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***Abstract*—The present letter proposes an alternate procedure that can be effectively employed to replace the essentially algorithmic sifting process in Huang's empirical mode decomposition (EMD) method. Recent works have demonstrated that EMD acts essentially as a dyadic filter bank that can be compared to wavelet decompositions. However, the origin of EMD is algorithmic in nature and, hence, lacks a solid theoretical framework. The present letter proposes to resolve the major problem in the EMD method—the mean envelope detection of a signal—by a parabolic partial differential equation (PDE)-based approach. The proposed approach is validated by employing several numerical studies where the PDE-based sifting process is applied to some synthetic composite signals.**

***Index Terms*—Empirical mode decomposition (EMD), mean envelope, parabolic equation.**

I. Introduction

THIS LETTER puts forth an alernative to the problem of mean-envelope estimation of a signal, which is a crucial step in the empirical mode decomposition (EMD) method, originaly proposed by Huang *et al.* [1]. Although it showed remarkable effectiveness in some applications [2]–[5], this method is essentially algorithmic in nature and, hence, suffers from the drawback that there is no well-established analytical formulation on the basis of which any theoretical analysis and performance evaluation can be carried out. The purpose of this letter, therefore, is to contribute an analytical framework for a better understanding of the EMD method. Therefore, we propose a fourth-order nonlinear partial differential equation system that can solve the problem of local mean estimation of a signal. The asymptotic solution of this system of coupled equations leads to the so-called "mean envelope" and intrinsic mode functions (IMFs) of the signal.

II. EMD

This section presents the EMD method in a nutshell. All the details regarding the implementation of the EMD algorithm and Matlab EMD codes are fully available in [6] and [7]. Essentials of the EMD method iteratively decomposes a complex signal (i.e., a signal with several characteristic time scales coexisting) into several elementary AM-FM-type components, called IMFs. The underlying principle of this decomposition is to locally identify the most rapid oscillations in the signal, defined as a waveform interpolating interwoven local maxima and minima. To do so, the local maxima points (and, respectively, the local minima points) are interpolated with a cubic spline, to determine the upper (and, respectively, the lower) envelope. The mean envelope (i.e., the half sum of the upper and the lower envelopes) is then subtracted from the initial signal, and the same interpolation scheme is reiterated on the remainder. The so-called sifting process terminates when the mean envelope is reasonably zero everywhere, and the resultant signal is designated as the first IMF. The higher order IMFs are iteratively extracted, applying the same procedure for the initial signal, after removing the previous IMFs. In the original definition of IMF [1], to be an IMF a signal must satisfy two criteria, the first one being that the number of local maxima and the number of local minima must differ by at most one and the second that the mean of its upper and lower envelopes must equal zero. So, for any one-dimensional discrete signal $s_n = s[n]$, EMD can finally be presented with the following representation:

$$s_n = \sum_{k=1}^{K} \mathrm{imf}_{k,n} + r_n \tag{1}$$

where imf_k is the kth mode (or IMF) of the signal, and r is the residual trend (a low-order polynomial component). The sifting procedure generates a finite (and limited) number of IMFs that are nearly orthogonal to each other [1]. In accordance with the nature of this decomposition procedure, the technique decomposes data into K fundamental components, each with a distinct time scale where the first component has the smallest time scale. As the decomposition proceeds, the time scale increases, and hence, the mean frequency of the mode decreases.

A. EMD-Related Works

Several research efforts on the EMD method have been specifically addressed to the algorithm improvement [8], experimental characterization of fractional Gaussian noise decomposition showing spontaneous emergence of a filter bank structure, almost dyadic and self-similar, and resulting in a possible Hurst's exponent estimation [9]–[12]. Several works have proposed different approaches for two-dimensional (2-D) extension of EMD, including a row-wise/columnwise decomposition, in the spirit of the popular nonstandard wavelet transform, or a truly two-dimensional version of EMD [13], [14] and more recently in [15]. If one is permitted to draw a

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fast conclusion, the EMD method will appear as a simple one, a local and fully data-driven approach, adapted to nonlinear oscillations. Moreover, the combination of the EMD method and the associated Hilbert spectral analysis can offer a powerful method for nonlinear and nonstationary data analysis [1].

B. Drawbacks of EMD

During the sifting process of the standard EMD [1], a cubic spline-based interpolation method is a crucial step to create the upper and the lower envelopes of the data set. If the cubic splines are fitted at the extreme points, that can produce several inconveniences: 1) problems can occur near the terminating points, 2) end swings can eventually propagate inward, and 3) the overshoots and the undershoots may become a common phenomenon. In 2-D versions, the main drawbacks of EMD are the definitions of the extrema of an image (or a surface) and the choice of the interpolation method for application on a set of scattering points. Moreover, such a decomposition in two dimensions [13], [14] is extremely time consuming. Note that a B-spline method is proposed in [15] and is $\mathcal{O}(N)$, where N is the number of elements in the 2-D image. The main drawback is that EMD is limited to the numerical simulations and suffers from the lack of a formal mathematical framework beyond it. Despite some works reported in [16], EMD formalism remains an exciting challenge.

III. Partial Differential Equation (PDE)-Based Formulation

In order to implement sifting procedure in a PDE-based framework, the following processes are based on the definition of characteristic points of a function: turning points and curvature points. The present letter focuses its interest on turning points, which are minima, maxima, and inflection points, defined by the zeros of their first and/or second derivatives, when these exist. The discussions in the following two subsections make use of fourth-order parabolic equations of the form

$$s_t = -g(x)s_{xxxx} \tag{2}$$

where $s \equiv s(x,t)$. Equation (2) can be viewed as a long-range diffusion (LRD) equation[1] with variable coefficient $g(x)$ that depends on position and more precisely on some characteristic points of the signal to be decomposed. As a first proposition, we define a coupled PDE's system in place of the sifting process to estimate lower and upper envelopes. To continue, a second formulation is given in order to directly estimate the mean envelope through the inflection points.

In the following, we use the notation $s_0(x) = s(x, t=0)$ for the initial condition and $s_\infty(x) = s(x, t=\infty)$ for the asymptotic solution of (2).

A. Coupled PDE's Formulation

A simple method to estimate the mean envelope is to formulate a coupled PDE's system to mimic the sifting process exactly, based on the estimation of the upper and the lower envelopes. Here, the turning points are constituted of the maxima

[1]LRD equations usually incorporate short- and long-range terms. See [17, p. 244] for further details.

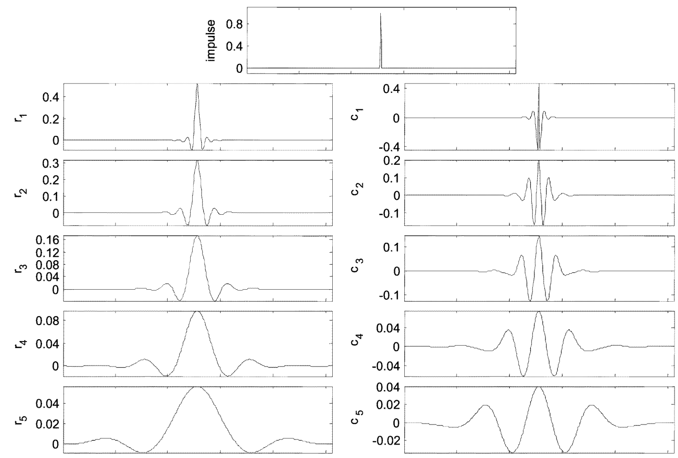

Fig. 1. Decomposition of a Dirac impulse. Five first components (right column) and residuals (left column) of the decomposition of an impulse signal (shown in the top plot). The components labeled c_1 to c_5 (right) are equivalent to impulse responses of a filter bank (compare with the results from EMD reported in [12]).

and the minima of the signal to be decomposed, respectively. This coupled PDE's system can be described as

$$\begin{cases} s_t^+ = -g^+((s_0)_x, (s_0)_{xx})s_{xxxx}^+ \\ s_t^- = -g^-((s_0)_x, (s_0)_{xx})s_{xxxx}^-. \end{cases} \tag{3}$$

When the system (3) described in the equation converges, the asymptotic solutions $s_\infty^+(x)$ and $s_\infty^-(x)$ represent the upper and the lower envelops of the signal s_0, respectively. Hence, the mean envelope is obtained by

$$s_\infty(x) = \frac{1}{2}[s_\infty^+(x) + s_\infty^-(x)]. \tag{4}$$

In (3), stopping functions $g^\pm$ depend on both the first- and the second-order signal derivatives, with $g^\pm \geq 0$. For example

$$g^\pm = \frac{1}{3}[|\text{sign}((s_0)_x)| \pm \text{sign}((s_0)_{xx}) + 1]. \tag{5}$$

In this manner, $g^+ = 0$ at maxima of s_0 and $g^- = 0$ at minima of s_0. So, LRD acts only between the two consecutive maxima (similarly minima) points until the fourth-order derivative of $s(x,t)$ gets canceled. Consequently, after convergence, the resulting signal $s_\infty^+(x)$ [similarly $s_\infty^-(x)$] is a piecewise cubic polynomial curve interpolating the successive maxima (similarly minima) of a signal.

B. Simple PDE Formulation

Here, the main challenge that lies ahead of us is a direct estimation of the mean envelope. Here, the function $g(x)$ is now a positive function of second derivative of the signal $s_0(x)$. For example, $g(x) = |(s_0)_{xx}|$ and the equation can be described as

$$s_t = -|(s_0)_{xx}|s_{xxxx}. \tag{6}$$

The process of LRD is stopped at characteristic points of the signal s where the second-order derivatives of $s_0(x)$ undergoes change of sign. These points are inflection points and are located between two successive extrema of $s_0(x)$. So, LRD acts only between two consecutive inflection points until the fourth-order

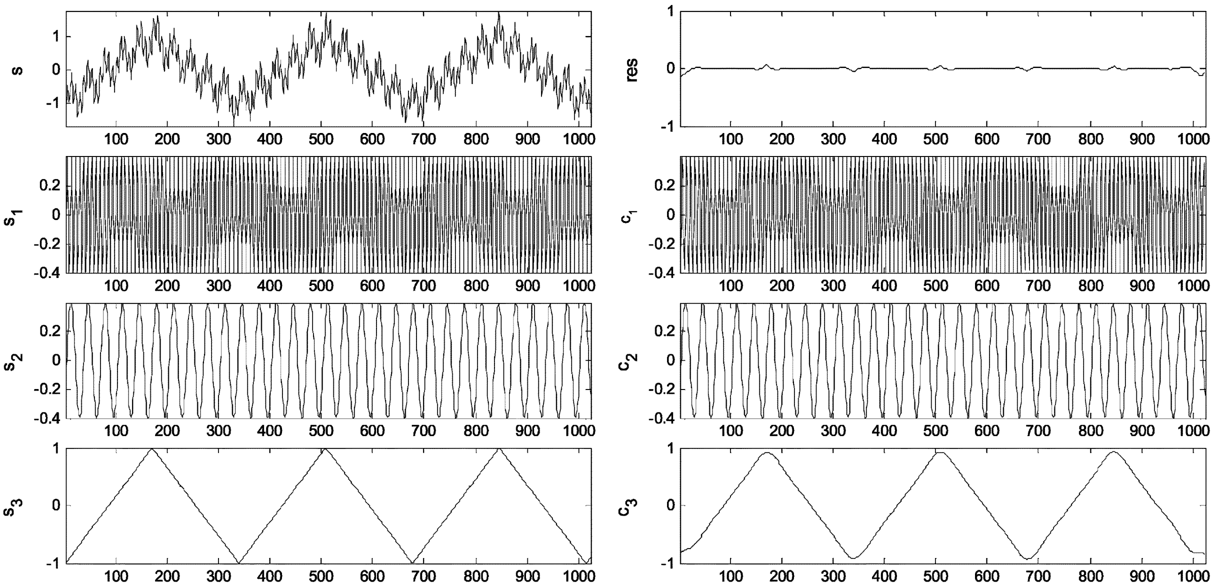

Fig. 2. PDE-based signal decomposition. In this first example, a composite signal (left, first row), resulting from the superposition of 2 triangular (s_1 and s_3) waveforms and one tone (s_2), is decomposed into its three expected elementary components labeled c_1 to c_3 (right, rows 2 to 4). Residual component of the decomposition is also given in the first row (compare these results with those obtained using EMD algorithm [8, Fig. 2]).

derivative of $s(x,t)$ gets canceled. Consequently, after convergence, the resulting signal $s_\infty(x)$ is a piecewise cubic polynomial curve interpolating the successive inflection points of $s_0(x)$. An interesting modification of (6) is

$$s_t = -|\text{sign}((s_0)_{xx})| s_{xxxx}. \quad (7)$$

In (5) and (7), the sign function $\text{sign}(z)$ is replaced by a regularized version. A possible expression can be given by $\text{sign}_\alpha(z) = (2/\pi)\text{atan}(\pi z/\alpha)$.

C. Numerical Solution

The numerical solution for the coupled PDE's system is implemented with a Crank–Nicolson scheme (with $\lambda = 0.5$ and Neumann boundary condition)

$$\frac{s_n^{k+1} - s_n^k}{\Delta t} = -g^{\pm}(D_1 s_{0n}, D_2 s_{0n}) \times D_4(\lambda s_n^{k+1} + (1-\lambda)s_n^k) \quad (8)$$

where $s = s^{\pm}$ stands for the upper or the lower envelope signal, and $D_2 = D^+D^-, D_4 = D_2 D_2$, where D^+ and D^- are forward and backward first difference operators on the x dimension. On other hand, the approximation of the stopping functions demands much attention and can be described by

$$g((s_0)_x, (s_0)_{xx}) \approx g(D_1 s_0, D_2 s_0) \quad (9)$$

with $D_1 z = m(D^+z, D^-z)$. Here, $m(a,b)$ stands for the *minmod* limiter, which can be expressed as

$$m(a,b) = \frac{1}{2}(\text{sign}(a) + \text{sign}(b)) \cdot \min(|a|, |b|). \quad (10)$$

For (7), the Crank–Nicolson implicit method is also used. Hence, the discretization of (7) (with $\lambda = 0.5$) can be represented as

$$\frac{s_n^{k+1} - s_n^k}{\Delta t} = -|\text{sign}_\alpha(D_2 s_{0n})| \times D_4(\lambda s_n^{k+1} + (1-\lambda)s_n^k). \quad (11)$$

IV. Results

To give a concise presentation, the following illustrating results were obtained using the coupled PDE's process (3). Equation (7) is also capable of producing similar results but requires a careful—and, hence, time consuming—attention to step-time value Δt. Contrary to the solutions of (11), solutions of (8) are less sensitive to step time. All IMFs were obtained in a small number (<8) of iterations in the PDE-based sifting procedure and with a constant time step (for example $\Delta t = 20$) to assure relatively fast convergence of the process. In order to make a proper comparison, the examples described in [8] and [12] are again considered. Fig. 1 illustrates the modewise decomposition of a Dirac impulse. It can be easily seen that the wavelet-like form of the successively extracted five first components are in agreement with the results reported in [12]. The first example of signal decomposition consists of a sum of two triangular waveforms and a tone (presented in Fig. 2). The second example is a composite signal originating from the superposition of two sinusoidal FM signals and one Gaussian logon[2] (presented in Fig. 3). In this case, the components overlap in both time and frequency, thus disabling the components to be separated by any nonadaptive filtering technique. In all these examples, both linear and nonlinear oscillations are effectively separated.

[2]A Gaussian logon is an FM tone modulated by a Gaussian, so it is defined on a portion of a time-frequency space.

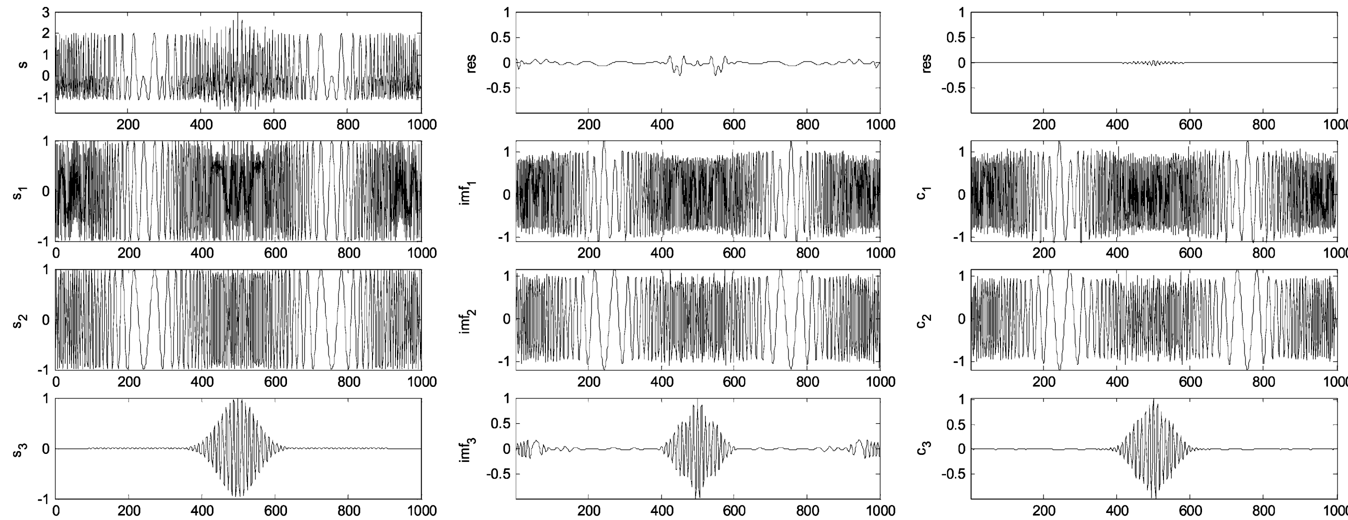

Fig. 3. PDE-based signal decomposition. In this second example, a composite signal (left column, first row), resulting from the superposition of two sinusoidal FM signals (s_1 and s_2) and one Gaussian logon (s_3), is decomposed in its three expected elementary components labeled c_1 to c_3 (right column, rows 2 to 4). The residual component of the decomposition is also given in the first row. For comparison, IMF and the residual resulting from classical EMD are also shown (IMFs are labeled as imf_1 to imf_3, shown in the middle column).

V. Conclusion

It is a well-known fact that the EMD method is developed on the basis of an algorithm, and hence, it suffers from a lack of a full, generally accepted theoretical framework. Hence, it is of immense importance that an analytical formulation for the so-called mean envelope be developed for characterization of this method. The main problems are associated with the fact that the local mean of a signal depends on its characteristic local time scales. A novel approach has been presented in this letter, which estimates the mean envelope of a signal in a PDE-based framework. The utility of the proposed method has been successfully demonstrated with the help of several synthetic signals, which demonstrate that this approach performs as well as the classical EMD method. The main objective of this formulation is to contribute analytically to a better understanding of the EMD method. Furthermore, PDE-based formulation for the sifting process results in an easier implementation, and it is less time consuming for 2-D extension of image decomposition than the method described in [13]. The PDE-based EMD codes are fully available in [7].

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