where the presence of jumps dominates the asymptotic behavior, for at-the-money options the presence or absence of a continuous martingale (Brownian) component dominates the asymptotic behavior.

For the finite variation case, we use a slightly modified version of Assumption 3.1:

Assumption 3.3 (Weak right-continuity of jump compensator). *For all $\varphi \in \mathcal{C}_0^b(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$,*

$$\lim_{t \to t_0, \, t > t_0} \mathbb{E}\left[\int_{\mathbb{R}} \left(e^{2y} \wedge |y| \right) \varphi(S_t, y) \, m(t, dy) | \mathcal{F}_{t_0} \right] = \int_{\mathbb{R}} \left(e^{2y} \wedge |y| \right) \varphi(S_{t_0}, y) \, m(t_0, dy).$$

Proposition 3.1 (Asymptotic for ATM call options for pure jump processes of finite variation). *Consider the process*

$$S_t = S_0 + \int_0^t r(s) S_{s-} \, ds + \int_0^t \int_{-\infty}^{+\infty} S_{s-}(e^y - 1)\tilde{M}(ds \, dy). \qquad (32)$$

Under the Assumptions 3.3 and 3.2 and the condition,

$$\forall t \in [t_0, T], \quad \int_{\mathbb{R}} |y| \, m(t, dy) < \infty,$$

$$\frac{1}{t - t_0} C_{t_0}(t, S_{t_0}) \underset{t \to t_0^+}{\longrightarrow}$$

$$1 \left\{ r(t_0) > \int_{\mathbb{R}} (e^y - 1) m(t_0, dy) \right\} S_{t_0} \left(r(t_0) + \int_{\mathbb{R}} (1 - e^y)^+ m(t_0, dy) \right) \qquad (33)$$

$$+ 1 \left\{ r(t_0) \leq \int_{\mathbb{R}} (e^y - 1) m(t_0, dy) \right\} S_{t_0} \int_{\mathbb{R}} (e^y - 1)^+ m(t_0, dy).$$

Proof. Replacing $\mathbb{P}$ by the conditional probability $\mathbb{P}_{\mathcal{F}_{t_0}}$, we may set $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. The Tanaka-Meyer formula applied to $(S_t - S_0)^+$ gives

$$(S_t - S_0)^+ = \int_0^t ds \, 1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1) \, m(s, dy) \right)$$

$$+ \sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+.$$

Hence, applying Fubini's theorem,

$$
\begin{aligned}
\mathbb{E}\left[(S_t - S_0)^+\right] &= \mathbb{E}\left[\int_0^t ds\, 1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy)\right)\right] \\
&+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \left[(S_{s-} e^y - S_0)^+ - (S_{s-} - S_0)^+\right] m(s, dy)\, ds\right] \\
&= \int_0^t ds\, \mathbb{E}\left[1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy)\right)\right] \\
&+ \int_0^t ds\, \mathbb{E}\left[\int_{\mathbb{R}} \left[(S_s e^y - S_0)^+ - (S_s - S_0)^+\right] m(s, dy)\right].
\end{aligned}
$$

Furthermore, under the Assumptions 2.1 and 2.2 for $X_t = \log(S_t)$ (see equation (27)), one may apply Theorem 2.1 to the function

$$
f : x \in \mathbb{R} \mapsto \exp(x),
$$

yielding

$$
\lim_{t \to 0^+} \frac{1}{t} \mathbb{E}\left[S_t - S_0\right] = S_0 \left(r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy)\right).
$$

Furthermore, observing that

$$
S_t\, 1_{\{S_t > S_0\}} = (S_t - S_0)^+ + S_0\, 1_{\{S_t > S_0\}},
$$

we write

$$
\begin{aligned}
\mathbb{E}\left[S_t\, 1_{\{S_t > S_0\}}\right] &= \mathbb{E}\left[(S_t - S_0)^+ + S_0\, 1_{\{S_t > S_0\}}\right] \\
&\leq \mathbb{E}\left[|S_t - S_0|\right] + S_0\, \mathbb{P}\left(S_t > S_0\right),
\end{aligned}
$$

using the Lipschitz continuity of $x \mapsto (x - S_0)_+$. Since $t \to \mathbb{E}\left[S_t\right]$ is right-continuous at 0, for t small enough :

$$
0 \leq \mathbb{E}\left[S_t\, 1_{\{S_t > S_0\}}\right] \leq \frac{1}{2} + S_0.
$$

Thus,

$$
\begin{aligned}
&\lim_{t \to 0^+} \frac{1}{t} \mathbb{E}\left[\int_0^t ds\, 1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy)\right)\right] \\
&= S_0 \left(r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy)\right) 1\{r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy) > 0\}.
\end{aligned}
$$

Let us now focus on the jump term and show that

$$
t \in [0, T[\mapsto \mathbb{E}\left[\int_{\mathbb{R}} \left[(S_t e^y - S_0)^+ - (S_t - S_0)^+\right] m(t, dy)\right],
$$

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is right-continuous at 0 with right-limit

$$S_0 \int_{\mathbb{R}} (e^y - 1)^+ m(0, dy).$$

One shall simply observes that

$$\left| (xe^y - S_0)^+ - (x - S_0)^+ - (S_0 e^y - S_0)^+ \right| \le (x + S_0) \left| e^y - 1 \right|,$$

using the Lipschitz continuity of $x \mapsto (x - S_0)_+$ and apply Assumption 3.3. To finish the proof, one shall gather both limits together :

$$
\begin{aligned}
= \quad & S_0 \left(r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy) \right) 1 \left\{ r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy) > 0 \right\} \\
+ \quad & S_0 \int_{\mathbb{R}} (e^y - 1)^+ m(0, dy).
\end{aligned}
$$

This ends the proof.

$\square$

Proposition 3.2 (Asymptotics of ATM call options for pure-jump martingales of infinite variation). *Consider a semimartingale whose continuous martingale part is zero:*

$$S_t = S_0 + \int_0^t r(s) S_{s-} \, ds + \int_0^t \int_{-\infty}^{+\infty} S_{s-} (e^y - 1) \tilde{M}(ds \ dy). \tag{34}$$

Under the Assumptions 3.1 and 3.2, if there exists $\alpha \in]1, 2[$ and a family $m^\alpha(t, dy)$ of positive measures such that

$$\forall t \in [t_0, T], \quad m(\omega, t, dy) = m^\alpha(\omega, t, dy) + 1_{|y| \le 1} \frac{c(y)}{|y|^{1+\alpha}} \, dy \text{ a.s.,} \tag{35}$$

where $c(.) > 0$ is continuous at 0 and

$$\forall t \in [t_0, T] \quad \int_{\mathbb{R}} |y| \, m^\alpha(t, dy) < \infty, \tag{36}$$

then

$$\frac{1}{(t - t_0)^{1/\alpha}} C_{t_0}(t, S_{t_0}) \xrightarrow[t \to t_0^+]{} S_{t_0} \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-c(0) \, |z|^\alpha} - 1}{z^2} \, dz. \tag{37}$$

Proof. Without loss of generality, we set $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. The at-the-money call price can be expressed as

$$C_0(t, S_0) = \mathbb{E}\left[(S_t - S_0)^+ \right] = S_0 \mathbb{E}\left[\left(\frac{S_t}{S_0} - 1 \right)^+ \right]. \tag{38}$$

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Define, for $f \in C_b^2(]0, \infty[, \mathbb{R})$

$$\mathcal{L}_0 f(x) = r(0) x f'(x) + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1).f'(x)] m(0, dy). \quad (39)$$

We decompose $\mathcal{L}_0$ as the sum $\mathcal{L}_0 = \mathcal{K}_0 + \mathcal{J}_0$ where

$$\mathcal{K}_0 f(x) = r(0) x f'(x) + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1).f'(x)] \, m^\alpha(0, dy),$$

$$\mathcal{J}_0 f(x) = \int_{-1}^{1} [f(xe^y) - f(x) - x(e^y - 1).f'(x)] \, \frac{c(y)}{|y|^{1+\alpha}} \, dy.$$

The term $\mathcal{K}_0$ may be be interpreted in terms of Theorem 2.1: if $(Z_t)_{[0,T]}$ is a *finite variation* semimartingale of the form (34) starting from $Z_0 = S_0$ with jump compensator $m^\alpha(t, dy)$, then by Theorem 2.1,

$$\forall f \in C_b^2(]0, \infty[, \mathbb{R}), \qquad \lim_{t \to 0^+} \frac{1}{t} e^{-\int_0^t r(s)\,ds} \mathbb{E}[f(Z_t)] = \mathcal{K}_0 f(S_0). \quad (40)$$

The idea is now to interpret $\mathcal{L}_0 = \mathcal{K}_0 + \mathcal{J}_0$ in terms of a *multiplicative decomposition* $S_t = Y_t Z_t$ where $Y = \mathcal{E}(L)$ is the stochastic exponential of a pure-jump Lévy process with Lévy measure $c(y)/|y|^{1+\alpha} \, dy$, which we can take independent from Z. Indeed, let $Y = \mathcal{E}(L)$ where L is a pure-jump Lévy martingale with Lévy measure $1_{|y| \le 1} \, c(y)/|y|^{1+\alpha} \, dy$, independent from Z, with infinitesimal generator $\mathcal{J}_0$. Then Y is a martingale and $[Y, Z] = 0$. Then $S = YZ$ and Y is an exponential Lévy martingale, independent from Z, with $E[Y_t] = 1$.

A result of Tankov [26, Proposition 5, Proof 2] for exponential Lévy processes then implies that

$$\frac{1}{t^{1/\alpha}} \, \mathbb{E}\left[(Y_t - 1)^+\right] \overset{t \to 0^+}{\to} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-c(0)|z|^\alpha} - 1}{z^2} \, dz. \quad (41)$$

We will show that the term (41) is the dominant term which gives the asymptotic behavior of $C_0(T, S_0)$.

Indeed, by the Lipschitz continuity of $x \mapsto (x - S_0)_+$,

$$|(S_t - S_0)_+ - S_0(Y_t - 1)_+| \le Y_t |Z_t - S_0|,$$

so, taking expectations and using that Y is independent from Z, we get

$$\underbrace{\mathbb{E}[e^{-\int_0^t r(s)\,ds} |(S_t - S_0)_+}_{C_0(t, S_0)} - S_0(Y_t - 1)_+|] \le \underbrace{\mathbb{E}(Y_t)}_{=1} \mathbb{E}[e^{-\int_0^t r(s)\,ds} |Z_t - S_0|].$$

To estimate the right hand side of this inequality note that $|Z_t - S_0| = (Z_t - S_0)_+ + (S_0 - Z_t)_+$. Since Z has finite variation, from Proposition 3.1

$$E[e^{-\int_0^t r(s)\,ds} (Z_t - S_0)_+] \overset{t \to 0^+}{\sim} t S_0 \int_0^\infty dx \, e^x m([x, +\infty[).$$

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Using the martingale property of $e^{-\int_0^t r(s)\,ds}Z_t$) yields

$$E[e^{-\int_0^t r(s)\,ds}(S_0 - Z_t)_+] \overset{t\to 0^+}{\sim} tS_0 \int_0^\infty dx\ e^x m([x, +\infty[).$$

Hence, dividing by $t^{1/\alpha}$ and taking $t \to 0^+$ we obtain

$$\frac{1}{t^{1/\alpha}}\, e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[|Z_t - S_0|^+\right] \overset{t\to 0^+}{\to} 0.$$

Thus, dividing by $t^{1/\alpha}$ the above inequality and using (41) yields

$$\frac{1}{t^{1/\alpha}}\, e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[(S_t - S_0)_+\right] \overset{t\to 0^+}{\to} S_0\, \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-c(0)|z|^\alpha} - 1}{z^2}\, dz.$$

$\square$

We now focus on a third case, when S is a continuous semimartingale, i.e. an Ito process. From known results in the diffusion case [8], we expect in this case a short-maturity behavior ins $O(\sqrt{t})$. We propose here a proof of this behavior in a semimartingale setting using the notion of semimartingale local time.

Proposition 3.3 (Asymptotic for at-the-money options for continuous semi-martingales). *Consider the process*

$$S_t = S_0 + \int_0^t r(s)S_s ds + \int_0^t S_s \delta_s dW_s. \tag{42}$$

Under the Assumptions 3.1 and 3.2 and the following non-degeneracy condition in the neighborhood of t_0,

$$\exists \epsilon > 0, \qquad \mathbb{P}\left(\forall t \in [t_0, T], \qquad \delta_t \geq \epsilon\right) = 1,$$

we have

$$\frac{1}{\sqrt{t - t_0}}\, C_{t_0}(t, S_{t_0}) \underset{t\to t_0^+}{\longrightarrow} \frac{S_{t_0}}{\sqrt{2\pi}}\, \delta_{t_0}. \tag{43}$$

Proof. Set $t_0 = 0$ and consider, without loss of generality, the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. Applying the Tanaka-Meyer formula to $(S_t - S_0)^+$, we have

$$(S_t - S_0)^+ = \int_0^t 1_{\{S_s > S_0\}} dS_s + \frac{1}{2}L_t^{S_0}(S).$$

where $L_t^{S_0}(S)$ corresponds to the semimartingale local time of S_t at level S_0 under $\mathbb{P}$. As noted in Section 3.1, Assumption 3.2 implies that the discounted price $\hat{S}_t = e^{-\int_0^t r(s)\,ds}S_t$ is a $\mathbb{P}$-martingale. So

$$dS_t = e^{\int_0^t r(s)\,ds}\left(r(t)S_t dt + d\hat{S}_t\right), \quad \text{and}$$

$$\int_0^t 1_{\{S_s > S_0\}} dS_s = \int_0^t e^{\int_0^s r(u)\, du}\, 1_{\{S_s > S_0\}} d\hat{S}_s + \int_0^t e^{\int_0^s r(u)\, du}\, r(s) S_s 1_{\{S_s > S_0\}} ds,$$

where the first term is a martingale. Taking expectations, we get:

$$C(t, S_0) = \mathbb{E}\left[e^{-\int_0^t r(s)\, ds} \int_0^t e^{\int_0^s r(u)\, du}\, r(s) S_s\, 1_{\{S_s > S_0\}} ds + \frac{1}{2} e^{-\int_0^t r(s)\, ds}\, L_t^{S_0}(S) \right].$$

Since $\hat{S}$ is a martingale,

$$\forall t \in [0, T] \quad \mathbb{E}\left[S_t \right] = e^{\int_0^t r(s)\, ds}\, S_0 < \infty. \tag{44}$$

Hence $t \to \mathbb{E}\left[S_t \right]$ is right-continuous at 0:

$$\lim_{t \to 0^+} \mathbb{E}\left[S_t \right] = S_0. \tag{45}$$

Furthermore, under the Assumptions 2.1 and 2.2 for $X_t = \log(S_t)$ (see equation (27)), one may apply Theorem 2.1 to the function

$$f : x \in \mathbb{R} \mapsto (\exp(x) - S_0)^2,$$

yielding

$$\lim_{t \to 0^+} \frac{1}{t} \mathbb{E}\left[(S_t - S_0)^2 \right] = \mathcal{L}_0 f(X_0),$$

where $\mathcal{L}_0$ is defined via equation (5) with $m \equiv 0$. Since $\mathcal{L}_0 f(X_0) < \infty$, then in particular,

$$t \mapsto \mathbb{E}\left[(S_t - S_0)^2 \right]$$

is right-continuous at 0 with right limit 0. Observing that

$$S_t 1_{\{S_t > S_0\}} = (S_t - S_0)^+ + S_0 1_{\{S_t > S_0\}},$$

we write

$$\mathbb{E}\left[S_t 1_{\{S_t > S_0\}} \right] = \mathbb{E}\left[(S_t - S_0)^+ + S_0 1_{\{S_t > S_0\}} \right]$$
$$\leq \mathbb{E}\left[|S_t - S_0| \right] + S_0 \mathbb{P}\left(S_t > S_0 \right),$$

using the Lipschitz continuity of $x \mapsto (x - S_0)_+$. Since $t \to \mathbb{E}\left[S_t \right]$ is right-continuous at 0, for t small enough :

$$0 \leq \mathbb{E}\left[S_t 1_{\{S_t > S_0\}} \right] \leq \frac{1}{2} + S_0.$$

Thus, applying Fubini,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\, du}\, r(s) S_s\, 1_{\{S_s > S_0\}} ds \right] = O(t),$$

a fortiori,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du} r(s)S_s 1_{\{S_s > S_0\}} ds\right] = o\left(\sqrt{t}\right).$$

Hence (if the limit exists)

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} C(t, S_0) = \lim_{t\to 0} \frac{1}{\sqrt{t}} e^{-\int_0^t r(s)\,ds} \mathbb{E}\left[\frac{1}{2} L_t^{S_0}(S)\right] = \lim_{t\to 0} \frac{1}{\sqrt{t}} \mathbb{E}\left[\frac{1}{2} L_t^{S_0}(S)\right].$$

(46)

By the Dubins-Schwarz theorem [24, Theorem 1.5], there exists a Brownian motion B such that

$$\forall t < [U]_\infty, \quad U_t = \int_0^t \delta_s\, dW_s = B_{[U]_t} = B_{\int_0^t \delta_s^2 ds}.$$

$$\text{So} \quad \forall t < [U]_\infty \quad S_t = S_0 \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\delta_s^2\right) ds + B_{[U]_t}\right)$$

$$= S_0 \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\delta_s^2\right) ds + B_{\int_0^t \delta_s^2\, ds}\right).$$

The occupation time formula then yields, for $\phi \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R})$,

$$\int_0^\infty \phi(K)L_t^K\left(S_0 \exp\left(B_{[U]}\right)\right) dK = \int_0^t \phi\left(S_0 \exp\left(B_{[U]_u}\right)\right) S_0^2 \exp\left(B_{[U]_u}\right)^2 \delta_u^2\, du$$

$$= \int_{-\infty}^\infty \phi(S_0 \exp(y))S_0^2 \exp(y)^2 L_t^y\left(B_{[U]}\right) dy,$$

where $L_t^K\left(S_0 \exp\left(B_{[U]}\right)\right)$ (resp. $L_t^y\left(B_{[U]}\right)$) denotes the semimartingale local time of the process $S_0 \exp\left(B_{[U]}\right)$ at K and (resp. $B_{[U]}$ at y). A change of variable leads to

$$\int_{-\infty}^\infty \phi(S_0 \exp(y))S_0 \exp(y)L_t^{S_0 e^y}\left(S_0 \exp\left(B_{[U]}\right)\right) dy$$

$$= \int_{-\infty}^\infty \phi(S_0 \exp(y))S_0^2 \exp(y)^2 L_t^y\left(B_{[U]}\right).$$

Hence

$$L_t^{S_0}\left(S_0 \exp\left(B_{[U]}\right)\right) = S_0 L_t^0\left(B_{[U]}\right).$$

We also have

$$L_t^0\left(B_{[U]}\right) = L_{\int_0^t \delta_s^2\, ds}^0(B),$$

where $L_{\int_0^t \delta_s^2\, ds}^0(B)$ denotes the semimartingale local time of B at time $\int_0^t \delta_s^2\, ds$ and level 0. Using the scaling property of Brownian motion,

$$\mathbb{E}\left[L_t^{S_0}\left(S_0 \exp\left(B_{[U]}\right)\right)\right] = S_0 \mathbb{E}\left[L_{\int_0^t \delta_s^2\, ds}^0(B)\right]$$

$$= S_0 \mathbb{E}\left[\sqrt{\int_0^t \delta_s^2\, ds}\, L_1^0(B)\right].$$

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Hence

$$\lim_{t \to 0^+} \frac{1}{\sqrt{t}} \mathbb{E}\left[L_t^{S_0}\left(S_0 \exp\left(B_{[U]}\right)\right)\right] = \lim_{t \to 0^+} \frac{1}{\sqrt{t}} S_0 \mathbb{E}\left[\sqrt{\int_0^t \delta_s^2 \, ds} \, L_1^0(B)\right]$$

$$= \lim_{t \to 0^+} S_0 \mathbb{E}\left[\sqrt{\frac{1}{t} \int_0^t \delta_s^2 \, ds} \, L_1^0(B)\right].$$

Let us show that

$$\lim_{t \to 0^+} S_0 \mathbb{E}\left[\sqrt{\frac{1}{t} \int_0^t \delta_s^2 \, ds} \, L_1^0(B)\right] = S_0 \, \delta_0 \, \mathbb{E}\left[L_1^0(B)\right]. \tag{47}$$

Using the Cauchy-Schwarz inequality,

$$\left|\mathbb{E}\left[\left(\sqrt{\frac{1}{t} \int_0^t \delta_s^2 \, ds} - \delta_0\right) L_1^0(B)\right]\right| \leq \mathbb{E}\left[L_1^0(B)^2\right]^{1/2} \mathbb{E}\left[\left(\sqrt{\frac{1}{t} \int_0^t \delta_s^2 \, ds} - \delta_0\right)^2\right]^{1/2}.$$

The Lipschitz property of $x \to (\sqrt{x} - \delta_0)^2$ on $[\epsilon, +\infty[$ yields

$$\mathbb{E}\left[\left(\sqrt{\frac{1}{t} \int_0^t \delta_s^2 \, ds} - \delta_0\right)^2\right] \leq c(\epsilon)\mathbb{E}\left[\left|\frac{1}{t} \int_0^t \left(\delta_s^2 - \delta_0^2\right) \, ds\right|\right]$$

$$\leq \frac{c(\epsilon)}{t} \int_0^t ds \, \mathbb{E}\left[\left|\delta_s^2 - \delta_0^2\right|\right].$$

where $c(\epsilon)$ is the Lipschitz constant of $x \to (\sqrt{x} - \delta_0)^2$ on $[\epsilon, +\infty[$. Assumption 3.1 and Lemma 2.1 then imply (47). By Lévy's theorem for the local time of Brownian motion, $L_1^0(B)$ has the same law as $|B_1|$, leading to

$$\mathbb{E}\left[L_1^0(B)\right] = \sqrt{\frac{2}{\pi}}.$$

Clearly, since $L_t^K(S) = L_t^K\left(S_0 \exp\left(B_{[U]}\right)\right)$,

$$\lim_{t \to 0} \frac{1}{\sqrt{t}} \mathbb{E}\left[\frac{1}{2}L_t^{S_0}(S)\right] = \frac{S_0}{\sqrt{2\pi}} \delta_0. \tag{48}$$

This ends the proof. □

We can now treat the case of a general Itô semimartingale with both a continuous martingale component and a jump component.

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Theorem 3.2 (Short-maturity asymptotics for at-the-money call options). *Consider the price process S whose dynamics is given by*

$$S_t = S_0 + \int_0^t r(s) S_{s-} ds + \int_0^t S_{s-} \delta_s dW_s + \int_0^t \int_{-\infty}^{+\infty} S_{s-}(e^y - 1) \tilde{M}(ds\ dy).$$

Under the Assumptions 3.1 and 3.2 and the folllowing non-degeneracy condition in the neighborhood of t_0

$$\exists \epsilon > 0, \qquad \mathbb{P}\left(\forall t \in [t_0, T], \quad \delta_t \geq \epsilon\right) = 1,$$

we have

$$\frac{1}{\sqrt{t - t_0}} C_{t_0}(t, S_{t_0}) \underset{t \to t_0^+}{\longrightarrow} \frac{S_{t_0}}{\sqrt{2\pi}} \delta_{t_0}. \tag{49}$$

Proof. Applying the Tanaka-Meyer formula to $(S_t - S_0)^+$, we have

$$(S_t - S_0)^+ = \int_0^t 1_{\{S_{s-} > S_0\}} dS_s + \frac{1}{2} L_t^{S_0}$$
$$+ \sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s. \tag{50}$$

As noted above, Assumption 3.2 implies that the discounted price $\hat{S}_t = e^{-\int_0^t r(s)\,ds} S_t$ is a martingale under $\mathbb{P}$. So we can write

$$dS_t = e^{\int_0^t r(s)\,ds}\left(r(t) S_{t-} dt + d\hat{S}_t\right), \qquad \text{and}$$

$$\int_0^t 1_{\{S_{s-} > S_0\}} dS_s = \int_0^t e^{\int_0^s r(u)\,du} 1_{\{S_{s-} > S_0\}} d\hat{S}_s + \int_0^t e^{\int_0^s r(u)\,du} r(s) S_{s-} 1_{\{S_{s-} > S_0\}} ds,$$

where the first term is a martingale. Taking expectations, we get

$$e^{\int_0^t r(s)\,ds} C(t, S_0) = \mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du} r(s) S_s 1_{\{S_{s-} > S_0\}} ds + \frac{1}{2} L_t^{S_0}\right]$$
$$+ \mathbb{E}\left[\sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s\right].$$

Since $\hat{S}$ is a martingale,

$$\forall t \in [0, T] \quad \mathbb{E}[S_t] = e^{\int_0^t r(s)\,ds} S_0 < \infty. \tag{51}$$

Hence $t \to \mathbb{E}[S_t]$ is right-continuous at 0:

$$\lim_{t \to 0^+} \mathbb{E}[S_t] = S_0. \tag{52}$$

Furthermore, under the Assumptions 2.1 and 2.2 for $X_t = \log(S_t)$ (see equation (27)), one may apply Theorem 2.1 to the function

$$f : x \in \mathbb{R} \mapsto (\exp(x) - S_0)^2,$$

yielding

$$\lim_{t\to 0^+} \frac{1}{t} \mathbb{E}\left[(S_t - S_0)^2\right] = \mathcal{L}_0 f(X_0),$$

where $\mathcal{L}_0$ is defined via equation (5). Since $\mathcal{L}_0 f(X_0) < \infty$, then in particular,

$$t \mapsto \mathbb{E}\left[(S_t - S_0)^2\right]$$

is right-continuous at 0 with right limit 0. Observing that

$$S_t \, 1_{\{S_t > S_0\}} = (S_t - S_0)^+ + S_0 \, 1_{\{S_t > S_0\}},$$

we write

$$
\begin{aligned}
\mathbb{E}\left[S_t \, 1_{\{S_t > S_0\}}\right] &= \mathbb{E}\left[(S_t - S_0)^+ + S_0 \, 1_{\{S_t > S_0\}}\right] \\
&\leq \mathbb{E}\left[|S_t - S_0|\right] + S_0 \, \mathbb{P}\left(S_t > S_0\right),
\end{aligned}
$$

using the Lipschitz continuity of $x \mapsto (x - S_0)_+$. Since $t \to \mathbb{E}[S_t]$ is right-continuous at 0, for t small enough :

$$0 \leq \mathbb{E}\left[S_t \, 1_{\{S_t > S_0\}}\right] \leq \frac{1}{2} + S_0.$$

Thus, applying Fubini,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du} r(s) S_s \, 1_{\{S_s > S_0\}} ds\right] = O(t),$$

a fortiori,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du} r(s) S_s \, 1_{\{S_s > S_0\}} ds\right] = o\left(\sqrt{t}\right).$$

Let us now focus on the jump part,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s\right] \\
&= \mathbb{E}\left[\int_0^t ds \int m(s, dx) \left(S_{s-} e^x - S_0\right)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} S_{s-}(e^x - 1)\right]
\end{aligned}
$$
(53)

Observing that

$$\left|(ze^x - S_0)^+ - (z - S_0)^+ - 1_{\{z > S_0\}} z(e^x - 1)\right| \leq C \left(S_0 e^x - z\right)^2,$$

then, together with Assumption 3.1 and Lemma 2.1 implies,

$$\mathbb{E}\left[\sum_{0 < s \leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-} > S_0\}} \Delta S_s\right] = O(t) = o(\sqrt{t}).$$

Since $\delta_0 \geq \epsilon$, equation (48) yields the result. $\qquad\square$

Remark 3.2. *As noted by Berestycki et al [7, 9] in the diffusion case, the regularity of f at S_{t_0} plays a crucial role in the asymptotics of $\mathbb{E}\left[f(S_t)\right]$. Theorem 2.1 shows that $\mathbb{E}\left[f(S_t)\right] \sim ct$ for smooth functions f, even if $f(S_{t_0}) = 0$, while for call option prices we have $\sim \sqrt{t}$ asymptotics at-the-money where the function $x \to (x - S_0)_+$ is not smooth.*

Remark 3.3. *In the particular case of a Lévy process, Proposition 3.1, Proposition 3.2 and Theorem 3.2 imply a recent result of Tankov [26, Proposition 5, Proof 2].*

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