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# Nesterenko's linear independence criterion for vectors

Stéphane Fischler

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## Abstract

In this paper we deduce a lower bound for the rank of a family of  $p$  vectors in  $\mathbb{R}^k$  (considered as a vector space over the rationals) from the existence of a sequence of linear forms on  $\mathbb{R}^p$ , with integer coefficients, which are small at  $k$  points. This is a generalization to vectors of Nesterenko's linear independence criterion (which corresponds to  $k = 1$ ). It enables one to make use of some known constructions of linear forms small at several points, related to Padé approximation. As an application, we prove that at least  $\frac{2 \log a}{1 + \log 2}(1 + o(1))$  odd integers  $i \in \{3, 5, \dots, a\}$  are such that either  $\zeta(i)$  or  $\zeta(i + 2)$  is irrational, where  $a$  is an odd integer,  $a \rightarrow \infty$ . This refines upon Ball-Rivoal's theorem, namely  $\zeta(i) \notin \mathbb{Q}$  for at least  $\frac{\log a}{1 + \log 2}(1 + o(1))$  such  $i$ .

**Keywords:** Linear independence ; Riemann zeta function ; Irrationality ; Measure of simultaneous approximation.

**Math. Subject Classification (2010):** 11J72 (Primary) ; 11J82, 11J13, 11M06 (Secondary).

## 1 Introduction

The motivation for this paper comes from irrationality results on values of Riemann zeta function  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  at odd integers  $s \geq 3$ . The first result is due to Apéry [1]:  $\zeta(3) \notin \mathbb{Q}$ . The next breakthrough in this topic is due to Rivoal [23] and Ball-Rivoal [2]:

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \zeta(7), \dots, \zeta(a)) \geq \frac{\log a}{1 + \log 2}(1 + o(1)) \quad (1.1)$$

as  $a \rightarrow \infty$ , where  $a$  is an odd integer; notice this is a lower bound on the rank of this family of real numbers, in  $\mathbb{R}$  considered as a vector space over the rationals. Conjecturally the left handside is equal to  $\frac{a+1}{2}$ , but even the constant  $\frac{1}{1 + \log 2}$  in Eq. (1.1) has never been improved. Actually, known refinements of Ball-Rivoal's proof provide new results only for fixed values of  $a$ : the improvement always lies inside the error term  $o(1)$  as  $a \rightarrow \infty$ .

In this text we prove the following result, which is a strict improvement upon Eq. (1.1).

**Theorem 1.** *In  $\mathbb{R}^2$  seen as a  $\mathbb{Q}$ -vector space, the family of vectors*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \zeta(3) \\ 6\zeta(5) \end{pmatrix}, \begin{pmatrix} \zeta(5) \\ 15\zeta(7) \end{pmatrix}, \dots, \begin{pmatrix} \zeta(a) \\ \frac{a(a+1)}{2}\zeta(a+2) \end{pmatrix}$$

*has rank greater than or equal to  $\frac{2\log a}{1+\log 2}(1+o(1))$  as  $a \rightarrow \infty$ , with  $a$  odd.*

It is clear that Theorem 1 implies Eq. (1.1): if  $E$  is a  $\mathbb{Q}$ -subspace of  $\mathbb{R}$  which contains 1,  $\zeta(3), \zeta(5), \dots, \zeta(a)$  then  $E \times E$  contains all vectors of Theorem 1. Using linear algebra it is not difficult (see §2.3) to prove that Theorem 1 is a strict improvement upon Ball-Rivoal's result (1.1), namely it provides more information of  $\zeta(3), \dots, \zeta(a)$ .

Theorem 1 can also be stated as follows (see §2.3): if  $F$  is a subspace of  $\mathbb{R}^{(a+3)/2}$ , defined over  $\mathbb{Q}$ , which contains both points  $e_1 = (1, 0, \zeta(3), \zeta(5), \dots, \zeta(a))$  and  $e_2 = (0, 1, \binom{4}{2}\zeta(5), \binom{6}{2}\zeta(7), \dots, \binom{a+1}{2}\zeta(a+2))$ , then  $\dim_{\mathbb{R}} F \geq \frac{2\log a}{1+\log 2}(1+o(1))$ .

Among the vectors  $\begin{pmatrix} \zeta(i) \\ \binom{i+1}{2}\zeta(i+2) \end{pmatrix}$  of Theorem 1, we may discard those such that both  $\zeta(i)$  and  $\zeta(i+2)$  are rational numbers. Accordingly there are at least  $\frac{2\log a}{1+\log 2}(1+o(1))$  odd integers  $i \in \{3, 5, \dots, a\}$  such that either  $\zeta(i)$  or  $\zeta(i+2)$  is irrational. This consequence of Theorem 1 can be stated as follows (see §4.2).

**Corollary 1.** *Let  $a \geq 3$  be an odd integer. Denote by  $n_1(a)$  the number of odd integers  $i \in \{3, 5, \dots, a\}$  such that  $\zeta(i) \notin \mathbb{Q}$ , and by  $n_2(a)$  the number of odd integers  $j \in \{5, 7, \dots, a\}$  such that  $\zeta(j) \notin \mathbb{Q}$  and  $\zeta(j-2) \in \mathbb{Q}$ . Then we have*

$$n_1(a) + n_2(a) \geq \frac{2\log a}{1+\log 2}(1+o(1))$$

*as  $a \rightarrow \infty$ .*

Since  $n_2(a) \leq n_1(a)$  by definition, this corollary immediately implies the lower bound  $n_1(a) \geq \frac{\log a}{1+\log 2}(1+o(1))$  which follows from Ball-Rivoal's result (1.1). In the case where  $n_1(a) < \frac{2\log a}{1+\log 2}(1+o(1))$ , it asserts also that the set of odd  $i \leq a$  such that  $\zeta(i) \notin \mathbb{Q}$  does not consist in few large blocks of consecutive odd integers (because  $n_2(a)$  is essentially the number of such blocks). For instance, if  $n_1(a) = \frac{\log a}{1+\log 2}(1+o(1))$  then only  $o(\log a)$  odd integers  $j \leq a$  satisfy both  $\zeta(j) \notin \mathbb{Q}$  and  $\zeta(j-2) \notin \mathbb{Q}$ : if there are very few irrational values they have to be mostly non-consecutive. Under the same assumption, there are  $\frac{\log a}{1+\log 2}(1+o(1))$  odd integers  $j \leq a$  such that  $\zeta(j) \notin \mathbb{Q}$  but  $\zeta(j-2), \zeta(j+2) \in \mathbb{Q}$ : "almost all" blocks consist in just one odd integer.

The linear forms we use to prove Theorem 1 provide also the following result:

**Theorem 2.** *Let  $\lambda \in \mathbb{R}$ . When  $a$  is odd and  $a \rightarrow \infty$ , the dimension of the  $\mathbb{Q}$ -vector space spanned by the numbers*

$$1, \lambda, \zeta(3) + 6\lambda\zeta(5), \zeta(5) + 15\lambda\zeta(7), \dots, \zeta(a) + \binom{a+1}{2}\lambda\zeta(a+2)$$

*is greater than or equal to  $\frac{\log a}{1+\log 2}(1+o(1))$ .*

The proof of Theorems 1 and 2 is based on the following variant of Ball-Rivoal's construction, to which a linear independence criterion is applied: Nesterenko's [21] for Theorem 2, and a generalization to vectors (namely Theorem 3 below) for Theorem 1.

Let  $a, r, n$  be positive integers such that  $a$  is odd and  $6r \leq a$ . Consider

$$S_n = (2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{(t - (2r + 1)n)_{2rn}^3 (t + n + 1)_{2rn}^3}{(t - n)_{2n+1}^a}$$

and

$$S_n'' = \frac{1}{2}(2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{d^2}{dt^2} \left( \frac{(t - (2r + 1)n)_{2rn}^3 (t + n + 1)_{2rn}^3}{(t - n)_{2n+1}^a} \right)$$

where Pochhammer's symbol is defined by  $(\alpha)_k = \alpha(\alpha + 1) \dots (\alpha + k - 1)$ . The same symmetry argument as in Ball-Rivoal's construction yields

$$S_n = \ell_{0,n} + \ell_{3,n}\zeta(3) + \ell_{5,n}\zeta(5) + \dots + \ell_{a,n}\zeta(a)$$

with  $\ell_{0,n}, \ell_{3,n}, \ell_{5,n}, \dots, \ell_{a,n} \in \mathbb{Q}$ ; the definition of  $S_n''$  (namely like  $S_n$ , but with a double derivation) implies

$$S_n'' = \ell_{0,n}'' + \ell_{3,n} \binom{4}{2} \zeta(5) + \ell_{5,n} \binom{6}{2} \zeta(7) + \dots + \ell_{a,n} \binom{a+1}{2} \zeta(a+2)$$

with the same coefficients  $\ell_{3,n}, \ell_{5,n}, \dots, \ell_{a,n}$ , except for  $\ell_{0,n}'' \in \mathbb{Q}$ . Therefore the linear form  $\ell_{0,n}X_0 + \ell_{0,n}''X_0'' + \ell_{3,n}X_3 + \ell_{5,n}X_5 + \dots + \ell_{a,n}X_a$  is very small at both points  $e_1 = (1, 0, \zeta(3), \zeta(5), \dots, \zeta(a))$  and  $e_2 = (0, 1, \binom{4}{2}\zeta(5), \binom{6}{2}\zeta(7), \dots, \binom{a+1}{2}\zeta(a+2))$ ; applying Theorem 3 below yields Theorem 1. The same construction yields Theorem 2 by applying Nesterenko's original linear independence criterion to  $S_n + \lambda S_n''$ ; see §4 for details.

The construction of  $S_n$  and  $S_n''$  is analogous to that of [17], where the analogue of Theorem 2 is proved for the numbers  $\zeta(j) + j\lambda\zeta(j+1)$  with  $j$  odd,  $3 \leq j \leq a$ . Theorem 3 applies to the construction of [17], but the (qualitative) result it provides is a triviality: since  $\zeta_{j+1} \in \mathbb{Q}^*\pi^{j+1}$  because  $j+1$  is even, the vectors  $(\zeta_{j+1}^{(j)})$  are obviously  $\mathbb{Q}$ -linearly independent in  $\mathbb{R}^2$  because their second coordinates are. The situation is exactly the same for Théorème 2 of [10], with the numbers  $\text{Li}_j(z) + \lambda \frac{\log^j(z)}{(j-1)!}$ , where  $z \in \mathbb{Q} \cap (0, 1)$ , because  $\log z$  is a transcendental number. However it should be emphasized that the quantitative form (iii) of Theorem 3 still provides new information in these settings: it enables one to take advantage really of the properties of the linear forms (see the remarks at the end of this introduction).

More generally, the kind of construction used in [17], [10] and here, namely using a derivation as in the definition of  $S_n''$  to obtain linear forms small at several points, is very classical (especially in connection to simultaneous Padé approximation): see for instance [3] or [13]. This idea is also involved in results of Gutnik and Hessami-Pilehrood stated in §2.4 below. Our main result, namely the following generalization of Nesterenko's linear independence criterion [21] to vectors, applies in these situations.

We let  $\mathbb{R}^p$  be endowed with its canonical scalar product and the corresponding norm.

**Theorem 3.** *Let  $1 \leq k \leq p - 1$ , and  $e_1, \dots, e_k \in \mathbb{R}^p$ .*

*Let  $\tau_1, \dots, \tau_k > 0$  be pairwise distinct real numbers.*

*Let  $\omega_1, \dots, \omega_k, \varphi_1, \dots, \varphi_k$  be real numbers such that there exist infinitely many integers  $n$  with the following property: for any  $j \in \{1, \dots, k\}$ ,  $n\omega_j + \varphi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$ .*

*Let  $(Q_n)_{n \geq 1}$  be an increasing sequence of positive integers, such that  $Q_{n+1} = Q_n^{1+O(1/n)}$ ; if  $\omega_1 = \dots = \omega_k = 0$ , this assumption can be weakened to  $Q_{n+1} = Q_n^{1+o(1)}$ .*

*For any  $n \geq 1$ , let  $L_n = \ell_{1,n}X_1 + \dots + \ell_{p,n}X_p$  be a linear form on  $\mathbb{R}^p$ , with integer coefficients  $\ell_{i,n}$  such that, as  $n \rightarrow \infty$ :*

$$|L_n(e_j)| = Q_n^{-\tau_j+o(1)} |\cos(n\omega_j + \varphi_j) + o(1)| \text{ for any } j \in \{1, \dots, k\}, \quad (1.2)$$

and

$$\max_{1 \leq i \leq p} |\ell_{i,n}| \leq Q_n^{1+o(1)}.$$

Then:

(i) *If  $F$  is a subspace of  $\mathbb{R}^p$  defined over  $\mathbb{Q}$  which contains  $e_1, \dots, e_k$  then*

$$\dim F \geq k + \tau_1 + \dots + \tau_k.$$

*In other words, letting  $C_1, \dots, C_p \in \mathbb{R}^k$  denote the columns of the matrix whose rows are  $e_1, \dots, e_k \in \mathbb{R}^p$ , we have*

$$\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p) \geq k + \tau_1 + \dots + \tau_k$$

*in  $\mathbb{R}^k$  seen as a  $\mathbb{Q}$ -vector space.*

(ii) *The vectors  $e_1, \dots, e_k$  are  $\mathbb{R}$ -linearly independent in  $\mathbb{R}^p$ , and the  $\mathbb{R}$ -subspace they span does not intersect  $\mathbb{Q}^p \setminus \{(0, \dots, 0)\}$ .*

(iii) *Let  $\varepsilon > 0$ , and  $Q$  be sufficiently large (in terms of  $\varepsilon$ ). Let  $\mathcal{C}$  denote the set of all vectors that can be written as  $\lambda_1 e_1 + \dots + \lambda_k e_k + u$  with:*

$$\begin{cases} \lambda_1, \dots, \lambda_k \in \mathbb{R} \text{ such that } |\lambda_j| \leq Q^{\tau_j} \text{ for any } j \in \{1, \dots, k\} \\ u \in (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^{\perp} \text{ such that } \|u\| \leq Q^{-1-\varepsilon} \end{cases}$$

*Then  $\mathcal{C} \cap \mathbb{Z}^p = \{(0, \dots, 0)\}$ .*

If  $k = 1$  and  $\omega_1 = \varphi_1 = 0$ , this is exactly Nesterenko's linear independence criterion [21]. With  $k = 1$ , the generalization to arbitrary values of  $\omega_1$  and  $\varphi_1$  has been proved in [8] when  $Q_n = \beta^n$  for some  $\beta > 1$ ; this is the most interesting case since it includes oscillating behaviours of the linear forms provided by the saddle point method (and  $S_n''$  defined above has this kind of asymptotics). Note that when  $\omega_j = 0$ , we assume  $\varphi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$  so that Eq. (1.2) reads  $|L_n(e_j)| = Q_n^{-\tau_j+o(1)}$ . At last, in general Nesterenko's criterion is stated under the assumption  $Q_{n+1} = Q_n^{1+o(1)}$ ; however assuming  $Q_{n+1} = Q_n^{1+O(1/n)}$  if some  $\omega_j$  is

non-zero is not a serious drawback since any sequence  $Q_n = \beta^{n^d}$  with  $\beta > 1$  and  $d > 0$  satisfies this property.

In the conclusions, (ii) is an easy result, and (iii) is the main part (it is a quantitative version of (ii)). We deduce (i) from (iii) using Minkowski's convex body theorem, thereby generalizing the proof given in [12] and [11] of Nesterenko's linear independence criterion. The equivalence between both statements of (i) comes from linear algebra; it is proved in §2.3.

A result analogous to Theorem 3, but in which  $p$  linearly independent linear forms like  $L_n$  appear in the assumption, is proved in §2.4. This linear independence criterion (in the style of Siegel's) is much easier to prove than Theorem 3. Both results can be thought of as transference principles. In this respect it is worth pointing out that in Theorem 3 we assume essentially that for any positive integer  $Q$  there is a linear form  $L_n$ : indeed this is  $L_n$ , where  $n$  is such that  $Q_n \leq Q < Q_{n+1}$  so that  $Q = Q_n^{1+o(1)}$  because  $Q_{n+1} = Q_n^{1+o(1)}$ .

The assumption that  $\tau_1, \dots, \tau_k$  are pairwise distinct is very important in Theorem 3, and it cannot be omitted. For instance, if  $\tau_1 = \tau_2$  then  $L_n(e_1 - e_2)$  could be very small: up to replacing  $(e_1, e_2)$  with  $(e_1 + e_2, e_1 - e_2)$ , this amounts to dropping the assumption that the linear forms  $L_n$  are not too small at the points  $e_j$ . Now this assumption is known to be essential, already in the classical case of Nesterenko's linear independence criterion (except for proving the linear independence of three numbers, see Theorem 2 of [12]). Actually, if  $\tau_1 = \tau_2$  then  $L_n(e_1 - e_2)$  could even vanish, so the possibility that  $e_1 = e_2$  cannot be eliminated: even assertion (ii) may fail to hold.

Under the assumptions of Theorem 3, it can be useful to apply this result (in the qualitative form (i)) to the point  $\lambda_1 e_1 + \dots + \lambda_k e_k$  with fixed  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  (which amounts to applying Nesterenko's original criterion, generalized [8] to allow oscillatory behaviours). This is how Theorem 2 is proved, and also the above-mentioned results of [17] and [10] (with  $k = 2$ ). However, the quantitative conclusion (iii) of Theorem 3 itself contains all quantitative results that can be obtained in this way (because the latter involve smaller convex bodies  $\mathcal{C}$ ), so that it implies the qualitative statement (i) for any linear combination  $\lambda_1 e_1 + \dots + \lambda_k e_k$  using Minkowski's convex body theorem.

At last, we remark that the lower bound  $k + \tau_1 + \dots + \tau_k$  in (i) is optimal (see [9] for a converse statement). It seems natural to imagine that (iii) is essentially optimal too: up to  $Q^\varepsilon$ ,  $\mathcal{C}$  is defined as the set of all  $x$  such that the assumptions on  $L_n$  imply  $|L_n(x)| < 1$ , so that  $L_n(x) = 0$  if  $x \in \mathcal{C} \cap \mathbb{Z}^p$ , where  $n$  is chosen such that  $Q \approx Q_n$ . When  $p = 2$ ,  $k = 1$ , and  $e_1 = (1, \xi)$ , Theorem 3 (iii) yields an upper bound  $\mu(\xi) \leq 1 + \frac{1}{\tau_1}$  on the irrationality exponent of  $\xi$ , and reduces essentially to Lemma 1 of [11]. A converse statement in this case is proved in [11] (Theorem 1).

The structure of this text is as follows. In §2 we give some corollaries of Theorem 3, in the setting of Padé approximation (§2.1) or in relation to the approximation of a fixed subspace by points of  $\mathbb{Z}^p$  (§2.2). We also study in §2.3 the rank of vectors in  $\mathbb{R}^k$  seen as a

vector space over the rationals; this allows us to prove the equivalence of both statements of (i) in Theorem 3, and the fact that Theorem 1 is a strict improvement on Ball-Rivoal's result (1.1) on zeta values. A linear independence criterion in the style of Siegel's is proved in §2.4.

Then we prove Theorem 3 in §3; the main new step is detailed in §3.4. At last, §4 is devoted to the applications to zeta values.

## 2 Consequences and related results

### 2.1 Connection with Padé approximation

To begin with, let us state Theorem 3 in terms of  $C_1, \dots, C_p \in \mathbb{R}^k$ .

**Theorem 4.** *Let  $C_1, \dots, C_p \in \mathbb{R}^k$ , with  $k, p \geq 1$ .*

*Let  $\tau_1, \dots, \tau_k, \omega_1, \dots, \omega_k, \varphi_1, \dots, \varphi_k$  and  $(Q_n)_{n \geq 1}$  be as in Theorem 3.*

*For any  $n \geq 1$ , let  $\ell_{1,n}, \dots, \ell_{p,n} \in \mathbb{Z}$  be such that, as  $n \rightarrow \infty$ :*

$$\max_{1 \leq i \leq p} |\ell_{i,n}| \leq Q_n^{1+o(1)}$$

and

$$\ell_{1,n}C_1 + \dots + \ell_{p,n}C_p = \begin{pmatrix} \pm Q_n^{-\tau_1+o(1)} (\cos(n\omega_1 + \varphi_1) + o(1)) \\ \vdots \\ \pm Q_n^{-\tau_k+o(1)} (\cos(n\omega_k + \varphi_k) + o(1)) \end{pmatrix} \quad (2.1)$$

where the  $\pm$  signs can be independent from one another. Then:

(i) *We have*

$$\mathrm{rk}_{\mathbb{Q}}(C_1, \dots, C_p) \geq k + \tau_1 + \dots + \tau_k,$$

where  $\mathrm{rk}_{\mathbb{Q}}(C_1, \dots, C_p)$  is the rank of the family of vectors  $C_1, \dots, C_p$  in  $\mathbb{R}^k$ , considered as a  $\mathbb{Q}$ -vector space.

(ii) *For any non-zero linear form  $\chi : \mathbb{R}^k \rightarrow \mathbb{R}$  there exists  $i \in \{1, \dots, p\}$  such that  $\chi(C_i) \notin \mathbb{Q}$ .*

(iii) *Let  $\varepsilon > 0$ , and  $Q$  be sufficiently large in terms of  $\varepsilon$ . Let  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ , not all zero, be such that  $|\lambda_j| \leq Q^{\tau_j}$  for any  $j \in \{1, \dots, k\}$ . Then denoting by  $\chi$  the linear map  $\mathbb{R}^k \rightarrow \mathbb{R}$  defined by  $\chi(x_1, \dots, x_k) = \lambda_1 x_1 + \dots + \lambda_k x_k$ , we have*

$$\mathrm{dist}\left((\chi(C_1), \dots, \chi(C_p)), \mathbb{Z}^p \setminus \{(0, \dots, 0)\}\right) \geq Q^{-1-\varepsilon}$$

where  $\mathrm{dist}(y, \mathbb{Z}^p \setminus \{(0, \dots, 0)\})$  is the minimal distance of  $y \in \mathbb{R}^p$  to a non-zero integer point.

This result is just a translation of Theorem 3. Indeed let us consider the matrix  $M \in \text{Mat}_{k,p}(\mathbb{R})$  of which  $C_1, \dots, C_p$  are the columns. We denote by  $e_1, \dots, e_k \in \mathbb{R}^p$  the rows of  $M$ . Then assumption (2.1) means that the linear form  $L_n = \ell_{1,n}X_1 + \dots + \ell_{p,n}X_p$  on  $\mathbb{R}^p$  is small at the points  $e_1, \dots, e_k$ . It is not difficult to see that (ii) and (iii) in Theorem 4 are respectively equivalent to (ii) and (iii) in Theorem 3, because  $(\chi(C_1), \dots, \chi(C_p)) = \lambda_1 e_1 + \dots + \lambda_k e_k$ . We remark also that assuming  $k \leq p-1$  in Theorem 3 is not necessary; it won't be used in the proof. Anyway this upper bound follows from (ii), so that it is actually a consequence of the other assumptions.

Let us focus now on an important special case of Theorem 4, related to Padé approximation: when  $C_1, \dots, C_k$  is the canonical basis of  $\mathbb{R}^k$ . This happens in all practical situations mentioned in this paper, including Theorem 1 and Eqns. (2.3), (2.4) and (2.5) in §2.4 below: indeed Padé approximation provides linear combinations of  $C_{k+1}, \dots, C_p$  which are very close to  $\mathbb{Z}^k$ . In this case, in (ii) the interesting point is when the linear form  $\chi(x_1, \dots, x_k) = \lambda_1 x_1 + \dots + \lambda_k x_k$  has rational coefficients  $\lambda_j$ ; then we have  $\chi(C_i) \notin \mathbb{Q}$  for some  $i \in \{k+1, \dots, p\}$ . An analogous remark holds for (iii); both are more easily stated as follows, in terms of  $e_1, \dots, e_k$ . We denote by  $\|\cdot\|$  any fixed norm on  $\mathbb{R}^{p-k}$ .

**Corollary 2.** *Under the assumptions of Theorem 3, suppose that for any  $j \in \{1, \dots, k\}$  we have  $e_j = (0, \dots, 0, 1, 0, \dots, 0, e'_j)$  with  $e'_j \in \mathbb{R}^{p-k}$ , where the 1 is in  $j$ -th position.*

*Then no non-trivial  $\mathbb{Q}$ -linear combination of  $e'_1, \dots, e'_k$  belongs to  $\mathbb{Q}^{p-k}$ . In addition, let  $\varepsilon > 0$ , and  $Q$  be sufficiently large in terms of  $\varepsilon$ . Let  $\lambda_1, \dots, \lambda_k \in \mathbb{Z}$ , not all zero, be such that  $|\lambda_j| \leq Q^{\tau_j}$  for any  $j \in \{1, \dots, k\}$ . Then for any  $S \in \mathbb{Z}^{p-k}$  we have*

$$\|\lambda_1 e'_1 + \dots + \lambda_k e'_k - S\| \geq Q^{-1-\varepsilon}.$$

This corollary is a measure of linear independence of the vectors  $e'_1, \dots, e'_k$  and those of the canonical basis of  $\mathbb{Z}^{p-k}$ . It can be weakened by assuming  $|\lambda_j| \leq Q^\tau$  for any  $j \in \{1, \dots, k\}$ , where  $\tau = \min(\tau_1, \dots, \tau_k)$  (as in Theorem 5 below). Then a *measure of non-discreteness* (in the sense of [14]) is obtained for the lattice  $\mathbb{Z}e'_1 + \dots + \mathbb{Z}e'_k + \mathbb{Z}^{p-k}$ , which has rank  $p$ . In the examples (2.3), (2.4) and (2.5) considered in §2.4 below, the matrix with columns  $C_{k+1}, \dots, C_p$  is symmetric (with  $p = 2k$ ), so that this lattice is exactly  $\mathbb{Z}C_1 + \dots + \mathbb{Z}C_p$  (using the fact that  $C_1, \dots, C_k$  is the canonical basis of  $\mathbb{R}^k$ ).

This case  $k = p/2$  lies “in the middle” between  $k = 1$ , which corresponds to type I Padé approximation and Nesterenko’s original criterion, and  $k = p-1$ , which corresponds to type II Padé approximation. In the latter case, Corollary 2 yields the following result by letting  $\xi_j = -e'_j$ .

**Corollary 3.** *Let  $k \geq 1$ , and  $\xi_1, \dots, \xi_k \in \mathbb{R}$ .*

*Let  $\tau_1, \dots, \tau_k > 0$  be pairwise distinct real numbers.*

*Let  $(Q_n)_{n \geq 1}$  be an increasing sequence of positive integers, such that  $Q_{n+1} = Q_n^{1+o(1)}$ .*

*For any  $n \geq 1$ , let  $\ell_{1,n}, \dots, \ell_{k,n}, \ell_{k+1,n} \in \mathbb{Z}$  be such that*

$$\max_{1 \leq i \leq k+1} |\ell_{i,n}| \leq Q_n^{1+o(1)}$$



and

$$|\ell_{k+1,n}\xi_j - \ell_{j,n}| = Q_n^{-\tau_j + o(1)} \text{ for any } j \in \{1, \dots, k\}.$$

Then:

(i) The numbers  $1, \xi_1, \dots, \xi_k$  are  $\mathbb{Q}$ -linearly independent.

(ii) Let  $\varepsilon > 0$ , and  $Q$  be sufficiently large (in terms of  $\varepsilon$ ). Then for any  $(a_0, a_1, \dots, a_k) \in \mathbb{Z}^{k+1} \setminus \{(0, \dots, 0)\}$  with  $|a_j| \leq Q^{\tau_j}$  for any  $j \in \{1, \dots, k\}$ , we have:

$$|a_0 + a_1\xi_1 + \dots + a_k\xi_k| \geq Q^{-1-\varepsilon}.$$

We have not found this statement in the literature; see however [7] (p. 98), [15] (Lemma 2.1) or [16] (Lemma 6.1) for related results, which are maybe closer to Siegel's criterion than to Nesterenko's.

A direct proof of Corollary 3 can be derived from the equality

$$|\ell_{k+1,n}| \cdot |a_0 + a_1\xi_1 + \dots + a_k\xi_k| = \left| \sum_{j=1}^k a_j(\ell_{k+1,n}\xi_j - \ell_{j,n}) + \ell_{k+1,n}a_0 + \sum_{j=1}^k a_j\ell_{j,n} \right|$$

by arguing as in §3.2 for (i), and by taking  $n$  maximal such that  $Q_n \cdot |a_0 + a_1\xi_1 + \dots + a_k\xi_k| < \frac{1}{2}$  for (ii).

## 2.2 Upper bound on a Diophantine exponent

Given a subspace  $F$  of  $\mathbb{R}^p$ , and a non-zero point  $P \in \mathbb{R}^p$ , we denote by  $\text{Dist}(P, F)$  the projective distance of  $P$  to  $F$ , seen in  $\mathbb{P}^p(\mathbb{R})$ . Several definitions may be given, all of them equivalent up to multiplicative constants (see for instance [25]); we choose  $\text{Dist}(P, F) = \frac{\|u\|}{\|P\|}$  where  $u$  is the orthogonal projection of  $P$  on  $F^\perp$  (that is,  $P$  can be written as  $u + f$  with  $u \in F^\perp$  and  $f \in F$ ), and  $\|\cdot\|$  is the Euclidean norm on  $\mathbb{R}^p$ .

The following result is a consequence of Theorem 3.

**Theorem 5.** *Under the assumptions of Theorem 3, let  $\tau = \min(\tau_1, \dots, \tau_k)$  and  $F = \text{Span}_{\mathbb{R}}(e_1, \dots, e_k)$ . Then for any  $\varepsilon > 0$  and any  $P \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$  we have:*

$$\text{Dist}(P, F) \geq \|P\|^{-1-\frac{1}{\tau}-\varepsilon}$$

provided  $\|P\|$  is sufficiently large in terms of  $\varepsilon$ .

It is important to notice that Theorem 5 is *not* optimal, since it involves only  $\min(\tau_1, \dots, \tau_k)$ . It is specially interesting when  $\tau_1, \dots, \tau_k$  are close to one another.

The interest of Theorem 5 is that it can be written as an upper bound on a Diophantine exponent which measures the approximation of  $F$  by points of  $\mathbb{Z}^p$  (see [25], [19], [5]).

**Proof** of Theorem 5: Using assertion (ii) of Theorem 3, we see that  $(e_1, \dots, e_k)$  is a basis of  $F$ . Since  $F$  is finite-dimensional, all norms on  $F$  are equivalent: there exists  $\kappa > 0$  such that, for any  $f = \lambda_1 e_1 + \dots + \lambda_k e_k \in F$  (with  $\lambda_j \in \mathbb{R}$ ), we have  $\max |\lambda_j| \leq \kappa \|f\|$ .

Let  $\varepsilon > 0$ . Let  $Q_0$  be such that assertion (iii) of Theorem 3 holds for any  $Q \geq Q_0$ ; we assume that  $\|P\| \geq Q_0^\tau / \kappa$ . Letting  $Q = (\kappa \|P\|)^{1/\tau}$  we have  $Q \geq Q_0$ . Since  $P \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$ ,  $P$  does not belong to the set  $\mathcal{C}$  defined in assertion (iii). Now writing  $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u$  with  $\lambda_j \in \mathbb{R}$  and  $u \in F^\perp$ , we have

$$\max_{1 \leq j \leq k} |\lambda_j| \leq \kappa \|\lambda_1 e_1 + \dots + \lambda_k e_k\| \leq \kappa \|P\| = Q^\tau$$

so that  $\|u\| > Q^{-1-\varepsilon}$ . Using the definition of  $Q$  and that of  $\text{Dist}(P, F)$ , this concludes the proof of Theorem 5.

## 2.3 Rational rank of vectors

In this section, we give some details about the conclusions of our criterion and prove that Theorem 1 is a strict improvement on Ball-Rivoal's result [2] on zeta values.

In Nesterenko's linear independence criterion, a lower bound is derived for the dimension of the  $\mathbb{Q}$ -subspace of  $\mathbb{R}$  spanned by  $\xi_0, \dots, \xi_r \in \mathbb{R}$ , that is, for the  $\mathbb{Q}$ -rank of  $\xi_0, \dots, \xi_r$  in  $\mathbb{R}$  considered as a vector space over  $\mathbb{Q}$ . This rank is equal to the dimension of the smallest subspace of  $\mathbb{R}^{r+1}$ , defined over the rationals, which contains the point  $(\xi_0, \dots, \xi_r)$ . We generalize in Lemma 1 below this equality to our setting.

Recall that a subspace  $F$  of  $\mathbb{R}^p$  is said to be *defined over*  $\mathbb{Q}$  if it is the zero locus of a family of linear forms with rational coefficients. This is equivalent to the existence of a basis (or a generating family) of  $F$ , as a vector space over  $\mathbb{R}$ , consisting in vectors of  $\mathbb{Q}^p$  (see for instance §8 of [4]). Since the intersection of a family of subspaces of  $\mathbb{R}^p$  defined over  $\mathbb{Q}$  is again defined over  $\mathbb{Q}$ , there exists for any subset  $S \subset \mathbb{R}^p$  a minimal subspace of  $\mathbb{R}^p$ , defined over  $\mathbb{Q}$ , which contains  $S$ : this is the intersection of all subspaces of  $\mathbb{R}^p$ , defined over  $\mathbb{Q}$ , which contain  $S$ .

Let  $M$  be a matrix with  $k \geq 1$  rows,  $p \geq 1$  columns, and real entries. Letting  $e_1, \dots, e_k \in \mathbb{R}^p$  denote the rows of  $M$ , we can consider as above the smallest subspace of  $\mathbb{R}^p$ , defined over  $\mathbb{Q}$ , which contains  $e_1, \dots, e_k$ . On the other hand, we denote by  $C_1, \dots, C_p \in \mathbb{R}^k$  the columns of  $M$  and consider  $\mathbb{R}^k$  as an infinite-dimensional vector space over  $\mathbb{Q}$ . Then  $\text{Span}_{\mathbb{Q}}(C_1, \dots, C_p)$  is the smallest  $\mathbb{Q}$ -vector subspace of  $\mathbb{R}^k$  containing  $C_1, \dots, C_p$ ; it consists in all linear combinations  $r_1 C_1 + \dots + r_p C_p$  with  $r_1, \dots, r_p \in \mathbb{Q}$ . Its dimension (as a  $\mathbb{Q}$ -vector space) is the rank (over  $\mathbb{Q}$ ) of  $C_1, \dots, C_p$ , denoted by  $\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p)$ .

**Lemma 1.** *Let  $M \in \text{Mat}_{k,p}(\mathbb{R})$  with  $k, p \geq 1$ . Denote by  $e_1, \dots, e_k \in \mathbb{R}^p$  the rows of  $M$ , and by  $C_1, \dots, C_p \in \mathbb{R}^k$  its columns. Then  $\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p)$  is the dimension of the smallest subspace of  $\mathbb{R}^p$ , defined over  $\mathbb{Q}$ , which contains  $e_1, \dots, e_k$ .*

When  $k = 1$ , this lemma means that the  $\mathbb{Q}$ -rank of  $\xi_0, \dots, \xi_r$  is equal to the dimension of the smallest subspace of  $\mathbb{R}^{r+1}$ , defined over the rationals, which contains the point  $(\xi_0, \dots, \xi_r)$ .

**Proof** of Lemma 1: Let  $G = (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^\perp$ , where  $\mathbb{R}^p$  is equipped with the usual scalar product. Let  $F$  denote the minimal subspace of  $\mathbb{R}^p$ , defined over  $\mathbb{Q}$ , which contains  $e_1, \dots, e_k$ . Then  $F^\perp$  is the maximal subspace of  $\mathbb{R}^p$ , defined over  $\mathbb{Q}$ , which is contained in  $G = \{e_1, \dots, e_k\}^\perp$ . Therefore  $F^\perp = \text{Span}_{\mathbb{R}}(G \cap \mathbb{Q}^p) = (G \cap \mathbb{Q}^p) \otimes_{\mathbb{Q}} \mathbb{R}$ : any basis of the  $\mathbb{Q}$ -vector space  $G \cap \mathbb{Q}^p$  is an  $\mathbb{R}$ -basis of  $F^\perp$ . Since  $G \cap \mathbb{Q}^p = \ker \psi$  where  $\psi : \mathbb{Q}^p \rightarrow \mathbb{R}^k$  is defined by  $\psi(r_1, \dots, r_p) = r_1 C_1 + \dots + r_p C_p$ , we have:

$$\dim_{\mathbb{R}} F = p - \dim_{\mathbb{R}} F^\perp = p - \dim_{\mathbb{Q}}(G \cap \mathbb{Q}^p) = \text{rk}_{\mathbb{Q}} \psi = \text{rk}_{\mathbb{Q}}(C_1, \dots, C_p).$$

This concludes the proof of Lemma 1.

Let us proceed now to prove that Theorem 1 is a strict improvement on Ball-Rivoal's result [2] on zeta values.

Let  $a \geq 3$  be an odd integer, and  $\xi_3, \xi_5, \dots, \xi_{a+2}$  be real numbers. We let  $\underline{\xi} = (\xi_3, \xi_5, \dots, \xi_{a+2})$ ,

$$n(\underline{\xi}) = \dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \xi_3, \xi_5, \dots, \xi_a),$$

and  $N(\underline{\xi})$  denote the dimension of the minimal subspace of  $\mathbb{R}^{(a+3)/2}$ , defined over  $\mathbb{Q}$ , which contains both points  $(1, 0, \xi_3, \xi_5, \dots, \xi_a)$  and  $(0, 1, \binom{4}{2}\xi_5, \binom{6}{2}\xi_7, \dots, \binom{a+1}{2}\xi_{a+2})$ . As Lemma 1 shows,  $N(\underline{\xi})$  is the  $\mathbb{Q}$ -rank of the family of vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \xi_3 \\ \binom{4}{2}\xi_5 \end{pmatrix}, \begin{pmatrix} \xi_5 \\ \binom{6}{2}\xi_7 \end{pmatrix}, \dots, \begin{pmatrix} \xi_a \\ \binom{a+1}{2}\xi_{a+2} \end{pmatrix}.$$

When  $\underline{\xi} = (\zeta(3), \zeta(5), \dots, \zeta(a+2))$  and  $a \rightarrow \infty$ , Ball-Rivoal's result reads  $n(\underline{\xi}) \geq \frac{\log a}{1+\log 2}(1+o(1))$  whereas Theorem 1 states that  $N(\underline{\xi}) \geq \frac{2 \log a}{1+\log 2}(1+o(1))$ . The latter is a strict improvement on the former, as the following lemma shows.

**Lemma 2.** *With the previous notation:*

(i) *For any  $\underline{\xi}$  we have  $n(\underline{\xi}) + 1 \leq N(\underline{\xi}) \leq 2n(\underline{\xi}) + 1$ .*

(ii) *For any positive integers  $n$  and  $N$  with  $N \leq \frac{a+3}{2}$  and  $n+1 \leq N \leq 2n+1$  there exists  $\underline{\xi}$  such that  $n(\underline{\xi}) = n$  and  $N(\underline{\xi}) = N$ .*

**Proof** of Lemma 2:

(i) Let  $F_1$  (resp.  $F_2$ , resp.  $F$ ) be the minimal subspace of  $\mathbb{R}^{(a+3)/2}$ , defined over  $\mathbb{Q}$ , which contains the point  $e_1 = (1, 0, \xi_3, \xi_5, \dots, \xi_a)$  (resp.  $e_2 = (0, 1, \binom{4}{2}\xi_5, \binom{6}{2}\xi_7, \dots, \binom{a+1}{2}\xi_{a+2})$ , resp. both  $e_1$  and  $e_2$ ). We have  $\dim F_1 = n(\underline{\xi})$  and  $\dim F_2 = \dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \xi_5, \xi_7, \dots, \xi_{a+2}) \leq n(\underline{\xi}) + 1$ . Since  $F_1 + F_2$  is a subspace of  $\mathbb{R}^{(a+3)/2}$  defined over  $\mathbb{Q}$  which contains both  $e_1$  and  $e_2$ , we have  $N(\underline{\xi}) \leq \dim(F_1 + F_2) \leq 2n(\underline{\xi}) + 1$ . On the other hand, denoting by  $H$  the hyperplane of  $\mathbb{R}^{(a+3)/2}$  defined by the vanishing of the second coordinate, we have  $F_1 \subset F \cap H \subsetneq F$  (by minimality of  $F_1$  and because  $e_2 \notin H$ ), so that  $n(\underline{\xi}) \leq N(\underline{\xi}) - 1$ .

(ii) Let  $i = 2n - 1$  and  $j = 2N - 3$ , so that  $1 \leq i \leq j \leq a$  and  $j \leq 2i + 1$ ; obviously  $i$  and  $j$  are positive odd integers. If  $i \geq 3$  we choose  $\xi_3, \xi_5, \dots, \xi_i \in \mathbb{R}$  such that  $1, \xi_3, \xi_5, \dots, \xi_i$

are  $\mathbb{Q}$ -linearly independent. If  $j \geq i + 2$  we let  $\xi_{i+2} = 1, \xi_{i+4} = \xi_3, \dots, \xi_j = \xi_{j-i-1}$  (where  $\xi_1 = 1$ ). At last, we let  $\xi_{j+2} = \xi_{j+4} = \dots = \xi_{a+2} = 0$ . Then we have  $n(\underline{\xi}) = n$  by construction.

For any odd integer  $k \in \{3, 5, \dots, a\}$ , let  $C_k = \begin{pmatrix} \xi_k \\ \binom{k+1}{2} \xi_{k+2} \end{pmatrix} \in \mathbb{R}^2$ ; let also  $C_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $C_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then  $N(\underline{\xi}) = \text{rk}_{\mathbb{Q}}(C_{-1}, C_1, C_3, C_5, \dots, C_a)$  using Lemma 1. Since  $C_{j+2} = C_{j+4} = \dots = C_a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , we have  $N(\underline{\xi}) = \text{rk}_{\mathbb{Q}}(C_{-1}, C_1, \dots, C_j)$ . Let us prove that these  $\frac{j+3}{2} = N$  vectors are  $\mathbb{Q}$ -linearly independent; this will conclude the proof that  $N(\underline{\xi}) = N$ .

Let  $\lambda_{-1}, \lambda_1, \lambda_3, \dots, \lambda_j \in \mathbb{Q}$  be such that  $\lambda_{-1}C_{-1} + \lambda_1C_1 + \lambda_3C_3 + \dots + \lambda_jC_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . This means

$$\begin{cases} \lambda_{-1} + \lambda_{i+2} + (\lambda_3 + \lambda_{i+4})\xi_3 + (\lambda_5 + \lambda_{i+6})\xi_5 + \dots + (\lambda_{j-i-1} + \lambda_j)\xi_{j-i-1} \\ \quad + \lambda_{j-i+1}\xi_{j-i+1} + \lambda_{j-i+3}\xi_{j-i+3} + \dots + \lambda_i\xi_i = 0 \\ \lambda_1 + \binom{i+1}{2}\lambda_i + \binom{i+3}{2}\lambda_{i+2}\xi_3 + [\binom{4}{2}\lambda_3 + \binom{i+5}{2}\lambda_{i+4}]\xi_5 + [\binom{6}{2}\lambda_5 + \binom{i+7}{2}\lambda_{i+6}]\xi_7 \\ \quad + \dots + [\binom{j-i-2}{2}\lambda_{j-i-3} + \binom{j-1}{2}\lambda_{j-2}]\xi_{j-i-1} \\ \quad + \binom{j-i}{2}\lambda_{j-i-1}\xi_{j-i+1} + \binom{j-i+2}{2}\lambda_{j-i+1}\xi_{j-i+3} + \dots + \binom{i-1}{2}\lambda_{i-2}\xi_i = 0. \end{cases}$$

Since  $1, \xi_3, \xi_5, \dots, \xi_i$  are  $\mathbb{Q}$ -linearly independent, this implies

$$\begin{cases} \lambda_{-1} + \lambda_{i+2} = \lambda_3 + \lambda_{i+4} = \lambda_5 + \lambda_{i+6} = \dots = \lambda_{j-i-1} + \lambda_j = 0 \\ \lambda_1 + \binom{i+1}{2}\lambda_i = \binom{i+3}{2}\lambda_{i+2} = \binom{4}{2}\lambda_3 + \binom{i+5}{2}\lambda_{i+4} = \binom{6}{2}\lambda_5 + \binom{i+7}{2}\lambda_{i+6} \\ \quad = \dots = \binom{j-i-2}{2}\lambda_{j-i-3} + \binom{j-1}{2}\lambda_{j-2} = 0 \\ \lambda_{j-i-1} = \lambda_{j-i+1} = \dots = \lambda_i = 0. \end{cases}$$

Since  $\binom{t}{2} \neq \binom{t+i+1}{2}$  for any  $t \in \{4, 6, \dots, j-i-2\}$ , this is easily seen to imply  $\lambda_{-1} = \lambda_1 = \lambda_3 = \dots = \lambda_j = 0$ . This concludes the proof of Lemma 2.

## 2.4 Connection with a Siegel-type criterion

The following result is analogous to Theorem 3, but its proof is much easier. It relies on Siegel's ideas for linear independence (see for instance [7], p. 81–82 and 215–216, or [20], Proposition 4.1).

**Proposition 1.** *Let  $1 \leq k \leq p - 1$ , and  $e_1, \dots, e_k \in \mathbb{R}^p$  be linearly independent vectors.*

*Let  $(Q_n)_{n \geq 1}$  be an increasing sequence of positive integers, and for any  $n \geq 1$ , let  $L_n^{(t)} = \ell_{1,n}^{(t)}X_1 + \dots + \ell_{p,n}^{(t)}X_p$  be  $p$  linearly independent linear forms on  $\mathbb{R}^p$  (for  $1 \leq t \leq p$ ), with integer coefficients  $\ell_{i,n}^{(t)}$  such that, as  $n \rightarrow \infty$ :*

$$|L_n^{(t)}(e_j)| \leq Q_n^{-\tau_j + o(1)} \text{ for any } j \in \{1, \dots, k\} \text{ and any } t \in \{1, \dots, p\},$$

where  $\tau_1, \dots, \tau_k > 0$  are real numbers, and

$$\max_{\substack{1 \leq i \leq p \\ 1 \leq t \leq p}} |\ell_{i,n}^{(t)}| \leq Q_n^{1+o(1)}.$$

Then:

(a) Conclusions (i) and (ii) of Theorem 3 hold.

(b) Let  $\varepsilon > 0$ , and  $n$  be sufficiently large (in terms of  $\varepsilon$ ). Let  $\mathcal{C}_n$  denote the set of all vectors that can be written as  $\lambda_1 e_1 + \dots + \lambda_k e_k + u$  with:

$$\begin{cases} \lambda_1, \dots, \lambda_k \in \mathbb{R} \text{ such that } |\lambda_j| \leq Q_n^{\tau_j - \varepsilon} \text{ for any } j \in \{1, \dots, k\} \\ u \in (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^\perp \text{ such that } \|u\| \leq Q_n^{-1 - \varepsilon} \end{cases}$$

Then  $\mathcal{C}_n \cap \mathbb{Z}^p = \{(0, \dots, 0)\}$ .

The main difference with Theorem 3 is that we require here  $p$  linearly independent linear forms for any  $n$ . This makes the proof much easier, and enables one to get rid of several important assumptions of Theorem 3 (namely  $Q_{n+1} = Q_n^{1+o(1)}$ ,  $\tau_1, \dots, \tau_k$  pairwise distinct, and  $|L_n(e_j)|$  not too small).

If  $Q_{n+1} = Q_n^{1+o(1)}$  in Proposition 1 then in (b) we may replace  $Q_n$  with any  $Q$ , by letting  $n$  be such that  $Q_n \leq Q < Q_{n+1}$ .

**Proof** of Proposition 1: To prove conclusion (i) of Theorem 3, let  $F$  be a subspace of  $\mathbb{R}^p$  defined over  $\mathbb{Q}$ , of dimension  $d$ , which contains  $e_1, \dots, e_k$ . Let  $n$  be sufficiently large. Up to reordering  $L_n^{(1)}, \dots, L_n^{(p)}$ , we may assume the restrictions of  $L_n^{(1)}, \dots, L_n^{(d)}$  to  $F$  to be linearly independent linear forms on  $F$ . Denoting by  $(u_1, \dots, u_d)$  a basis of  $F$  consisting in vectors of  $\mathbb{Z}^p$ , the matrix  $[L_n^{(t)}(u_j)]_{1 \leq t, j \leq d}$  has a non-zero integer determinant. By making suitable linear combinations of the columns, the values  $L_n^{(t)}(e_1), \dots, L_n^{(t)}(e_k)$  appear and lead to the upper bound  $Q_n^{d-k-\tau_1-\dots-\tau_k+o(1)}$  on the absolute value of this determinant. This concludes the proof of (i) of Theorem 3.

To prove part (b) of Proposition 1 (which implies conclusion (ii) of Theorem 3), we let  $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathcal{C}_n \cap \mathbb{Z}^p$  be non-zero; then  $L_n^{(t)}(P) \neq 0$  for some  $t$ , but  $L_n^{(t)}(P) \in \mathbb{Z}$  and  $|L_n^{(t)}(P)| < 1$ . This concludes the proof of Proposition 1.

This criterion is used (in its qualitative part (a), and in the special case of  $k = 1$  point) by Nikishin [22] to prove that  $1, \text{Li}_1(z), \text{Li}_2(z), \dots, \text{Li}_a(z)$  are  $\mathbb{Q}$ -linearly independent provided that  $z = p/q < 0$  where  $p$  and  $q$  are integers and  $q$  is sufficiently large in terms of  $|p|$  and  $a$ ; here  $\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$  is the  $s$ -th polylogarithm, for  $s \geq 1$ . Marcovecchio has given [20] a proof of Rivoal's result [24]

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \text{Li}_1(z), \text{Li}_2(z), \dots, \text{Li}_a(z)) \geq \frac{\log a}{1 + \log 2} (1 + o(1)) \quad (2.2)$$

as  $a \rightarrow \infty$ , for any  $z \in \mathbb{Q}$  such that  $|z| < 1$ , based on Siegel's ideas (whereas Rivoal used Nesterenko's criterion). However no such proof is known for Ball-Rivoal's theorem (1.1).

Using essentially Proposition 1 with  $k = 2$ , Gutnik proved [13] that the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \log 2 \\ \zeta(2) \end{pmatrix}, \begin{pmatrix} \zeta(2) \\ -3\zeta(3) \end{pmatrix} \quad (2.3)$$

are  $\mathbb{Q}$ -linearly independent in  $\mathbb{R}^2$  (so that, for any  $r \in \mathbb{Q}^*$ , at least one number among  $\zeta(2) - 2r \log 2$  and  $3\zeta(3) - r\zeta(2)$  is irrational). More recently he obtained also [14] the  $\mathbb{Q}$ -linear independence of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2\zeta(3) \\ 3\zeta(4) \end{pmatrix}, \begin{pmatrix} 3\zeta(4) \\ 6\zeta(5) \end{pmatrix}. \quad (2.4)$$

Such results of linear independence are known only for at most 4 vectors (analogously to the irrationality of odd zeta values, known only for  $\zeta(3)$ ). To obtain the linear independence of more than 4 vectors, it is necessary (with the present methods) to consider polylogarithms at points  $z$  close to 0, as in Nikishin's result. In this direction, T. Hessami-Pilehrood has proved [18] that if  $q$  is greater than some explicit function of  $k$  then the following  $2k$  vectors are  $\mathbb{Q}$ -linearly independent in  $\mathbb{R}^k$ :

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad (2.5)$$

$$\begin{pmatrix} \text{Li}_1\left(\frac{-1}{q}\right) \\ \text{Li}_2\left(\frac{-1}{q}\right) \\ \vdots \\ \text{Li}_k\left(\frac{-1}{q}\right) \end{pmatrix}, \dots, \begin{pmatrix} \binom{j-1}{j-1} \text{Li}_j\left(\frac{-1}{q}\right) \\ \binom{j}{j-1} \text{Li}_{j+1}\left(\frac{-1}{q}\right) \\ \vdots \\ \binom{j+k-2}{j-1} \text{Li}_{j+k-1}\left(\frac{-1}{q}\right) \end{pmatrix}, \dots, \begin{pmatrix} \binom{k-1}{k-1} \text{Li}_k\left(\frac{-1}{q}\right) \\ \binom{k}{k-1} \text{Li}_{k+1}\left(\frac{-1}{q}\right) \\ \vdots \\ \binom{2k-2}{k-1} \text{Li}_{2k-1}\left(\frac{-1}{q}\right) \end{pmatrix}.$$

The same result holds with  $1/q$  instead of  $-1/q$ ; see also Gutnik's preprints cited in [18].

The proofs of (2.3), (2.4) and (2.5) rely on the same principle: in addition to a "derivation procedure" as in the definition of  $S_n''$  (see the introduction), they involve constructing several sequences of linearly independent small linear forms, as in Nikishin's proof, and using essentially Proposition 1. All of them lead to the  $\mathbb{Q}$ -linear independence of the whole set of  $p$  vectors under consideration (with strict restrictions: few vectors or a point  $z$  close to 0), because Proposition 1 is applied with  $k + \tau_1 + \dots + \tau_k > p - 1$  so that conclusion (a) yields  $\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p) \geq p$ . However the proofs can probably be modified to provide lower bounds such as Eq. (2.2).

### 3 Proof of the criterion

This section is devoted to proving Theorem 3. Reindexing  $e_1, \dots, e_k$  is necessary, we assume that  $\tau_1 > \dots > \tau_k > 0$ . This will be used in §§3.2 and 3.5.

#### 3.1 Reduction to the non-oscillatory case

In this subsection, we deduce the general case of Theorem 3 from the special case where  $\omega_1 = \dots = \omega_k = 0$ ; notice that in this case we have  $\phi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$  for any  $j \in \{1, \dots, k\}$ ,

so that Eq. (1.2) reads  $|L_n(e_j)| = Q_n^{-\tau_j+o(1)}$ . This special case will be proved in the following subsections, under the assumption that  $Q_{n+1} = Q_n^{1+o(1)}$  (which is weaker than the assumption  $Q_{n+1} = Q_n^{1+O(1/n)}$  we make when  $\omega_1, \dots, \omega_k$  may be non-zero).

Let  $\omega_1, \dots, \omega_k, \varphi_1, \dots, \varphi_k$ , and  $(Q_n)$  be as in Theorem 3, with  $Q_{n+1} = Q_n^{1+O(1/n)}$ . Since there are infinitely many integers  $n$  such that, for any  $j \in \{1, \dots, k\}$ ,  $n\omega_j + \varphi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$ , Proposition 1 of [8] provides  $\varepsilon, \lambda > 0$  and an increasing function  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\lim_{n \rightarrow \infty} \frac{\psi(n)}{n} = \lambda$  and, for any  $n$  and any  $j \in \{1, \dots, k\}$ ,  $|\cos(\psi(n)\omega_j + \varphi_j)| \geq \varepsilon$ . Let  $L'_n = L_{\psi(n)}$  and  $Q'_n = Q_{\psi(n)}$  for any  $n \geq 1$ . Then we have  $|L'_n(e_j)| = Q'_n^{-\tau_j+o(1)}$  because  $|\cos(\psi(n)\omega_j + \varphi_j)| = Q_{\psi(n)}^{o(1)}$ . Let us check that  $Q'_{n+1} = Q_n^{1+o(1)}$ ; then the special case of Theorem 3 will apply to the sequences  $(L'_n)_{n \geq 1}$  and  $(Q'_n)_{n \geq 1}$ , with the same  $e_1, \dots, e_k, \tau_1, \dots, \tau_k$ : this will conclude the proof.

Since  $Q_{n+1} = Q_n^{1+O(1/n)}$  there exists  $M > 0$  such that, for any  $n \geq 1$ ,  $Q_{n+1} \leq Q_n^{1+M/n}$ ; this implies

$$Q_{n+\ell} \leq Q_n^{(1+M/n)^\ell}$$

for any  $\ell \geq 0$ . Letting  $\delta_n = \psi(n+1) - \psi(n) \geq 1$ , we have:

$$Q'_{n+1} = Q_{\psi(n)+\delta_n} \leq Q_{\psi(n)}^{(1+M/\psi(n))^{\delta_n}} \leq Q_{\psi(n)}^{\exp(M\delta_n/\psi(n))} = Q_n^{1+o(1)}$$

by writing the exponent  $\delta_n$  as  $\frac{\psi(n)}{M} \frac{M\delta_n}{\psi(n)}$  with  $(1 + M/\psi(n))^{\psi(n)/M} \leq e$ , and using the fact that  $\delta_n = o(n)$  since  $\psi(n) = \lambda n + o(n)$ . This concludes the reduction to the case where  $\omega_1 = \dots = \omega_k = 0$  and  $Q_{n+1} = Q_n^{1+o(1)}$ .

### 3.2 Proof of (ii)

Let us start with the easiest part of Theorem 3, namely (ii).

We shall prove simultaneously that  $e_1, \dots, e_k$  are linearly independent in  $\mathbb{R}^p$ , and that  $F \cap \mathbb{Q}^p = \{(0, \dots, 0)\}$  where  $F = \text{Span}_{\mathbb{R}}(e_1, \dots, e_k)$ . With this aim in mind, we assume (by contradiction) that there exist real numbers  $\lambda_1, \dots, \lambda_k$ , not all zero, such that  $\sum_{j=1}^k \lambda_j e_j \in \mathbb{Q}^p$ ; multiplying all  $\lambda_j$  by a common denominator of the coordinates, we may assume  $\sum_{j=1}^k \lambda_j e_j \in \mathbb{Z}^p$ . Then  $\kappa_n = L_n(\sum_{j=1}^k \lambda_j e_j) = \sum_{j=1}^k \lambda_j L_n(e_j)$  is an integer for any  $n \geq 1$ . Now if  $n$  is sufficiently large then  $|\kappa_n| \leq \sum_{j=1}^k |\lambda_j| |L_n(e_j)| < 1$ , so that  $\kappa_n = 0$ . Let  $j_0$  denote the largest integer  $j$  such that  $\lambda_j \neq 0$ . Then for any  $n$  sufficiently large, the fact that  $\kappa_n = 0$  implies  $|\lambda_{j_0} L_n(e_{j_0})| = |\sum_{j=1}^{j_0-1} \lambda_j L_n(e_j)|$  so that

$$|\lambda_{j_0}| \leq \sum_{j=1}^{j_0-1} |\lambda_j| \frac{|L_n(e_j)|}{|L_n(e_{j_0})|} \leq \sum_{j=1}^{j_0-1} |\lambda_j| Q_n^{\tau_{j_0} - \tau_j + o(1)}$$

as  $n \rightarrow \infty$ . Now the right handside tends to 0 as  $n \rightarrow \infty$  because we have assumed that  $\tau_1 > \dots > \tau_k$ , so that  $\lambda_{j_0} = 0$ : this contradicts the definition of  $\lambda_{j_0}$ .

Therefore such real numbers  $\lambda_1, \dots, \lambda_k$  cannot exist, and this concludes the proof of (ii).

### 3.3 Proof that (ii) and (iii) imply (i)

Before proceeding in §§3.4 and 3.5 to the proof of (iii), which is the main part, we deduce (i) from (ii) and (iii). Recall that the second statement of (i) is equivalent to the first one (which we shall prove now) thanks to Lemma 1 proved in §2.3.

Let  $F$  be a subspace of  $\mathbb{R}^p$ , defined over  $\mathbb{Q}$ , which contains  $e_1, \dots, e_k$ . Letting  $s = \dim F$ , we have  $s > k$  using (ii). Assertion (iii) yields, for any  $\varepsilon > 0$  and any  $Q$  sufficiently large (in terms of  $\varepsilon$ ), a subset  $\mathcal{C}$  of  $\mathbb{R}^s$  with no integer point other than  $(0, \dots, 0)$ . Now  $\mathcal{C} \cap F$  is a convex body, compact and symmetric with respect to the origin, in the Euclidean space  $F$ . On the other hand,  $\mathbb{Z}^p \cap F$  is a lattice in  $F$  because  $F$  is defined over  $\mathbb{Q}$ . Therefore Minkowski's convex body theorem (see for instance Chapter III of [6]) implies that  $\mathcal{C} \cap F$  has volume less than  $2^s \det(\mathbb{Z}^p \cap F)$ . Now this volume is equal to  $Q^{\tau_1 + \dots + \tau_k - (1+\varepsilon)(s-k)}$ , up to a multiplicative constant which depends only on  $s, k, e_1, \dots, e_k$ . Since  $Q$  can be chosen arbitrarily large, we have  $\tau_1 + \dots + \tau_k - (1+\varepsilon)(s-k) \leq 0$ . Now  $\varepsilon$  can be any positive real number, so that  $\tau_1 + \dots + \tau_k \leq s - k$ , thereby concluding the proof of (i).

### 3.4 The main step

We state and prove in this section the main tool in the proof of Theorem 3, namely Proposition 2.

Recall that Nesterenko's linear independence criterion is much easier to prove if the linear forms  $L_n, L_{n+1}, \dots, L_{n+p-1}$  are linearly independent (see §2.3 of [12] or the references to Siegel's criterion in the introduction). Of course this is not always the case, but Proposition 2 is a step in this direction. Actually letting  $F = \text{Span}_{\mathbb{R}}(e_1, \dots, e_k)$ , we consider the restrictions  $L_n|_F$  of the linear forms to  $F$ ; recall that  $\dim F = k$  thanks to (ii) proved in §3.2. It is not true in general that  $L_n|_F, L_{n+1}|_F, \dots, L_{n+k-1}|_F$  are linearly independent linear forms on  $F$ : for instance, the equality  $L_n = L_{n+1}$  might hold for any even integer  $n$  (because of the error terms  $o(1)$  in the assumptions of Theorem 3). To make this statement correct, we introduce a function  $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$  such that  $\varphi(n) \geq n + 1$  for any  $n \geq 1$ : the integer  $\varphi(n)$  plays the role of  $n + 1$ , that is: applying  $\varphi$  corresponds to "taking the next integer". The idea is that  $\varphi(n)$  will be large enough (in comparison to  $n$ ) to avoid obvious counter-examples as above coming from error terms. In the situation of Theorem 3,  $\varphi(n)$  will be defined by the property  $Q_{\varphi(n)-1} \leq Q_n^{1+\varepsilon_1} < Q_{\varphi(n)}$  (where  $\varepsilon_1$  is a small positive real number); in this way, the error terms  $o(1)$  in the assumptions of Theorem 3 will not be a problem any more.

With this definition, we are able to prove that for any  $n$  sufficiently large, the linear forms  $L_n|_F, L_{\varphi(n)}|_F, L_{\varphi^2(n)}|_F, \dots, L_{\varphi^{k-1}(n)}|_F$  on  $F$  are linearly independent (where  $\varphi^i = \varphi \circ \dots \circ \varphi$ ), so that they make up a basis of the dual vector space  $F^*$ . In the proof of Theorem 3 we shall need the following quantitative version of this property: in writing the linear form  $e_j^*$  (defined by  $e_j^*(\lambda_1 e_1 + \dots + \lambda_k e_k) = \lambda_j$ ) as a linear combination of  $\frac{1}{L_n(e_j)} L_n|_F, \frac{1}{L_{\varphi(n)}(e_j)} L_{\varphi(n)}|_F, \dots, \frac{1}{L_{\varphi^{k-1}(n)}(e_j)} L_{\varphi^{k-1}(n)}|_F$ , the coefficients that appear are bounded independently from  $n$  (actually they are between  $-3$  and  $3$ ).

We state and prove Proposition 2 in a general setting, with assumptions on  $\varphi$  that



make the proof work, because this result might be of independent interest. Its proof relies on estimating a  $k \times k$  determinant.

**Proposition 2.** *Let  $p > k > 0$ ,  $e_1, \dots, e_k \in \mathbb{R}^p$  and  $(L_n)_{n \geq 1}$  be a sequence of linear forms on  $\mathbb{R}^p$ . For any  $n \geq 1$  and any  $j \in \{1, \dots, k\}$ , let  $\varepsilon_{j,n} = |L_n(e_j)|$ . Let  $n_0 \geq 1$  be such that  $\varepsilon_{j,n} \neq 0$  for any  $j$  and any  $n \geq n_0$ . Let also  $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$  be such that*

$$\forall n \geq 1 \quad \varphi(n) \geq n + 1 \quad (3.1)$$

and

$$\forall i \in \{1, \dots, k-1\}, \forall n \geq n_0, \forall n' \geq \varphi(n) \quad \frac{\varepsilon_{i,n'}}{\varepsilon_{i,n}} \leq \frac{1}{(k+1)!} \frac{\varepsilon_{i+1,n'}}{\varepsilon_{i+1,n}}. \quad (3.2)$$

Let  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  and  $M = \sum_{j=1}^k \lambda_j e_j$ . Then for any  $n \geq n_0$  and any  $j \in \{1, \dots, k\}$  we have:

$$|\lambda_j| \leq \left(1 + \frac{1}{k} + \frac{1}{k^2}\right) \sum_{i=1}^k \frac{|L_{\varphi^{i-1}(n)}(M)|}{|L_{\varphi^{i-1}(n)}(e_j)|}.$$

Taking  $M = 0$  implies that, under the assumptions of this proposition,  $e_1, \dots, e_k$  are linearly independent ; in our case this was proved already (in a simpler way) in §3.2.

Proposition 2 is optimal up to the value of the constant  $1 + \frac{1}{k} + \frac{1}{k^2}$ : it would be false with a constant less than  $1/k$  instead (this is immediately seen by taking  $M = e_j$ ). We did not try to improve on the constant  $1 + \frac{1}{k} + \frac{1}{k^2}$ , but anyway it could easily be made smaller by replacing  $\frac{1}{(k+1)!}$  in (3.2) with a smaller constant.

Assumption (3.1) has been explained before the statement of Proposition 2. Assumption (3.2) means that  $\frac{\varepsilon_{i,n}}{\varepsilon_{i+1,n}} \rightarrow 0$  as  $n \rightarrow \infty$ , and  $\varphi$  provides a control upon the rate of decrease of this sequence. In the proof of Theorem 3, this assumption will be checked in §3.5 using the assumption  $\tau_i > \tau_{i+1}$ .

In the proof of Proposition 2 we shall use the following lemma.

**Lemma 3.** *Under the assumptions of Proposition 2, for any  $\sigma \in \mathfrak{S}_k$  and any  $n \geq n_0$  we have:*

$$\prod_{j=1}^k \varepsilon_{j, \varphi^{\sigma(j)-1}(n)} \leq \eta_\sigma \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)} \quad (3.3)$$

where  $\eta_\sigma = \frac{1}{(k+1)!}$  for  $\sigma \neq \text{Id}$ , and  $\eta_{\text{Id}} = 1$ .

**Proof** of Lemma 3: First of all, let us notice that assumption (3.2) implies:

$$\forall i, j \in \{1, \dots, k\} \text{ with } i < j, \forall n \geq n_0, \forall n' \geq \varphi(n) \quad \varepsilon_{i,n'} \varepsilon_{j,n} \leq \frac{1}{(k+1)!} \varepsilon_{i,n} \varepsilon_{j,n'}. \quad (3.4)$$

For  $\sigma \neq \text{Id}$  let  $\kappa_\sigma$  denote the largest integer  $j \in \{1, \dots, k\}$  such that  $\sigma(j) \neq j$ ; put also  $\kappa_{\text{Id}} = 0$ . We are going to prove Eq. (3.3) by induction on  $\kappa_\sigma$ . If  $\kappa_\sigma \leq 1$  then  $\sigma = \text{Id}$ , so that Eq. (3.3) holds trivially. Let  $\sigma \in \mathfrak{S}_k$  be such that  $\kappa_\sigma \geq 2$ , and assume that Eq.

(3.3) holds for any  $\sigma'$  such that  $\kappa_{\sigma'} < \kappa_\sigma$ . We have  $\sigma(j) = j$  for any  $j \in \{\kappa_\sigma + 1, \dots, k\}$ , and  $\sigma(\kappa_\sigma) < \kappa_\sigma$ . Let  $j_0 = \sigma^{-1}(\kappa_\sigma)$ ; then  $j_0 < \kappa_\sigma$ . Let  $\sigma' = \sigma \circ \tau_{j_0, \kappa_\sigma}$  where  $\tau_{j_0, \kappa_\sigma}$  is the transposition that exchanges  $j_0$  and  $\kappa_\sigma$ . Then  $\sigma'(j) = j$  for any  $j \in \{\kappa_\sigma, \dots, k\}$  so that  $\kappa_{\sigma'} < \kappa_\sigma$  and Eq. (3.3) holds for  $\sigma'$ . Since  $\sigma'(j) = \sigma(j)$  for  $j \notin \{j_0, \kappa_\sigma\}$ ,  $\sigma'(j_0) = \sigma(\kappa_\sigma)$  and  $\sigma'(\kappa_\sigma) = \kappa_\sigma$ , this implies (using the fact that  $\eta_{\sigma'} \leq 1$ )

$$\varepsilon_{j_0, \varphi^{\sigma(\kappa_\sigma)-1}(n)} \varepsilon_{\kappa_\sigma, \varphi^{\kappa_\sigma-1}(n)} \prod_{\substack{1 \leq j \leq k \\ j \notin \{j_0, \kappa_\sigma\}}} \varepsilon_{j, \varphi^{\sigma(j)-1}(n)} \leq \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)}.$$

On the other hand, Eq. (3.4) implies

$$\varepsilon_{j_0, \varphi^{\kappa_\sigma-1}(n)} \varepsilon_{\kappa_\sigma, \varphi^{\sigma(\kappa_\sigma)-1}(n)} \leq \frac{1}{(k+1)!} \varepsilon_{j_0, \varphi^{\sigma(\kappa_\sigma)-1}(n)} \varepsilon_{\kappa_\sigma, \varphi^{\kappa_\sigma-1}(n)}$$

because  $j_0 < \kappa_\sigma$  and  $\sigma(\kappa_\sigma) < \kappa_\sigma$  (so that  $\varphi^{\sigma(\kappa_\sigma)-1}(n) < \varphi^{\kappa_\sigma-1}(n)$ ). Multiplying out the previous two inequalities yields Eq. (3.3) for  $\sigma$ , since  $\sigma(j_0) = \kappa_\sigma$ . This concludes the proof of Lemma 3.

**Proof** of Proposition 2: Let  $n \geq n_0$ . Denote by  $\Delta$  the determinant of the matrix  $[L_{\varphi^{i-1}(n)}(e_j)]_{1 \leq i, j \leq k}$ . We have

$$|\Delta| \geq \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)} - \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma \neq \text{Id}}} \prod_{j=1}^k \varepsilon_{j, \varphi^{\sigma(j)-1}(n)}$$

so that Lemma 3 applied to all  $\sigma \neq \text{Id}$  yields

$$|\Delta| \geq \left(1 - \frac{1}{k+1}\right) \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)}.$$

On the other hand, let  $j \in \{1, \dots, k\}$  and  $\Delta_j$  be the determinant of the matrix with columns  $C_1, \dots, C_k$  given by:

$$\begin{cases} C_{j'} = [L_{\varphi^{i-1}(n)}(e_{j'})]_{1 \leq i \leq k} \text{ for } j' \neq j, \\ C_j = [L_{\varphi^{i-1}(n)}(M)]_{1 \leq i \leq k}. \end{cases}$$

Since  $M = \lambda_1 e_1 + \dots + \lambda_k e_k$ , the lower bound proved for  $|\Delta|$  yields:

$$|\Delta_j| = |\lambda_j| \cdot |\Delta| \geq \frac{k}{k+1} |\lambda_j| \prod_{j'=1}^k \varepsilon_{j', \varphi^{j'-1}(n)}. \quad (3.5)$$

Now expanding  $\Delta_j$  with respect to the  $j$ -th column yields

$$|\Delta_j| \leq \sum_{i=1}^k |L_{\varphi^{i-1}(n)}(M)| \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(j)=i}} \prod_{\substack{1 \leq j' \leq k \\ j' \neq j}} \varepsilon_{j', \varphi^{\sigma(j')-1}(n)}$$

so that, using Lemma 3:

$$|\Delta_j| \leq \left( \prod_{j'=1}^k \varepsilon_{j', \varphi^{j'-1}(n)} \right) \sum_{i=1}^k \frac{|L_{\varphi^{i-1}(n)}(M)|}{\varepsilon_{j, \varphi^{i-1}(n)}} \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(j)=i}} \eta_\sigma.$$

Now we have  $\eta_\sigma = 1$  for at most one  $\sigma$ , and  $\eta_\sigma = \frac{1}{(k+1)!}$  for all other permutations  $\sigma$  among the  $(k-1)!$  such that  $\sigma(j) = i$ , so that

$$\sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(j)=i}} \eta_\sigma \leq 1 + \frac{(k-1)!}{(k+1)!} = \frac{k+1 + \frac{1}{k}}{k+1},$$

thereby completing the proof of the upper bound

$$|\Delta_j| \leq \frac{k+1 + \frac{1}{k}}{k+1} \left( \prod_{j'=1}^k \varepsilon_{j', \varphi^{j'-1}(n)} \right) \sum_{i=1}^k \frac{|L_{\varphi^{i-1}(n)}(M)|}{\varepsilon_{j, \varphi^{i-1}(n)}}.$$

Comparing this with the lower bound (3.5) concludes the proof of Proposition 2.

### 3.5 Proof of (iii)

We are now in position to prove the remaining part of Theorem 3, namely (iii). We assume  $\tau_1 > \dots > \tau_k > 0$  and  $\omega_1 = \dots = \omega_k = 0$  (see §3.1), so that  $|L_n(e_j)| = Q_n^{-\tau_j + o(1)}$ .

Let  $\varepsilon > 0$ . We choose  $\varepsilon_1 > 0$  sufficiently small, so that

$$((1 + \varepsilon_1)^{k-1} - 1)\tau_1 < \varepsilon/4 \text{ and } (1 + \varepsilon_1)^{k-1} \leq 1 + \varepsilon/2. \quad (3.6)$$

If  $k = 1$  there is no assumption on  $\varepsilon_1$ , because it does not really appear in the proof: Proposition 2 is a triviality in this case, and the proof essentially reduces to that of [12].

For any  $n \geq 1$ , we define  $\varphi(n)$  by  $Q_{\varphi(n)-1} \leq Q_n^{1+\varepsilon_1} < Q_{\varphi(n)}$ , because the sequence  $(Q_n)$  is increasing and we may assume  $Q_n \geq 1$  for any  $n$ . Then we have  $\varphi(n) \geq n + 1$ . This implies  $\lim_{n \rightarrow +\infty} \varphi(n) = +\infty$ , so that  $Q_{\varphi(n)} = Q_{\varphi(n)-1}^{1+o(1)}$  (because we assume  $Q_{n+1} = Q_n^{1+o(1)}$ ) and

$$Q_{\varphi(n)} = Q_n^{1+\varepsilon_1+o(1)}; \quad (3.7)$$

here  $o(1)$  denotes any sequence that tends to 0 as  $n \rightarrow \infty$ . Letting  $\varepsilon_{j,n} = |L_n(e_j)|$  for any  $j \in \{1, \dots, k\}$  and any  $n \geq 1$ , the assumption  $\varepsilon_{j,n} = Q_n^{-\tau_j + o(1)}$  implies  $\varepsilon_{j,n} \neq 0$  for any  $j, n$  with  $n$  sufficiently large. Let  $i \in \{1, \dots, k-1\}$ ,  $n_0$  be sufficiently large, and let  $n \geq n_0$ ,  $n' \geq \varphi(n)$ . Then we have  $\varepsilon_{i,n'} = Q_{n'}^{-\tau_i + o(1)}$  (and the analogous property for  $\varepsilon_{i+1,n'}$ ), where  $o(1)$  denotes a sequence that tends to 0 as  $n \rightarrow +\infty$ , because we assume that  $n' \geq \varphi(n) \geq n + 1$ . Using Eq. (3.7) and the assumption  $\tau_i > \tau_{i+1}$  we obtain

$$\frac{\varepsilon_{i,n'} \varepsilon_{i+1,n}}{\varepsilon_{i,n} \varepsilon_{i+1,n'}} = \frac{Q_n^{\tau_i - \tau_{i+1} + o(1)}}{Q_{n'}^{\tau_i - \tau_{i+1} + o(1)}} \leq \frac{Q_n^{\tau_i - \tau_{i+1} + o(1)}}{Q_{\varphi(n)}^{\tau_i - \tau_{i+1} + o(1)}} = Q_n^{-(\tau_i - \tau_{i+1})\varepsilon_1 + o(1)} \leq \frac{1}{(k+1)!}$$

if  $n_0$  is sufficiently large, so that assumption (3.2) of Proposition 2 holds.

Let  $Q$  be sufficiently large in terms of  $\varepsilon$ . Let  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  be such that  $|\lambda_j| \leq Q^{\tau_j}$  for any  $j$ , and let  $u \in (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^{\perp}$  with  $\|u\| \leq Q^{-1-\varepsilon}$ . Assume that the point  $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u$  belongs to  $\mathbb{Z}^p \setminus \{(0, \dots, 0)\}$ .

Let  $n$  be the least integer such that:

$$\text{for any } n' \geq n \text{ and any } j \in \{1, \dots, k\}, \text{ we have } |L_{n'}(e_j)| \leq \frac{Q^{-\tau_j}}{2k}.$$

Of course  $n$  depends on  $Q$ , and  $n$  tends to  $\infty$  as  $Q \rightarrow \infty$ . Therefore if  $u_n = o(1)$ , that is  $u_n \rightarrow 0$  as  $n \rightarrow \infty$ , then  $u_n$  tends also to 0 as  $Q \rightarrow \infty$ .

By minimality of  $n$  we have for some  $j \in \{1, \dots, k\}$ :

$$Q_n^{-\tau_j+o(1)} = |L_n(e_j)| \leq \frac{Q^{-\tau_j}}{2k} < |L_{n-1}(e_j)| = Q_{n-1}^{-\tau_j+o(1)} = Q_n^{-\tau_j+o(1)}$$

so that  $Q = Q_n^{1+o(1)}$ .

Let  $\ell$  denote the least integer such that

$$\text{for any } \ell' \geq \ell \text{ and any } j \in \{1, \dots, k\}, \text{ we have } |\lambda_j L_{\ell'}(e_j)| \leq \frac{1}{2k}. \quad (3.8)$$

Since this upper bound holds for any  $\ell' \geq n$ , this integer exists and we have  $\ell \leq n$ .

The integer  $\ell$  depends on  $Q$  and on the choice of  $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$ . Let us prove that  $\ell \rightarrow \infty$  as  $Q \rightarrow \infty$ , independently on the choice of  $P$ . Let  $\ell_0 \geq 1$ , and  $K_{\ell_0}$  denote the set of all points  $M = \lambda'_1 e_1 + \dots + \lambda'_k e_k$  with  $|\lambda'_j| \leq \frac{1}{2k|L_{\ell_0}(e_j)|}$  for any  $j \in \{1, \dots, k\}$  (by making  $\ell_0$  larger if necessary, we may assume  $L_{\ell_0}(e_j) \neq 0$ ). Then assertion (ii) proved in §3.2 yields  $K_{\ell_0} \cap (\mathbb{Z}^p \setminus \{(0, \dots, 0)\}) = \emptyset$ . Since  $K_{\ell_0}$  is compact and  $\mathbb{Z}^p \setminus \{(0, \dots, 0)\}$  is discrete, there exists  $\eta > 0$  such that the distance of any point of  $K_{\ell_0}$  to any point of  $\mathbb{Z}^p \setminus \{(0, \dots, 0)\}$  is greater than  $\eta$ . If  $Q$  is sufficiently large, namely such that  $Q^{-1-\varepsilon} < \eta$ , then for any choice of  $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$  with  $\|u\| \leq Q^{-1-\varepsilon} < \eta$  we have  $\lambda_1 e_1 + \dots + \lambda_k e_k \notin K_{\ell_0}$  so that  $|\lambda_j L_{\ell_0}(e_j)| > \frac{1}{2k}$  for some  $j \in \{1, \dots, k\}$ . By definition of  $\ell$  this implies  $\ell > \ell_0$ , and concludes the proof that  $\ell \rightarrow \infty$  as  $Q \rightarrow \infty$ . In what follows, a sequence denoted by  $o(1)$  will tend to 0 as  $n$ ,  $\ell$  or  $Q$  tends to  $\infty$ ; therefore in any case, it tends to 0 as  $Q \rightarrow \infty$ .

Now, writing  $u = (u_1, \dots, u_p)$  we have  $\max_{1 \leq h \leq p} |u_h| \leq \|u\|$  so that for any  $i \in \{1, \dots, k\}$ :

$$\begin{aligned} |L_{\varphi^{i-1}(\ell)}(u)| &\leq \sum_{h=1}^p |u_h| |\ell_{h, \varphi^{i-1}(\ell)}| \leq p \|u\| Q_{\varphi^{i-1}(\ell)}^{1+o(1)} \\ &\leq p \|u\| Q_{\ell}^{(1+\varepsilon_1)^{i-1}(1+o(1))} \text{ using Eq. (3.7)} \\ &\leq p Q^{-1-\varepsilon} Q_n^{(1+\varepsilon_1)^{i-1}(1+o(1))} \text{ since } \ell \leq n \\ &\leq Q^{-1-\varepsilon+(1+\varepsilon_1)^{i-1}(1+o(1))} \text{ since } Q = Q_n^{1+o(1)} \\ &\leq Q^{-\varepsilon/2} \text{ thanks to the choice of } \varepsilon_1. \end{aligned}$$

On the other hand, Eq. (3.8) yields for any  $i \in \{1, \dots, k\}$ :

$$|L_{\varphi^{i-1}(\ell)}(\sum_{j=1}^k \lambda_j e_j)| \leq \sum_{j=1}^k \frac{1}{2k} = \frac{1}{2}.$$

Therefore we obtain for the point  $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathbb{Z}P$ :

$$|L_{\varphi^{i-1}(\ell)}(P)| \leq Q^{-\varepsilon/2} + \frac{1}{2} < 1.$$

Now  $L_{\varphi^{i-1}(\ell)}(P)$  is an integer for any  $i \in \{1, \dots, k\}$ , so it is zero. This yields the following upper bound on  $|L_{\varphi^{i-1}(\ell)}(M)|$  (where we let  $M = \sum_{j=1}^k \lambda_j e_j$ ):

$$|L_{\varphi^{i-1}(\ell)}(M)| = |L_{\varphi^{i-1}(\ell)}(u)| \leq Q^{-\varepsilon/2}.$$

Combining this upper bound with Proposition 2 yields, for any  $j \in \{1, \dots, k\}$ :

$$\begin{aligned} |\lambda_j L_{\ell-1}(e_j)| &\leq \left(1 + \frac{1}{k} + \frac{1}{k^2}\right) \sum_{i=1}^k Q^{-\varepsilon/2} Q^{\tau_j + o(1)} Q_{\varphi^{i-1}(\ell)}^{-\tau_j + o(1)} Q_{\ell-1} \\ &\leq Q_{\ell}^{\tau_j((1+\varepsilon_1)^{i-1}-1)+o(1)} Q^{-\varepsilon/2} \text{ using Eq. (3.7)} \\ &\leq Q_{\ell}^{\varepsilon/4+o(1)} Q^{-\varepsilon/2} \text{ using the assumption } \tau_j \leq \tau_1 \text{ and Eq. (3.6)} \\ &\leq Q_{\ell}^{-\varepsilon/4+o(1)} \leq \frac{1}{2k} \text{ since } Q_{\ell} \leq Q_n = Q^{1+o(1)}. \end{aligned}$$

This contradicts the minimality of  $\ell$  in Eq. (3.8), thereby concluding the proof of (iii).

## 4 Application to zeta values

### 4.1 Construction of the linear forms

Let  $a, r, n$  be positive integers such that  $a$  is odd and  $6r \leq a$ . Let

$$S_n = (2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{(t - (2r+1)n)_{2rn}^3 (t+n+1)_{2rn}^3}{(t-n)_{2n+1}^a}$$

and

$$S_n'' = \frac{1}{2} (2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{d^2}{dt^2} \left( \frac{(t - (2r+1)n)_{2rn}^3 (t+n+1)_{2rn}^3}{(t-n)_{2n+1}^a} \right)$$

where the derivative is taken at  $t$ . As usual, Pochhammer's symbol is defined by  $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$ .

Letting  $k = t - n$ , we have obviously

$$S_n = (2n)!^{a-6r} \sum_{k=1}^{\infty} \frac{(k - 2rn)_{2rn}^3 (k + 2n + 1)_{2rn}^3}{(k)_{2n+1}^a}$$

(where the sum actually starts at  $k = 2rn + 1$ ), and a similar expression for  $S_n''$ .

It is well known (see for instance Théorème 5 of [10], Lemme 1 of [2] or Lemma 1.1 of [26]) that we have

$$S_n = \ell_{0,n} + \ell_{3,n}\zeta(3) + \ell_{5,n}\zeta(5) + \dots + \ell_{a,n}\zeta(a)$$

and

$$S_n'' = \ell_{0,n}'' + \ell_{3,n} \binom{4}{2} \zeta(5) + \ell_{5,n} \binom{6}{2} \zeta(7) + \dots + \ell_{a,n} \binom{a+1}{2} \zeta(a+2)$$

with  $\ell_{0,n}, \ell_{0,n}'', \ell_{3,n}, \ell_{5,n}, \dots, \ell_{a,n} \in \mathbb{Q}$ . Moreover  $d_{2n}^{a+2}$  is a common denominator of these rational numbers (see [26], Lemma 1.4), where  $d_k$  is the least common multiple of  $1, 2, \dots, k$ . Recall (for ulterior use) that  $d_k = e^{k+o(k)}$  as  $k \rightarrow \infty$ , an equivalent form of the Prime Number Theorem.

We shall also need an upper bound on the coefficients of the linear forms  $S_n$  and  $S_n''$ , namely

$$\max \left( |\ell_{0,n}|, |\ell_{0,n}'|, |\ell_{3,n}|, |\ell_{5,n}|, \dots, |\ell_{a,n}| \right) \leq \left[ 2^{2(a-6r)} (2r+1)^{6(2r+1)} \right]^{n+o(n)}$$

as  $n \rightarrow \infty$  (see Proposition 3.1 of [26]).

The asymptotic behaviour of  $|S_n|$  and  $|S_n''|$  is more subtle. From a technical point of view, this is the main difference with T. and K. Hessami-Pilehrood's construction [17] involving analogues of  $S_n$  and  $S_n''$ : as  $n \rightarrow \infty$ ,  $S_n''$  oscillates whereas their analogue of  $S_n''$  doesn't (see Lemma 6 of [17]). We state in the following lemma the properties we need, and postpone its proof until §4.3.

**Lemma 4.** *Assume that  $a$  is sufficiently large, and  $r$  is the integer part of  $a \exp(-\sqrt{\log a})$ . Then we have, as  $n \rightarrow \infty$ :*

$$|S_n| = \varepsilon_a^{n+o(n)} \quad \text{and} \quad |S_n''| = \varepsilon_a^{m+o(n)} |\cos(n\omega_a + \varphi_a)|$$

with

$$0 < \varepsilon_a'' < \varepsilon_a \leq \frac{2^{6(r+1)}}{r^{2(a-6r)}} < 1 \quad \text{and} \quad \varphi_a \not\equiv \frac{\pi}{2} \pmod{\pi}.$$

The strict inequality  $\varepsilon_a'' < \varepsilon_a$  is very important for applying Theorem 3 (see §4.2), because it yields  $\tau_1 \neq \tau_2$ . This point is the main difficulty when trying to generalize Theorem 1 to vectors in  $\mathbb{R}^k$  involving  $\zeta(j), \zeta(j+2), \dots, \zeta(j+2(k-1))$  for odd  $j \leq a$ .

## 4.2 Proof of the Diophantine results

To prove Theorem 1, we apply Theorem 3 to the linear forms  $d_{2n}^{a+2} S_n$  and  $d_{2n}^{a+2} S_n''$  where  $a$  is sufficiently large, and  $r$  is the integer part of  $a \exp(-\sqrt{\log a})$ . We use the following parameters:  $k = 2$ ,  $e_1 = (1, 0, \zeta(3), \zeta(5), \dots, \zeta(a))$ ,  $e_2 = (0, 1, \binom{4}{2}\zeta(5), \binom{6}{2}\zeta(7), \dots, \binom{a+1}{2}\zeta(a+2))$ ,  $Q_n = \beta^n$  with

$$\beta = e^{2(a+2)} 2^{2(a-6r)} (2r+1)^{6(2r+1)},$$

$\omega_1 = \varphi_1 = 0$ ,  $\omega_2 = \omega_a$ ,  $\varphi_2 = \varphi_a$ ,  $\tau_1 = \frac{-\log(e^{2(a+2)}\varepsilon_a)}{\log\beta}$ ,  $\tau_2 = \frac{-\log(e^{2(a+2)}\varepsilon_a'')}{\log\beta}$ . This gives the following lower bound for the rank of the family in Theorem 1 (using Lemmas 1 and 4), as  $a \rightarrow \infty$ :

$$2 + \tau_1 + \tau_2 \geq \frac{-2}{\log\beta} \log\left(\frac{e^{2(a+2)}2^{6(r+1)}}{r^{2(a-6r)}}\right) = \frac{4a \log r}{2a(1 + \log 2)}(1 + o(1)) = \frac{2 \log a}{1 + \log 2}(1 + o(1))$$

since  $r \log r = o(\log a)$  and  $\log r = (\log a)(1 + o(1))$ . This concludes the proof of Theorem 1, except for Lemma 4 which we shall prove in the next subsection.

Let us deduce Corollary 1 from Theorem 1. Let  $\tilde{n}_2(a)$  denote the number of odd integers  $i \in \{3, 5, \dots, a\}$  such that  $\zeta(i) \in \mathbb{Q}$  and  $\zeta(i+2) \notin \mathbb{Q}$ ; then we have  $\tilde{n}_2(a) = n_2(a)$  or  $\tilde{n}_2(a) = n_2(a) + 1$ . Moreover  $n_1(a) + \tilde{n}_2(a)$  is the number of odd  $i \in \{3, 5, \dots, a\}$  such that  $\zeta(i)$  or  $\zeta(i+2)$  is irrational. Since the vector  $\begin{pmatrix} \zeta(i) \\ \binom{i+1}{2} \zeta(i+2) \end{pmatrix}$  can be removed from the family in Theorem 1 as soon as both  $\zeta(i)$  and  $\zeta(i+2)$  are rational numbers, the rank involved in this result is less than or equal to  $n_1(a) + \tilde{n}_2(a) + 2 \leq n_1(a) + n_2(a) + 3$ . Therefore Corollary 1 follows from Theorem 1.

To prove Theorem 2, we apply Theorem 3 with  $k = 1$ ,  $\omega_1 = \varphi_1 = 0$  (that is, Nesterenko's original linear independence criterion) to the linear forms  $S_n + \lambda S_n''$ . We take  $Q_n = \beta^n$  as above, and  $\tau = \tau_1 = \frac{-\log(e^{2(a+2)}\varepsilon_a)}{\log\beta}$  because  $\varepsilon_a'' < \varepsilon_a$ . This yields  $\tau + 1 \geq \frac{\log a}{1 + \log 2}(1 + o(1))$  as a lower bound for the dimension, thereby proving Theorem 2.

### 4.3 Asymptotic properties

We now prove Lemma 4 stated in §4.1; this completes the proofs given in the previous subsection. The most difficult point (from a technical point of view) is the asymptotic behaviour of  $|S_n''|$ . It has been determined by Zudilin (§2 of [26]) in a general setting, using the saddle point method. We adapt this proof here, because we have to choose different parameters. Our notation is essentially taken from [26].

We keep the notation and assumptions of §4.1 (except for now that of Lemma 4); for instance we assume that  $a$  is odd and  $6r \leq a$ . We shall use the following polynomial:

$$Q(X) = (X + 2r + 1)^3(X - 1)^{a+3} - (X - 2r - 1)^3(X + 1)^{a+3} \in \mathbb{Q}[X]. \quad (4.1)$$

The complex roots of  $Q$  are localized in [26]; we are specially interested in two of them, denoted by  $\tau_0$  and  $\mu_1$ . The root  $\tau_0$  (resp.  $\mu_1$ ) is the unique root of  $Q$  in the domain  $\operatorname{Re} z > 0$ ,  $\operatorname{Im} z > 0$  (resp.  $z \in \mathbb{R}$ ,  $z > 2r + 1$ ) of  $\mathbb{C}$ . Of course  $Q$ ,  $\tau_0$  and  $\mu_1$  depend on  $a$  and  $r$  (but not on  $n$ ).

Let us consider the complex plane with cuts along the rays  $(-\infty, 1]$  and  $[2r + 1, +\infty)$ . We let for  $\tau \in \mathbb{C} \setminus ((-\infty, 1] \cup [2r + 1, +\infty))$ :

$$\begin{aligned} f(\tau) &= 3(\tau + 2r + 1) \log(\tau + 2r + 1) + 3(2r + 1 - \tau) \log(2r + 1 - \tau) \\ &\quad + (a + 3)(\tau - 1) \log(\tau - 1) - (a + 3)(\tau + 1) \log(\tau + 1) + 2(a - 6r) \log(2). \end{aligned} \quad (4.2)$$

In this formula all logarithms are evaluated at positive real numbers if  $\tau$  belongs to the real interval  $(1, 2r + 1)$ , and we choose the determinations so that all of them take real values in this case. We also let, for  $\tau \in \mathbb{C} \setminus ((-\infty, 1] \cup [2r + 1, +\infty))$ ,

$$f_0(\tau) = f(\tau) - \tau f'(\tau).$$

For  $\tau \in (2r + 1, +\infty)$ , let us denote by  $\tau + i0$  the corresponding point on the upper bank of the cut  $[2r + 1, +\infty)$ . Since  $\log(2r + 1 - (\tau + i0)) = \log(\tau - (2r + 1)) - i\pi$  with  $\tau - (2r + 1) > 0$ , we have for  $\tau \in (2r + 1, +\infty)$ :

$$\begin{aligned} f(\tau + i0) &= 3(\tau + 2r + 1) \log(\tau + 2r + 1) + 3(2r + 1 - \tau) \log(\tau - 2r - 1) \\ &\quad + (a + 3)(\tau - 1) \log(\tau - 1) - (a + 3)(\tau + 1) \log(\tau + 1) + 2(a - 6r) \log(2) - 3(2r + 1 - \tau)i\pi. \end{aligned}$$

This function of  $\tau \in (2r + 1, +\infty)$  is increasing on  $(2r + 1, \mu_1)$ , assumes a maximal value at  $\tau = \mu_1$ , and is decreasing on  $(\mu_1, +\infty)$  (see Eq. (2.34) and Corollary 2.2 of [26]). Following the second proof of Lemma 3 in [2], we obtain:

$$\lim_{n \rightarrow \infty} \frac{\log |S_n|}{n} = \operatorname{Re} f(\mu_1 + i0) = \operatorname{Re} f_0(\mu_1 + i0) \quad (4.3)$$

since  $f'(\mu_1 + i0) = 3i\pi \in i\mathbb{R}$ . The main difference with the proof of [2] is the term  $2(a - 6r) \log 2$  in (4.2), which comes from the fact that the integer denoted here by  $n$  is actually  $2n$  with the notation of [2]; this has an effect because of the normalization factor  $(2n)!^{a-6r}$ .

By applying the saddle point method, Zudilin proves the following result, where  $\tau_0$  is defined (as above) to be unique root of  $Q$  in the domain  $\operatorname{Re} z > 0$ ,  $\operatorname{Im} z > 0$ , and  $g$  is defined on the cut plane  $\mathbb{C} \setminus ((-\infty, 1] \cup [2r + 1, +\infty))$  by

$$g(\tau) = \frac{(\tau + 2r + 1)^{3/2} (2r + 1 - \tau)^{3/2}}{(\tau + 1)^{(a+3)/2} (\tau - 1)^{(a+3)/2}}. \quad (4.4)$$

**Lemma 5.** *Assume that  $a$  is odd,  $6r \leq a$ , and*

$$\mu_1 \leq 2r + 1 + \min \left( \frac{3r(r+1)}{2(a+3)}, \frac{r(r+1)}{3(2r+1)} \right). \quad (4.5)$$

Let  $\varepsilon''_{r,a} = \exp \operatorname{Re} f_0(\tau_0)$ ,  $\omega_{r,a} = \operatorname{Im} f_0(\tau_0)$ , and  $\varphi_{r,a} = -\frac{1}{2} \arg f''(\tau_0) + \arg g(\tau_0)$ . If

$$\text{either } \varphi_{r,a} \not\equiv \frac{\pi}{2} \pmod{\pi} \text{ or } \omega_{r,a} \not\equiv 0 \pmod{\pi} \quad (4.6)$$

then we have, as  $n \rightarrow \infty$ ,

$$|S_n''| = \varepsilon''_{r,a}{}^{n+o(n)} |\cos(n\omega_{r,a} + \varphi_{r,a}) + o(1)|.$$



We shall now deduce Lemma 4 from Lemma 5, thereby concluding the proof of Theorem 1. With this aim in mind, we assume from now on that  $a$  is sufficiently large, and  $r$  is the integer part of  $a \exp(-\sqrt{\log a})$ . To begin with, let us check the assumptions of Lemma 5 in this case. From now on we denote by  $o(1)$  any sequence that depends only on  $a$  and tends to 0 as  $a \rightarrow \infty$ .

The polynomial  $Q$  defined in (4.1) satisfies  $Q(2r+1) > 0$ . Let  $\nu(a) = \exp(-\exp \sqrt[3]{\log a})$ ; then we have also

$$Q(2r+1+\nu(a)) = (2r)^{a+3} \left[ \left(4r+2+\nu(a)\right)^3 \left(1+\frac{\nu(a)}{2r}\right)^{a+3} - \nu(a)^3 \left(1+\frac{1}{r}+\frac{\nu(a)}{2r}\right)^{a+3} \right] < 0$$

provided  $a$  is sufficiently large, because (since  $\log r = o(a/r)$ )

$$3 \log(4r+2+\nu(a)) + (a+3) \log\left(1+\frac{\nu(a)}{2r}\right) \leq 4 \log(r) + \frac{(a+3)\nu(a)}{2r} \leq \frac{a+3}{6r},$$

and also, since  $|\log \nu(a)| = o(a/r)$ :

$$3 \log \nu(a) + (a+3) \log\left(1+\frac{1}{r}+\frac{\nu(a)}{2r}\right) \geq 3 \log \nu(a) + \frac{a+3}{2r} \geq \frac{a+3}{4r}.$$

Therefore the root  $\mu_1$  of  $Q$  (which is the only one in the real interval  $(2r+1, +\infty)$ ) satisfies  $2r+1 < \mu_1 < 2r+1+\nu(a)$ . As a corollary, the assumption (4.5) of Lemma 5 holds. Moreover we have  $|\tau_0 - 2r - 1| < |\mu_1 - 2r - 1|$  (see the proof of Lemma 2.7 (e) of [26]), so that

$$|\tau_0 - 2r - 1| < \nu(a). \quad (4.7)$$

This upper bound will be used now to check assumption (4.6) of Lemma 5. Following the notation of [26] we let

$$\alpha_+ = \arg(\tau_0 - 1), \alpha_- = \arg(\tau_0 + 1), \beta_+ = -\arg(2r+1-\tau_0), \beta_- = \arg(\tau_0 + 2r+1),$$

where these arguments belong to  $(-\pi, \pi)$  and would be 0 if  $\tau_0$  were in the real interval  $(1, 2r+1)$ . Then Eq. (2.26) in the proof of Lemma 2.7 of [26] reads (with  $\tau = \tau_0$  and  $f'(\tau_0) = i\pi$ ):

$$3(\beta_- + \beta_+) + (a+3)(\alpha_+ - \alpha_-) = \pi. \quad (4.8)$$

Now Eq. (4.7) yields

$$0 \leq \sin \alpha_+ = \frac{\operatorname{Im} \tau_0}{|\tau_0 - 1|} \leq \frac{|\tau_0 - (2r+1)|}{|\tau_0 - 1|} \leq \frac{\nu(a)}{r}$$

so that  $a\alpha_+$  tends to 0 as  $a \rightarrow \infty$ . The same property can be proved in the same way for  $\alpha_-$  and  $\beta_-$ , so that Eq. (4.8) yields  $\beta_+ = \pi/3 + o(1)$ .

Now let us deduce that assumption (4.6) of Lemma 5 holds. We have

$$f''(\tau_0) = \frac{3}{\tau_0 + 2r + 1} + \frac{3}{2r + 1 - \tau_0} + \frac{a+3}{\tau_0 - 1} - \frac{a+3}{\tau_0 + 1}$$

(see at the end of the proof of Lemma 2.9 of [26]), so that  $f''(\tau_0) = \frac{3(1+o(1))}{2r+1-\tau_0}$  and

$$\arg f''(\tau_0) \equiv -\arg(2r+1-\tau_0) + o(1) \equiv \beta_+ + o(1) \equiv \pi/3 + o(1) \pmod{2\pi}.$$

In the same way the definition of  $g$  yields

$$\arg g(\tau_0) \equiv \frac{3}{2}\beta_- - \frac{3}{2}\beta_+ - \frac{a+3}{2}\alpha_- - \frac{a+3}{2}\alpha_+ \equiv -\frac{\pi}{2} + o(1) \pmod{2\pi}$$

so that

$$-\frac{1}{2} \arg f''(\tau_0) + \arg g(\tau_0) \equiv \frac{-2\pi}{3} + o(1) \not\equiv \frac{\pi}{2} \pmod{\pi}$$

if  $a$  is sufficiently large, so that Eq. (4.6) holds.

We have proved that Lemma 5 applies when  $r$  is the integer part of  $a \exp(-\sqrt{\log a})$  and  $a$  is sufficiently large. It provides  $\varepsilon''_{r,a}$ ,  $\omega_{r,a}$  and  $\varphi_{r,a}$ , which we denote respectively by  $\varepsilon''_a$ ,  $\omega_a$  and  $\varphi_a$ . We also let  $\varepsilon_a = \lim_{n \rightarrow \infty} \frac{\log |S_n|}{n} = \operatorname{Re} f_0(\mu_1 + i0)$  (see Eq. (4.3)).

The inequality  $\varepsilon''_a < \varepsilon_a$  follows from Lemma 2.10 of [26]. To prove that  $\varepsilon_a \leq \frac{2^{6(r+1)}}{r^{2(a-6r)}}$ , we follow [2] (end of the second proof of Lemme 3). In precise terms, for any  $k > 2rn$  we have  $k + (2r+1)n < 2^{1+1/r}k$ , so that

$$\begin{aligned} (2n)!^{a-6r} \frac{(k-2rn)_{2rn}^3 (k+2n+1)_{2rn}^3}{(k+1)_{2n}^a} &< (2n)^{2n(a-6r)} \frac{k^{6rn} (2^{1+1/r}k)^{6rn}}{k^{2an}} \\ &= \left(\frac{2n}{k}\right)^{2n(a-6r)} 2^{6rn(1+1/r)} < \left[\frac{2^{6(r+1)}}{r^{2(a-6r)}}\right]^n. \end{aligned}$$

This yields

$$|S_n| \leq \left[\frac{2^{6(r+1)}}{r^{2(a-6r)}}\right]^n \sum_{k=2rn+1}^{+\infty} \frac{1}{k^a}$$

so that  $\varepsilon_a \leq \frac{2^{6(r+1)}}{r^{2(a-6r)}}$ . This concludes the proof of Lemma 4.

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S. Fischler, Equipe d’Arithmétique et de Géométrie Algébrique, Université Paris-Sud, Bâtiment 425, 91405 Orsay Cedex, France