On the well-posedness of the full low-Mach number limit system in general critical Besov spaces

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Abstract

This work is devoted to the well-posedness issue for the low-Mach number limit system obtained from the full compressible Navier-Stokes system, in the whole space $\mathbb{R}^d$ with $d \geq 2$.

In the case where the initial temperature (or density) is close to a positive constant, we establish the local existence and uniqueness of a solution in critical homogeneous Besov spaces of type $\dot{B}^s_{p,1}$. If, in addition, the initial velocity is small then we show that the solution exists for all positive time. In the fully nonhomogeneous case, we establish the local well-posedness in nonhomogeneous Besov spaces $B^s_{p,1}$ (still with critical regularity) for arbitrarily large data with positive initial temperature.

Our analysis strongly relies on the use of a *modified* divergence-free velocity which allows to reduce the system to a nonlinear coupling between a parabolic equation and some evolutionary Stokes system. As in the recent work by Abidi-Paicu [1] concerning the density dependent incompressible Navier-Stokes equations, the Lebesgue exponents of the Besov spaces for the temperature and the (modified) velocity, need not be the same. This enables us to consider initial data in Besov spaces with a *negative* index of regularity.

1 Introduction

The full Navier-Stokes system

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) & = & 0, \\ \partial_t(\rho v) + \mathrm{div}\,(\rho v \otimes v) - \mathrm{div}\,\sigma + \nabla p & = & 0, \\ \partial_t(\rho e) + \mathrm{div}\,(\rho v e) - \mathrm{div}\,(k\nabla\vartheta) + p\,\mathrm{div}\,v & = & \sigma\cdot Dv, \end{cases} \tag{1.1}$$

governs the free evolution of a viscous and heat conducting compressible Newtonian fluid.

In the above system, $\rho = \rho(t,x)$ stands for the mass density, $v = v(t,x)$, for the velocity field and $e = e(t,x)$, for the internal energy per unit mass. The time variable t belongs to $\mathbb{R}^+$ or to $[0,T]$ and the space variable x is in $\mathbb{R}^d$ with $d \geq 2$. The scalar functions $p = p(t,x)$ and $\vartheta = \vartheta(t,x)$ denote the pressure and temperature respectively, and σ is the viscous strain tensor, given by

$$\sigma = 2\zeta Sv + \eta\,\mathrm{div}\,v\,\mathrm{Id}\,,$$

where Id is the $d\times d$ identity matrix, $Sv := \frac{1}{2}(\nabla v + Dv)$, the so-called deformation tensor[1]. The heat conductivity k and the Lamé (or viscosity) coefficients ζ and η may depend smoothly on ρ and on ϑ. The above system has to be supplemented with two *state equations* involving p, ρ, e and ϑ.

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[1]In all the paper, we agree that for $v = (v^1,\cdots,v^d)$ a vector field, $(Dv)_{ij} := \partial_j v^i$ and $(\nabla v)_{ij} := \partial_i v^j$.

In the present paper, we want to consider the low Mach number limit of the full Navier-Stokes System (1.1). From an heuristic viewpoint, this amounts to neglecting the compression due to pressure variations, a common assumption when describing highly subsonic flows. Following the introduction of P.-L. Lions's book [24] (see also the physics book by Zeytounian [30]), we here explain how the low Mach number limit system may be derived *formally* from (1.1). For simplicity we restrict ourselves to the case of an *ideal* gas, namely we assume that

$$p = R\rho\vartheta, \quad e = C_v\vartheta, \tag{1.2}$$

where R, C_v denote the ideal gas constant and the specific heat constant, respectively.

Let us define the (dimensionless) Mach number ε as the ratio of the reference velocity over the reference sound speed of the fluid, then suppose that (ρ, v, ϑ) is some given classical solution of System (1.1) corresponding to the small ε. Then the rescaled triplet

$$\Big(\rho_\varepsilon(t,x) = \rho\Big(\frac{t}{\varepsilon}, x\Big), \quad v_\varepsilon(t,x) = \frac{1}{\varepsilon}v\Big(\frac{t}{\varepsilon}, x\Big), \quad \vartheta_\varepsilon(t,x) = \vartheta\Big(\frac{t}{\varepsilon}, x\Big)\Big)$$

satisfies

$$\left\{\begin{array}{ccc} \partial_t\rho_\varepsilon + \mathrm{div}\,(\rho_\varepsilon v_\varepsilon) & = & 0, \\ \partial_t(\rho_\varepsilon v_\varepsilon) + \mathrm{div}\,(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) - \mathrm{div}\,\sigma_\varepsilon + \frac{\nabla p_\varepsilon}{\varepsilon^2} & = & 0, \\ \frac{1}{\gamma-1}(\partial_t p_\varepsilon + \mathrm{div}\,(p_\varepsilon v_\varepsilon)) - \mathrm{div}\,(k_\varepsilon\nabla\vartheta_\varepsilon) + p_\varepsilon \mathrm{div}\, v_\varepsilon & = & \varepsilon^2\sigma_\varepsilon\cdot Dv_\varepsilon, \end{array}\right. \tag{1.3}$$

with

$$\begin{gathered} \sigma_\varepsilon = 2\zeta_\varepsilon Sv_\varepsilon + \eta_\varepsilon \mathrm{div}\, v_\varepsilon \mathrm{Id}\,, \quad p_\varepsilon = R\rho_\varepsilon\vartheta_\varepsilon, \quad \gamma = 1 + R/C_v, \\ \zeta_\varepsilon = \frac{1}{\varepsilon}\zeta\Big(\frac{t}{\varepsilon}, x\Big), \quad \eta_\varepsilon = \frac{1}{\varepsilon}\eta\Big(\frac{t}{\varepsilon}, x\Big) \text{ and } k_\varepsilon = \frac{1}{\varepsilon}k\Big(\frac{t}{\varepsilon}, x\Big). \end{gathered}$$

By letting the Mach number ε go to 0, the momentum equation of (1.3) implies that

$$p_\varepsilon = P(t) + \Pi(t,x)\varepsilon^2 + o(\varepsilon^2).$$

Plugging this formula into the energy equation of (1.3) entails that $P(t)$ is independent of t, provided v_ε and $\nabla\vartheta_\varepsilon$ vanish at infinity. From now on, we shall denote this constant by P_0.

Bearing in mind the equation of state given in (1.2), we deduce that $\rho = P_0/(R\vartheta)$. Therefore, denoting $C_p = \gamma C_v = \gamma R/(\gamma-1)$, the low Mach number limit system reads

$$\left\{\begin{array}{ccc} \rho C_P(\partial_t\vartheta + v\cdot\nabla\vartheta) - \mathrm{div}\,(k\nabla\vartheta) & = & 0, \\ \rho(\partial_t v + v\cdot\nabla v) - \mathrm{div}\,\sigma + \nabla\Pi & = & 0, \\ \gamma P_0 \mathrm{div}\, v - (\gamma-1)\mathrm{div}\,(k\nabla\vartheta) & = & 0. \end{array}\right. \tag{1.4}$$

A number of mathematical results concerning the low Mach number limit from (1.3) to (1.4) have been obtained in the past three decades, most of them being dedicated to the isentropic or to the barotropic isothermal cases (that is $\vartheta \equiv const$ and $p = p(\rho)$) under the assumption that the viscosity coefficients are independent of ρ. From a technical viewpoint, those latter cases are easier to deal with inasmuch as only the first two equations of System (1.3) have to be considered. As a consequence, the expected limit system is the standard incompressible Navier-Stokes or Euler system. In the inviscid case, the first mathematical results concerning the incompressible limit go back to the eighties with the works by Klainerman and Majda [22], Isozaki [20, 21] and Ukai [29]. In the viscous case, the justification of the *incompressible Navier-Stokes* equation as the zero-Mach limit of the *compressible Navier-Stokes* equation, has been done in different contexts by e.g. Danchin [9, 10], Desjardins and Grenier [14], Hagström and Lorentz [18], Hoff [19], Lions [25], and Lions and Masmoudi [26]. In contrast, there are quite a few results for the full system. The inviscid case – the *non-isentropic Euler*, has been considered in the whole space and periodic settings by Métivier and Schochet [27, 28]. Recently Alazard performed a rigorous analysis for the *full Navier-Stokes* equations with large temperature variations in the *Sobolev* spaces H^s with s large enough, see [3]. In the framework of variational solutions with finite energy, this asymptotics has been justified in the somewhat different regime of Oberbeck-Boussinesq approximation (see the book [17] by Feireisl-Novotný).

To our knowledge, the only work dedicated to the limit system (1.4) in the general case where ϑ is not a constant or the conductivity k is not zero (note that if $k \equiv 0$ then the system reduces to the nonhomogeneous incompressible Navier-Stokes equations studied in e.g. [1, 11]) is the paper [15] by P. Embid. There, local-in-time existence of smooth solutions (in Sobolev spaces) is established not only for (1.4) but also for a more complicated system of reacting flows.

The present paper is to study the *well-posedness* issue for the full low Mach number limit system (1.4) in the critical Besov spaces, locally and globally. We expect our work to be the first step of justifying rigorously the limit process in the critical Besov spaces, a study that we plan to do in the future.

In what follows, we assume that the coefficients (ζ, η, k) in (1.4) are C^∞ functions of the temperature ϑ, and we consider only the viscous and heat-conducting case, namely

$$k(\vartheta) > 0, \qquad \zeta(\vartheta) > 0 \qquad \text{and} \qquad \eta(\vartheta) + 2\zeta(\vartheta) > 0.$$

At first sight, this assumption ensures that System (1.4) is of parabolic type, up to the pressure term that may be seen as the Lagrange multiplier corresponding to the constraint given in the last equation. Handling this relation between $\mathrm{div}\, v$ and the temperature is the first difficulty that has to be faced. In order to reduce the study to a system which looks more like the incompressible Navier-Stokes equations, it is natural to perform the following change of velocity:

$$u = v - \alpha k \nabla\vartheta \quad \text{with} \quad \alpha := \frac{\gamma - 1}{\gamma P_0} = \frac{R}{C_p P_0} = \frac{1}{C_p \rho \vartheta}\cdot \tag{1.5}$$

We claim that (ϑ, v) satisfies (1.4) (for some $\nabla\Pi$) if and only there exists some function Q so that (ϑ, u) fulfills

$$\left\{\begin{array}{rcl} \partial_t\vartheta + u\cdot\nabla\vartheta - \mathrm{div}\,(\kappa\nabla\vartheta) & = & f(\vartheta), \\ \partial_t u + u\cdot\nabla u - \mathrm{div}\,(\mu\nabla u) + \vartheta\nabla Q & = & h(\vartheta, u), \\ \mathrm{div}\, u & = & 0, \end{array}\right. \tag{1.6}$$

with $\kappa(\vartheta) = \alpha k \vartheta$, $\mu(\vartheta) = \alpha C_p \zeta\vartheta$, $f(\vartheta) = -2\alpha k|\nabla\vartheta|^2$ and

$$h(\vartheta, u) = A_1|\nabla\vartheta|^2\nabla\vartheta + A_2\Delta\vartheta\nabla\vartheta + A_3\nabla\vartheta\cdot\nabla^2\vartheta + A_4\nabla u\cdot\nabla\vartheta + A_5 Du\cdot\nabla\vartheta,$$

where the coefficients A_i $(i = 1, \cdots 5)$ are functions of ϑ the value of which is given in (1.8) below.

In order to derive (1.6), we first notice that

$$\begin{aligned} \partial_t(\rho v) + \mathrm{div}\,(\rho v\otimes v) &= \partial_t(\rho u) + \mathrm{div}\,(\rho v\otimes u) + \partial_t(\rho\alpha k\nabla\vartheta) + \mathrm{div}\,(\rho\alpha k v\otimes\nabla\vartheta), \\ &= \rho(\partial_t u + v\cdot\nabla u) + \partial_t(C_p^{-1}\vartheta^{-1}k\nabla\vartheta) + \mathrm{div}\,(C_p^{-1}\vartheta^{-1}kv\otimes\nabla\vartheta). \end{aligned}$$

Hence, given that k is a function of ϑ we deduce that there exists some function Q_1 so that

$$\partial_t(\rho v) + \mathrm{div}\,(\rho v\otimes v) = \rho(\partial_t u + u\cdot\nabla u) + \rho\alpha k Du\cdot\nabla\vartheta + \mathrm{div}\,(C_p^{-1}\vartheta^{-1}kv\otimes\nabla\vartheta) + \nabla Q_1.$$

Next, using the fact that

$$Sv = \frac{1}{2}(\nabla u - Du) + Dv,$$

we may write that

$$\begin{aligned} -\mathrm{div}\,\sigma &= -\mathrm{div}\,(\zeta\nabla u) + \mathrm{div}\,(\zeta Du) - 2\mathrm{div}\,(\zeta Dv) - \nabla(\eta\mathrm{div}\, v), \\ &= -\mathrm{div}\,(\zeta\nabla u) + \nabla u\cdot\nabla\zeta + 2\mathrm{div}\,(v\otimes\nabla\zeta) - \nabla(\eta\mathrm{div}\, v + 2\mathrm{div}\,(\zeta v)). \end{aligned}$$

Therefore, (ρ, v) satisfies (1.4) (for some suitable Π) if and only if there exists some Q_2 so that

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho - \mathrm{div}\,(\rho\alpha k\nabla\vartheta) = 0, \\ \rho(\partial_t u + u\cdot\nabla u) - \mathrm{div}\,(\zeta\nabla u) + \rho\alpha k Du\cdot\nabla\vartheta + \nabla u\cdot\nabla\zeta + \mathrm{div}\,(\beta v\otimes\nabla\vartheta) + \nabla Q_2 = 0, \\ \mathrm{div}\, u = 0, \end{cases}$$

with

$$\beta := C_p^{-1}\vartheta^{-1}k + 2\zeta'. \tag{1.7}$$

Finally, using that $\mathrm{div}\, u = 0$ we get (denoting by B a primitive of β)

$$\begin{aligned} \mathrm{div}\,(\beta v\otimes\nabla\vartheta) &= \mathrm{div}\,(\beta u\otimes\nabla\vartheta) + \mathrm{div}\,(\beta\alpha k\nabla\vartheta\otimes\nabla\vartheta), \\ &= -\beta\nabla u\cdot\nabla\vartheta + \nabla\mathrm{div}\,(B(\vartheta)u) + \mathrm{div}\,(\beta\alpha k\nabla\vartheta\otimes\nabla\vartheta). \end{aligned}$$

So after multiplying the equation for u by $\rho^{-1} = \alpha C_p\vartheta$ and using the fact that

$$-\alpha C_p\vartheta\mathrm{div}\,(\zeta\nabla u) = -\mathrm{div}\,(\mu\nabla u) + \alpha C_p\zeta Du\cdot\nabla\vartheta \quad \text{with} \quad \mu(\vartheta) := \alpha C_p\vartheta\zeta(\vartheta),$$

we get (1.6) with one new Q and

$$\begin{gathered} A_1 = -\alpha^2 C_p\vartheta(\beta k)', \quad A_2 = A_3 = -\alpha^2\beta k C_p\vartheta, \\ A_4 = -\rho^{-1}\zeta' + \rho^{-1}\beta = k\alpha + \alpha C_p\vartheta\zeta', \quad A_5 = -\alpha C_p\zeta - \alpha k. \end{gathered} \tag{1.8}$$

Motivated by prior works on incompressible or compressible Navier-Stokes equations with variable density (see in particular [6, 7, 8, 11]), we shall use scaling arguments so as to determine the optimal functional framework for solving the above system.

Here we notice that if $(\vartheta, u, \nabla Q)$ is a solution of (1.6), then so does

$$(\vartheta(\ell^2 t, \ell x), \ell u(\ell^2 t, \ell x), \ell^3\nabla Q(\ell^2 t, \ell x)) \quad \text{for all} \quad \ell > 0. \tag{1.9}$$

Therefore, critical spaces for the initial data (ϑ_0, u_0) must be norm invariant by the transform

$$(\vartheta_0, u_0)(x) \to (\vartheta_0(\ell x), \ell u_0(\ell x)) \quad \text{for all} \quad \ell > 0. \tag{1.10}$$

Let us first consider the easier case where the initial temperature ϑ_0 is close to a constant (say 1 to simplify the presentation). Then, setting $\theta = \vartheta - 1$, System (1.6) recasts in

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta - \bar\kappa\Delta\theta &= a(\theta), \\ \partial_t u + u\cdot\nabla u - \bar\mu\Delta u + \nabla Q &= c(\theta, u, \nabla Q), \\ \mathrm{div}\, u &= 0, \end{cases} \tag{1.11}$$

where $\bar\kappa = \kappa(1)$, $\bar\mu = \mu(1)$ and

$$\begin{aligned} &a(\theta) = \mathrm{div}\,((\kappa(1+\theta) - \bar\kappa)\nabla\theta) + f(1+\theta), \\ &c(\theta, u, \nabla Q) = \mathrm{div}\,((\mu(1+\theta) - \bar\mu)\nabla u) - \theta\nabla Q + h(1+\theta, u). \end{aligned}$$

Let us notice that the following functional space[2]:

$$\Big(L^\infty(\mathbb{R}^+; \dot B^{d/p_1}_{p_1,r_1}) \cap L^1(\mathbb{R}^+; \dot B^{d/p_1+2}_{p_1,r_1})\Big) \times \Big(L^\infty(\mathbb{R}^+; \dot B^{d/p_2-1}_{p_2,r_2}) \cap L^1(\mathbb{R}^+; \dot B^{d/p_2+1}_{p_2,r_2})\Big)^d \times \Big(L^1(\mathbb{R}^+; \dot B^{d/p_3-1}_{p_3,r_3})\Big)^d$$

satisfies the scaling condition (1.9) for any $1 \le p_1, p_2, p_3, r_1, r_2, r_3 \le \infty$.

However, as a L^∞ control over θ is needed in order to keep the ellipticity of the second order operators of the system and since $\dot B^{d/p}_{p,r} \hookrightarrow L^\infty$ if and only if $r = 1$, we shall assume that $r_1 = 1$.

[2] The reader is referred to Definition 2.3 for the definition of homogeneous Besov spaces.

In addition, in many places, having $\nabla u\in L^1([0,T];L^\infty)$ will be needed. This will induce us to choose $r_2=1$ and $r_3=1$ as well. So finally, we plan to solve System (1.11) in the space

$$\dot F_T^{p_1,p_2}:=\Big(\widetilde C_T(\dot B^{d/p_1}_{p_1,1})\cap L^1_T(\dot B^{d/p_1+2}_{p_1,1})\Big)\times\Big(\widetilde C_T(\dot B^{d/p_2-1}_{p_2,1})\cap L^1_T(\dot B^{d/p_2+1}_{p_2,1})\Big)^d\times\Big(L^1_T(\dot B^{d/p_2-1}_{p_2,1})\Big)^d$$

where $\widetilde C_T(\dot B^\sigma_{p,1})$ is a (large) subspace of $\mathcal C([0,T];\dot B^\sigma_{p,1})$ (see Definition 2.2).

In what follows, we shall denote

$$\begin{aligned}
\|\theta\|_{\dot X^{p_1}(T)}&=\|\theta\|_{\widetilde L^\infty_T(\dot B^{d/p_1}_{p_1,1})}+\|\theta\|_{L^1_T(\dot B^{d/p_1+2}_{p_1,1})},\\
\|u\|_{\dot Y^{p_2}(T)}&=\|u\|_{\widetilde L^\infty_T(\dot B^{d/p_2-1}_{p_2,1})}+\|u\|_{L^1_T(\dot B^{d/p_2+1}_{p_2,1})},\\
\|\nabla Q\|_{\dot Z^{p_2}(T)}&=\|\nabla Q\|_{L^1_T(\dot B^{d/p_2-1}_{p_2,1})},
\end{aligned}$$

and we will drop T in $\dot X^{p_1}(T)$, $\dot Y^{p_2}(T)$, $\dot Z^{p_3}(T)$ if $T=+\infty$.

Let us now state our main result for (1.11) in the slightly nonhomogeneous case (that is under a smallness condition for θ_0).

Theorem 1.1. *Let $\theta_0\in\dot B^{d/p_1}_{p_1,1}$ and $u_0\in\dot B^{d/p_2-1}_{p_2,1}$ with $\operatorname{div}u_0=0$. Assume that*

$$1\le p_1<2d,\quad 1\le p_2<\infty,\quad p_1\le 2p_2,\quad \frac1{p_1}+\frac1{p_2}>\frac1d,\quad \frac1{p_2}+\frac1d\ge\frac1{p_1}\quad\text{and}\quad \frac1{p_1}+\frac1d\ge\frac1{p_2}\cdot \tag{1.12}$$

There exist two constants τ and K depending only on the coefficients of System (1.11) and on d,p_1,p_2, and satisfying the following properties:

- *If*
$$\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}}\le\tau, \tag{1.13}$$
then there exists $T\in(0,+\infty]$ such that System (1.11) has a unique solution $(\theta,u,\nabla Q)$ in $\dot F_T^{p_1,p_2}$, which satisfies
$$\|\theta\|_{\dot X^{p_1}(T)}\le K\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}}\quad\text{and}\quad \|u\|_{\dot Y^{p_2}(T)}+\|\nabla Q\|_{\dot Z^{p_2}(T)}\le K\big(\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}}+\|u_0\|_{\dot B^{d/p_2-1}_{p_2,1}}\big).$$
- *If*
$$\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}}+\|u_0\|_{\dot B^{d/p_2-1}_{p_2,1}}\le\tau, \tag{1.14}$$
then $T=+\infty$ and the unique global solution satisfies
$$\|\theta\|_{\dot X^{p_1}}+\|u\|_{\dot Y^{p_2}}+\|\nabla Q\|_{\dot Z^{p_2}}\le K\big(\|\theta_0\|_{\dot B^{d/p_1}_{p_1,1}}+\|u_0\|_{\dot B^{d/p_2-1}_{p_2,1}}\big). \tag{1.15}$$

In addition, the flow map $(\theta_0,u_0)\mapsto(\theta,u,\nabla Q)$ is Lipschitz continuous from $\dot B^{d/p_1}_{p_1,1}\times\dot B^{d/p_2-1}_{p_2,1}$ to $\dot F_T^{p_1,p_2}$.

Remark 1.1. A similar statement for the nonhomogeneous incompressible Navier-Stokes equations has been obtained by Abidi-Paicu in [1]. There, the conditions over p_1 and p_2 (which stem from the structure of the nonlinearities) are not exactly the same as ours for there is no gain of regularity over the density and the right-hand side of the momentum equation in (1.11) is simpler.

Let us stress that in the above statement, the homogeneous Besov spaces for the velocity are almost the same as for the standard incompressible Navier-Stokes equation (except that in this latter case, one may take *any* space $\dot B^{d/p_2-1}_{p_2,r}$ with $1\le p_2<\infty$ and $1\le r\le\infty$). In particular, here one may take p_2 as large as we want hence the regularity exponent d/p_2-1 may be negative and our result ensures that suitably oscillating *large* velocities give rise to a global solution.

The important observation for solving (1.11) is that all the "source terms" (that is the terms on the right-hand side) are at least quadratic. In a suitable functional framework –the one given by our scaling considerations, we thus expect them to be negligible if the initial data are small. Hence, appropriate a priori estimates for the linearized system pertaining to (1.11) and suitable product estimates suffice to control the solution for all time if the data are small. This will enable us to prove the global existence. In addition, a classical argument borrowed from the one that is used in the constant density case will enable us to consider large u_0.

Let us now turn to the fully nonhomogeneous case. Then, in order to ensure the ellipticity of the second order operators in the left-hand side of (1.6), we have to assume that ϑ_0 is bounded by below by some positive constant. Proving a priori estimates for the heat or Stokes equations with variable time-dependent rough coefficients will be the key to our local existence statement. Bounding the gradient of the pressure (namely ∇Q) is the main difficulty. To achieve it, we will have to consider the elliptic equation

$$\operatorname{div}(\vartheta\nabla Q)=\operatorname{div}L\quad\text{with}\quad L:=-u\cdot\nabla u+Du\cdot\nabla\mu+h. \tag{1.16}$$

In the energy framework (that is in Sobolev spaces H^s or in Besov spaces $B^{d/p_2}_{p_2,1}$ with $p_2=2$), this is quite standard. At the same time, if $p_2\neq 2$, estimating ∇Q in $B^{d/p_2-1}_{p_2,1}$ requires some low order information in L^{p_2} over ∇Q. Thanks to suitable functional embedding, we shall see that if $p_2\geq 2$ then it suffices to bound ∇Q in L^2, an information which readily stems from the standard L^2 elliptic theory. As a consequence, we will have to restrict our attention to a functional framework which ensures that L belongs to L^2. This will induce us to make further assumptions on p_1 and p_2 (compared to (1.12) in Theorem 1.1) so as to ensure in particular that ∇Q is in L^2. Consequently, the homogeneous critical framework is no longer appropriate since some additional control will be required over the low frequencies of the solution.

More precisely, we shall prove the existence of a solution in the following nonhomogeneous space $F_T^{p_1,p_2}$:

$$\Big(\widetilde{C}_T(B^{d/p_1}_{p_1,1})\cap L^1_T(B^{d/p_1+2}_{p_1,1})\Big)\times\Big(\widetilde{C}_T(B^{d/p_2-1}_{p_2,1})\cap L^1_T(B^{d/p_2+1}_{p_2,1})\Big)^d\times\Big(L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)\Big)^d,$$

which are critical in terms of regularity but more demanding concerning the behavior at infinity.

In what follows, we denote

$$\begin{aligned}
\|\theta\|_{X^{p_1}(T)}&=\|\theta\|_{\widetilde{L}^\infty_T(B^{d/p_1}_{p_1,1})}+\|\theta\|_{L^1_T(B^{d/p_1+2}_{p_1,1})},\\
\|u\|_{Y^{p_2}(T)}&=\|u\|_{\widetilde{L}^\infty_T(B^{d/p_2-1}_{p_2,1})}+\|u\|_{L^1_T(B^{d/p_2+1}_{p_2,1})},\\
\|\nabla Q\|_{Z^{p_2}(T)}&=\|\nabla Q\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}.
\end{aligned}$$

Let us state our main local-in-time existence result for the fully nonhomogeneous case.

Theorem 1.2. *Let (p_1,p_2) satisfy*

$$1<p_1\leq 4,\quad 2\leq p_2\leq 4,\quad \frac{1}{p_2}+\frac{1}{d}>\frac{1}{p_1} \tag{1.17}$$

with in addition

$$p_1<4\ \text{ if }\ d=2,\quad\text{and}\quad \frac{1}{p_1}+\frac{1}{d}>\frac{1}{p_2}\ \text{ if }\ d\geq 3.$$

For any initial temperature $\vartheta_0=1+\theta_0$ and velocity field u_0 which satisfy

$$0<m\leq\vartheta_0,\quad \operatorname{div}u_0=0\quad\text{and}\quad \|\theta_0\|_{B^{d/p_1}_{p_1,1}}+\|u_0\|_{B^{d/p_2-1}_{p_2,1}}\leq M, \tag{1.18}$$

for some positive constants m, M, *there exists a positive time* T *depending only on* m, M, p_1, p_2, d, *and on the parameters of the system such that* (1.6) *has a unique solution* $(\vartheta, u, \nabla Q)$ *with* $(\theta, u, \nabla Q)$ *in* $F_T^{p_1,p_2}$. *Furthermore, for some constant* $C = C(d, p_1, p_2)$, *we have*

$$m \le \vartheta \quad \text{and} \quad \|\theta\|_{X^{p_1}(T)} + \|u\|_{Y^{p_2}(T)} + \|\nabla Q\|_{Z^{p_2}(T)} \le CM, \tag{1.19}$$

and the flow map $(\theta_0, u_0) \mapsto (\theta, u, \nabla Q)$ *is Lipschitz continuous.*

Remark 1.2. *The above theorems 1.1 1.2 and the transformation* (1.5) *ensure that the original system* (1.4) *is well-posed. More precisely, in the case* $1 \le p_1 = p_2 < 2d$ *for the initial data* (ϑ_0, v_0) *satisfying the third equation, and* $(\vartheta_0 - 1, v_0)$ *in* $\dot{B}^{d/p_1}_{p_1,1} \times \dot{B}^{d/p_1-1}_{p_1,1}$ *with* (1.13), *we get a local solution* $(\vartheta, v, \nabla \Pi)$ *of* (1.4) *such that* $(\theta, u, \nabla Q) \in \dot{F}_T^{p_1,p_1}$ *and the solution is global if* (1.14) *holds. Under the same regularity assumptions with in addition* $2 \le p_1 \le 4$ *then if* ϑ_0 *is just bounded from below, we get a local solution in* $F_T^{p_1,p_1}$. *In the case* $p_1 \ne p_2$, *a similar result holds true. It is more complicated to state, though.*

Let us end this section with a few comments and a short list of open questions that we plan to address in the future.

- To simplify the presentation, we restricted to the *free* evolution of a solution to (1.6). As in e.g. [8, 11], our methods enable us to treat the case where the fluid is subject to some external body force.

- We expect similar results for equations of state such as those that have been considered by Alazard in the Appendix of [2] or, more generally, for reacting flows as in [15] (as it only introduces coupling with parabolic equations involving reactants, the scaling of which is the same as that of ϑ). We here restricted our analysis to ideal gases for simplicity only.

- In the two-dimensional case, unless $k = \eta = 0$ and ζ is a positive constant (that is for the incompressible Navier-Stokes equations), the question of global existence for large data is widely open. Note however that our derivation of (1.4) highlights the important role of the parameter β defined in (1.7). As a matter of fact, it has been discovered very recently by the second author in [23] that global existence holds true in dimensional two (even for large data) if $\beta = 0$.

- As for the classical incompressible Euler equations, working in a critical functional framework is no longer relevant in the inviscid case. However,the approach proposed in [13] carries out to our system (see the forthcoming paper [16]).

- Granted with the above results, it is natural to study the asymptotics ε going to 0 in the above functional framework. This would extend some of the results of Alazard in [3] to the case of rough data.

The rest of the paper unfolds as follows. In the next section, we introduce the main tool for the proof –the Littlewood-Paley decomposition– and define Besov spaces and some related functional spaces. In passing, we state product laws in those spaces and commutator estimates. In Section 3, we focus on the proof of our first well-posedness result (pertaining to the case where the initial temperature is close to a constant) whereas our second well-posedness result is proved in Section 4. The proof of a commutator estimate in postponed in Appendix.

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2 Tools

Let us first fix some notation.

- Throughout this paper, C represents some "harmless" constant, which can be understood from the context. In some places, we shall alternately use the notation $A \lesssim B$ instead of $A \leq CB$, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.
- If $p \in [1, +\infty]$ then we denote by p' the *conjugated* exponent of p defined by $1/p + 1/p' = 1$.
- If X is a Banach space, $T > 0$ and $p \in [1, +\infty]$ then $L^p_T(X)$ stands for the set of Lebesgue measurable functions f from $[0, T)$ to X such that $t \mapsto \|f(t)\|_X$ belongs to $L^p([0, T))$. If $T = +\infty$, then the space is merely denoted by $L^p(X)$. Finally, if I is some interval of $\mathbb{R}$ then the notation $\mathcal{C}(I; X)$ stands for the set of continuous functions from I to X.
- We shall keep the same notation X to designate vector-fields with components in X.

2.1 Basic results on Besov spaces

First of all we recall briefly the definition of the so-called nonhomogeneous *Littlewood-Paley* decomposition: a dyadic partition of unity with respect to the Fourier variable. More precisely, fix a smooth nonincreasing radial function χ, which is supported in the ball $B(0, \frac{4}{3})$ and equals to 1 in a neighborhood of $B(0, 1)$. Set $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$, then we have

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1.$$

Let $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, $h_j = \mathcal{F}^{-1}\varphi_j$, and $\check{h} = \mathcal{F}^{-1}\chi$. The *dyadic blocks* $(\Delta_j)_{j \in \mathbb{Z}}$ are defined by

$$\begin{aligned}
&\Delta_j u = 0 \quad \text{if} \quad j \leq -2, \\
&\Delta_{-1} u = \chi(D)u = \int_{\mathbb{R}^d} \check{h}(y) u(x - y)\, dy, \\
&\Delta_j u = \varphi_j(D) u = \int_{\mathbb{R}^d} h_j(y) u(x - y)\, dy \quad \text{if} \quad j \geq 0,
\end{aligned}$$

and we also introduce the low-frequency cut-off:

$$S_j u = \sum_{k \leq j-1} \Delta_k u.$$

Note that $S_j u = \chi(2^{-j} D) u$ if $j \geq 0$.

As shown in e.g. [4], the above dyadic decomposition satisfies

$$\Delta_k \Delta_j u \equiv 0 \quad \text{if} \quad |k - j| \geq 2 \quad \text{and} \quad \Delta_k(S_{j-1} u \Delta_j u) \equiv 0 \quad \text{if} \quad |k - j| \geq 5.$$

In addition, for any tempered distribution u, one may write

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u,$$

and, owing to Bersntein's inequalities (see e.g. [4], Chap. 2),

$$\begin{aligned}
&\|\Delta_j u\|_{L^{p_1}} \lesssim 2^{d(\frac{1}{p_2} - \frac{1}{p_1})} \|\Delta_j u\|_{L^{p_2}} \quad \text{if } p_1 \geq p_2, \\
&\|D^k(\Delta_j u)\|_{L^p} \lesssim 2^{kj} \|\Delta_j u\|_{L^p}, \quad \forall j \geq -1, \\
&\|D^k(\Delta_j u)\|_{L^p} \approx 2^{kj} \|\Delta_j u\|_{L^p}, \quad \forall j \geq 0.
\end{aligned}$$

We can now define the nonhomogeneous Besov spaces $B^s_{p,r}$ as follows:

Definition 2.1. *For $s\in\mathbb{R}$, $(p,r)\in[1,+\infty]^2$, and $u\in\mathcal{S}'(\mathbb{R}^d)$, we set*

$$\|u\|_{B^s_{p,r}}=\Bigg(\sum_{j\geq-1}2^{jsr}\|\Delta_j u\|^r_{L^p}\Bigg)^{1/r}\quad\text{if}\ \ r<\infty,\qquad\text{and}\qquad \|u\|_{B^s_{p,\infty}}:=\sup_{j\geq-1}\big\{2^{js}\|\Delta_j u\|_{L^p}\big\}.$$

We then define

$$B^s_{p,r}=B^s_{p,r}(\mathbb{R}^d):=\big\{u\in\mathcal{S}'(\mathbb{R}^d),\|u\|_{B^s_{p,r}}<\infty\big\}.$$

Throughout, we shall use freely the following classical properties for Besov spaces.

Proposition 2.1. *The following properties hold true:*

(i) Action of derivatives: $\|\nabla u\|_{B^{s-1}_{p,r}}\lesssim\|u\|_{B^s_{p,r}}$.

(ii) Embedding: $B^s_{p_1,r_1}\hookrightarrow B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$ *if* $p_1\leq p_2$, $r_1\leq r_2$, *and* $B^{\frac{d}{p}}_{p,1}\hookrightarrow L^\infty$ *for all* $p\in[1,\infty]$.

(iii) Real interpolation: $(B^{s_1}_{p,r_1},B^{s_2}_{p,r_2})_{\theta,r'}=B^{(1-\theta)s_1+\theta s_2}_{p,r'}$.

When dealing with product of functions in Besov spaces, it is often convenient to use paradifferential calculus, a tool that has been introduced by J.-M. Bony in [5]. Recall that the paraproduct between u and v is defined by

$$T_uv=\sum_j S_{j-1}u\,\Delta_jv,$$

and that the remainder of u and v is defined by

$$R(u,v)=\sum_j\Delta_ju\,\widetilde{\Delta_j}v\quad\text{with}\quad\widetilde{\Delta_j}v=(\Delta_{j-1}+\Delta_j+\Delta_{j+1})v.$$

Then we have the following so-called Bony's decomposition for the product between u and v:

$$uv=T_uv+R(u,v)+T_vu=T'_uv+T_vu. \tag{2.1}$$

We shall often use the following estimates in Besov spaces for the paraproduct and remainder operators.

Proposition 2.2. *Let* $1\leq r,r_1,r_2,p,p_1,p_2\leq\infty$ *with* $\frac{1}{r}\leq\min\{1,\frac{1}{r_1}+\frac{1}{r_2}\}$ *and* $\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{p_2}\cdot$

- *If* $p\leq p_2$ *then we have:*

$$\|T_uv\|_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,r}}\lesssim\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}\quad\text{if}\ \ s_1<d\big(\tfrac{1}{p_1}+\tfrac{1}{p_2}-\tfrac{1}{p}\big), \tag{2.2}$$

$$\|T_uv\|_{B^{s_2}_{p,r}}\lesssim\|u\|_{L^{p_3}}\|v\|_{B^{s_2}_{p_2,r}}\ \ \text{if}\ \ \tfrac{1}{p_3}=\tfrac{1}{p}-\tfrac{1}{p_2}\cdot \tag{2.3}$$

- *If* $s_1+s_2+d\min\{0,1-\frac{1}{p_1}-\frac{1}{p_2}\}>0$, *then*

$$\|R(u,v)\|_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,r}}\lesssim\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}; \tag{2.4}$$

- *if* $s_1+s_2+d\min\{0,1-\frac{1}{p_1}-\frac{1}{p_2}\}=0$ *and* $\frac{1}{r_1}+\frac{1}{r_2}\geq1$ *then*

$$\|R(u,v)\|_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,\infty}}\lesssim\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}. \tag{2.5}$$

Proof. Most of these results are classical (see e.g. [4]). We just prove (2.2) and (2.3). which is a slight generalization of Prop. 2.3 in [1]. We write that

$$T_u v=\sum_{j\geq 1}T_j(u,v)\quad\text{with}\quad T_j(u,v)=S_{j-1}u\Delta_j v.$$

Since $\Delta_{j'}(S_{j-1}u\Delta_j v)\equiv 0$ for $|j'-j|>4$, it suffices to show that, for some sequence $(c_j)_{j\in\mathbb{N}}$ such that $\|(c_j)\|_{\ell^r}=1$, we have

$$\begin{aligned}\|T_j(u,v)\|_{L^p}&\lesssim c_j2^{-js}\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}\quad&\text{if}\quad s_1<d/p_1+d/p_2-d/p,\\ \|T_j(u,v)\|_{L^p}&\lesssim c_j2^{-js}\|u\|_{L^{p_3}}\|v\|_{B^{s_2}_{p_2,r_2}}\quad&\text{if}\quad s_1=d/p_1+d/p_2-d/p,\end{aligned}$$

with $s=s_1+s_2+\frac{d}{p}-\frac{d}{p_1}-\frac{d}{p_2}$.

According to Hölder's inequality, we have

$$\|T_j(u,v)\|_{L^p}\leq\|S_{j-1}u\|_{L^{p_3}}\|\Delta_j v\|_{L^{p_2}}\quad\text{with}\quad\frac{1}{p_3}=\frac{1}{p}-\frac{1}{p_2}\cdot\tag{2.6}$$

Hence, using the definition of S_{j-1} and Bernstein's inequality (here we notice that $p_1\leq p_3$, a consequence of $\frac1p\leq\frac{1}{p_1}+\frac{1}{p_2}$):

$$\|T_j(u,v)\|_{L^p}\lesssim\sum_{j'\leq j-2}2^{j'(\frac{d}{p_1}-\frac{d}{p_3})}\|\Delta_{j'}u\|_{L^{p_1}}\|\Delta_j v\|_{L^{p_2}},$$

whence

$$2^{js}\|T_j(u,v)\|_{L^p}\leq\sum_{j'\leq j-2}2^{(j-j')(s_1+\frac{d}{p_3}-\frac{d}{p_1})}\big(2^{j's_1}\|\Delta_{j'}u\|_{L^{p_1}}\big)\big(2^{js_2}\|\Delta_j v\|_{L^{p_2}}\big).$$

Therefore, if $s_1+d/p_3-d/p_1<0$ then the result stems from convolution and Hölder inequalities for series. In the case where $s_1+d/p_3-d/p_1=0$, we just have to use that $\|S_{j-1}u\|_{L^{p_3}}\leq C\|u\|_{L^{p_3}}$ in (2.6).

The proof of (2.4) goes from similar arguments and is thus left to the reader (see also [1]). □

From the above proposition and (2.1), one may deduce a number of estimates in Besov spaces for the product of two functions. We shall use the following result:

Proposition 2.3. *The following estimates hold true:*

(i) $\|uv\|_{B^s_{p,r}}\lesssim\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}\|v\|_{L^\infty}$ *if* $s>0$.

(ii) If $s_1<\frac{d}{p_1}$, $s_2<d\min\{\frac{1}{p_1},\frac{1}{p_2}\}$, $s_1+s_2+d\min\{0,1-\frac{1}{p_1}-\frac{1}{p_2}\}>0$ *and* $\frac1r\leq\min\{1,\frac{1}{r_1}+\frac{1}{r_2}\}$ *then*

$$\|uv\|_{B^{s_1+s_2-\frac{d}{p_1}}_{p_2,r}}\lesssim\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.\tag{2.7}$$

(iii) We also have the following limit cases:

- *if* $s_1=d/p_1$, $s_2<\min(d/p_1,d/p_2)$ *and* $s_2+d\min(1/p_1,1/p_2')>0$ *then*

$$\|uv\|_{B^{s_2}_{p_2,r}}\lesssim\|u\|_{B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}\|v\|_{B^{s_2}_{p_2,r}};\tag{2.8}$$

- *if* $s_2=\min(d/p_1,d/p_2)$, $s_1<d/p_1$ *and* $s_1+s_2+d\min\{0,1-\frac{1}{p_1}-\frac{1}{p_2}\}>0$ *then*

$$\|uv\|_{B^{s_1+s_2-\frac{d}{p_1}}_{p_2,r}}\lesssim\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,1}};$$

- *if* $1/r_1+1/r_2\geq 1,\ s_1<\frac{d}{p_1},\ s_2<d\min\{\frac{1}{p_1},\frac{1}{p_2}\}$ *and* $s_1+s_2+d\min(0,1-\frac{1}{p_1}-\frac{1}{p_2})=0$ *then*
$$\|uv\|_{B^{s_1+s_2-\frac{d}{p_1}}_{p_2,\infty}}\lesssim\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.$$

The following commutator estimates (see the proof in Appendix) will be also needed:

Proposition 2.4. *Let* $\frac{1}{r}=\min\{1,\frac{1}{r_1}+\frac{1}{r_2}\}$ *and* $(s,\nu)\in\mathbb{R}\times\mathbb{R}$ *satisfying*
$$-d\min\Big\{\frac{1}{p_1'},\frac{1}{p_2}\Big\}<s<\nu+d\min\Big\{\frac{1}{p_1},\frac{1}{p_2}\Big\}\quad\text{and}\quad -d\min\Big\{\frac{1}{p_1},\frac{1}{p_2}\Big\}<\nu<1.\tag{2.9}$$
For $j\geq -1$*, denote* $R_j(u,v):=[u,\Delta_j]v$*. We have*
$$\|\left(2^{js}\|R_j(u,v)\|_{L^{p_1}}\right)_{j\geq -1}\|_{\ell^r}\lesssim\|\nabla u\|_{B^{\frac{d}{p_2}+\nu-1}_{p_2,r_2}}\|v\|_{B^{s-\nu}_{p_1,r_1}}.\tag{2.10}$$

The following limit cases also hold true:

- *if* $s=\nu+d\min\{\frac{1}{p_1},\frac{1}{p_2}\},\ r_1=1$ *and* $r_2=r$ *then we have*
$$\|\left(2^{js}\|R_j(u,v)\|_{L^{p_1}}\right)_{j\geq -1}\|_{\ell^r}\lesssim\|\nabla u\|_{B^{\frac{d}{p_2}+\nu-1}_{p_2,r}}\|v\|_{B^{s-\nu}_{p_1,1}};\tag{2.11}$$

- *if* $\nu=1,\ r_1=r$ *and* $r_2=\infty$ *then we have*
$$\|\left(2^{js}\|R_j(u,v)\|_{L^{p_1}}\right)_{j\geq -1}\|_{\ell^r}\lesssim\|\nabla u\|_{B^{\frac{d}{p_2}}_{p_2,\infty}\cap L^\infty}\|v\|_{B^{s-1}_{p_1,r}}.\tag{2.12}$$

Finally, if in addition to (2.9)*, we have* $\nu>1-d\min(1/p_1,1/p_2)$ *then*
$$\|\big(2^{j(s-1)}\|\partial_k R_j(u,v)\|_{L^{p_1}}\big)_{j\geq -1}\|_{\ell^r}\lesssim\|\nabla u\|_{B^{\frac{d}{p_2}+\nu-1}_{p_2,r_2}}\|v\|_{B^{s-\nu}_{p_1,r_1}}\quad\text{for all }\ k\in\{1,\cdots,d\},\tag{2.13}$$
with the above changes in the limit cases.

We shall also use the following result for the action of smooth functions (see e.g. [4]):

Proposition 2.5. *Let* $(p,r)\in[1,+\infty]^2$ *and* $s>0$*. Let* f *be a smooth function from* $\mathbb{R}$ *to* $\mathbb{R}$*.*

- *If* $f(0)=0$ *then for all* $u\in B^s_{p,r}\cap L^\infty$ *we have*
$$\|f\circ u\|_{B^s_{p,r}}\leq C(f',\|u\|_{L^\infty})\|u\|_{B^s_{p,r}}.\tag{2.14}$$

- *If* $f'(0)=0$ *then for all* u *and* v *in* $B^s_{p,r}\cap L^\infty$*, we have*
$$\|f\circ v-f\circ u\|_{B^s_{p,r}}\leq C(f'',\|u\|_{L^\infty\cap B^s_{p,r}},\|v\|_{L^\infty\cap B^s_{p,r}})\|v-u\|_{B^s_{p,r}}.\tag{2.15}$$

When solving evolutionary PDEs, it is natural to use spaces of type $L^\rho_T(X)=L^\rho(0,T;X)$ with X denoting some Banach space. In our case, X will be a Besov space so that we will have to localize the equations through Littlewood-Paley decomposition. This will provide us with estimates of the Lebesgue norm of each dyadic block *before* performing integration in time. This leads to the following definition:

Definition 2.2. *For* $s\in\mathbb{R}$*,* $(\rho,p,r)\in[1,+\infty]^3$ *and* $T\in[0,+\infty]$*, we set*
$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}=\left(\sum_{j\geq -1}2^{rjs}\left(\int_0^T\|\Delta_j u(t)\|^\rho_{L^p}\,dt\right)^{\frac{r}{\rho}}\right)^{\frac{1}{r}},$$
with the usual change if $r=+\infty$ *or* $\rho=+\infty$*.*

We also set $\widetilde{C}_T(B^s_{p,r})=\widetilde{L}^\infty_T(B^s_{p,r})\cap\mathcal{C}([0,T];B^s_{p,r})$*.*

Let us remark that, by virtue of Minkowski's inequality, we have

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \leq \|u\|_{L^\rho_T(B^s_{p,r})} \text{ if } \rho \leq r,$$
$$\|u\|_{L^\rho_T(B^s_{p,r})} \leq \|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \text{ if } \rho \geq r,$$

and hence in particular $\|u\|_{\widetilde{L}^1_T(B^s_{p,1})} = \|u\|_{L^1_T(B^s_{p,1})}$ holds.

Let $\theta \in [0,1]$, $\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}$, and $s = \theta s_1 + (1-\theta)s_2$, then the following interpolation inequality holds true:

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \leq \|u\|^\theta_{\widetilde{L}^{\rho_1}_T(B^{s_1}_{p,r})} \|u\|^{1-\theta}_{\widetilde{L}^{\rho_2}_T(B^{s_2}_{p,r})}.$$

In this framework, one may get product or composition estimates similar to those that have been stated above. The general rule is that the Lebesgue exponents pertaining to the time variable behave according to Hölder's inequality. For instance, one has:

$$\|uv\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \lesssim \|u\|_{L^{\rho_1}_T(L^\infty)} \|v\|_{\widetilde{L}^{\rho_4}_T(B^s_{p,r})} + \|u\|_{\widetilde{L}^{\rho_2}_T(B^s_{p,r})} \|v\|_{L^{\rho_3}_T(L^\infty)}, \tag{2.16}$$

whenever $s > 0$, $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_4} = \frac{1}{\rho_2} + \frac{1}{\rho_3}$, and

$$\|uv\|_{\widetilde{L}^\rho_T(B^{s_1+s_2-\frac{d}{p}}_{p,r})} \lesssim \|u\|_{\widetilde{L}^{\rho_1}_T(B^{s_1}_{p,r})} \|v\|_{\widetilde{L}^{\rho_2}_T(B^{s_2}_{p,\infty})}, \tag{2.17}$$

if $s_1 + s_2 + d\min\{0, 1 - \frac{2}{p}\} > 0$, $s_1, s_2 < \frac{d}{p}$, $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$.

As pointed out in the introduction, scaling invariant spaces have to be homogeneous. As a consequence, the optimal framework for proving our first well-posedness result (namely Theorem 1.1) turns to be *homogeneous* Besov spaces. For completeness, we here define those spaces. We first need to introduce *homogeneous* dyadic blocks

$$\dot{\Delta}_j u = \int_{\mathbb{R}^d} h_j(y) u(x-y)\,dy, \quad \forall j \in \mathbb{Z}$$

and the homogeneous low-frequency truncation operator

$$\dot{S}_j := \chi(2^{-j}D), \quad \forall j \in \mathbb{Z}. \tag{2.18}$$

We then define homogeneous semi-norms:

$$\|u\|_{\dot{B}^s_{p,r}} = \big\|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_{j\in\mathbb{Z}}\big\|_{\ell^r}.$$

Note that, for $u \in \mathcal{S}'(\mathbb{R}^d)$, the equality

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$$

holds true modulo polynomials only. Hence, the functional spaces related to the above semi-norm cannot be defined without care. Following [4], we shall define homogeneous Besov spaces as follows:

Definition 2.3. *The homogeneous Besov space $\dot{B}^s_{p,r}$ is the set of tempered distributions u such that*

$$\|u\|_{\dot{B}^s_{p,r}} < \infty \quad \textit{and} \quad \lim_{j\to-\infty} \|\dot{S}_j u\|_{L^\infty} = 0.$$

The above definition ensures that $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is a Banach space provided that

$$s < d/p \quad \text{or} \quad s \leq d/p \ \text{ if } \ r = 1. \tag{2.19}$$

All the above estimates remain true in homogeneous spaces. In addition, if $u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u$ and $\|u\|_{\dot{B}^s_{p,r}}$ is finite for some (s,p,r) satisfying (2.19) then u belongs to $\dot{B}^s_{p,r}(\mathbb{R}^d)$, owing to the aforementioned Bernstein's inequalities. This fact will be used repeatedly.

3 The proof of Theorem 1.1

This section is devoted to the well-posedness issue for System (1.11) in the slightly nonhomogeneous case. The proof strongly relies on a priori estimates for the linearized equations about 0 which will be recalled in the first part of this section. The proof of existence and uniqueness will be carried out in the second part.

3.1 The linearized equations

In the case of a given velocity field w, the linearized temperature equation about 0 reads

$$\begin{cases} \partial_t\theta + w\cdot\nabla\theta - \bar\kappa\Delta\theta = f, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{3.1}$$

Obviously, the convection term $w\cdot\nabla\theta$ is of lower order so that it may be included in the "source terms" if it is only a matter of solving (1.11). However, considering the above convection-diffusion equation (3.1) rather than the standard heat equation will enable us to get more accurate estimates. The same remark holds for the linearized momentum equation (3.2):

$$\begin{cases} \partial_t u + w\cdot\nabla u - \bar\mu\Delta u + \nabla Q = h, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{3.2}$$

The reader is referred to [12] for the proof of the following two results.

Proposition 3.1. *Let $1\le p\le p_1\le\infty$ and $1\le r\le\infty$. Let $s\in\mathbb{R}$ satisfy*

$$\begin{cases} s<1+\dfrac{d}{p_1}, \quad \text{or} \quad s\le 1+\dfrac{d}{p_1} \quad \textit{if} \;\; r=1, \\ s>-d\min\Big\{\dfrac{1}{p_1},\dfrac{1}{p'}\Big\}, \quad \text{or} \quad s>-1-d\min\Big\{\dfrac{1}{p_1},\dfrac{1}{p'}\Big\} \;\; \text{if} \;\; \operatorname{div} w=0. \end{cases} \tag{3.3}$$

There exists a constant C depending only on $d,$ $r,$ s and $s-1-\frac{d}{p_1}$ such that for any smooth solution θ of (3.1) *with $\bar\kappa\ge 0,$ and $\rho\in[1,\infty],$ we have the following a priori estimate:*

$$\bar\kappa^{\frac{1}{\rho}}\|\theta\|_{\widetilde L^\rho_T(\dot B^{s+\frac{2}{\rho}}_{p,r})} \le e^{CW_{p_1}(T)}\Big(\|\theta_0\|_{\dot B^s_{p,r}} + \|f\|_{\widetilde L^1_T(\dot B^s_{p,r})}\Big)$$

with
$$\begin{cases} W_{p_1}(T) = \displaystyle\int_0^T \|\nabla w(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}\,dt & \textit{if} \;\; s<\frac{d}{p_1}+1, \\ W_{p_1}(T) = \displaystyle\int_0^T \|\nabla w(t)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\,dt & \textit{if} \;\; s=\frac{d}{p_1}+1. \end{cases}$$

Proposition 3.2. *Let $p,$ $p_1,$ $r,$ s and W_{p_1} be as in Proposition 3.1. There exists a constant C depending only on $d,$ $r,$ s and $s-1-\frac{d}{p_1}$ such that for any smooth solution $(u,\nabla Q)$ of* (3.2) *with $\bar\mu\ge 0,$ and $\rho\in[1,\infty],$ we have the following a priori estimate:*

$$\bar\mu^{\frac{1}{\rho}}\|u\|_{\widetilde L^\rho_T(\dot B^{s+\frac{2}{\rho}}_{p,r})} \le e^{CW_{p_1}(T)}\Big(\|u_0\|_{\dot B^s_{p,r}} + \|\mathcal{P}h\|_{\widetilde L^1_T(\dot B^s_{p,r})}\Big),$$
$$\|\nabla Q-\mathcal{Q}h\|_{\widetilde L^1_T(\dot B^s_{p,r})} \le C\Big(e^{CW_{p_1}(T)}-1\Big)\Big(\|u_0\|_{\dot B^s_{p,r}} + \|\mathcal{P}h\|_{\widetilde L^1_T(\dot B^s_{p,r})}\Big).$$

Above, $\mathcal{P}$ and $\mathcal{Q}$ stand for the orthogonal projectors over divergence-free and potential vector-fields, respectively.

3.2 The well-posedness issue in the slightly nonhomogeneous case

For proving existence, we will follow a standard procedure, first we construct a sequence of approximate solutions, second, we prove uniform bounds for them, and finally we show the convergence to some solution of the system. In the case of *large* initial velocity, we will have to split the constructed velocity into the free solution of the Stokes system with initial data u_0, and the discrepancy to this free velocity. Stability estimates and uniqueness will be obtained afterward by the same argument as the convergence of the sequence.

Step 1. Approximate solutions

Solving System (1.11) will be based on an iterative scheme: first we set $(\theta^0, u^0, \nabla Q^0) \equiv 0$ then, once $(\theta^n, u^n, \nabla Q^n)$ has been defined over $\mathbb{R}^+ \times \mathbb{R}^d$, we define $(\theta^{n+1}, u^{n+1}, \nabla Q^{n+1})$ as the solution to the following linear system[3] :

$$\left\{\begin{array}{ccc} \partial_t \theta^{n+1} + u^n \cdot \nabla \theta^{n+1} - \bar\kappa \Delta \theta^{n+1} & = & a^n, \\ \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} - \bar\mu \Delta u^{n+1} + \nabla Q^{n+1} & = & c^n, \\ \operatorname{div} u^{n+1} & = & 0, \\ (\theta^{n+1}, u^{n+1})_{|t=0} & = & (\dot S_{n+1}\theta_0, \dot S_{n+1} u_0), \end{array}\right. \tag{3.4}$$

with $\dot S_{n+1}$ defined in (2.18) and, denoting $\vartheta^n := 1 + \theta^n$,

$$\begin{aligned} a^n &:= a^n(\theta^n) = \operatorname{div}\left((\kappa(\vartheta^n) - \bar\kappa)\nabla\theta^n\right) - \kappa'(\vartheta^n)|\nabla\theta^n|^2, \\ c^n &:= c^n(\theta^n, u^n, \nabla Q^n) = \operatorname{div}\left((\mu(\vartheta^n) - \bar\mu)\nabla u^n\right) - \theta^n \nabla Q^n + A_1^n |\nabla\theta^n|^2 \nabla\theta^n \\ &\qquad + A_2^n \Delta\theta^n \nabla\theta^n + A_3^n \nabla^2\theta^n \cdot \nabla\theta^n + A_4^n \nabla u^n \cdot \nabla\theta^n + A_5^n Du^n \cdot \nabla\theta^n. \end{aligned}$$

Above, it is understood that $A_i^n := A_i(\vartheta^n)$ with A_i defined by (1.8).

Step 2. Uniform bounds

In order to bound $(\theta^{n+1}, u^{n+1}, \nabla Q^{n+1})$, one may take advantage of Proposition 3.1 with $s = d/p_1$ and Lebesgue exponents (p_1, p_2) (here comes the assumption that $1/p_1 \le 1/d + 1/p_2$), and of Proposition 3.2 with $s = d/p_2 - 1$ and exponents (p_2, p_2). Concerning θ^{n+1}, if $p_2 \le p_1$ then we use the embedding $\dot B^{\frac{d}{p_2}+1}_{p_2,1} \hookrightarrow \dot B^{\frac{d}{p_1}+1}_{p_1,1}$. We eventually get

$$\|\theta^{n+1}\|_{\dot X^{p_1}(t)} \lesssim e^{\|u^n\|_{L^1_t(\dot B^{d/p_2+1}_{p_2,1})}} \Big(\|\dot S_{n+1}\theta_0\|_{\dot B^{d/p_1}_{p_1,1}} + \|a^n\|_{L^1_t(\dot B^{d/p_1}_{p_1,1})} \Big), \tag{3.5}$$

$$\|u^{n+1}\|_{\dot Y^{p_2}(t)} + \|\nabla Q^{n+1}\|_{\dot Z^{p_2}(t)} \lesssim e^{\|u^n\|_{L^1_t(\dot B^{d/p_2+1}_{p_2,1})}} \Big(\|\dot S_{n+1} u_0\|_{\dot B^{d/p_2-1}_{p_2,1}} + \|c^n\|_{L^1_t(\dot B^{d/p_2-1}_{p_2,1})} \Big). \tag{3.6}$$

Let us now bound a^n and c^n. Using Propositions 2.3 and 2.5, we easily get

$$\|a^n\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} \lesssim \big(1 + \|\theta^n\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\big)\big(\|\theta^n\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} \|\theta^n\|_{\dot B^{\frac{d}{p_1}+2}_{p_1,1}} + \|\nabla\theta^n\|^2_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\big). \tag{3.7}$$

As regards c^n, it is mostly a matter of bounding the following terms in $L^1_t(\dot B^{\frac{d}{p_2}-1}_{p_2,1})$ (keeping in mind that $1/p_2 \le 1/d + 1/p_1$):

$$\nabla^2\theta^n \cdot \nabla\theta^n, \quad |\nabla\theta^n|^2\nabla\theta^n, \quad \operatorname{div}(\theta^n \nabla u^n), \quad \nabla\theta^n \otimes \nabla u^n \quad \text{and} \quad \theta^n \nabla Q^n.$$

Indeed, on any interval $[0, T]$, taking the ϑ^n dependency of the coefficients into account will only multiply the estimates by some continuous function of $\|\theta^n\|_{L^\infty_T(\dot B^{\frac{d}{p_1}}_{p_1,1})}$. In what follows, this function will be denoted by C_{θ^n}.

[3]Note that the existence of solution for this system may be deduced from the case with no convection. Indeed, considering the convection terms as source terms, it is not difficult to construct an iterative scheme the convergence of which is based on the estimates of the previous subsection.

Now, if $p_1 < 2d$ then Proposition 2.3 ensures that the usual product maps $\dot{B}^{\frac{d}{p_1}-1}_{p_1,1} \times \dot{B}^{\frac{d}{p_1}}_{p_1,1}$ in $\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}$. Therefore, if $p_1 \le p_2$, then functional embedding implies that

$$\|\nabla^2\theta^n \cdot \nabla\theta^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|\nabla^2\theta^n\|_{\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}} \|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}. \tag{3.8}$$

To deal with the more complicated case where $p_1 > p_2$, we use the following Bony's decomposition:

$$\nabla^2\theta^n \cdot \nabla\theta^n = T_{\nabla^2\theta^n}\nabla\theta^n + R(\nabla^2\theta^n, \nabla\theta^n) + T_{\nabla\theta^n}\nabla^2\theta^n.$$

Finally, Proposition 2.2 enables to conclude that under conditions

$$p_1 < 2d, \quad p_1 \le 2p_2 \quad \text{and} \quad \frac{1}{p_2} \le \frac{1}{p_1} + \frac{1}{d},$$

we have

$$\|\nabla^2\theta^n \cdot \nabla\theta^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|\nabla^2\theta^n\|_{\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}} \|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + \|\nabla^2\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} \|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}}. \tag{3.9}$$

Bounding $|\nabla\theta^n|^2\nabla\theta^n$ stems from similar arguments. Under the above conditions, it is found that

$$\||\nabla\theta^n|^2\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}} \|\nabla\theta^n\|^2_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}. \tag{3.10}$$

We also easily get

$$\|\mathrm{div}\,(\theta^n\nabla u^n)\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \lesssim \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} \|\nabla u^n\|_{\dot{B}^{\frac{d}{p_2}}_{p_2,1}} + \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}+1}_{p_1,1}} \|\nabla u^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}. \tag{3.11}$$

Finally, according to Inequality (2.8),

$$\begin{aligned} \|\nabla u^n \otimes \nabla\theta^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} &\lesssim \|\nabla\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} \|\nabla u^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}, \\ \|\theta^n\nabla Q^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} &\lesssim \|\theta^n\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} \|\nabla Q^n\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}, \end{aligned}$$

provided that

$$\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{d} \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{d} \ge \frac{1}{p_2}. \tag{3.12}$$

So, plugging all the above inequalities in (3.5),(3.6) finally implies that

$$\|\theta^{n+1}\|_{\dot{X}^{p_1}(t)} \lesssim e^{C\|u^n\|_{L^1_t(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})}} \Big(\|\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + C_{\theta^n}\|\theta^n\|^2_{\dot{X}^{p_1}(t)}\Big), \tag{3.13}$$

$$\begin{aligned} \|u^{n+1}\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^{n+1}\|_{\dot{Z}^{p_2}(t)} \lesssim e^{C\|u^n\|_{L^1_t(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})}} \Big(\|u_0\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \\ + C_{\theta^n}\|\theta^n\|_{\dot{X}^{p_1}(t)}\big(\|\theta^n\|_{\dot{X}^{p_1}(t)} + \|u^n\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^n\|_{\dot{Z}^{p_2}(t)}\big)\Big). \end{aligned} \tag{3.14}$$

Note that the right-hand sides involves only initial data and at least quadratic combinations of the norms of $(\theta^n, u^n, \nabla Q^n)$. From a standard induction argument, it is thus easy to find some small constant τ such that if

$$\|\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + \|u_0\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}} \le \tau \tag{3.15}$$

then, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$, we have for some $K > 0$ depending only on the parameters of the system and on d, p_1, p_2,

$$\|\theta^n\|_{\dot{X}^{p_1}(t)} + \|u^n\|_{\dot{Y}^{p_2}(t)} + \|\nabla Q^n\|_{\dot{Z}^{p_2}(t)} \le K\big(\|\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + \|u_0\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}\big). \tag{3.16}$$

This completes the proof of uniform estimates in the case where both θ_0 and u_0 are small.

Let us now concentrate on the case where only θ_0 is small. Assuming that T has been chosen so that

$$\exp\biggl(C\int_0^T \|u^n\|_{\dot B^{\frac{d}{p_2}+1}_{p_2,1}}\,dt\biggr)\leq 2, \tag{3.17}$$

and that θ_0 is small enough, Inequality (3.13) still implies that

$$\|\theta^{n+1}\|_{\dot X^{p_1}(T)}\leq K\|\theta_0\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} \tag{3.18}$$

if (1.13) is satisfied and if θ^n also satisfies (3.18).

However, if u_0 is large then Inequality (3.14) is not enough to bound u^{n+1}. Therefore we introduce the "free" solution u_L to the heat equation

$$\left\{\begin{array}{rcl}\partial_t u_L-\bar\mu\Delta u_L &=& 0,\\ u_L|_{t=0} &=& u_0,\end{array}\right. \tag{3.19}$$

and define $u_L^n:=\dot S_n u_L$. Of course that $\operatorname{div} u_0\equiv 0$ implies that $\operatorname{div} u_L\equiv 1$. Now, $\bar u^{n+1}:=u^{n+1}-u_L^{n+1}$ satisfies

$$\left\{\begin{array}{rcl}\partial_t\bar u^{n+1}+u^n\cdot\nabla\bar u^{n+1}-\bar\mu\Delta\bar u^{n+1}+\nabla Q^{n+1} &=& \bar c^n,\\ \operatorname{div}\bar u^{n+1} &=& 0,\\ \bar u^{n+1}|_{t=0} &=& 0,\end{array}\right.$$

with $\bar c^n=c^n-u^n\cdot\nabla u_L^{n+1}$.

Note that $u^n\cdot\nabla u_L^{n+1}=\operatorname{div}(u^n\otimes u_L^{n+1})$. Hence, as $\dot B^{\frac{d}{p_2}}_{p_2,1}$ is an algebra for $p_2<\infty$, we have

$$\|\bar c^n\|_{\dot B^{\frac{d}{p_2}-1}_{p_2,1}}\lesssim\|c^n\|_{\dot B^{\frac{d}{p_2}-1}_{p_2,1}}+\|u^n\|_{\dot B^{\frac{d}{p_2}}_{p_2,1}}\|u_L^{n+1}\|_{\dot B^{\frac{d}{p_2}}_{p_2,1}}. \tag{3.20}$$

Hence, bounding c^n as above but splitting u^n into $\bar u^n+u_L^n$ when dealing with the terms $\nabla u^n\cdot\nabla\theta^n$ or $Du^n\cdot\nabla\theta^n$, we get under hypothesis (3.17), for all $t\in[0,T]$,

$$\begin{aligned}\|\bar u^{n+1}\|_{\dot Y^{p_2}(t)}+\|\nabla Q^{n+1}\|_{\dot Z^{p_2}(t)}\leq C\Bigl(&\|\theta^n\|_{\dot X^{p_1}(t)}\bigl(\|\theta^n\|_{\dot X^{p_1}(t)}+\|\bar u^n\|_{\dot Y^{p_2}(t)}+\|\nabla\bar Q^n\|_{\dot Z^{p_2}(t)}\bigr)\\ &+\|u_L\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}\bigl(\|u^n\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}+\|\nabla\theta^n\|_{L^2_t(\dot B^{\frac{d}{p_1}}_{p_1,1})}\bigr)+\|\theta^n\|_{L^\infty_t(\dot B^{\frac{d}{p_1}}_{p_1,1})}\|\nabla u_L\|_{L^1_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}\Bigr).\end{aligned}$$

Therefore, if we assume in addition that T has been chosen so that

$$\|u_L\|_{L^2_T(\dot B^{\frac{d}{p_2}}_{p_2,1})\cap L^1_T(\dot B^{\frac{d}{p_2}+1}_{p_2,1})}+\|\nabla Q_L\|_{L^1_T(\dot B^{\frac{d}{p_2}-1}_{p_2,1})}\leq\tau \tag{3.21}$$

and if

$$\|\bar u^n\|_{\dot Y^{p_2}(T)}+\|\nabla Q^n\|_{\dot Z^{p_2}(T)}\leq\tau \tag{3.22}$$

then we have also (taking τ smaller if needed) (3.17) and

$$\|\bar u^{n+1}\|_{\dot Y^{p_2}(T)}+\|\nabla Q^{n+1}\|_{\dot Z^{p_2}(T)}\leq\tau.$$

Now, an elementary induction argument enables us to conclude that both (3.17) and (3.22) are satisfied (for all $n\in\mathbb N$) if T has been chosen so that (3.21) holds.

Step 3. Convergence of the scheme

Let us just treat the case where only local existence is expected (that is u_0 may be large). We fix some time T such that (3.21) is fulfilled. Let $(\delta\theta^n, \delta u^n, \delta Q^n) := (\theta^{n+1} - \theta^n, u^{n+1} - u^n, Q^{n+1} - Q^n)$. We have

$$\begin{cases} \partial_t \delta\theta^n + u^n \cdot \nabla \delta\theta^n - \bar{\kappa}\Delta\delta\theta^n & = & -\delta u^{n-1} \cdot \nabla\theta^n + a^{n+1} - a^n, \\ \partial_t \delta u^n + u^n \cdot \nabla \delta u^n - \bar{\mu}\Delta\delta u^n + \nabla\delta Q^n & = & -\delta u^{n-1} \cdot \nabla u^n + c^{n+1} - c^n, \\ (\delta\theta^n, \delta u^n)|_{t=0} & = & (\dot{\Delta}_n\theta_0, \dot{\Delta}_n u_0). \end{cases}$$

By arguing exactly as in the proof of the stability estimates below, it is not difficult to establish that if τ has been chosen small enough in (3.21) then for all $n \geq 1$,

$$\|\delta\theta^n\|_{\dot{X}^{p_1}(T)} + \|\delta u^n\|_{\dot{Y}^{p_2}(T)} + \|\nabla\delta Q^n\|_{\dot{Z}^{p_2}(T)} \leq C\big(2^{n\frac{d}{p_1}}\|\dot{\Delta}_n\theta_0\|_{L^{p_1}} + 2^{n(\frac{d}{p_2}-1)}\|\dot{\Delta}_n u_0\|_{L^{p_2}}\big)$$
$$+\frac{1}{2}\big(\|\delta\theta^{n-1}\|_{\dot{X}^{p_1}(T)} + \|\delta u^{n-1}\|_{\dot{Y}^{p_2}(T)} + \|\nabla\delta Q^{n-1}\|_{\dot{Z}^{p_2}(T)}\big).$$

Hence $(\theta^n, u^n, \nabla Q^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\dot{F}_T^{p_1,p_2}$. The limit $(\theta, u, \nabla Q)$ belongs to $\dot{F}_T^{p_1,p_2}$ and obviously satisfies System (1.11).

Step 4. Uniqueness and stability estimates

Let us consider two solutions $(\theta^1, u^1, \nabla Q^1)$ and $(\theta^2, u^2, \nabla Q^2)$ of System (1.11), in the space $\dot{F}_T^{p_1,p_2}$ with (p_1, p_2) satisfying (1.12). The difference $(\delta\theta, \delta u, \delta Q) := (\theta^2 - \theta^1, u^2 - u^1, Q^2 - Q^1)$ between these two solutions satisfies

$$\begin{cases} \partial_t\delta\theta + u^1 \cdot \nabla\delta\theta - \bar{\kappa}\Delta\delta\theta = -\delta u \cdot \nabla\theta^2 + a(\theta^2) - a(\theta^1), \\ \partial_t\delta u + u^1 \cdot \nabla\delta u - \bar{\mu}\Delta\delta u + \nabla\delta Q = -\delta u \cdot \nabla u^2 + c(\theta^2, u^2, \nabla Q^2) - c(\theta^1, u^1, \nabla Q^1), \\ \operatorname{div}\delta u = 0. \end{cases}$$

Therefore, according to Propositions 3.1 and 3.2, we have for all $t \in [0, T]$,

$$\|\delta\theta\|_{\dot{X}^{p_1}(t)} \lesssim e^{\|u^1\|_{L^1_t(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})}}\Big(\|\delta\theta_0\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}} + \|\delta u \cdot \nabla\theta^2\|_{L^1_t(\dot{B}^{\frac{d}{p_1}}_{p_1,1})} + \|a(\theta^2) - a(\theta^1)\|_{L^1_t(\dot{B}^{\frac{d}{p_1}}_{p_1,1})}\Big),$$

$$\|\delta u\|_{\dot{Y}^{p_2}(t)} + \|\nabla\delta Q\|_{\dot{Z}^{p_2}(t)} \lesssim e^{\|u^1\|_{L^1_t(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})}}\Big(\|\delta u_0\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_1,1}} + \|\delta u \cdot \nabla u^2\|_{L^1_t(\dot{B}^{\frac{d}{p_2}-1}_{p_2,1})}$$
$$+\|c(\theta^2, u^2, \nabla Q^2) - c(\theta^1, u^1, \nabla Q^1)\|_{L^1_t(\dot{B}^{\frac{d}{p_2}-1}_{p_2,1})}\Big).$$

The nonlinear terms in the right-hand side may be handled exactly as in the proof of the uniform estimates (as the norms which are involved are the same, there are no further conditions on p_1 and p_2). For instance, we have for $\frac{1}{p_1} \leq \frac{1}{p_2} + \frac{1}{d}$,

$$\|\delta u \cdot \nabla\theta^2\|_{L^1_t(\dot{B}^{\frac{d}{p_1}}_{p_1,1})} \lesssim \|\delta u\|_{L^2_t(\dot{B}^{\frac{d}{p_2}}_{p_2,1})}\|\nabla\theta^2\|_{L^2_t(\dot{B}^{\frac{d}{p_1}}_{p_1,1})} + \|\delta u\|_{L^1_t(\dot{B}^{\frac{d}{p_2}+1}_{p_2,1})}\|\nabla\theta^2\|_{L^\infty_t(\dot{B}^{\frac{d}{p_1}-1}_{p_1,1})},$$

and, because

$$a(\theta^2) - a(\theta^1) = \operatorname{div}\Big(\big(\kappa(\vartheta^2) - \kappa(\vartheta^1)\big)\nabla\theta^2 + \big(\kappa(\vartheta^1) - \bar{\kappa}\big)\nabla\delta\theta\Big)$$
$$-\big(\kappa'(\vartheta^2) - \kappa'(\vartheta^1)\big)|\nabla\theta^2|^2 - \kappa'(\vartheta^1)\big(\nabla(\theta^1 + \theta^2) \cdot \nabla\delta\theta\big),$$

we have, according to Propositions 2.3 and 2.5,

$$\|a(\theta^2)-a(\theta^1)\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} \lesssim C_{\theta^1,\theta^2}\Big(\big(\|\nabla\theta^1\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}+\|\nabla\theta^2\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\big)\|\nabla\delta\!\theta\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}} \\ +\big(\|\nabla\theta^2\|^2_{\dot B^{\frac{d}{p_1}}_{p_1,1}}+\|\nabla^2\theta^2\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\big)\|\delta\!\theta\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}+\|\theta^1\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\|\Delta\delta\!\theta\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}\Big).$$

We may proceed similarly in order to bound the right-hand side of the inequality for δu. We eventually get for all $t\in[0,T]$,

$$\|\delta\!\theta\|_{\dot X^{p_1}(t)} \leq C_{\theta^1,\theta^2,u^1}\Big(\|\delta\!\theta_0\|_{\dot B^{\frac{d}{p_1}}_{p_1,1}}+\|\theta^2\|_{\dot X^{p_1}(t)}\|\delta u\|_{\dot Y^{p_2}(t)}+(\|\theta^1\|_{\dot X^{p_1}(t)}+\|\theta^2\|_{\dot X^{p_1}(t)})\|\delta\!\theta\|_{\dot X^{p_1}(t)}\Big),$$

$$\|\delta u\|_{\dot Y^{p_2}(t)}+\|\nabla\delta Q\|_{\dot Z^{p_2}(t)} \leq C_{\theta^1,\theta^2,u^1}\Big(\|\delta u_0\|_{\dot B^{\frac{d}{p_2}-1}_{p_2,1}}+\|u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}(\|\delta\!\theta\|_{\dot X^{p_1}(t)}+\|\delta u\|_{\dot Y^{p_2}(t)}) \\ +\big(\|\nabla u^2\|_{L^1_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}+\|\nabla Q^2\|_{L^1_t(\dot B^{\frac{d}{p_2}-1}_{p_2,1})}\big)\|\delta\!\theta\|_{\dot X^{p_1}(t)} \\ +\big(\|\theta^1\|_{\dot X^{p_1}(t)}+\|\theta^2\|_{\dot X^{p_1}(t)}\big)\big(\|\delta\!\theta\|_{\dot X^{p_1}(t)}+\|\delta u\|_{\dot Y^{p_2}(t)}+\|\nabla\delta Q\|_{\dot Z^{p_2}(t)}\big)\Big).$$

In the case where $(\theta^1,u^1,\nabla Q^1)$ and $(\theta^2,u^2,\nabla Q^2)$ are small enough on $[0,T]$, all the terms involving $(\delta\!\theta,\delta u)$ in the right-hand side may be absorbed by the left-hand side. This yields stability estimates on the whole interval $[0,T]$, and implies uniqueness.

The case where the velocity is large requires more care for it is not clear that the terms corresponding to $\operatorname{div}(\delta\!\theta\cdot\nabla u^2)$, $\nabla u^2\cdot\nabla\delta\!\theta$, $\delta u\cdot\nabla u^2$ and $\delta\!\theta\nabla Q^2$ are small compared to $\|\delta\!\theta\|_{\dot X^{p_1}(t)}+\|\delta u\|_{\dot Y^{p_2}(t)}+\|\nabla\delta Q\|_{\dot Z^{p_2}(t)}$. However, we notice that they may be bounded in $L^1_t(\dot B^{\frac{d}{p_2}-1}_{p_2,1})$ by

$$(\|\nabla\delta\!\theta\|_{L^2_t(\dot B^{\frac{d}{p_1}}_{p_1,1})}+\|\delta u\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})})\|u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}+\|\delta\!\theta\|_{L^\infty_t(\dot B^{\frac{d}{p_1}}_{p_1,1})}\big(\|\nabla u^2\|_{L^1_t(\dot B^{\frac{d}{p_2}}_{p_2,1})}+\|\nabla Q^2\|_{L^1_t(\dot B^{\frac{d}{p_2}-1}_{p_2,1})}\big).$$

Obviously the terms corresponding to u^2 and ∇Q^2 go to zero when t tends to 0. If both solutions coincide initially, this implies uniqueness on a small enough time interval. Then uniqueness on the whole interval $[0,T]$ follows from standard continuity arguments.

For proving stability estimates, one may further decompose u^1 and u^2 into

$$u^1=\bar u^1+u_L \quad\text{and}\quad u^2=\bar u^2+u_L,$$

where u_L stands for the free solution to the Stokes system that has been defined in (3.19). We can thus write

$$\|u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})\cap L^1_t(\dot B^{\frac{d}{p_2}+1}_{p_2,1})} \;\leq\; \|u_L\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})\cap L^1_t(\dot B^{\frac{d}{p_2}+1}_{p_2,1})}+\|\bar u^2\|_{L^2_t(\dot B^{\frac{d}{p_2}}_{p_2,1})\cap L^1_t(\dot B^{\frac{d}{p_2}+1}_{p_2,1})}.$$

If T has been chosen so that (3.21) holds true and if $(\theta^2,u^2,\nabla Q^2)$ is the solution that has been constructed above then we conclude that the above terms may be bounded by τ. So they may be absorbed by the left-hand side, and it is thus possible to get the continuity of the flow map on $[0,T]$ for T satisfying (3.21). The details are left to the reader.

4 The proof of Theorem 1.2

In this section we establish local well-posedness results in the fully nonhomogeneous case: we just assume that the initial temperature is positive and tends to some positive constant at infinity (we take 1 for notational simplicity). In this framework, the estimates for the linear equations considered in Section 3 are not sufficient to bound the solutions to (1.6) even at small time. The reason why is that some quadratic terms such as $\nabla u\cdot\nabla\theta$ or $\theta\nabla Q$ may be of the same order as the terms of the left-hand side hence cannot be absorbed any longer. In the fully nonhomogeneous case, the appropriate linear equations that have to be considered have *variable coefficients* in their main order terms.

The first part of this section is devoted to the presentation and the proof of new a priori estimates for these linear equations. As we believe this type of estimates to be of interest in other contexts, we provide the statements for a wider range of Lebesgue and regularity exponents than those which will be needed to establish the well-posedness of (1.6) in our functional framework. The second part of this section is devoted to the proof of Theorem 1.2.

4.1 The linearized equations

In order to bound the temperature, we shall establish a priori estimates in nonhomogeneous Besov norms for the solutions to

$$\left\{\begin{array}{lcl}\partial_t\theta+q\cdot\nabla\theta-\mathrm{div}\,(\kappa\nabla\theta)&=&f,\\ \theta|_{t=0}&=&\theta_0.\end{array}\right.\tag{4.1}$$

Our main result (which extends the corresponding one in [8]) reads:

Proposition 4.1. *Let θ satisfy (4.1) on $[0,T]\times\mathbb{R}^d$. Let $(p_1,p_2)\in(1,\infty)^2$ and $s\in\mathbb{R}$ fulfill*

$$-1-d\min\left\{\frac{1}{p_1},\frac{1}{p_1'},\frac{1}{p_2}\right\}<s\leq d\min\left\{\frac{1}{p_1},\frac{1}{p_2}+\frac{1}{d}\right\}\cdot\tag{4.2}$$

Suppose that the conductivity function κ, the divergence free vector-field q, the initial data θ_0 and the source term f are smooth enough and decay at infinity, and that

$$m:=\min_{(t,x)\in[0,T]\times\mathbb{R}^d}\kappa(t,x)>0.\tag{4.3}$$

Then there exist constants $c_{1,p_1}(d,p_1,m),C_{1,p_1}(d,p_1,p_2,s,m),\widetilde{C}_{1,p_1}(d,p_1,s,m)$ such that the solution to (4.1) satisfies for all $t\in[0,T]$:

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\infty_t(B^s_{p_1,1})}+c_{1,p_1}\|\theta\|_{L^1_t(B^{s+2}_{p_1,1})}\leq & e^{C_{1,p_1}(\|\nabla q\|_{L^1_t(B^{d/p_2}_{p_2,1})}+\|\nabla\kappa\|^2_{L^2_t(B^{d/p_1}_{p_1,1})})}\\ &\times(\|\theta_0\|_{B^s_{p_1,1}}+\widetilde{C}_{1,p_1}\|\Delta_{-1}\theta\|_{L^1_t(L^{p_1})}+\|f\|_{L^1_t(B^s_{p_1,1})}).\end{aligned}\tag{4.4}$$

Proof. As a warm up, we focus on the special case $p_1=p_2=2$ and $s\in(-d/2,d/2]$ which may be achieved by classical energy arguments. Applying Δ_j to (4.1) yields for all $j\geq-1$,

$$\partial_t\theta_j+q\cdot\nabla\theta_j-\mathrm{div}\,(\kappa\nabla\theta_j)=f_j+R^1_j-\mathrm{div}\,R^2_j,\tag{4.5}$$

where

$$\theta_j=\Delta_j\theta,\quad f_j=\Delta_jf,\quad R^1_j=[q,\Delta_j]\cdot\nabla\theta\ \text{ and }\ R^2_j=[\kappa,\Delta_j]\nabla\theta.$$

Taking the L^2 inner product of the above equation with θ_j and integrating by parts (recall that $\mathrm{div}\,q=0$), we get

$$\frac{1}{2}\frac{d}{dt}\|\theta_j\|^2_{L^2}+\int\kappa|\nabla\theta_j|^2\leq\|\theta_j\|_{L^2}(\|f_j\|_{L^2}+\|R^1_j\|_{L^2}+\|\mathrm{div}\,R^2_j\|_{L^2}).$$

Notice that we have $\|\nabla\theta_j\|_{L^2}\approx2^j\|\theta_j\|_{L^2}$ for $j\geq0$. Hence, dividing formally both sides of the inequality by $\|\theta_j\|_{L^2}$ and integrating with respect to the time variable, we get for some constant c_1 depending only on d,

$$\begin{aligned}\|\theta_j\|_{L^\infty_t(L^2)}+c_1m2^{2j}\|\theta_j\|_{L^1_t(L^2)}\leq&\|(\theta_0)_j\|_{L^2}+\delta_j^{-1}c_1m2^{2j}\|\Delta_{-1}\theta\|_{L^1_t(L^2)}\\ &+\|f_j\|_{L^1_t(L^2)}+\|R^1_j\|_{L^1_t(L^2)}+\|\mathrm{div}\,R^2_j\|_{L^1_t(L^2)},\end{aligned}$$

where

$$\delta_j^{-1} = 1 \text{ if } j = -1 \quad \text{and} \quad \delta_j^{-1} = 0 \text{ if } j \neq -1.$$

Applying Inequality (2.12) with regularity index s and Inequality (2.13) with regularity index $s+1$ and $\nu = 1$ yields

$$\|R_j^1\|_{L_t^1(L^2)} \lesssim 2^{-js} c_j \int_0^t \|\nabla q\|_{B^{d/2}_{2,1}} \|\nabla\theta\|_{B^{s-1}_{2,1}}\, dt' \quad \text{if} \quad -d/2 < s \leq d/2+1,$$
$$\|\mathrm{div}\, R_j^2\|_{L_t^1(L^2)} \lesssim 2^{-js} c_j \int_0^t \|\nabla\kappa\|_{B^{d/2}_{2,1}} \|\nabla\theta\|_{B^{s}_{2,1}}\, dt' \quad \text{if} \quad -d/2-1 < s \leq d/2.$$

Now multiplying both sides by 2^{js}, summing up over j and taking advantage of the interpolation inequality $\|\cdot\|_{B^{s+1}_{2,1}} \lesssim \|\cdot\|^{1/2}_{B^s_{2,1}} \|\cdot\|^{1/2}_{B^{s+2}_{2,1}}$ in Proposition 2.1 yields

$$\|\theta\|_{\widetilde{L}_t^\infty(B^s_{2,1})} + c_1 m \|\theta\|_{L_t^1(B^{s+2}_{2,1})} \leq \|\theta_0\|_{B^s_{2,1}} + c_1 m \|\Delta_{-1}\theta\|_{L_t^1(L^2)} + \|f\|_{L_t^1(B^s_{2,1})}$$
$$+ C_1 \int_0^t (\|\nabla q\|_{B^{d/2}_{2,1}} + \|\nabla\kappa\|^2_{B^{d/2}_{2,1}}) \|\theta\|_{B^s_{2,1}}\, dt'.$$

Then applying Gronwall's inequality leads to Inequality (4.4).

To treat the general case $1 < p_1 < \infty$ we multiply (4.5) by $|\theta_j|^{p_1-2}\theta_j$. We arrive at

$$\frac{1}{p_1}\frac{d}{dt}\int |\theta_j|^{p_1}\, dx + (p_1-1)\int \kappa|\theta_j|^{p_1-2}|\nabla\theta_j|^2\, dx \leq \|\theta_j\|^{p_1-1}_{L^{p_1}} (\|f_j\|_{L^{p_1}} + \|R_j^1\|_{L^{p_1}} + \|\mathrm{div}\, R_j^2\|_{L^{p_1}}).$$

Next, we use (bearing in mind that $1 < p_1 < \infty$) the following Bernstein type inequality (see Appendix B in [13]):

$$\int |\theta_j|^{p_1-2}|\nabla\theta_j|^2 \gtrsim 2^{2j}\int |\theta_j|^{p_1}\, dx \quad \text{for} \quad j \geq 0. \tag{4.6}$$

Hence we get

$$\frac{d}{dt}\|\theta_j\|^{p_1}_{L^{p_1}} + 2^{2j} m \|\theta_j\|^{p_1}_{L^{p_1}} \lesssim \|\theta_j\|^{p_1-1}_{L^{p_1}} (\|f_j\|_{L^{p_1}} + \|R_j^1\|_{L^{p_1}} + \|\mathrm{div}\, R_j^2\|_{L^{p_1}}).$$

Therefore dividing both sides by $\|\theta_j\|^{p_1-1}_{L^{p_1}}$ and using that, according to (2.12) and (2.13),

$$\|R_j^1\|_{L^{p_1}} \lesssim 2^{-js} c_j \|\nabla q\|_{B^{d/p_2}_{p_2,1}} \|\nabla\theta\|_{B^{s-1}_{p_1,1}} \quad \text{if} \quad -d\min\left(\frac{1}{p_1'}, \frac{1}{p_2}\right) < s \leq 1 + d\min\left(\frac{1}{p_1}, \frac{1}{p_2}\right),$$
$$\|\mathrm{div}\, R_j^2\|_{L^{p_1}} \lesssim 2^{-js} c_j \|\nabla\kappa\|_{B^{d/p_1}_{p_1,1}} \|\nabla\theta\|_{B^{s}_{p_1,1}} \quad \text{if} \quad -d\min\left(\frac{1}{p_1'} \frac{1}{p_1}\right) < s+1 \leq 1 + \frac{d}{p_1},$$

integrating in time, multiplying both sides with 2^{js}, summing up over $j \in \mathbb{Z}$ and performing an interpolation inequality, one arrives at

$$\|\theta\|_{\widetilde{L}_t^\infty(B^s_{p_1,1})} + c_{1,p_1} m \|\theta\|_{L_t^1(B^{s+2}_{p_1,1})} \leq \|\theta_0\|_{B^s_{p_1,1}} + c_{1,p_1} 2^{-(s+2)} m \|\Delta_{-1}\theta\|_{L_t^1(L^{p_1})} + \|f\|_{L_t^1(B^s_{p_1,1})}$$
$$+ C_{1,p_1} \int_0^t (\|\nabla q\|_{B^{d/p_2}_{p_2,1}} + \|\nabla\kappa\|^2_{B^{d/p_1}_{p_1,1}}) \|\theta\|_{B^s_{p_1,1}}\, dt',$$

which yields (4.4) by Gronwall inequality, except in the case where s is too negative.

To improve the condition over s for s negative, it suffices to use the fact that, owing to $\mathrm{div}\, q = 0$, one has

$$R_j^1 = \mathrm{div}\,([q, \Delta_j]\theta).$$

Then one may apply Inequality (2.13) to $\mathrm{div}\,[q, \Delta_j]\theta$ with $s+1$ instead of s. The details are left to the reader. $\square$

Remark 4.1. *Let us further remark that*

$$\operatorname{div} R_j^2 = [\nabla\kappa, \Delta_j]\cdot\nabla\theta + [\kappa,\Delta_j]\Delta\theta.$$

This decomposition allows to improve the condition (4.2) *for positive* s*: if we only assume that* $s \leq 1 + d\min(1/p_1, 1/p_2)$ *then Inequality* (4.4) *holds true with the additional term* $\|\nabla^2\kappa\|_{L^1_t(B^{d/p_1}_{p_1,1})}$ *in the exponential. As only the case* $s = d/p_1$ *is needed for proving Theorem 1.2, we do not provide more details here.*

Note also that, for $s = d/p_1$, *Condition* (4.2) *holds if and only if* $1/p_1 \leq 1/p_2 + 1/d$.

In the fully nonhomogeneous case, the appropriate linearized momentum equation turns out to be

$$\begin{cases} \partial_t u + q\cdot\nabla u - \operatorname{div}(\mu\nabla u) + P\nabla Q &= h, \\ \operatorname{div} u &= 0, \\ u|_{t=0} &= u_0, \end{cases} \tag{4.7}$$

where q is a given divergence free vector-field, and (μ, P) are given positive functions. Let us first consider the case $p_1 = p_2 = 2$ which may be handled by standard energy arguments.

Proposition 4.2. *Let* $P,\ \mu,\ q,\ h,\ u_0$ *be smooth and decay sufficiently at infinity with* $\operatorname{div} q = 0$. *Let* $(u, \nabla Q)$ *satisfy* (4.7) *on* $[0,T]\times\mathbb{R}^d$. *Let* $s\in[0,d/2]$. *Suppose that, for some positive constants* m *and* M, *we have*

$$\|\nabla P\|_{L^\infty_T(B^{d/2-1}_{2,1})} + \|P\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq M \quad \text{and} \quad \min(P,\mu)\geq m. \tag{4.8}$$

There exists a constant $c_P = c_P(d,s)$ *such that, if for some integer* N, *one has*

$$\inf_{x\in\mathbb{R}^d, t\in[0,T]} S_N P(t,x) \geq m/2, \quad \|P - S_N P\|_{L^\infty_T(B^{d/2}_{2,1})} \leq c_P m, \tag{4.9}$$

then there exist constants $c_2(d,m)$, $C_2(d,s,m,M,N)$, $\widetilde{C}_2(d,s,m)$, $C_3(d,s,m,M,N)$, *such that for all* $t\in[0,T]$,

$$\begin{aligned} \|u\|_{\widetilde{L}^\infty_t(B^s_{2,1})} + c_2\|u\|_{L^1_t(B^{s+2}_{2,1})} \leq & e^{C_2(\|\nabla q\|_{L^1_t(B^{d/2}_{2,1})} + \|\nabla\mu\|^2_{L^2_t(B^{d/2}_{2,1})})} \\ & \times\big(\|u_0\|_{B^s_{2,1}} + \widetilde{C}_2\|\Delta_{-1}u\|_{L^1_t(L^2)} + C_2\|h\|_{L^1_t(B^s_{2,1})}\big), \end{aligned} \tag{4.10}$$

$$\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq C_3\int_0^t \left(\|h\|_{B^s_{2,1}} + \|\nabla q\|_{B^{d/2}_{2,1}}\|u\|_{B^s_{2,1}} + \|\nabla\mu\|_{B^{d/2}_{2,1}}\|\nabla u\|_{B^s_{2,1}}\right) d\tau. \tag{4.11}$$

Proof. Following the proof of Proposition 4.1, we apply Δ_j to (4.7). This yields

$$\partial_t u_j + q\cdot\nabla u_j - \operatorname{div}(\mu\nabla u_j) = h_j + R_j^1 - \operatorname{div} R_j^2 - \Delta_j(P\nabla Q),$$

where

$$u_j := \Delta_j u, \quad h_j := \Delta_j h, \quad R_j^1 := [q,\Delta_j]\cdot\nabla u, \quad R_j^2 := [\mu,\Delta_j]\cdot\nabla u.$$

As above, we thus get if $-d/2 < s \leq d/2$,

$$\begin{aligned} \|u\|_{\widetilde{L}^\infty_t(B^s_{2,1})} + c_2(d)m\|u\|_{L^1_t(B^{s+2}_{2,1})} \leq & \|u_0\|_{B^s_{2,1}} + \widetilde{C}_2(c_2,s)\|\Delta_{-1}u\|_{L^1_t(L^2)} + \|h\|_{L^1_t(B^s_{2,1})} \\ & + C(d,s,m)\int_0^t \big(\|\nabla q\|_{B^{d/2}_{2,1}}\|u\|_{B^s_{2,1}} + \|\nabla\mu\|^2_{B^{d/2}_{2,1}}\|\nabla u\|_{B^s_{2,1}}\big)\,dt' + \|P\nabla Q\|_{L^1_t(B^s_{2,1})}. \end{aligned} \tag{4.12}$$

We now have to bound ∇Q. Applying the divergence operator to the first equation, we then arrive at the following elliptic equation *with variable coefficients*[4]:

$$\operatorname{div}(P\nabla Q) = \operatorname{div} L \quad \text{with} \quad L := -u\cdot\nabla q + \nabla\mu\cdot(\nabla u)^T + h. \tag{4.13}$$

First we take the L^2 inner product to the equation (4.13) with Q to get

$$m\|\nabla Q\|_{L^2} \leq \|L\|_{L^2}. \tag{4.14}$$

Next, applying Δ_j to (4.13) yields (with obvious notation)

$$\operatorname{div}(P\nabla Q_j) = \operatorname{div} L_j + \operatorname{div}([P,\Delta_j]\nabla Q).$$

Hence, taking the L^2 inner product with Q_j and integrating by parts yields

$$m\|\nabla Q_j\|_{L^2} \leq \|L_j\|_{L^2} + \|[P,\Delta_j]\nabla Q\|_{L^2}.$$

So using the commutator estimate (2.10), we easily get if $-d/2 < \nu \leq 1$ and $-d/2 < s \leq \nu + d/2$,

$$m\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq \|L\|_{L^1_t(B^s_{2,1})} + C_Q(d,s,\nu)\|\nabla P\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|\nabla Q\|_{L^1_t(B^{s-\nu}_{2,1})}. \tag{4.15}$$

Now we consider two cases:

- Case $0 < s \leq d/2$. Let us first assume that ∇P has some extra regularity: suppose for instance that it belongs to $L^\infty_T(B^{d/2+\nu-1}_{2,1})$ for some ν such that $\nu + d/2 \geq s > \nu > 0$. As $\|\cdot\|_{B^0_{2,2}} = \|\cdot\|_{L^2}$ we arrive (by interpolation) at

$$\|\nabla Q\|_{B^{s-\nu}_{2,1}} \lesssim \|\nabla Q\|^{\nu/s}_{L^2}\|\nabla Q\|^{1-\nu/s}_{B^s_{2,1}} \lesssim \|L\|^{\nu/s}_{L^2}\|\nabla Q\|^{1-\nu/s}_{B^s_{2,1}}. \tag{4.16}$$

  Hence (4.15) implies that

$$\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq C(d,s,\nu,m)(1+\|\nabla P\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})})^{s/\nu}\|L\|_{L^1_t(B^s_{2,1})}. \tag{4.17}$$

  Now, if P satisfies only Conditions (4.8) and (4.9) then we decompose it into

$$P = P_N + (P - P_N) \quad \text{with } P_N := S_N P.$$

  Note that $\nabla P_N \in H^\infty$ and that the equation for Q recasts in

$$\operatorname{div}(P_N\nabla Q) = \operatorname{div}(L + E_N), \quad \text{where} \quad E_N = (P_N - P)\cdot\nabla Q.$$

  Therefore, following the procedure leading to (4.15) and bearing the first part of Condition (4.9) in mind, yields

$$\frac{m}{2}\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq \|L+E_N\|_{L^1_t(B^s_{2,1})} + C_Q(d,s,\nu)\|\nabla P_N\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|\nabla Q\|_{L^1_t(B^{s-\nu}_{2,1})}. \tag{4.18}$$

  We notice that for $-d/2 < s \leq d/2$,

$$\begin{aligned}\|\nabla P_N\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})} &\leq C_P(d)2^{N\nu}\|\nabla P\|_{L^\infty_t(B^{d/2-1}_{2,1})},\\ \|E_N\|_{L^1_t(B^s_{2,1})} &\leq C_P(d,s)\|P_N - P\|_{L^\infty_t(B^{d/2}_{2,1})}\|\nabla Q\|_{L^1_t(B^s_{2,1})},\end{aligned}$$

[4]Here we use that $\operatorname{div}(q\cdot\nabla u) = \operatorname{div}(u\cdot\nabla q)$ and $\operatorname{div}\big(\operatorname{div}(\mu\nabla u)\big) = \operatorname{div}\big(\nabla u\cdot\nabla\mu\big)$ owing to $\operatorname{div} u = \operatorname{div} q = 0$.

hence the term pertaining to E_N may be absorbed by the left-hand side of (4.18) if c_P is small enough in (4.9). Then using the same interpolation argument as above, we end up with

$$m\|\nabla Q\|_{L^1_t(B^s_{2,1})} \leq C_Q(d,s,\nu)2^{N\nu}\big(1+\|\nabla P\|_{L^\infty_t(B^{d/2-1}_{2,1})}\big)^{s/\nu}\|L\|_{L^1_t(B^s_{2,1})}. \tag{4.19}$$

Now, in order to complete the proof of Inequality (4.11), it is only a matter of using the product estimates (ii) stated in Proposition 2.3 for bounding L, which implies that

$$\|L\|_{B^s_{2,1}} \lesssim \|\nabla q\|_{B^{\frac d2}_{2,1}}\|u\|_{B^s_{2,1}} + \|\nabla\mu\|_{B^{\frac d2}_{2,1}}\|\nabla u\|_{B^s_{2,1}} + \|h\|_{B^s_{2,1}} \quad \text{if} \quad -d/2 < s \leq d/2.$$

- Case $s=0$: in this case, the interpolation inequality (4.16) fails, so that we have to modify the proof accordingly. First we apply Inequality (4.15) for some $0<\nu<1$, and Inequality (2.10), to get (by virtue of (4.14)):

$$\begin{aligned}\|[P,\Delta_j]\nabla Q\|_{L^1_t(B^0_{2,1})} &\lesssim c_j\|\nabla P\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|\nabla Q\|_{L^1_t(B^{0-\nu}_{2,1})}\\ &\lesssim c_j\|\nabla P\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|L\|_{L^1_t(L^2)},\end{aligned}$$

hence

$$\begin{aligned}\|\nabla Q\|_{L^1_t(B^0_{2,1})} &\lesssim \|L\|_{L^1_t(B^0_{2,1})} + \|\nabla P\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})}\|L\|_{L^1_t(L^2)}\\ &\leq C(d,\nu,m)(1+\|\nabla P\|_{L^\infty_t(B^{d/2+\nu-1}_{2,1})})\|L\|_{L^1_t(B^0_{2,1})},\end{aligned} \tag{4.20}$$

which is quite similar as (4.17) and hence the same procedure implies also (4.19).

In order to prove (4.10), it suffices to plug the above estimate for the pressure in (4.12). The main point is that, if $-d/2<s\leq d/2$ then we have

$$\|P\nabla Q\|_{B^s_{2,1}} \lesssim \big(\|P\|_{L^\infty}+\|\nabla P\|_{B^{\frac d2-1}_{2,1}}\big)\|\nabla Q\|_{B^s_{2,1}},$$

as may be easily seen by decomposing $P\nabla Q$ into $\Delta_{-1}P\,\nabla Q + (\mathrm{Id}-\Delta_{-1})P\,\nabla Q$ and using the product estimates of Proposition 2.3. Then Gronwall lemma leads to Inequality (4.10). □

Remark 4.2. *In Proposition 4.2 we need the assumption $s\geq 0$ to get the necessary L^2 estimate for ∇Q. However, some negative indices may be achieved by duality arguments. As the corresponding estimates are not needed in our paper, we here do not give more details on that issue.*

We now want to extend Proposition 4.2 to more general Besov spaces which are not directly related to the energy space. To simplify the presentation, we focus on the regularity exponent $s=d/p_2-1$ which is the only one that we will have to consider in the proof of Theorem 1.2. Our main result reads:

Proposition 4.3. *Let $T>0$ and $(u,\nabla Q)$ be a solution to (4.7) on $[0,T]\times\mathbb{R}^d$. Suppose that the given functions P, μ, that the divergence free vector-field q, the initial data u_0 and the source term h are smooth and decay at infinity. Let p_1 be in $[1,\infty)$ and $p_2\in[2,4]$ satisfy*

$$p_2 \leq \frac{2p_1}{p_1-2} \quad \textit{if} \quad p_1>2, \qquad \frac{1}{p_2}\leq\frac{1}{p_1}+\frac1d, \quad \textit{and} \quad (p_1,p_2)\neq(4,4) \quad \textit{if} \quad d=2. \tag{4.21}$$

Assume that there exist some constants $0<m<M$, $c_{P,p_1,p_2}(d,p_1,p_2)$ small enough, and $N\in\mathbb{N}$ such that

$$\begin{aligned}&\min(\mu,P)\geq m, \quad \|P\|_{L^\infty([0,T]\times\mathbb{R}^d)}+\|\nabla P\|_{L^\infty_T(B^{d/p_1-1}_{p_1,1})}\leq M,\\ &\inf_{x\in\mathbb{R}^d,t\in[0,T]} S_NP(t,x)\geq m/2, \quad \|P-S_NP\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\leq c_{P,p_1,p_2}m.\end{aligned} \tag{4.22}$$

Then there exist constants $c_{2,p_2}(d,p_2,m), C_{2,p_1,p_2}(d,p_1,p_2,m,M,N), \widetilde{C}_{2,p_2}(d,p_2,m)$, $C_{3,p_1,p_2}(d,p_1,p_2,m,M,N)$ *such that the following a priori estimates hold:*

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_t(B^{d/p_2-1}_{p_2,1})} + c_{2,p_2}\|u\|_{L^1_t(B^{d/p_2+1}_{p_2,1})}\\
&\qquad\leq e^{C_{2,p_1,p_2}(\|\nabla q\|_{L^1_t(B^{d/p_2}_{p_2,1})}+\|\nabla q\|^{4/3}_{L^{4/3}_t(B^{d/p_2-1/2}_{p_2,1})}+\|\nabla\mu\|^2_{L^2_t(B^{d/p_1}_{p_1,1})}+\|\nabla\mu\|^{2/(1-\eta)}_{L^{2/(1-\eta)}_t(B^{d/p_1-\eta}_{p_1,1})})}\\
&\qquad\times\big(\|u_0\|_{B^{d/p_2}_{p_2,1}} + \widetilde{C}_{2,p}\|\Delta_{-1}u\|_{L^1_t(L^{p_2})} + C_{2,p_1,p_2}\|h\|_{L^1_t(L^2\cap B^{d/p_2-1}_{p_2,1})}\big),
\end{aligned}\tag{4.23}$$

with $\eta=\min(1/2,d/p_1)$, *and*

$$\begin{aligned}
\|\nabla Q\|_{L^1_t(B^{\frac{d}{p_2}-1}_{p_2,1}\cap L^2)} \leq C_{3,p_1,p_2}\int_0^t\bigg(&\|h\|_{L^2\cap B^{d/p_2-1}_{p_2,1}} + \|\nabla q\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}} + \|u\cdot\nabla q\|_{L^2}\\
&+\|\nabla\mu\|_{B^{d/p_1}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1}_{p_2,1}} + \|\nabla u\cdot\nabla\mu\|_{L^2}\bigg)d\tau.
\end{aligned}\tag{4.24}$$

Proof. With the notation of Proposition 4.1, we have

$$\|(2^{j(d/p_2-1)}\|R^1_j\|_{L^{p_2}})_{j\in\mathbb{Z}}\|_{\ell^1} \lesssim \|\nabla q\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}},\tag{4.25}$$

$$\|(2^{j(d/p_2-1)}\|\mathrm{div}\,R^2_j\|_{L^{p_2}})_{j\in\mathbb{Z}}\|_{\ell^1} \lesssim \|\nabla\mu\|_{B^{d/p_1}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1}_{p_2,1}},\tag{4.26}$$

$$\|P\nabla Q\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \big(\|P\|_{L^\infty}+\|\nabla P\|_{B^{d/p_1-1}_{p_1,1}}\big)\|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}}.\tag{4.27}$$

Indeed[5] Inequality (4.25) follows from (2.10) with "p_1" = "p_2" = p_2, "ν" = 1, "s" = d/p_2-1 (note that the condition $p_2<2d$ is not required for $\mathrm{div}\,q=0$, a consequence of (2.13) with "s" = d/p_2 and "ν" = 1) while (4.26) stems from (2.13) with "p_1" = p_2, "p_2" = p_1, "ν" = 1, "s" = d/p_2 (here we need that $1/p_2\leq 1/p_1+1/d$); and (4.27) is a consequence of the decomposition $P=\Delta_{-1}P+(\mathrm{Id}-\Delta_{-1})P$ and of (2.7) with "s_1" = d/p_1, "s_2" = d/p_2-1 (which requires that $1/p_2\leq 1/p_1+1/d$ and $1/p_1+1/p_2>1/d$).

Now, granted with the above inequalities, the same procedure as in Proposition 4.1 yields

$$\begin{aligned}
\|u\|_{\widetilde{L}^\infty_t(B^{d/p_2-1}_{p_2,1})}+c_{2,p_2}\|u\|_{L^1_t(B^{d/p_2+1}_{p_2,1})} &\leq \|u_0\|_{B^{d/p_2-1}_{p_2,1}} + c_{2,p_2}2^{-(d/p_2+1)}\|\Delta_{-1}u\|_{L^1_t(L^{p_2})}\\
&+C(d,p_1,p_2,m)\int_0^t(\|\nabla q\|_{B^{d/p_2}_{p_2,1}}+\|\nabla\mu\|^2_{B^{d/p_1}_{p_1,1}})\|u\|_{B^{d/p_2-1}_{p_2,1}}\,dt'\\
&+\|h\|_{L^1_t(B^{d/p_2-1}_{p_2,1})} + \big(\|P\|_{L^\infty_t(L^\infty)}+\|\nabla P\|_{L^\infty_t(B^{d/p_1-1}_{p_1,1})}\big)\|\nabla Q\|_{L^1_t(B^{d/p_2-1}_{p_2,1})}.
\end{aligned}\tag{4.28}$$

So bounding ∇Q is our next task. First of all, using the fact that Q satisfies the elliptic equation (4.13), we still have

$$m\|\nabla Q\|_{L^2}\leq\|L\|_{L^2}\quad\text{with}\quad L=h+u\cdot\nabla q+\nabla u\cdot\nabla\mu.$$

Hence, given that $L^2\hookrightarrow B^{d/p_2-d/2}_{p_2,2}$ (here comes that $p_2\geq 2$), we deduce that

$$m\|\nabla Q\|_{B^{d/p_2-d/2}_{p_2,2}}\lesssim\|L\|_{L^2}.\tag{4.29}$$

Of course, this implies that

$$m\|\nabla\Delta_{-1}Q\|_{L^{p_2}}\lesssim\|L\|_{L^2}.\tag{4.30}$$

[5]Here it is understood that the quote marks designate the indices in the original inequalities (2.7), (2.10) and (2.13).

In order to bound ∇Q in $B^{d/p_2-1}_{p_2,1}$, we use again the fact that

$$\operatorname{div}(P\nabla Q_j) = \nabla\cdot L_j + \nabla\cdot[P,\Delta_j]\nabla Q.$$

Therefore,

$$m\int |Q_j|^{p_2-2}|\nabla Q_j|^2\,dx \lesssim \int |Q_j|^{p_2-2}|\nabla Q_j\cdot(L_j+[P,\Delta_j]\nabla Q)|\,dx.$$

Taking advantage of (4.6), we get after a few computations:

$$\|\nabla Q_j\|_{L^{p_2}} \lesssim \|L_j\|_{L^{p_2}} + \|[P,\Delta_j]\nabla Q\|_{L^{p_2}} \quad \text{for } \ j\geq 0.$$

Applying Inequality (2.10) with

$$\text{“}u\text{”} = P,\quad \text{“}v\text{”} = \nabla Q,\quad \text{“}p_1\text{”} = p_2,\quad \text{“}p_2\text{”} = p_1,\quad \text{“}\nu\text{”} = 1/4,\quad \text{“}s\text{”} = d/p_2-1,$$

(which is possible provided $1/p_1+1/p_2>1/d$ and $1/p_2\leq 1/p_1+5/(4d)$), we get

$$\|[P,\Delta_j]\nabla Q\|_{L^{p_2}} \lesssim 2^{-j(d/p_2-1)}c_j\|\nabla P\|_{B^{d/p_1-3/4}_{p_1,1}}\|\nabla Q\|_{B^{d/p_2-5/4}_{p_2,1}} \quad \text{with} \quad \sum_j c_j = 1. \tag{4.31}$$

In the case $d\geq 3$, arguing by interpolation, we get

$$\|\nabla Q\|_{B^{d/p_2-5/4}_{p_2,1}} \lesssim \|\nabla Q\|^{\frac{2d-5}{2d-4}}_{B^{d/p_2-1}_{p_2,1}}\|\nabla Q\|^{\frac{1}{2d-4}}_{B^{d/p_2-d/2}_{p_2,2}}. \tag{4.32}$$

Therefore together with (4.29) and (4.30), this implies that

$$m\|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \|L\|_{B^{d/p_2-1}_{p_2,1}} + \big(1+\|\nabla P\|_{B^{d/p_1-3/4}_{p_1,1}}\big)^{2d-4}\|L\|_{L^2}. \tag{4.33}$$

In the case $d=2$, the interpolation inequality (4.32) fails. However, from (4.29) and (4.31) we directly get for p_1,p_2 satisfying (4.21),

$$\begin{aligned} m\|\nabla Q\|_{B^{2/p_2-1}_{p_2,1}} &\lesssim \|L\|_{B^{2/p_2-1}_{p_2,1}} + \|\nabla P\|_{B^{2/p_1-3/4}_{p_1,1}}\|\nabla Q\|_{B^{2/p_2-5/4}_{p_2,1}}\\ &\lesssim \|L\|_{B^{2/p_2-1}_{p_2,1}} + (1+\|\nabla P\|_{B^{2/p_1-3/4}_{p_1,1}})\|L\|_{L^2}. \end{aligned} \tag{4.34}$$

Therefore, in any dimension $d\geq 2$, we have

$$m\|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}\cap L^2} \leq C_{Q,d,p_1,p_2}\big(1+\|\nabla P\|_{B^{d/p_1-3/4}_{p_1,1}}\big)^{\max(1,2d-4)}\|L\|_{B^{d/p_2-1}_{p_2,1}\cap L^2}. \tag{4.35}$$

In order to treat the case where ∇P is only in $L^\infty_T(B^{d/p_1-1}_{p_1,1})$, we proceed exactly as in the proof of Proposition 4.2, decomposing P into two parts, the smooth large part S_NP and the small rough part $P-S_NP$. Under the same assumptions as in (4.27), we find that

$$\|E_N\|_{B^{d/p_2-1}_{p_2,1}} \leq C_{P,p_1,p_2}\|P-P_N\|_{B^{d/p_1}_{p_1,1}}\|\nabla Q\|_{B^{d/p_2-1}_{p_2,1}}.$$

Therefore, if c_{P,p_1,p_2} is small enough in (4.22), we get

$$\|\nabla Q\|_{L^1_t(B^{d/p_2-1}_{p_2,1}\cap L^2)} \leq C_{Q,d,p_1,p_2}2^{N/4}\big(1+\|\nabla P\|_{L^\infty_T(B^{d/p_1-1}_{p_1,1})}\big)^{\max(1,2d-4)}\|L\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}. \tag{4.36}$$

In order to complete the proof of (4.24), we now have to bound L. First, we notice that, applying (2.7) with “p_1” = “p_2” = p_2, “s_1” = d/p_2 and “s_2” = d/p_2-1 yields, if $p_2<2d$,

$$\|u\cdot\nabla q\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \|\nabla q\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}}. \tag{4.37}$$

If $2d \le p_2 < \infty$ then, owing to $\operatorname{div} u = 0$, the same inequality is true. Indeed applying Bony's decomposition, we discover that

$$u\cdot\nabla q = T_u\nabla q + T_{\nabla q}u + \operatorname{div} R(u, q-\Delta_{-1}q) + R(u,\Delta_{-1}\nabla q).$$

The first two terms may be bounded as in (4.37). For the third one, one has (because $\mathcal{F}(q-\Delta_{-1}q)$ is supported away from the origin)

$$\begin{aligned}\|\operatorname{div} R(u,q-\Delta_{-1}q)\|_{B^{d/p_2-1}_{p_2,1}} &\lesssim \|R(u,q-\Delta_{-1}q)\|_{B^{d/p_2}_{p_2,1}}\\ &\lesssim \|u\|_{B^{d/p_2-1}_{p_2,1}}\|q-\Delta_{-1}q\|_{B^{d/p_2+1}_{p_2,1}} \lesssim \|u\|_{B^{d/p_2-1}_{p_2,1}}\|\nabla q\|_{B^{d/p_2}_{p_2,1}}.\end{aligned}$$

And finally, we have

$$R(u,\Delta_{-1}\nabla q) = \sum_{-1\le j\le 0}\Delta_j\Delta_{-1}\nabla q\,(\Delta_{j-1}+\Delta j+\Delta_{j+1})u,$$

so it is clear that we have

$$\|R(u,\Delta_{-1}\nabla q)\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \|R(u,\Delta_{-1}\nabla q)\|_{L^{p_2}} \lesssim \|\nabla q\|_{L^\infty}\|S_2u\|_{L^{p_2}} \lesssim \|\nabla q\|_{B^{d/p_2}_{p_2,1}}\|u\|_{B^{d/p_2-1}_{p_2,1}}.$$

Next, just as in (4.27), under the conditions $1/p_2 \le 1/p_1 + 1/d$ and $1/p_1 + 1/p_2 > 1/d$, we have

$$\|\nabla u\cdot\nabla\mu\|_{B^{d/p_2-1}_{p_2,1}} \lesssim \|\nabla\mu\|_{B^{d/p_1}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1}_{p_2,1}}. \tag{4.38}$$

This gives (4.24).

In order to complete the proof of the lemma, we still have to bound $u\cdot\nabla q$ and $\nabla u\cdot\nabla\mu$ in L^2. To handle the former term, we just use the fact that

$$B^{d/p_2-1/2}_{p_2,1} \hookrightarrow L^4 \quad\text{if}\quad p_2\le 4.$$

Hence, by virtue of Hölder's inequality,

$$\|u\cdot\nabla q\|_{L^2} \lesssim \|u\|_{B^{d/p_2-1/2}_{p_2,1}}\|\nabla q\|_{B^{d/p_2-1/2}_{p_2,1}}. \tag{4.39}$$

Concerning the latter term, if both p_1 and p_2 are less than or equal to 4 then one may merely use the embedding

$$B^{d/p_1-1/2}_{p_1,1}\hookrightarrow L^4 \quad\text{and}\quad B^{d/p_2-1/2}_{p_2,1}\hookrightarrow L^4,$$

hence

$$\|\nabla u\cdot\nabla\mu\|_{L^2} \lesssim \|\nabla\mu\|_{B^{d/p_1-1/2}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1/2}_{p_2,1}}$$

Now, if $p_1>4$ then we first write

$$\|\nabla u\cdot\nabla\mu\|_{L^2} \le \|\nabla\mu\|_{L^{p_1}}\|\nabla u\|_{L^{\widetilde p_1}} \quad\text{with}\quad \widetilde p_1 = \frac{2p_1}{p_1-2}.$$

Let $\eta = \min(1/2, d/p_1)$. Then we notice that if $p_2\le\widetilde p_1$ then

$$B^{d/p_1-\eta}_{p_1,1}\hookrightarrow L^{p_1} \quad\text{and}\quad B^{d/p_2-1+\eta}_{p_2,1}\hookrightarrow L^{\widetilde p_1}.$$

Therefore

$$\|\nabla u\cdot\nabla\mu\|_{L^2} \lesssim \|\nabla\mu\|_{B^{d/p_1-\eta}_{p_1,1}}\|\nabla u\|_{B^{d/p_2-1+\eta}_{p_2,1}}.$$

Together with (4.28), interpolation inequalities and Gronwall lemma, this enables us to complete the proof of (4.23). □

Remark 4.3. *The quantities* $\widetilde{C}_{1,p_1}\|\Delta_{-1}\theta\|_{L^1_t(L^{p_1})}$, $\widetilde{C}_2\|\Delta_{-1}u\|_{L^1_t(L^2)}$ *and* $\widetilde{C}_{2,p_2}\|\Delta_{-1}u\|_{L^1_t(L^{p_2})}$ *in the a priori estimates* (4.4), (4.10) *and* (4.23) *respectively can be absorbed if the time* t *is small. Indeed, for instance, one has for any* $s\in\mathbb{R}$,

$$\|\Delta_{-1}\theta\|_{L^1_t(L^{p_1})}\lesssim\|\theta\|_{L^1_t(B^s_{p_1,1})}\leq t\|\theta\|_{L^\infty_t(B^s_{p_1,1})},$$

hence $\widetilde{C}_{1,p_1}\|\Delta_{-1}\theta\|_{L^1_t(L^{p_1})}$ *can be absorbed by the left-hand side if* t *is small.*

In the case of the linearized momentum equation, plugging (4.10) *in* (4.24), *we thus deduce that, for small enough time, one has for some constant* C *depending only on* d,p_1,p_2,m,M,N,

$$\|\nabla Q\|_{L^1_t(B^{\frac{d}{p_2}-1}_{p_2,1}\cap L^2)}\leq C\|h\|_{L^1_t(L^2\cap B^{d/p_2-1}_{p_2,1})}$$
$$+\Big(\|u_0\|_{B^{d/p_2}_{p_2,1}}+\|h\|_{L^1_t(L^2\cap B^{d/p_2-1}_{p_2,1})}\Big)\Big(e^{C\int_0^t(\|\nabla q\|_{B^{d/p_2}_{p_2,1}}+\|\nabla q\|^{4/3}_{B^{d/p_2-1/2}_{p_2,1}}+\|\nabla\mu\|^2_{B^{d/p_1}_{p_1,1}}+\|\nabla\mu\|^{2/(1-\eta)}_{B^{d/p_1-\eta}_{p_1,1}})\,d\tau}-1\Big).$$

Remark 4.4. *Compared to the statement of Proposition 3.2 in the case* $s=d/p_2-1$ *one has to assume in addition that* $p_2\leq 4$ *and also that* $p_2\leq 2p_1/(p_1-2)$ *(if* $p_1\geq 2$*). This is due to the fact that bounding* ∇Q, *through the elliptic equation* (4.13) *requires a* L^2 *information over the right-hand side, that is on* h *and on quadratic terms. The naive idea is just that, according to Hölder's inequality* L^4 *bounds over* ∇q, u, ∇u *and* $\nabla\mu$ *provides this* L^2 *bound. This is the key to go beyond the energy framework for* (4.7). *At the same time, we do not know how to treat the case* $p_2>4$.

4.2 The proof of the well-posedness in the fully nonhomogeneous case

We follow the same procedure as in the proof of Theorem 1.1: first we construct a sequence of approximate solutions, then we prove uniform bounds for this sequence and finally, we show the convergence to some solution of (1.6). Compared to the almost homogeneous case, the main difference is that our estimates rely mostly on Propositions 4.1 and 4.3. Furthermore, in order to handle large data, we will have to introduce the "free solution" (θ_L,u_L) corresponding to data (θ_0,u_0), namely the solution to

$$\left\{\begin{array}{rcl}\partial_t\theta_L-\bar\kappa\Delta\theta_L&=&0,\\ \partial_t u_L-\bar\mu\Delta u_L&=&0,\\ (\theta_L,u_L)|_{t=0}&=&(\theta_0,u_0),\end{array}\right.\tag{4.40}$$

with $\bar\kappa=\kappa(1)$ and $\bar\mu=\mu(1)$.

Step 1. Construction of a sequence of approximate solutions

As θ_0 is in $B^{d/p_1}_{p_1,1}$ and u_0, in $B^{d/p_2-1}_{p_2,1}$, the above System (4.40) has a unique global solution (θ_L,u_L) with (see e.g. [4])

$$\theta_L\in\widetilde{C}_T(B^{d/p_1}_{p_1,1})\cap L^1_T(B^{d/p_1+2}_{p_1,1})\quad\text{and}\quad u_L\in\widetilde{C}_T(B^{d/p_2-1}_{p_2,1})\cap L^1_T(B^{d/p_2+1}_{p_2,1})\quad\text{for all }T>0,$$

and we have (if T is small enough and with C depending only on d,p_1,p_2)

$$\begin{array}{rl}\|\theta_L\|_{\widetilde{L}^\infty_T(B^{d/p_1}_{p_1,1})}+\bar\kappa\|\theta_L\|_{L^1_T(B^{d/p_1+2}_{p_1,1})}&\leq C\|\theta_0\|_{B^{d/p_1}_{p_1,1}},\\ \|u_L\|_{\widetilde{L}^\infty_T(B^{d/p_2-1}_{p_2,1})}+\bar\mu\|u_L\|_{L^1_T(B^{d/p_2+1}_{p_2,1})}&\leq C\|u_0\|_{B^{d/p_2-1}_{p_2,1}}.\end{array}\tag{4.41}$$

Note also that the divergence free property for the initial velocity is conserved during the evolution. Another important feature is that, owing to $\theta_L\in\widetilde{L}^\infty_T(B^{d/p_1}_{p_1,1})$, we have, for any $T>0$,

$$\lim_{N\to+\infty}\|\theta_L-S_N\theta_L\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}=0.\tag{4.42}$$

Let us fix some small enough positive time T. Given (4.42), we see that for any positive constant c, there exists some positive integer N_0 so that

$$\|\theta_L - S_{N_0}\theta_L\|_{L^\infty_T(B^{d/p_1}_{p_1,1})} \leq cm. \tag{4.43}$$

In addition, if the data satisfy (1.18) then one may assume that we have (changing N_0 and C if need be)

$$\frac{m}{2} \leq S_n\vartheta_0 \leq CM, \quad \|S_n\theta_0\|_{B^{d/p_1}_{p_1,1}} + \|S_nu_0\|_{B^{d/p_2-1}_{p_2,1}} \leq CM \quad \text{for all} \quad n \geq N_0. \tag{4.44}$$

In order to define our approximate solutions, we use the following iterative scheme: first we set $(\theta^0, u^0, \nabla Q^0) = (S_{N_0}\theta_0, S_{N_0}u_0, 0)$ (this is obviously a smooth stationary function with decay at infinity) then, assuming that the approximate solution $(\theta^n, u^n, \nabla Q^n)$ has been constructed over $\mathbb{R}^+ \times \mathbb{R}^d$, we set $\vartheta^n = 1 + \theta^n$ and define $(\theta^{n+1}, u^{n+1}, \nabla Q^{n+1})$ to be the unique solution of the system

$$\left\{\begin{array}{ccc} \partial_t\theta^{n+1} + u^n\cdot\nabla\theta^{n+1} - \operatorname{div}(\kappa^n\nabla\theta^{n+1}) & = & f^n, \\ \partial_t u^{n+1} + u^n\cdot\nabla u^{n+1} - \operatorname{div}(\mu^n\nabla u^{n+1}) + \vartheta^n\nabla Q^{n+1} & = & h^n, \\ \operatorname{div} u^{n+1} & = & 0, \\ (\theta^{n+1}, u^{n+1})|_{t=0} & = & (S_{N_0+n+1}\theta_0, S_{N_0+n+1}u_0), \end{array}\right.$$

where

$$\kappa^n = \lambda k(\vartheta^n), \quad \mu^n = \beta\zeta(\vartheta^n), \quad f^n = f(1+\theta^n, u^n) \ \text{ and } \ h^n = h(1+\theta^n, u^n).$$

Note that the existence and uniqueness of a global smooth solution for the above system is ensured by the standard theory of parabolic equations (concerning θ^{n+1}) and by (a slight modification of) Theorem 2.10 in [3] (concerning u^{n+1}) whenever (θ^n, u^n) is suitably smooth and the coefficients κ^n, μ^n are bounded by above and by below. In fact, given (4.44), the maximum principle ensures that

$$m/2 \leq \vartheta^n \leq CM. \tag{4.45}$$

Hence κ^n and μ^n are bounded by above and from below independently of n.

Next, we notice that if we set

$$(\theta^n_L, u^n_L, \nabla Q^n_L) = (S_{N_0+n}\theta_L, S_{N_0+n}u_L, 0)$$

then the equation for $(\bar\theta^{n+1}, \bar u^{n+1}, \nabla\bar Q^{n+1}) = (\theta^{n+1} - \theta^{n+1}_L, u^{n+1} - u^{n+1}_L, \nabla Q^{n+1} - 0)$ reads

$$\left\{\begin{array}{ccc} \partial_t\bar\theta^{n+1} + u^n\cdot\nabla\bar\theta^{n+1} - \operatorname{div}(\kappa^n\nabla\bar\theta^{n+1}) & = & F^n, \\ \partial_t\bar u^{n+1} + u^n\cdot\nabla\bar u^{n+1} - \operatorname{div}(\mu^n\nabla\bar u^{n+1}) + (1+\theta^n_L)\nabla\bar Q^{n+1} & = & H^n, \\ \operatorname{div}\bar u^{n+1} & = & 0, \\ (\bar\theta^{n+1}, \bar u^{n+1})|_{t=0} & = & (0,0), \end{array}\right.$$

where

$$F^n = -u^n\cdot\nabla\theta^{n+1}_L + \operatorname{div}((\kappa^n - \bar\kappa)\nabla\theta^{n+1}_L) + f^n,$$
$$H^n = -u^n\cdot\nabla u^{n+1}_L + \operatorname{div}((\mu^n - \bar\mu)\nabla u^{n+1}_L) - \bar\theta^n\nabla\bar Q^{n+1} + h^n.$$

Let us point out that, given (4.43), we have (up to a harmless change of c)

$$\|\theta^n_L - S_{N_0}\theta^n_L\|_{L^\infty_T(B^{d/p_1}_{p_1,1})} \leq cm \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.46}$$

Step 2. Uniform bounds

Bounding $(\bar\theta^{n+1},\bar u^{n+1},\nabla\bar Q^{n+1})$ in terms of the free solution (θ_L,u_L) and of $(\bar\theta^n,\bar u^n,\nabla\bar Q^n)$ relies on Propositions 4.1 and 4.3 with $s=d/p_1$ and $s=d/p_2-1$, respectively, and $N=N_0$ (here we have to take c small enough in (4.46)). Using the fact that, as pointed out by Remark 4.3, taking T smaller if needed allows to discard $\widetilde C_1\|\Delta_{-1}\bar\theta^{n+1}\|_{L^1_t(L^{p_1})}$ and $\widetilde C_2\|\Delta_{-1}\bar u^{n+1}\|_{L^1_t(L^{p_2})}$ in the estimates, we get, under the condition (1.17),

$$\|\bar\theta^{n+1}\|_{X^{p_1}(T)}\leq Ce^{C_1\left(\|\nabla u^n\|_{L^1_T(B^{d/p_2}_{p_2,1})}+\|\nabla\kappa^n\|^2_{L^2_T(B^{d/p_1}_{p_1,1})}\right)}\|F^n\|_{L^1_T(B^{d/p_1}_{p_1,1})},$$

$$\|\bar u^{n+1}\|_{Y^{p_2}(T)}+\|\nabla\bar Q^{n+1}\|_{Z^{p_2}(T)}\leq C\|H^n\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}$$
$$\times e^{C_2\left(\|\nabla u^n\|_{L^1_T(B^{d/p_2}_{p_2,1})}+\|\nabla u^n\|^{4/3}_{L^{4/3}_T(B^{d/p_2-1/2}_{p_2,1})}+\|\nabla\mu^n\|^2_{L^2_T(B^{d/p_1}_{p_1,1})}+\|\nabla\mu^n\|^4_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}\right)}.$$

From Proposition 2.5 and elementary interpolation inequalities, we gather that all the terms in the exponential may be bounded by $\|\theta^n\|_{X^{p_1}(T)}+\|u^n\|_{Y^{p_2}(T)}$ to some power. Therefore, if we assume that

$$\|\theta^n\|_{X^{p_1}(T)}+\|u^n\|_{Y^{p_2}(T)}\leq 2CM \tag{4.47}$$

then we have[6]

$$\|\bar\theta^{n+1}\|_{X^{p_1}(T)}\leq C_M\Big(\|f^n\|_{L^1_T(B^{d/p_1}_{p_1,1})}+\|u^n\cdot\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1}_{p_1,1})}+\|(\kappa^n-\bar\kappa)\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1+1}_{p_1,1})}\Big), \tag{4.48}$$

$$\begin{aligned}\|\bar u^{n+1}\|_{Y^{p_2}(T)}+\|\nabla\bar Q^{n+1}\|_{Z^{p_2}(T)}\leq C_M\Big(&\|h^n\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}+\|u^n\cdot\nabla u_L^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\\&+\|\mathrm{div}\,((\mu^n-\bar\mu)\nabla u_L^{n+1})\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}+\|\bar\theta^n\nabla\bar Q^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\Big).\end{aligned} \tag{4.49}$$

So bounding the right-hand sides of (4.48) and of (4.49) is our next task. Given (4.47), we easily get from Propositions 2.3 and 2.5:

$$\begin{aligned}\|f^n\|_{L^1_T(B^{d/p_1}_{p_1,1})}&\leq C_M\|\nabla\theta^n\|^2_{L^2_T(B^{d/p_1}_{p_1,1})},\\ \|(\kappa^n-\bar\kappa)\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1+1}_{p_1,1})}&\leq C_M\big(\|\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\|\nabla\theta_L\|_{L^1_T(B^{d/p_1+1}_{p_1,1})}\\&\qquad+\|\nabla\theta_L\|_{L^2_T(B^{d/p_1}_{p_1,1})}\|\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big).\end{aligned}$$

If $p_2\leq p_1$, the space $B^{d/p_2}_{p_2,1}$ is embedded in the Banach algebra $B^{d/p_1}_{p_1,1}$. Hence

$$\|u^n\cdot\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1}_{p_1,1})}\leq C\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\|\nabla\theta_L\|_{L^2_T(B^{d/p_1}_{p_1,1})}. \tag{4.50}$$

If $p_1<p_2$ then (4.50) is no longer true. However, from Bony's decomposition and Proposition 2.2, it is not difficult to get that

$$\|u^n\cdot\nabla\theta_L^{n+1}\|_{L^1_T(B^{d/p_1}_{p_1,1})}\lesssim\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\|\nabla\theta_L\|_{L^2_T(B^{d/p_1}_{p_1,1})}+\|u^n\|_{L^{2/(1-\varepsilon)}_T(B^{d/p_2-\varepsilon}_{p_2,1})}\|\nabla\theta_L\|_{L^{2/(1+\varepsilon)}_T(B^{d/p_1+\varepsilon}_{p_1,1})}$$

whenever $\varepsilon\in[0,1]$ and $d/p_1\leq 1-\varepsilon+d/p_2$.

[6]In all that follows, we denote by C_M a suitable increasing function of M. To simplify the notation, we omit the dependency with respect to d, N, p_1, p_2, etc.

Next, computations similar to those that enable us to bound h^n in the homogeneous framework lead to

$$\|h^n\|_{L^1_T(B^{d/p_2-1}_{p_2,1})}\leq C_M\Big(\|\nabla\theta^n\|^2_{L^2_T(B^{d/p_1}_{p_1,1})}+\|\nabla^2\theta^n\|_{L^2_T(B^{d/p_1-1}_{p_1,1})}\|\nabla\theta^n\|_{L^2_T(B^{d/p_1}_{p_1,1})}$$
$$+\|\nabla^2\theta^n\|_{L^{2/(2-\varepsilon)}_T(B^{d/p_1-\varepsilon}_{p_1,1})}\|\nabla\theta^n\|_{L^{2/\varepsilon}_T(B^{d/p_1-1+\varepsilon}_{p_1,1})}+\|\nabla\theta^n\|_{L^2_T(B^{d/p_1}_{p_1,1})}\|\nabla u^n\|_{L^2_T(B^{d/p_2-1}_{p_2,1})}\Big),$$

provided

$$p_1<2d,\quad p_1\leq 2p_2,\quad \frac{1}{p_1}+\frac{1}{p_2}>\frac{1}{d}\ \text{ and }\ \frac{1}{p_2}\leq\frac{1}{p_1}+\frac{1}{d}-\frac{\varepsilon}{d}\ \text{ for some }\ \varepsilon\in[0,1],$$

and for $p_1\leq 4$,

$$\|h^n\|_{L^1_T(L^2)}\leq C_M\Big(\|\nabla\theta^n\|_{L^2_T(L^\infty)}\|\nabla\theta^n\|^2_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}$$
$$+\|\nabla^2\theta^n\|_{L^{4/3}_T(B^{d/p_1-1/2}_{p_1,1})}\|\nabla\theta^n\|_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}+\|\nabla\theta^n\|_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}\|\nabla u^n\|_{L^{4/3}_T(B^{d/p_2-1/2}_{p_2,1})}\Big).$$

Under Condition (1.17), Hölder inequality, Propositions 2.3 and 2.5 also give

$$\|u^n\cdot\nabla u^{n+1}_L\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\lesssim\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\|u_L\|_{L^2_T(B^{d/p_2}_{p_2,1})}+\|u^n\|_{L^4_T(B^{d/p_2-1/2}_{p_2,1})}\|\nabla u_L\|_{L^{4/3}_T(B^{d/p_2-1/2}_{p_2,1})}$$

$$\|\mathrm{div}\,\big((\mu^n-\bar\mu)\nabla u^{n+1}_L\big)\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\lesssim\|\nabla\theta^n\|_{L^2_T(B^{d/p_1}_{p_1,1})}\|\nabla u_L\|_{L^2_T(B^{d/p_2-1}_{p_2,1})}$$
$$+\|\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\|u_L\|_{L^1_T(B^{d/p_2+1}_{p_2,1})}+\|\nabla\theta^n\|_{L^4_T(B^{d/p_1-1/2}_{p_1,1})}\|\nabla u_L\|_{L^{4/3}_T(B^{d/p_2-1/2}_{p_2,1})},$$

$$\|\bar\theta^n\nabla\bar Q^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}\lesssim\|\bar\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\|\nabla\bar Q^{n+1}\|_{L^1_T(B^{d/p_2-1}_{p_2,1}\cap L^2)}.$$

Let us fix some small positive constant τ_M that we shall specify later on and let us assume that T has been chosen so that

$$\|\theta_L\|_{L^{2/\varepsilon}_T(B^{d/p_1+\varepsilon}_{p_1,1})\cap L^4_T(B^{d/p_1+1/2}_{p_1,1})\cap L^1_T(B^{d/p_1+2}_{p_1,1})}+\|u_L\|_{L^4_T(B^{d/p_2-1/2}_{p_2,1})\cap L^1_T(B^{d/p_2+1}_{p_2,1})}\leq\tau_M.\tag{4.51}$$

Note that, in order that the above condition is satisfied for some positive T even if θ_0 is large, we have to rule out the case $\varepsilon=0$. This accounts for the strict inequality in the conditions

$$d/p_1<1+d/p_2\quad\text{and}\quad d/p_2<1+d/p_1,$$

that we did not have in the statement of Theorem 1.2.

Now, plugging all the above estimates in (4.48) and (4.49) yields (up to a harmless change of C_M)

$$\|\bar\theta^{n+1}\|_{X^{p_1}(T)}+\|\bar u^{n+1}\|_{Y^{p_2}(T)}+\|\nabla\bar Q^{n+1}\|_{Z^{p_2}(T)}$$
$$\leq C_M\Big(\|\bar\theta^n\|_{X^{p_1}(T)}(\|\bar\theta^n\|_{X^{p_1}(T)}+\|\bar u^n\|_{Y^{p_2}(T)}+\|\nabla\bar Q^{n+1}\|_{Z^{p_2}(T)})$$
$$+\tau_M\big(\|\bar\theta^n\|_{X^{p_1}(T)}+\|\bar u^n\|_{Y^{p_2}(T)}\big)+\tau_M^2+\tau_M\|\theta_L\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\Big).$$

Using (4.41) so as to bound the last term, we see that if we assume that

$$\|\bar\theta^n\|_{X^{p_1}(T)}+\|\bar u^n\|_{Y^{p_2}(T)}+\|\nabla\bar Q^n\|_{Z^{p_2}(T)}\leq K\tau_M\tag{4.52}$$

for some $K=K(M)$ that we shall choose below, and take τ_M so that

$$C_M\tau_MK\leq 1/2$$

then

$$\|\bar\theta^{n+1}\|_{X^{p_1}(T)}+\|\bar u^{n+1}\|_{Y^{p_2}(T)}+\|\nabla\bar Q^{n+1}\|_{Z^{p_2}(T)}\le 2C_M\tau_M\big(M+(K^2+K+1)\tau_M\big).$$

Hence $(\bar\theta^{n+1},\bar u^{n+1},\nabla\bar Q^{n+1})$ satisfies (4.52) too if we take $K=2C_M(1+M)$ and assume that τ_M also satisfies

$$(1+K+K^2)\tau_M\le 1.$$

This completes the proof of a priori estimates on any interval $[0,T]$ such that (4.51) is fulfilled.

Step 3. Convergence

The equation for $(\delta\theta^{n+1},\delta u^{n+1},\nabla\delta Q^{n+1})=(\theta^{n+1}-\theta^n,u^{n+1}-u^n,\nabla Q^{n+1}-\nabla Q^n)$ reads

$$\left\{\begin{array}{ccc}\partial_t\delta\theta^{n+1}+u^n\cdot\nabla\delta\theta^{n+1}-\mathrm{div}\,(\kappa^n\nabla\delta\theta^{n+1}) & = & I^n,\\ \partial_t\delta u^{n+1}+u^n\cdot\nabla\delta u^{n+1}-\mathrm{div}\,(\mu^n\nabla\delta u^{n+1})+(1+\theta_L^n)\nabla\delta Q^{n+1} & = & J^n,\\ \mathrm{div}\,\delta u^{n+1} & = & 0,\\ (\delta\theta^{n+1},\delta u^{n+1})|_{t=0} & = & (\Delta_{N_0+n}\theta_0,\Delta_{N_0+n}u_0),\end{array}\right.$$

where

$$\begin{aligned}I^n&=-\,\delta u^n\cdot\nabla\theta^n+\mathrm{div}\,(\delta\kappa^n\nabla\theta^n)+f^n-f^{n-1},\\ J^n&=-\,\delta u^n\cdot\nabla u^n+\mathrm{div}\,(\delta\mu^n\nabla u^n)-\delta\theta^n\nabla Q^n-\bar\theta^n\nabla\delta Q^{n+1}+h^n-h^{n-1}.\end{aligned}$$

Let $b_n=2^{(N_0+n)d/p_1}\|\Delta_{N_0+n}\theta_0\|_{L^{p_1}}$ and $d_n=2^{(N_0+n)(d/p_2-1)}\|\Delta_{N_0+n}u_0\|_{L^{p_2}}$. Since $\theta_0\in B^{d/p_1}_{p_1,1}$ and $u_0\in B^{d/p_2-1}_{p_2,1}$, we have $(b_n)\in\ell^1$ and $(d_n)\in\ell^1$.

To simplify the presentation, we assume that $p_1\ge p_2$. Then, applying Propositions 4.1 and 4.3 and using the bounds of the previous step, we get (bearing in mind that if τ_M is sufficiently small in (4.52) then one may absorb $\bar\theta^n\nabla\delta Q^{n+1}$):

$$\begin{aligned}\|\delta\theta^{n+1}\|_{X^{p_1}(T)}\le Cb_n+C_M\Big(&\big(\|\delta u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}+\|\delta\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big)\|\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\\ &+\|\delta\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\big(\|\theta^n\|_{L^1_T(B^{d/p_1+2}_{p_1,1})}+\|\theta^n\|^2_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big)\\ &+\|\delta\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\|\theta^{n-1}\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\Big),\end{aligned}$$

$$\begin{aligned}\|\delta u^{n+1}\|_{Y^{p_2}(T)}+\|\nabla\delta Q^{n+1}\|_{Z^{p_2}(T)}\le{}& Cd_n\\ +C_M\Big(&\|\delta\theta^n\|_{L^\infty_T(B^{d/p_1}_{p_1,1})}\big(\|(\theta^{n-1},\theta^n)\|^2_{L^2_T(B^{d/p_1+1}_{p_1,1})}+\|\theta^{n-1}\|_{L^1_T(B^{d/p_1+2}_{p_1,1})}+\|\nabla Q^n\|_{Z^{p_2}(T)}\big)\\ &+\|\delta\theta^n\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big(\|(\theta^n,\theta^{n-1})\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}+\|(\theta^n,\theta^{n-1})\|^2_{L^4_T(B^{d/p_1+1/2}_{p_1,1})}+\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\big)\\ &+\|\delta\theta^n\|_{L^4_T(B^{d/p_1+1/2}_{p_1,1})}\big(\|u^n\|_{L^{4/3}_T(B^{d/p_2+1/2}_{p_2,1})}+\|\theta^{n-1}\|_{L^{4/3}_T(B^{d/p_1+3/2}_{p_1,1})}\big)\\ &+\|\delta\theta^n\|_{L^{4/3}_T(B^{d/p_1+3/2}_{p_1,1})}\|\theta^n\|_{L^4_T(B^{d/p_1+1/2}_{p_1,1})}+\|\delta\theta^n\|_{L^{2/(2-\varepsilon)}_T(B^{d/p_1+2-\varepsilon}_{p_1,1})}\|\theta^n\|_{L^{2/\varepsilon}_T(B^{d/p_1+\varepsilon}_{p_1,1})}\\ &+\|\delta u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}\big(\|u^n\|_{L^2_T(B^{d/p_2}_{p_2,1})}+\|\theta^{n-1}\|_{L^2_T(B^{d/p_1+1}_{p_1,1})}\big)\\ &+\|\delta u^n\|_{L^4_T(B^{d/p_2-1/2}_{p_2,1})}\|u^n\|_{L^{4/3}_T(B^{d/p_2+1/2}_{p_2,1})}+\|\delta u^n\|_{L^{4/3}_T(B^{d/p_2+1/2}_{p_2,1})}\|\theta^{n-1}\|_{L^4_T(B^{d/p_1+1/2}_{p_1,1})}\Big)\end{aligned}$$

where $C=C(d,p_1,p_2)$.

Let us emphasize that, according to (4.51) and (4.52), the previous inequalities imply that, up to a change of C_M, we have

$$B^{n+1}(T)\le C_M\tau_MB^n(T)+C(b_n+d_n).$$

with $B^n(T)=\|\delta\theta^n\|_{X^{p_1}(T)}+\|\delta u^n\|_{Y^{p_2}(T)}+\|\nabla\delta Q^n\|_{Z^{p_2}(T)}$.

Therefore, taking τ_M small enough, we end up with

$$B^{n+1}(T) \le \frac{1}{2}B^n(T) + C(b_n+d_n).$$

As (b_n) and (d_n) are in ℓ^1, one may thus conclude that $\sum(B^n(T)) < \infty$, which is to say $(\theta^n, u^n, \nabla Q^n)_{n\in\mathbb{N}}$ is a Cauchy sequence and converges to a solution $(\theta, u, \nabla Q)$ of the system (1.6) in the space $F_T^{p_1,p_2}$ which also satisfies the estimates (1.19) (that $\vartheta \ge m$ is a consequence of the maximum principle for the parabolic equation satisfied by ϑ).

Step 4. Stability estimates and uniqueness

To prove the stability, i.e. the continuity of the flow map, and the uniqueness, we consider two solutions $(\theta^1, u^1, \nabla Q^1)$ and $(\theta^2, u^2, \nabla Q^2)$ of System (1.6) in $F_T^{p_1,p_2}$ with initial data (θ_0^1, u_0^1) and (θ_0^2, u_0^2), respectively. Let $\vartheta^1 = 1+\theta^1$ and $\vartheta^2 = 1+\theta^2$. We assume in addition that

$$\vartheta^1, \vartheta^2 \ge m$$

and we fix some large enough integer N_1 so that

$$m/2 \le 1 + S_{N_1}\theta^1 \quad \text{and} \quad \|\theta^1 - S_{N_1}\theta^1\|_{L_T^\infty(B^{d/p_1}_{p_1,1})} \le cm$$

with c given by Condition (4.22). Finally, we denote by M a common bound for the two solutions in $F_T^{p_1,p_2}$.

The proof goes from arguments similar to those of the previous step: we notice that the difference of the two solutions $(\delta\theta, \delta u, \nabla\delta Q) := (\theta^1-\theta^2, u^1-u^2, \nabla Q^1 - \nabla Q^2)$ satisfies

$$\left\{\begin{array}{ccc} \partial_t\delta\theta + u^1\cdot\nabla\delta\theta - \mathrm{div}\,(\kappa(\theta^1)\nabla\delta\theta) & = & I, \\ \partial_t\delta u + u^1\cdot\nabla\delta u - \mathrm{div}\,(\mu(\theta^1)\nabla\delta u) + \vartheta^1\nabla\delta Q & = & J, \\ \mathrm{div}\,\delta u & = & 0, \\ (\delta\theta, \delta u)|_{t=0} & = & (\delta\theta_0, \delta u_0), \end{array}\right.$$

where

$$\begin{aligned} I &= -\,\delta u\cdot\nabla\theta^2 + \nabla\cdot(\delta\kappa\nabla\theta^2) + f(1+\theta^1) - f(1+\theta^2), \\ J &= -\,\delta u\cdot\nabla u^2 + \nabla\cdot(\delta\mu\nabla u^2) - \delta\theta\nabla Q^2 + h(1+\theta^1, u^1) - h(1+\theta^2, u^2). \end{aligned}$$

Let $B(t) = \|\delta\theta\|_{X^{p_1}(t)} + \|\delta u\|_{Y^{p_2}(t)} + \|\nabla\delta Q\|_{Z^{p_2}(t)}$. Then arguing exactly as in the previous step, we get for small enough t,

$$\begin{aligned} B(t) \le C_{M,N_1}\Big(&\|\delta\theta_0\|_{B^{d/p_1}_{p_1,1}} + \|\delta u_0\|_{B^{d/p_2-1}_{p_2,1}} + \big(\|(\theta^1,\theta^2)\|_{L_t^2(B^{d/p_1+1}_{p_1,1})\cap L_t^4(B^{d/p_1+1/2}_{p_1,1})} \\ &+\|\theta^1\|_{L_t^{4/3}(B^{d/p_1+3/2}_{p_1,1})\cap L_t^4(B^{d/p_1+1/2}_{p_1,1})} + \|\theta^2\|_{L_t^1(B^{d/p_1+2}_{p_1,1})\cap L_t^{2/\varepsilon}(B^{d/p_1+\varepsilon}_{p_1,1})} \\ &+\|u^2\|_{L_t^2(B^{d/p_2}_{p_2,1})\cap L_t^{4/3}(B^{d/p_2+1/2}_{p_2,1})} + \|\nabla Q^2\|_{Z^{p_2}(t)}\big)B(t)\Big). \end{aligned}$$

If the initial data coincide then we have $B(0) = 0$. Given that the factors of $B(t)$ in the right-hand side go to 0 when t goes to 0, we thus get $B\equiv 0$ on a small enough time interval. Then, from standard continuation arguments, we conclude to uniqueness on the whole time interval $[0,T]$.

In the more general case where the initial data do not coincide, then one may split both solutions into

$$\theta^i = \theta^i_L + \bar\theta^i \quad \text{and} \quad u^i = u^i_L + \bar u^i,$$

where (θ^i_L, u^i_L) stands for the free solution of (4.40) pertaining to data (θ^i_0, u^i_0).

If $\|\delta\theta_0\|_{B^{d/p_1}_{p_1,1}} + \|\delta u_0\|_{B^{d/p_2-1}_{p_2,1}} \leq \delta$, then we have

$$\|\theta^2_L - \theta^1_L\|_{X^{p_1}(T)} + \|u^2_L - u^1_L\|_{Y^{p_2}(T)} \leq C\delta.$$

By arguing as in step two, one may also prove that if T is so small as to satisfy (4.51) for (say) (θ^1_L, u^1_L) and $\tau_M = \delta$ (with δ small enough) then $(\bar\theta^1, \bar u^1, \nabla\bar Q^1)$ satisfies (4.52). Therefore, from the above inequality, one may conclude that $B(T) \leq 2C_{M,N_1}\delta$. This completes the proof of the continuity of the flow map.

A Appendix

For the sake of completeness, we here prove the commutator estimates stated in Proposition 2.4. Throughout, it will be understood that $\|(c_j)_{j\in\mathbb{Z}}\|_{\ell^{r_2}} = \|(d_j)_{j\in\mathbb{Z}}\|_{\ell^{r_1}} = 1$ and that

$$\frac{1}{r} = \min\Big\{1, \frac{1}{r_1} + \frac{1}{r_2}\Big\}\cdot \tag{A.1}$$

Let $R_j(u,v) := [u, \Delta_j]v$ and $\widetilde u = u - \Delta_{-1}u$. Then we write the decomposition

$$R_j(u,v) = R^1_j(u,v) + R^2_j(u,v) + R^3_j(u,v) + R^4_j(u,v) + R^5_j(u,v)$$

with

$$R^1_j(u,v) := [T_{\widetilde u}, \Delta_j]v, \quad R^2_j(u,v) := T'_{\Delta_j v}\widetilde u, \quad R^3_j(u,v) := -\Delta_j T_v \widetilde u,$$
$$R^4_j(u,v) := -\Delta_j R(\widetilde u, v), \qquad R^5_j(u,v) := [\Delta_{-1}u, \Delta_j]v.$$

Let us first prove inequalities (2.10), (2.11) and (2.12). By virtue of the first-order Taylor's formula, we have

$$\begin{aligned} R^1_j(u,v) &= \sum_{|j-j'|\leq 4} [S_{j'-1}\widetilde u, \Delta_j]\Delta_{j'} v \\ &= 2^{-j'}\int_{\mathbb{R}^d}\int_0^1 h(y) y\cdot\nabla S_{j'-1}\widetilde u(x - 2^{-j}t'y)\Delta_{j'}v(x-2^{-j}y)\,dt'\,dy, \end{aligned}$$

hence

$$\|R^1_j(u,v)\|_{L^{p_1}} \lesssim 2^{-j}\|\nabla S_{j-1}\widetilde u\|_{L^\infty}\|\Delta_j v\|_{L^{p_1}},$$

whence

$$\big\|(2^{js}\|R^1_j(u,v)\|_{L^{p_1}})_{j\in\mathbb{N}}\big\|_{\ell^r} \lesssim \big\|(2^{j(\nu-1)}\|\nabla S_{j-1}\widetilde u\|_{L^\infty} 2^{j(s-\nu)}\|\Delta_j v\|_{L^{p_1}})\big\|_{\ell^r}.$$

So in the case $\nu = 1$, we readily get

$$\|(2^{js}\|R^1_j(u,v)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\nabla\widetilde u\|_{L^\infty}\|v\|_{B^{s-1}_{p_1,r}}. \tag{A.2}$$

To handle the case $\nu < 1$, we use the fact that

$$2^{j(\nu-1)}\|\nabla S_{j-1}\widetilde u\|_{L^\infty} 2^{j(s-\nu)}\|\Delta_j v\|_{L^{p_1}} \leq \sum_{j'\leq j-2} 2^{(j-j')(\nu-1)}\big(2^{j'(\nu-1)}\|\nabla\Delta_{j'}\widetilde u\|_{L^\infty}\big)\big(2^{j(s-\nu)}\|\Delta_j v\|_{L^{p_1}}\big).$$

Thus from Hölder and convolution inequalities for series and under assumption (A.1), one may conclude that

$$\|(2^{js}\|R^1_j\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\nabla\widetilde u\|_{B^{\nu-1}_{\infty,r_2}}\|v\|_{B^{s-\nu}_{p_1,r_1}} \quad \text{if } \nu < 1. \tag{A.3}$$

Concerning $R^2_j(u,v)$, we have

$$\|R^2_j(u,v)\|_{L^{p_1}} \leq \sum_{j'\geq j-3}\|\Delta_{j'}\widetilde u\, S_{j'+2}\Delta_j v\|_{L^{p_1}},$$

and we consider the following two cases (still under Condition (A.1)):

- $p_1 \geq p_2$:

$$\begin{aligned}
2^{js}\|R_j^2(u,v)\|_{L^{p_1}} &\leq 2^{js}\sum_{j'\geq j-3}\|\Delta_{j'}\widetilde{u}\|_{L^{p_1}}\|S_{j'+2}\,\Delta_j v\|_{L^\infty}\\
&\lesssim 2^{js}\sum_{j'\geq j-3}2^{j'd(\frac{1}{p_2}-\frac{1}{p_1})}\|\Delta_{j'}\widetilde{u}\|_{L^{p_2}}\|\Delta_j v\|_{L^\infty}\\
&\lesssim \sum_{j'\geq j-3}2^{(j-j')(\nu+\frac{d}{p_1})}c_{j'}\|\widetilde{u}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}d_j\|v\|_{B^{s-\nu-\frac{d}{p_1}}_{\infty,r_1}},
\end{aligned}$$

- $p_1 < p_2$:

$$\begin{aligned}
2^{js}\|R_j^2(u,v)\|_{L^{p_1}} &\leq 2^{js}\sum_{j'\geq j-3}\|\Delta_{j'}\widetilde{u}\|_{L^{p_2}}\|S_{j'+2}\Delta_j v\|_{L^{\frac{p_1p_2}{p_2-p_1}}}\\
&\lesssim \sum_{j'\geq j-3}2^{(j-j')(\nu+\frac{d}{p_2})}c_{j'}\|\widetilde{u}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}d_j\|v\|_{B^{s-\nu-\frac{d}{p_2}}_{\frac{p_1p_2}{p_2-p_1},r_1}}.
\end{aligned}$$

Hence for $\nu > -d\min\{\frac{1}{p_1},\frac{1}{p_2}\}$,

$$\|(2^{js}\|R_j^2(u,v)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\widetilde{u}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}\|v\|_{B^{s-\nu}_{p_1,r_1}}. \tag{A.4}$$

In the case $p_1 \leq p_2$, bounding $R_j^3(u,v)$ stems from (2.2), (2.3) (and an obvious embedding in the limit case). We get

$$\|(2^{js}\|R_j^3(u,v)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \begin{cases} \|\widetilde{u}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}\|v\|_{B^{s-\nu}_{p_1,r_1}}, & \text{if } s < \nu + d/p_2,\\ \|\widetilde{u}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r}}\|v\|_{B^{s-\nu}_{p_1,1}}, & \text{if } s = \nu + d/p_2. \end{cases} \tag{A.5}$$

To deal with the case $p_1 > p_2$, we just have to notice that, according to (2.2), (2.3), the paraproduct operator maps $B^{s-\nu-\frac{d}{p_1}}_{\infty,r_1} \times B^{\frac{d}{p_1}+\nu}_{p_1,r_2}$ in $B^s_{p_1,r}$ provided that $s < \nu + d/p_1$ (and $L^\infty \times B^{\frac{d}{p_1}+\nu}_{p_1,r}$ in $B^s_{p_1,r}$ if $s = \nu + d/p_1$). So we still get (A.5) provided $s < \nu + d/p_1$ and $s \leq \nu + d/p_1$, respectively.

As for the fourth term, it is only a matter of applying Inequality (2.4). We get

$$\|(2^{js}\|R_j^4(u,v)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\widetilde{u}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}}\|v\|_{B^{s-\nu}_{p_1,r_1}}, \text{ if } s > -d\min\Big\{\frac{1}{p_1'},\frac{1}{p_2}\Big\}. \tag{A.6}$$

The term $R_j^5(u,v)$ may be treated by arguing like in the proof of (A.2). One ends up with

$$\|(2^{js}\|R_j^5(u,v)\|_{L^{p_1}})_{j\in\mathbb{N}}\|_{\ell^r} \lesssim \|\nabla\Delta_{-1}u\|_{L^\infty}\|v\|_{B^{s-\nu}_{p_1,r}}, \qquad \text{if } \nu \leq 1. \tag{A.7}$$

Given that for any (s,p,r), one has (owing to the low-frequency cut-off)

$$\|\widetilde{u}\|_{B^s_{p,r}} \lesssim \|\nabla u\|_{B^{s-1}_{p,r}},$$

putting together (A.2), (A.3), (A.4), (A.5), (A.6) and (A.7) completes the proof of (2.10), (2.11) and (2.12).

In order to establish (2.13), we notice that the terms $R_j^i(u,v)$ with $i \neq 2$ are spectrally localized in balls of size 2^j. Hence Bernstein inequality together with (A.2), (A.3), (A.5), (A.6) and (A.7) ensures that they satisfy the desired inequality under Condition (2.9).

On the other hand $R_j^2(u,v)$ does not have this spectral localization property. Let us just treat the case $p_1 \geq p_2$ to simplify the presentation. We have

$$\|\partial_k R_j^2(u,v)\|_{L^{p_1}} \leq \sum_{j'\geq j-3}\Big(\|\Delta_{j'}\widetilde{u}\|_{L^{p_1}}\|\partial_k S_{j'+2}\Delta_j v\|_{L^\infty} + \|\partial_k\Delta_{j'}\widetilde{u}\|_{L^{p_1}}\|S_{j'+2}\Delta_j v\|_{L^\infty}\Big). \tag{A.8}$$

According to Bernstein's inequality, we have

$$\sum_{j'\geq j-3} \|\Delta_{j'}\widetilde{u}\|_{L^{p_1}} \|\partial_k S_{j'+2}\Delta_j v\|_{L^\infty} \leq C2^j \sum_{j'\geq j-3} \|\Delta_{j'}\widetilde{u}\|_{L^{p_1}} \|S_{j'+2}\Delta_j v\|_{L^\infty}.$$

Hence this term may be bounded as desired (just follow the previous computations).

In order to handle the second term of (A.8), we write that, according to Bersntein's inequality,

$$\|\partial_k \Delta_{j'}\widetilde{u}\|_{L^{p_1}} \|S_{j'+2}\Delta_j v\|_{L^\infty} \leq C2^{j'} \|\Delta_{j'}\widetilde{u}\|_{L^{p_1}} \|S_{j'+2}\Delta_j v\|_{L^\infty}.$$

Hence

$$2^{j(s-1)} \sum_{j'\geq j-3} \|\partial_k \Delta_{j'}\widetilde{u}\|_{L^{p_1}} \|S_{j'+2}\Delta_j v\|_{L^\infty} \leq C \sum_{j'\geq j-3} 2^{(j-j')(\nu+\frac{d}{p_1}-1)} c_{j'} \|\widetilde{u}\|_{B^{\frac{d}{p_2}+\nu}_{p_2,r_2}} d_j \|v\|_{B^{s-\nu-\frac{d}{p_1}}_{\infty,r_1}},$$

which leads to the desired inequality provided that $\nu + \frac{d}{p_1} - 1 > 0$. $\square$

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