T-coercivity: application to the discretization of Helmholtz-like problems

Patrick Ciarlet

T-coercivity: application to the discretization of Helmholtz-like problems

Patrick Ciarlet Jr.

Laboratoire POEMS, UMR 7231 CNRS/ENSTA/INRIA
ENSTA ParisTech, 32, boulevard Victor, 75739 Paris Cedex 15, France

Abstract

To solve variational indefinite problems, a celebrated tool is the Banach–Nečas–Babuška theory, which relies on the inf-sup condition. Here, we choose an alternate theory, T-coercivity. This theory relies on explicit inf-sup operators, both at the continuous and discrete levels. It is applied to solve Helmholtz-like problems in acoustics and electromagnetics. We provide simple proofs to solve the exact and discrete problems, and to show convergence under fairly general assumptions. We also establish sharp estimates on the convergence rates.

Keywords: inf-sup condition, T-coercivity, Helmholtz-like problems

1. Introduction

A few years ago, we proposed the T-coercivity theory with co-authors [4], to solve problems with sign-changing coefficients. It had already been used to solve other problems, such as boundary integral equations (see for instance [6]). It so happens that this T-coercivity theory is a reformulation of the Banach–Nečas–Babuška theory. Whereas the so-called BNB theory relies on an abstract inf-sup condition, T-coercivity uses explicit inf-sup operators, both at the continuous and discrete levels.

In this paper, we apply this theory to solve some very well-known Helmholtz problems: the acoustics problem, with a scalar unknown, and time-harmonic problems in electromagnetics, with vector unknowns. For the acoustics problem, convergence proofs are usually obtained by contradiction [2, 12]. Here we build a constructive proof of the result. Similarly, for time-harmonic problems in electromagnetics, convergence proofs usually rely on complex arguments, such as collectively compact families of discrete operators (see for instance [18], or [17], pp. 166-188): we again propose a constructive proof, slightly more involved than in the scalar case. In both cases, we discuss in some details the assumptions one has to make – when necessary – on the coefficients that characterize

the materials. Moreover, the proofs that we provide are much simpler than the ones already available in the literature, and we supply some sharp convergence estimates.

The outline of the paper is as follows. In the next section, we recall some well-known results on the well-posedness of variational problems, which we reformulate with the help of the theory of T-coercivity, and we derive results on the approximation of the problems within the same framework. In sections §§3-4, we apply the T-coercivity theory first to the scalar Helmholtz equation in acoustics, which we discretize using conforming Lagrange finite elements, and then to a (vector) electromagnetic wave equation in the time-frequency domain, which we discretize using edge finite elements. Finally, in an appendix, we briefly recall some salient results concerning those edge finite elements.

2. General framework

2.1. Starting point

Let V and W be two Hilbert spaces with scalar product $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$. We denote $\| \cdot \|_V$ and $\| \cdot \|_W$ the associated norms. Let us introduce $a(\cdot, \cdot)$ a continuous sesquilinear form over $V \times W$ and $f \in W'$. Here, W' refers to the topological dual space of W. The duality pairing is denoted $\langle \cdot, \cdot \rangle$ and the norm is defined by

$$\|f\|_{W'} := \sup_{w \in W \setminus \{0\}} \frac{|\langle f, w \rangle|}{\|w\|_W}.$$

We consider the variational problem

$$\begin{cases} Find\ u \in V\ such\ that \\ \forall w \in W,\ a(u, w) = \langle f, w \rangle. \end{cases} \tag{1}$$

First, let us recall a classical definition below.

Definition 1. Problem (1) is *well-posed* if, and only if, for all f, it has one and only one solution u, with continuous dependence:

$$\exists C > 0,\ \forall f \in W',\ \|u\|_V \leq C\|f\|_{W'}.$$

We define the operator $A \in \mathcal{L}(V, W')$ (the set of bounded operators from V to W') such that $\langle Au, w \rangle = a(u, w)$ for all $w \in W$. It is possible to reformulate Problem (1) as follows

$$\begin{cases} Find\ u \in V\ such\ that \\ Au = f\ in\ W'. \end{cases} \tag{2}$$

Problem (1) is well-posed if, and only if A is an isomorphism from V to W'. To address the solution of Problem (1), one can assume a *stability condition*, also called an *inf-sup condition*.

Definition 2. Let $a(\cdot, \cdot)$ be a continuous sesquilinear form over $V \times W$. It verifies an *inf-sup condition* if

$$\exists \alpha' > 0, \ \forall v \in V, \ \sup_{w \in W \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha' \|v\|_V. \tag{3}$$

This condition is supplemented with another one, see Theorem 1 below.

Let us now introduce another condition. As we shall see below, this amounts to using *explicit* inf-sup operators, i.e. operators that map each element of V to a suitable element w realizing the inf-sup condition.

Remark 1. Obviously, using an explicit inf-sup operator is standard. However, following [4], the originality of the method lies in a similar approach to solve the discrete problems, and also to prove convergence of the approximation, see §2.2.

Definition 3. Let $a(\cdot, \cdot)$ be a continuous sesquilinear form over $V \times W$. It is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V, W), \ \text{bijective}, \ \exists \underline{\alpha} > 0, \ \forall v \in V, \ |a(v, \mathtt{T}v)| \geq \underline{\alpha} \|v\|_V^2. \tag{4}$$

Theorem 1. (Well-posedness) *Let $a(\cdot, \cdot)$ be a continuous and sesquilinear form. Then the four assertions below are equivalent:*

(i)   the Problem (1) is well-posed;

(ii)   the form a satisfies an inf-sup condition and $R(A) = W'$;

(iii) the form a satisfies an inf-sup condition and the only element $w \in W$ which satisfies $a(v, w) = 0$ for all $v \in V$ is $w = 0$;

(iv)   the form a is T-coercive.

Remark 2. Assume that $W = V$.

If the form a is hermitian, that is if $a(v, w) = \overline{a(w, v)}$ for all $v, w \in V$, the inf-sup condition (3) is sufficient to ensure well-posedness.

In the same spirit, for a hermitian form a, Definition 3 can be simplified to: $a(\cdot, \cdot)$ is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V), \ \exists \underline{\alpha} > 0, \ \forall v \in V, \ |a(v, \mathtt{T}v)| \geq \underline{\alpha} \|v\|_V^2.$$

In other words, the fact that T be bijective is not required. Indeed, the previous condition implies that T is injective. Moreover, for all $v \in V \setminus \{0\}$, one has

$$\frac{|a(v, \mathtt{T}v)|}{\|\mathtt{T}v\|_V} \geq \underline{\alpha} \frac{\|v\|_V}{\|\mathtt{T}v\|_V} \|v\|_V \geq \frac{\underline{\alpha}}{|||\mathtt{T}|||} \|v\|_V.$$

Hence condition (3) holds.

2.2. Discretization of Problem (1)

Let us turn our attention to the approximation of the solution to Problem (1), which we assume to be well-posed. According to Theorem 1, there exists an inf-sup operator $\mathtt{T} \in \mathcal{L}(V, W)$ such that the form a is $\mathtt{T}$-coercive. To approximate this Problem, we let $(V_h)_h$ and $(W_h)_h$ be two infinite sequences of finite dimensional vector spaces. The parameter h takes strictly positive values, and it is destined to go to 0: if $n(h)$ denotes the dimension of V_h, then one has $\lim_{h \to 0} n(h) = +\infty$, so that V_h can "approximate" V. This also holds for the sequence of spaces $(W_h)_h$. When, for all h, $V_h \subset V$ and $W_h \subset W$, the approximation is a conforming discretization. In the sequel, we will always make this assumption.

Remark 3. For a nonconforming discretization of a problem (with sign-changing coefficients) solved by $\mathtt{T}$-coercivity, see [7]. For the classical Helmholtz-type problems we focus on, the tools we develop hereafter should be applicable to nonconforming discretizations, for instance with the popular Discontinuous Galerkin methods.

The discretization of problem (1) writes

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ \forall w_h \in W_h, \ a_h(u_h, w_h) = \langle f_h, w_h \rangle, \end{cases} \tag{5}$$

with discrete forms a_h and f_h (possibly) different respectively from a and f. In operator form, it writes

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ A_h u_h = f_h \text{ in } (W_h)', \end{cases} \tag{6}$$

with $A_h \in \mathcal{L}(V_h, (W_h)')$ defined by $\langle A_h v_h, w_h \rangle = a_h(v_h, w_h)$ for all $(v_h, w_h) \in V_h \times W_h$.

Below, we address the well-posedness of the discrete Problems (5) and we propose error estimates. To be able to solve (5), a necessary condition is $\dim V_h = \dim W_h$: we make this assumption from now on.

Definition 4. The family of sesquilinear forms $(a_h)_h$ is said to be *uniformly* $V_h \times W_h$-*stable* if

$$\exists \alpha_\dagger > 0, \ \forall h > 0, \ \forall v_h \in V_h, \quad \sup_{w_h \in W_h \setminus \{0\}} \frac{|a_h(v_h, w_h)|}{\|w_h\|_W} \geq \alpha_\dagger \|v_h\|_V. \tag{7}$$

As for the continuous problem (cf. [4]), we give an *a priori* intermediate condition to (7).

Definition 5. The family of sesquilinear forms $(a_h)_h$ is said to be *uniformly* $\mathtt{T}_h$-*coercive* if

$$\exists \alpha^\star, \beta^\star > 0, \ \forall h > 0, \ \exists \mathtt{T}_h \in \mathcal{L}(V_h, W_h), \ \forall v_h \in V_h, \\ |a_h(v_h, \mathtt{T}_h v_h)| \geq \alpha^\star \|v_h\|_V^2 \text{ and } |||\mathtt{T}_h||| \leq \beta^\star. \tag{8}$$

Next, introduce, for any $h > 0$ and any $v_h \in V_h$,

$$Cons_{f,h} \quad = \quad \sup_{w_h \in W_h \setminus \{0\}} \frac{|\langle f - f_h, w_h \rangle|}{\|w_h\|_V}, \tag{9}$$

$$Cons_{a,h}(v_h) \quad = \quad \sup_{w_h \in W_h \setminus \{0\}} \frac{|(a - a_h)(v_h, w_h)|}{\|w_h\|_V}. \tag{10}$$

These are consistency terms, in the sense that they express the discrepancies between the exact forms (a and f) and discrete forms (resp. a_h and f_h). One can obtain an error estimate including these consistency terms.

In $V_h \times W_h$, one can apply Theorem 1 to prove that Problem (5) is well-posed.

Theorem 2. (Well-posedness of the discrete problems) *Assume that $dim\,V_h = dim\,W_h$, and that the sesquilinear forms $(a_h)_h$ are uniformly bounded. Then the three assertions below are equivalent:*

(i)   *the Problem (5) is well-posed and $(A_h^{-1})_h$ is uniformly bounded;*

(ii)   *the family $(a_h)_h$ is uniformly $V_h \times W_h$-stable;*

(iii)   *the family $(a_h)_h$ is uniformly T_h-coercive.*

Moreover, if these conditions are satisfied, the error $\|u - u_h\|_V$ is bounded by

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \left(\|u - v_h\|_V + Cons_{f,h} + Cons_{a,h}(v_h) \right), \tag{11}$$

with $C := \max \left(\frac{1}{\alpha_\dagger}, \frac{|||a|||}{\alpha_\dagger} + 1 \right) > 0$ independent of h.

PROOF. (i) $\Longrightarrow$ (iii): define $\mathsf{t_h} := A_h^{-1} \circ I_{W_h \to W_h'}$ where $I_{W_h \to W_h'}$ is the isometry from W_h to W_h'. Since $(A_h^{-1})_h$ is uniformly bounded, there exists a constant C_1 such that, for all $h > 0$, $|||\mathsf{t_h}||| \leq C_1$. The inverse mapping $\mathsf{T}_h := \mathsf{t}_h^{-1}$ then belongs to $\mathcal{L}(V_h, W_h)$, and the family $(a_h)_h$ is uniformly T_h-coercive. Indeed, given $v_h \in V_h$, if we let $w_h = \mathsf{T}_h v_h$, we have $\|v_h\|_V = \|\mathsf{t}_h w_h\|_W \leq |||\mathsf{t}_h||| \, \|w_h\|_W$, so

$$a_h(v_h, \mathsf{T}_h v_h) = a_h(\mathsf{t}_h w_h, w_h) = \|w_h\|_W^2 \geq \frac{1}{|||\mathsf{t}_h|||^2} \|v_h\|_V^2 \geq \frac{1}{C_1^2} \|v_h\|_V^2.$$

Then, one has $\mathsf{T}_h = (A_h^{-1} \circ I_{W_h \to W_h'})^{-1} = I_{W_h' \to W_h} \circ A_h$, which yields $|||\mathsf{T_h}||| \leq |||A_h|||$: as the forms $(a_h)_h$ are uniformly bounded, so are the operators $(\mathsf{T}_h)_h$.

(iii) $\Longrightarrow$ (ii): for $v_h \in V_h \setminus \{0\}$, one has

$$\sup_{w_h \in W_h \setminus \{0\}} \frac{|a_h(v_h, w_h)|}{\|w_h\|_W} \geq \frac{|a_h(v_h, \mathsf{T}_h v_h)|}{\|\mathsf{T}_h v_h\|_W} \geq \alpha^\star \frac{\|v_h\|_V^2}{\|\mathsf{T}_h v_h\|_W} \geq \frac{\alpha^\star}{\beta^\star} \|v_h\|_V.$$

Hence, $(a_h)_h$ is uniformly $V_h \times W_h$-stable.

(ii) $\Longrightarrow$ (i): According to Theorem 1, if the family $(a_h)_h$ is uniformly $V_h \times W_h$-stable, Problem (5) is well-posed. Moreover, A_h^{-1} is uniformly bounded. Indeed,

5

$$\|A_h^{-1}f\| \leq \|f\|/\alpha_\dagger.$$

Now, let us focus on the error estimation (this part is very standard. It is kept here for the sake of completeness). By assumption, (7) holds for some $\alpha_\dagger > 0$. Given any $v_h \in V_h$, there exists $w_h \in W_h$ such that

$$
\begin{aligned}
\alpha_\dagger \|u_h - v_h\|_V \|w_h\|_V &\leq |a_h(u_h - v_h, w_h)|, \text{ and one can check that} \\
a_h(u_h - v_h, w_h) &= \langle f_h - f, w_h \rangle + a(u - v_h, w_h) + (a - a_h)(v_h, w_h).
\end{aligned}
$$

It follows that

$$\|u_h - v_h\|_V \leq \frac{1}{\alpha_\dagger}(Cons_{f,h} + |||a||| \, \|u - v_h\|_V + Cons_{a,h}(v_h)),$$

which leads to (11), since $\|u - u_h\|_V \leq \|u - v_h\|_V + \|u_h - v_h\|_V$. □

Corollary 1. *Assume there exists an isomorphism* $\mathtt{T} \in \mathcal{L}(V, W)$ *such that* $(v, v') \mapsto a(v, \mathtt{T}v')$ *is coercive on* $V \times V$. *Assume also* $\lim_{h \to 0} |||a_h - a||| = 0$ *and, finally, that there exists* $(\mathtt{T}_h)_h$, $\mathtt{T}_h \in \mathcal{L}(V_h, W_h)$ *such that* $\lim_{h \to 0} |||\mathtt{T}_h - \mathtt{T}||| = 0$. *Then, the family* $(a_h)_h$ *is uniformly* $\mathtt{T}_h$-*coercive for* h *small enough so estimate* (11) *holds true.*

PROOF. Indeed, one has, for any $v_h \in V_h$:

$$
\begin{aligned}
|a_h(v_h, \mathtt{T}_h v_h)| &= |a(v_h, \mathtt{T}_h v_h) + (a_h - a)(v_h, \mathtt{T}_h v_h)| \\
&= |a(v_h, \mathtt{T}v_h) - a(v_h, (\mathtt{T} - \mathtt{T}_h)v_h) + (a_h - a)(v_h, \mathtt{T}_h v_h)| \\
&\geq |a(v_h, \mathtt{T}v_h)| - |a(v_h, (\mathtt{T} - \mathtt{T}_h)v_h)| - |(a_h - a)(v_h, \mathtt{T}_h v_h)| \\
&\geq (\underline{\alpha} - |||a||| \, |||\mathtt{T} - \mathtt{T}_h||| - |||a_h - a||| \, |||\mathtt{T}_h|||) \, \|v_h\|_V^2.
\end{aligned}
$$

But $(|||\mathtt{T}_h|||)_h$ is bounded, hence the uniform $\mathtt{T}_h$-coercivity of the family $(a_h)_h$ is achieved (for h small enough). □

3. Helmholtz equation in acoustics

Consider a bounded domain Ω of $\mathbb{R}^d$, with $d = 1, 2, 3$. The model problem we study is a scalar wave equation in the time-frequency domain, e.g.

$$
\begin{cases}
\text{Find } u \in H^1(\Omega) \text{ such that} \\
\text{div}\,(\sigma \nabla u) + \omega^2 \eta u = f \text{ in } \Omega \\
u = 0 \text{ on } \partial\Omega.
\end{cases}
\tag{12}
$$

Above, f is a source, $\omega > 0$ is the given pulsation, and σ, η, for instance, stand respectively for the inverse of the mass density, and the inverse of the bulk/compressibility modulus. Assuming that f belongs to the dual space of $H_0^1(\Omega)$, called $H^{-1}(\Omega)$, the equivalent variational formulation is

$$
\begin{cases}
\text{Find } u \in H_0^1(\Omega) \text{ such that} \\
\displaystyle\int_\Omega \sigma \nabla u \cdot \nabla v \, d\Omega - \omega^2 \int_\Omega \eta u v \, d\Omega = -\langle f, v \rangle, \ \forall v \in H_0^1(\Omega).
\end{cases}
\tag{13}
$$

Remark 4. In the model scalar problem (12), we choose a homogeneous Dirichlet boundary condition. With this choice of the boundary condition, it is well-known that one can use real-valued fields, and find separately the real and imaginary parts of the solution. Also, other boundary conditions can be handled similarly: non-homogeneous Dirichlet, Neumann, Fourier on $\partial\Omega$, or mixed boundary conditions, i.e. different boundary conditions on different parts of the boundary.

The associated bilinear form is denoted by $a^a(\cdot,\cdot)$.

3.1. Well-posedness of the Helmholtz equation

To fix ideas, we assume that σ,η belong to $L^\infty(\Omega)$, and that there exist $\sigma_-,\eta_- > 0$ such that $\sigma > \sigma_-$ and $\eta > \eta_-$ almost everywhere in Ω. Then, we can endow $L^2(\Omega)$, respectively $H_0^1(\Omega)$, with the scalar products

$$(v,w)_{0,\eta} := \int_\Omega \eta v w\, d\Omega, \text{ resp. } (v,w)_{1,\sigma} := \int_\Omega \sigma\nabla v\cdot\nabla w\, d\Omega,$$

and associated norms. We also define the full $H^1(\Omega)$-scalar product: $(v,w)_1 := (v,w)_{0,\eta} + (v,w)_{1,\sigma}$ and its associated norm $\|\cdot\|_1$. Thanks to the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ one can apply the spectral theorem: there exists a Hilbert basis $(v_\ell)_{\ell\geq 0}$ of $L^2(\Omega)$ made up of eigenfunctions

$$\begin{cases} \textit{Find } (v_\ell,\lambda_\ell)\in H_0^1(\Omega)\times\mathbb{R} \textit{ such that } v_\ell\neq 0 \textit{ and} \\ (v_\ell,w)_{1,\sigma} = \lambda_\ell\,(v_\ell,w)_{0,\eta}, \ \forall w\in H_0^1(\Omega). \end{cases} \tag{14}$$

In addition, $(v_\ell)_{\ell\geq 0}$ is also an orthogonal basis of $H_0^1(\Omega)$. Moreover, all eigenvalues are of finite multiplicity, and $\lim_{\ell\to\infty}\lambda_\ell = +\infty$. To suit our purpose, for all $\ell\geq 0$, we prefer to *scale* the eigenfunction v_ℓ by a factor $(1+\lambda_\ell)^{-1/2}$, so that $\|v_\ell\|_1 = 1$. Hence, given $v\in H_0^1(\Omega)$, we write $v = \sum_{\ell\geq 0}\alpha_\ell v_\ell$, with $\alpha_\ell := (v,v_\ell)_1$ for $\ell\geq 0$, and $\|v\|_1 = (\sum_{\ell\geq 0}\alpha_\ell^2)^{1/2}$. Finally, the eigenpairs are ordered by increasing values of the eigenvalues.

Using a decomposition of the solution u over the basis $(v_\ell)_{\ell\geq 0}$, one finds easily that the acoustics problem is well-posed for all sources f if, and only if, $\omega^2 \notin \{\lambda_\ell\}_{\ell\geq 0}$. We make this assumption from now on.

Below, we first recover well-posedness with the help of the T-coercivity theory for the exact problem, and then we study its approximation with the same tool, in §3.2. Indeed, it is possible to define a suitable operator $\mathtt{T}^a$ for this problem. For that, let ℓ_{max} denote the largest index[1] $\ell\geq 0$ such that $\lambda_\ell < \omega^2$, and introduce the finite dimensional vector subspace[1] of $H_0^1(\Omega)$ defined by

$$V^- := \text{span}_{0\leq\ell\leq\ell_{max}}(v_\ell),$$

---

[1] When ω^2 is smaller than λ_0, $\ell_{max} = -1$, $V^- = \{0\}$ and $\mathtt{P}^- = 0$.

and finally the orthogonal projection operator[1] P^- from $H_0^1(\Omega)$ to V^-. By construction, the rank of the projection operator P^- is finite. The operator T^a is then defined either as $T^a := I_{H_0^1(\Omega)} - 2P^-$, or by its action on the basis vectors:

$$T^a v_\ell := \begin{cases} -v_\ell \text{ if } 0 \le \ell \le \ell_{max} \\ +v_\ell \text{ if } \ell > \ell_{max}. \end{cases}$$

Obviously, $(T^a)^2 = I_{H_0^1(\Omega)}$, so it is bijective.

Proposition 1. *The form $a^a(\cdot, \cdot)$ is T^a-coercive.*

PROOF. Given $v \in H_0^1(\Omega)$, one finds that

$$
\begin{aligned}
a^a(v, T^a v) &= \sum_{0 \le \ell \le \ell_{max}} \alpha_\ell [\omega^2 (v, v_\ell)_{0,\eta} - (v, v_\ell)_{1,\sigma}] + \sum_{\ell > \ell_{max}} \alpha_\ell [(v, v_\ell)_{1,\sigma} - \omega^2 (v, v_\ell)_{0,\eta}] \\
&= \sum_{0 \le \ell \le \ell_{max}} \left(\frac{\omega^2 - \lambda_\ell}{1 + \lambda_\ell} \right) \alpha_\ell^2 + \sum_{\ell > \ell_{max}} \left(\frac{\lambda_\ell - \omega^2}{1 + \lambda_\ell} \right) \alpha_\ell^2 \\
&\ge \underline{\alpha}^{\star, a} \|v\|_1^2,
\end{aligned}
$$

where $\underline{\alpha}^{\star, a} := \min \left(\min_{0 \le \ell \le \ell_{max}} \left(\frac{\omega^2 - \lambda_\ell}{1 + \lambda_\ell} \right), \min_{\ell > \ell_{max}} \left(\frac{\lambda_\ell - \omega^2}{1 + \lambda_\ell} \right) \right) = \min_{\ell \ge 0} \left| \frac{\lambda_\ell - \omega^2}{1 + \lambda_\ell} \right| > 0$.

Hence, the form $a^a(\cdot, \cdot)$ is T^a-coercive. $\qquad\square$

Thanks to Theorem 1, we conclude that the acoustics problem is well-posed when $\omega^2 \notin \{\lambda_\ell\}_{\ell \ge 0}$.

3.2. Discretization of the Helmholtz equation

Let us consider finite dimensional subspaces $(V_h^+)_h$ of $H^1(\Omega)$, and set $V_h := V_h^+ \cap H_0^1(\Omega)$. They can be obtained for instance with the help of the Lagrange finite elements on meshes of Ω made up of segments $(d = 1)$, triangles and/or quadrilaterals $(d = 2)$, tetrahedra, prisms and/or hexahedra $(d = 3)$ [8, 5, 14]. Classically, the index h is the meshsize. The discrete acoustics problems writes

$$
\begin{cases}
\text{Find } u_h \in V_h \text{ such that} \\
\int_{\Omega_h} \sigma \nabla u_h \cdot \nabla v_h \, d\Omega - \omega^2 \int_{\Omega_h} \eta u_h v_h \, d\Omega = -\langle f_h, v_h \rangle, \ \forall v_h \in V_h,
\end{cases}
\tag{15}
$$

where $\int_{\Omega_h} \cdot \, d\Omega$ stands for integrals possibly computed numerically with the help of quadratures, and similarly for $\langle f_h, \cdot \rangle$. Our goal, to prove convergence of the finite element discretization, is to apply Theorem 2, together with its Corollary 1.

Remark 5. On the matter of the threshold value of the meshsize (results hold for 'h small enough') which we do not discuss here, we refer to [12, 13].

We define the discrete forms $a_h^a(v_h, w_h) := \int_{\Omega_h} \sigma \nabla v_h \cdot \nabla w_h \, d\Omega - \omega^2 \int_{\Omega_h} \eta v_h w_h \, d\Omega$. Concerning the study of the consistency terms and of $|||a_h^a - a^a|||$, they can be derived from the classical properties of the quadratures: we refer again to [8, 5, 14] for extensive results on these topics. We assume that all terms go to 0 when h goes to 0.

On the other hand, we address the uniform $\mathtt{T}$-coercivity of the discrete forms below. To that aim, we shall define suitable discrete operators $(\mathtt{T}_h^a)_h$, in the same spirit as for the (exact) $\mathtt{T}^a$ operator.

If $\ell_{max} = -1$, then the result is obvious: $\mathtt{T}_h^a := \mathtt{I}_{V_h}$ works.

Consider from now on that $\ell_{max} \geq 0$.

The key idea is that, because the vector space V^- is of finite dimension, one is able to build a suitable approximation of this space in V_h by choosing approximations $(v_{\ell,h})_{0 \leq \ell \leq \ell_{max}}$ of the basis vectors $(v_\ell)_{0 \leq \ell \leq \ell_{max}}$, and then defining

$$V_h^- := \mathrm{span}_{0 \leq \ell \leq \ell_{max}}(v_{\ell,h}).$$

Indeed, the basic approximability property for the Lagrange finite element writes

$$\lim_{h \to 0} \left(\inf_{v_h \in V_h} \|v - v_h\|_1 \right) = 0, \ \forall v \in H_0^1(\Omega). \tag{16}$$

Hence, we can find, for all h and for $0 \leq \ell \leq \ell_{max}$, $v_{\ell,h} \in V_h$ such that $\|v_\ell - v_{\ell,h}\|_1 \leq \delta(h)$, with δ depending only on ℓ_{max} and $\lim_{h \to 0} \delta(h) = 0$.

Using standard linear algebra techniques, one obtains (by contradiction) that the finite element space V_h^- is of dimension $\ell_{max} + 1$ when h is small enough. Next, using for instance the Gram-Schmidt orthogonalization, one can build an orthonormal basis of V_h^-, still denoted by $(v_{\ell,h})_{0 \leq \ell \leq \ell_{max}}$ and in the process (by induction on ℓ), one checks that $\|v_\ell - v_{\ell,h}\|_1 \leq \tilde{\delta}(h)$, with an upper bound comparable to the previous one and still denoted by $\delta(h)$, $\lim_{h \to 0} \delta(h) = 0$. Last, defining the orthogonal projection operator $\mathtt{P}_h^-$ from V_h to V_h^-, one computes directly that there holds

$$|||\mathtt{P}^- - \mathtt{P}_h^-||| \leq \delta(h), \ \lim_{h \to 0} \delta(h) = 0. \tag{17}$$

Finally, we introduce the operator $\mathtt{T}_h^a := \mathtt{I}_{V_h} - 2\mathtt{P}_h^-$ of $\mathcal{L}(V_h)$.

Theorem 3. *The discrete solution u_h converges to the exact solution u of the acoustics problem, with a convergence rate that is governed by (11).*

PROOF. Given $v_h \in V_h$, we have

$$(\mathtt{T}^a - \mathtt{T}_h^a)v_h = v_h - 2\mathtt{P}^- v_h - v_h + 2\mathtt{P}_h^- v_h = 2(\mathtt{P}_h^- - \mathtt{P}^-)v_h.$$

Thanks to (17), one has $\lim_{h \to 0} |||\mathtt{T}^a - \mathtt{T}_h^a||| = 0$. According to Corollary 1, the family $(a_h^a)_h$ is uniformly $\mathtt{T}_h^a$-coercive, for h small enough. This ensures the existence and uniqueness of the discrete solution u_h to (15), for h small enough. Moreover, one concludes from Theorem 2 that u_h converges to the exact solution u, with a convergence rate that is governed by (11). $\square$

Remark 6. The multiplicative constant appearing in (11) behaves like $1/\alpha^{\star,a} = \max_{\ell \geq 0} \left| \frac{\lambda_\ell + 1}{\lambda_\ell - \omega^2} \right|$. As noted for instance in [12, 13], this constant cannot be better than the exact one.

3.3. Discussions on the convergence rate for the Helmholtz equation

In (11), we focus on providing an upper bound for $\inf_{v_h \in V_h} \|u - v_h\|_1$.
In the general case, the data f belongs to $H^{-1}(\Omega)$, and the basic approximability property (16) only yields convergence.
Consider from now on that f belongs to $L^2(\Omega)(^2)$. In this case, the solution u automatically belongs to the functional space

$$\Psi(\sigma) := \{v \in H_0^1(\Omega) \ : \ \mathrm{div}(\sigma \nabla v) \in L^2(\Omega)\}.$$

How can this property help obtain an upper bound?
To fix ideas, let us assume that Ω is a Lipschitz polyhedron, made up of composite materials. We assume moreover that σ is a piecewise constant function$(^3)$, which defines a partition $\mathcal{P} := \mathcal{P}(\sigma)$ of Ω into a finite number of subdomains $(\Omega_m)_{m=1\cdots M}$ such that, on each Ω_m, one has $\sigma(\boldsymbol{x}) = \sigma_m > 0$ a.e. In this case, we choose compatible meshes, in the sense that all tetrahedra, prisms and/or hexahedra lie exactly in one $\overline{\Omega_m}$, $m = 1 \cdots M$. We introduce:

$$PH^t(\Omega, \mathcal{P}) := \{v \in L^2(\Omega) \ : \ v_{|\Omega_j} \in H^t(\Omega_j), \ j = 1 \cdots M\}, \ t > 0.$$

In this setting, we obtain some extra regularity of u, as we know that $\Psi(\sigma)$ (endowed with the graph norm) is continuously embedded into a Sobolev space $PH^{1+\mathsf{s}}(\Omega)$, for some $\mathsf{s} := \mathsf{s}(\Omega, \sigma) > 0$ which depends only on the geometry Ω and on the piecewise coefficient σ [11, 10]. Hence, using the (modified) Clément, or the Scott-Zhang, interpolation operators [5, 14] with values in V_h, together with the continuous embedding property, we conclude that

$$\inf_{v_h \in V_h} \|u - v_h\|_1 \leq C \, \|f\|_{L^2(\Omega)} \, h^{\mathsf{s}}, \ C > 0 \text{ independent of } f \text{ and } u. \tag{18}$$

Remark 7. As mentioned in [11], the limiting value of the exponent s can be arbitrarily close to zero, even when Ω is a Lipschitz polyhedron.

On the other hand, if the coefficient is smooth, i.e. $\sigma \in W^{1,\infty}(\Omega)$, then one checks easily that $\sigma \nabla u$ belongs to $\boldsymbol{H}_0(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\mathrm{div}; \Omega)$. Now, according for instance to [10] and References therein, one has the continuous embedding of this functional space into $\boldsymbol{H}^{\mathsf{s}}(\Omega)$ for all $\mathsf{s} < \mathsf{s}_{\max}$, with $\mathsf{s}_{\max} = 1/2$ when the boundary $\partial \Omega$ is Lipschitz, respectively with $\mathsf{s}_{\max} := \mathsf{s}_{\max}(\Omega) > 1/2$ when Ω is a Lipschitz polyhedron, and finally for all $\mathsf{s} \leq 1$ when Ω is a convex polyhedron. Hence, estimate (18) holds with this exponent when σ is smooth.

---

[2]We could also consider that $f \in H^{-s}(\Omega)$, $s \in]0, 1[$, and then derive convergence rates with the help of *a priori* regularity estimates, in the spirit of [16], for instance when the coefficient σ is smooth.

[3]We could also consider a piecewise smooth coefficient σ over $\overline{\Omega}$.

4. Time-harmonic problems in electromagnetics

Consider again a bounded domain Ω of $\mathbb{R}^3$. The second model problem we study is an electromagnetic wave equation in the time-frequency domain, e.g. expressed in the electric field $\boldsymbol{e}$,

$$\begin{cases} Find \ \boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl};\Omega) \ such \ that \\ -\omega^2 \varepsilon \boldsymbol{e} + \mathbf{curl}(\nu \, \mathbf{curl} \, \boldsymbol{e}) = \boldsymbol{f} \ in \ \Omega \\ \boldsymbol{e} \times \boldsymbol{n} = 0 \ on \ \partial\Omega. \end{cases} \tag{19}$$

Above, $\boldsymbol{f}$ is a vector source, $\omega > 0$ is the given pulsation, and ε, ν are respectively the electric permittivity and the inverse of magnetic permeability. One usually assumes that $\boldsymbol{f}$ belongs to $\boldsymbol{L}^2(\Omega)$, so the equivalent variational formulation writes

$$\begin{cases} Find \ \boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \ such \ that \\ \displaystyle\int_\Omega \nu \, \mathbf{curl} \, \boldsymbol{e} \cdot \mathbf{curl} \, \boldsymbol{v} \, d\Omega \\ \qquad -\omega^2 \displaystyle\int_\Omega \varepsilon \boldsymbol{e} \cdot \boldsymbol{v} \, d\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\Omega, \ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega). \end{cases} \tag{20}$$

Remark 8. Again, with this choice of the boundary condition, it is well-known that one can use real-valued fields. Other boundary conditions can be handled similarly, and in particular a vanishing normal trace for the magnetic field. Also, the study can be extended to suitable boundary sources $\boldsymbol{f}$.

The associated bilinear form is denoted by $a^e(\cdot, \cdot)$. Classical configurations for Maxwell's equations include non-topologically trivial domains, and/or domains with a non-connected boundary. We recall hereafter some basic results concerning these configurations, before solving the electromagnetic wave equation in the time-frequency domain.

4.1. Preliminaries

We recall first the notion of trivial topology: given a vector field $\boldsymbol{v}$ defined over Ω such that $\mathbf{curl} \, \boldsymbol{v} = 0$ in Ω, does there exist a *continuous, single-valued* function p such that $\boldsymbol{v} = \nabla p$? The answer to this question can be found in (co)homology theory [15]:

either '*given any curl-free vector field $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega)$, there exists $p \in C^0(\Omega)$ such that $\boldsymbol{v} = \nabla p$ over Ω*';

or '*there exist I non-intersecting manifolds, $\Sigma_1, \ldots, \Sigma_I$, with boundaries $\partial\Sigma_i \subset \partial\Omega$, such that, if we let $\dot{\Omega} = \Omega \setminus \bigcup_{i=1}^I \Sigma_i$, given any curl-free vector field $\boldsymbol{v}$, there exists $\dot{p} \in C^0(\dot{\Omega})$ such that $\boldsymbol{v} = \nabla \dot{p}$ over $\dot{\Omega}$*'.

The domain Ω is said to be topologically trivial when $I = 0$.

Second, when the boundary $\partial\Omega$ is not connected, we let $(\Gamma_k)_{k=0\cdots K}$ be its (maximal) connected components.

In these configurations, one can build scalar potentials for curl-free elements of $\boldsymbol{H}(\mathbf{curl};\Omega)$, and also vector potentials for divergence-free elements of $\boldsymbol{H}(\mathrm{div};\Omega)$, under some compatibility conditions. We refer to [1] for details. Below, we provide explicit mentions of the results we use.

4.2. Well-posedness of the electromagnetic wave equation

To fix ideas, we assume now that ε, ν belong to $L^\infty(\Omega)$, and that there exist $\varepsilon_-, \nu_- > 0$ such that $\varepsilon > \varepsilon_-$ and $\nu > \nu_-$ almost everywhere in Ω. As previously, $\boldsymbol{L}^2(\Omega)$ is endowed with the scalar product $(\cdot, \cdot)_{0,\varepsilon}$.

We would like to mimic the process proposed in §3. In order to build a suitable Hilbert basis of the functional space $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$, let us begin by an orthogonal decomposition into two subspaces, with respect to the scalar product

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathbf{curl}} := (\boldsymbol{v}, \boldsymbol{w})_{0,\varepsilon} + (\mathbf{curl}\, \boldsymbol{v}, \mathbf{curl}\, \boldsymbol{w})_{0,\nu}.$$

We denote by $\| \cdot \|_{\mathbf{curl}}$ the associated norm.

Proposition 2. *There holds*

$$\boldsymbol{H}_0(\mathbf{curl}; \Omega) \;\; = \;\; \boldsymbol{G} \overset{\perp_{\mathbf{curl}}}{\oplus} \boldsymbol{W}_\varepsilon$$
$$where \qquad \boldsymbol{G} := \nabla H_0^1(\Omega), \;\; \boldsymbol{W}_\varepsilon := \{\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \; : \; \mathrm{div}(\varepsilon \boldsymbol{w}) = 0\}.$$

PROOF. This very standard result is usually obtained in two steps.
Given $\varphi \in H_0^1(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{W}_\varepsilon$, one finds that $\nabla\varphi$ and $\boldsymbol{w}$ are orthogonal by integration by parts:

$$(\nabla\varphi, \boldsymbol{w})_{\mathbf{curl}} = \int_\Omega \varepsilon \nabla\varphi \cdot \boldsymbol{w} \, d\Omega = -\int_\Omega \varphi \, \mathrm{div}(\varepsilon \boldsymbol{w}) \, d\Omega = 0.$$

Next, given $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$, one can solve the Dirichlet problem

$$\begin{cases} \text{Find } \varphi \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \varepsilon \nabla\varphi \cdot \nabla\psi \, d\Omega = \int_\Omega \varepsilon \boldsymbol{v} \cdot \nabla\psi \, d\Omega, \;\; \forall \psi \in H_0^1(\Omega). \end{cases}$$

By construction, $\nabla\varphi \in \boldsymbol{G}$ and $\boldsymbol{w} = \boldsymbol{v} - \nabla\varphi \in \boldsymbol{W}_\varepsilon$, so the conclusion follows. $\square$

Remark 9. In the previous proof, note that $\mathbf{curl}\, \boldsymbol{w} = \mathbf{curl}\, \boldsymbol{v}$.

Due to the above result, if we build Hilbert bases of the two vector subspaces $\boldsymbol{W}_\varepsilon$ and $\boldsymbol{G}$, they can be combined to form a Hilbert basis of $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$.

Next, we build a Hilbert basis of $\boldsymbol{W}_\varepsilon$. For that, we recall that $\boldsymbol{W}_\varepsilon$ is compactly embedded into $\boldsymbol{L}^2(\Omega)$. This result was first proven by Weber [20], and it holds under general assumptions on ε (see also [11]). As a consequence, $\boldsymbol{W}_\varepsilon$ is also compactly embedded into $\boldsymbol{H}(\mathrm{div}\, \varepsilon 0; \Omega) := \{\boldsymbol{w} \in \boldsymbol{H}(\mathrm{div}\, \varepsilon; \Omega) \; : \; \mathrm{div}(\varepsilon \boldsymbol{w}) = 0\}$, endowed with the scalar product $(\cdot, \cdot)_{0,\varepsilon}$. Moreover, we have the

Proposition 3. $\boldsymbol{W}_\varepsilon$ *is dense in* $\boldsymbol{H}(\mathrm{div}\, \varepsilon 0; \Omega)$.

PROOF. It is enough to check that any element of the dual space $(\boldsymbol{H}(\mathrm{div}\, \varepsilon 0; \Omega))'$ that vanishes over $\boldsymbol{W}_\varepsilon$ is actually equal to 0. Thanks to the Riesz theorem,

any such element can be represented by $\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}\varepsilon 0; \Omega)$, and its action by $\boldsymbol{w} \mapsto (\boldsymbol{v}, \boldsymbol{w})_{0,\varepsilon}$. Now, for $1 \leq k \leq K$, let $q_k \in H^1(\Omega)$ be such that

$$\operatorname{div}(\varepsilon \nabla q_k) = 0, \ q_k = \delta_{kk'} \text{ on } \Gamma_{kk'}, \ 0 \leq k' \leq K.$$

By construction, for $1 \leq k \leq K$, ∇q_k belongs to $\boldsymbol{W}_\varepsilon$, and $(\boldsymbol{v}, \nabla q_k)_{0,\varepsilon} = 0$ yields $\langle \varepsilon \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_k} = 0$. According to Theorem 3.12 in [1], there exists one, and only one $\boldsymbol{z} \in \boldsymbol{H}_0(\operatorname{div}; \Omega)$ such that

$$\varepsilon \boldsymbol{v} = \operatorname{\mathbf{curl}} \boldsymbol{z}, \ \operatorname{div} \boldsymbol{z} = 0, \ \langle \boldsymbol{z} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_i} = 0, \ 1 \leq i \leq I.$$

Thus, given any $\boldsymbol{w} \in \boldsymbol{W}_\varepsilon$, one finds by integration by parts

$$0 = (\boldsymbol{v}, \boldsymbol{w})_{0,\varepsilon} = (\operatorname{\mathbf{curl}} \boldsymbol{z}, \boldsymbol{w})_0 = (\boldsymbol{z}, \operatorname{\mathbf{curl}} \boldsymbol{w})_0.$$

But, we know from Theorem 3.17 in [1] that the mapping $\boldsymbol{w} \mapsto \operatorname{\mathbf{curl}} \boldsymbol{w}$ is surjective from $\boldsymbol{W}_{\varepsilon=1}$ onto $\{\boldsymbol{y} \in \boldsymbol{H}_0(\operatorname{div}; \Omega) \ : \ \operatorname{div} \boldsymbol{y} = 0, \ \langle \boldsymbol{y} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_i} = 0, \ 1 \leq i \leq I\}$. The surjectivity also holds from $\boldsymbol{W}_\varepsilon$ onto the same functional space, if one corrects the fields as in the proof Proposition 2 to recover $\operatorname{div}\varepsilon$-free fields, without modifying their curl. Hence there exists $\boldsymbol{w}' \in \boldsymbol{W}_\varepsilon$ such that $\boldsymbol{z} = \operatorname{\mathbf{curl}} \boldsymbol{w}'$ and it follows that $\boldsymbol{z} = 0$, and so $\boldsymbol{v} = 0$. $\qquad\square$

Therefore, using again the spectral theorem, we can build a Hilbert basis $(\boldsymbol{e}_\ell)_{\ell \geq 0}$ of $\boldsymbol{H}(\operatorname{div}\varepsilon 0; \Omega)$ made up of eigenfunctions

$$\begin{cases} \text{Find } (\boldsymbol{e}_\ell, \mu_\ell) \in \boldsymbol{W}_\varepsilon \times \mathbb{R} \text{ such that } \boldsymbol{e}_\ell \neq 0 \text{ and} \\ (\boldsymbol{e}_\ell, \boldsymbol{w})_{\operatorname{\mathbf{curl}}} = (1 + \mu_\ell) (\boldsymbol{e}_\ell, \boldsymbol{w})_{0,\varepsilon}, \ \forall \boldsymbol{w} \in \boldsymbol{W}_\varepsilon. \end{cases} \tag{21}$$

Note that, by construction, one has $\mu_\ell \geq 0$, for all $\ell \geq 0$. All eigenvalues are of finite multiplicity, and $\lim_{\ell \to \infty} \mu_\ell = +\infty$. In addition, $(\boldsymbol{e}_\ell)_{\ell \geq 0}$ is also an orthogonal basis of $\boldsymbol{W}_\varepsilon(\Omega)$. Hence, with the help of an appropriate scaling (by a factor $(1 + \mu_\ell)^{-1/2}$ for $\ell \geq 0$), $(\boldsymbol{e}_\ell)_{\ell \geq 0}$ is a Hilbert basis of the subspace $\boldsymbol{W}_\varepsilon$ with respect to the scalar product $(\cdot, \cdot)_{\operatorname{\mathbf{curl}}}$, ordered by increasing values of μ_ℓ. Furthermore, using Proposition 2(^{4}), one has actually, for all $\ell \geq 0$,

$$(\boldsymbol{e}_\ell, \boldsymbol{v})_{\operatorname{\mathbf{curl}}} = (1 + \mu_\ell) (\boldsymbol{e}_\ell, \boldsymbol{v})_{0,\varepsilon}, \ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{\mathbf{curl}}; \Omega). \tag{22}$$

Finally, recall that we built an orthogonal basis $(v_\ell)_{\ell \geq 0}$ of $H_0^1(\Omega)$, cf. (14). Then, if we scale $(v_\ell)_{\ell \geq 0}$ and replace σ by ε, we can define a Hilbert basis $(\boldsymbol{e}_\ell)_{\ell < 0}$ of the subspace $\boldsymbol{G}$ with respect to the scalar product $(\cdot, \cdot)_{\operatorname{\mathbf{curl}}}$, by setting $\boldsymbol{e}_\ell := \nabla v_{-(1+\ell)}$ for $\ell < 0$. We note that given any $\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{\mathbf{curl}}; \Omega)$ one has $(\boldsymbol{e}_\ell, \boldsymbol{v})_{\operatorname{\mathbf{curl}}} = (\boldsymbol{e}_\ell, \boldsymbol{v})_{0,\varepsilon}$ i.e. $\mu_\ell = 0$, for all $\ell < 0$ (compare to (22)).

---

^{4}Also, due to Proposition 2, one checks easily that there holds $\operatorname{\mathbf{curl}}(\nu \operatorname{\mathbf{curl}} \boldsymbol{e}_\ell) = \mu_\ell \boldsymbol{e}_\ell$ for $\ell \geq 0$ and that $\mu_\ell = 0$ amounts to $\operatorname{\mathbf{curl}} \boldsymbol{e}_\ell = 0$. Last, $\mu_\ell = 0$ holds for exactly K values of ℓ, with eigenfunctions in $\operatorname{span}_{1 \leq k \leq K}(\nabla q_k)$ as in the proof of Proposition 3 (we refer to Proposition 3.18 in [1] for the last property).

Thanks to Proposition 2, we deduce that $(e_\ell)_\ell$ defines a Hilbert basis of $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$. Given $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$, we write $\boldsymbol{v} = \sum_\ell \alpha_\ell e_\ell$, with $\alpha_\ell := (\boldsymbol{v}, e_\ell)_{\mathbf{curl}}$ for all ℓ, and $\|\boldsymbol{v}\|_{\mathbf{curl}} = (\sum_\ell \alpha_\ell^2)^{1/2}$.

In particular, using a decomposition of the solution e over the Hilbert basis $(e_\ell)_\ell$, one concludes that the electromagnetic wave equation is well-posed for all sources $\boldsymbol{f}$ if, and only if, $\omega^2 \notin \{\mu_\ell\}_\ell$. We make this assumption from now on.

Remark 10. Note that we can perform a similar construction to obtain a Hilbert basis of $\boldsymbol{L}^2(\Omega)$, starting from the orthogonal decomposition

$$\boldsymbol{L}^2(\Omega) = \boldsymbol{G} \overset{\perp}{\oplus} \boldsymbol{H}(\operatorname{div} \varepsilon 0; \Omega),$$

with respect to the scalar product $(\cdot, \cdot)_{0,\varepsilon}$.

We are now in a position to recover well-posedness for the (exact) electromagnetic wave equation, with the help of the $\mathtt{T}$-coercivity theory. For that, we define an operator $\mathtt{T}^e$: we let ℓ_{max} denote the largest index[1] $\ell \geq 0$ such that $\mu_\ell < \omega^2$, and introduce the finite dimensional vector subspace[1] of $\boldsymbol{W}_\varepsilon$ defined by

$$\boldsymbol{V}^- := \operatorname{span}_{0 \leq \ell \leq \ell_{max}}(e_\ell),$$

and the orthogonal projection operator[1] $\mathtt{P}^-$ from $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$ to $\boldsymbol{V}^-$. The rank of the operator $\mathtt{P}^-$ is finite. The operator $\mathtt{T}^e$ is then defined either as $\mathtt{T}^e := -\mathtt{i}_{\boldsymbol{G}} + \mathtt{i}_{\boldsymbol{W}_\varepsilon} - 2\mathtt{P}^-$, with $\mathtt{i}_{\boldsymbol{G}}$ (resp. $\mathtt{i}_{\boldsymbol{W}_\varepsilon}$), the canonical embedding of $\boldsymbol{G}$ (resp. $\boldsymbol{W}_\varepsilon$), into $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$; or by its action on the basis vectors:

$$\mathtt{T}^e e_\ell := \begin{cases} -e_\ell & \text{if } \ell \leq \ell_{max} \\ +e_\ell & \text{if } \ell > \ell_{max}. \end{cases}$$

By construction, $\mathtt{T}^e$ is a bijection, as $(\mathtt{T}^e)^2 = \mathtt{I}_{\boldsymbol{H}_0(\mathbf{curl};\Omega)}$.

Proposition 4. *The form* $a^e(\cdot, \cdot)$ *is* $\mathtt{T}^e$-*coercive.*

PROOF. Given $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$, one finds that

$$
\begin{aligned}
a^e(\boldsymbol{v}, \mathtt{T}^e \boldsymbol{v}) &= \omega^2 \sum_{\ell < 0} \alpha_\ell(\boldsymbol{v}, e_\ell)_{0,\varepsilon} + \sum_{0 \leq \ell \leq \ell_{max}} \alpha_\ell[\omega^2(\boldsymbol{v}, e_\ell)_{0,\varepsilon} - (\mathbf{curl}\,\boldsymbol{v}, \mathbf{curl}\,e_\ell)_{0,\nu}] \\
&\quad + \sum_{\ell > \ell_{max}} \alpha_\ell[(\mathbf{curl}\,\boldsymbol{v}, \mathbf{curl}\,e_\ell)_{0,\nu} - \omega^2(\boldsymbol{v}, e_\ell)_{0,\varepsilon}] \\
&= \sum_{\ell \leq \ell_{max}} \left(\frac{\omega^2 - \mu_\ell}{1 + \mu_\ell}\right) \alpha_\ell^2 + \sum_{\ell > \ell_{max}} \left(\frac{\mu_\ell - \omega^2}{1 + \mu_\ell}\right) \alpha_\ell^2 \\
&\geq \underline{\alpha}^{\star,e} \|\boldsymbol{v}\|_{\mathbf{curl}}^2,
\end{aligned}
$$

where $\underline{\alpha}^{\star,e} := \min\left(\min_{\ell \leq \ell_{max}}\left(\frac{\omega^2 - \mu_\ell}{1 + \mu_\ell}\right), \min_{\ell > \ell_{max}}\left(\frac{\mu_\ell - \omega^2}{1 + \mu_\ell}\right)\right) = \min_\ell \left|\frac{\mu_\ell - \omega^2}{1 + \mu_\ell}\right| > 0.$

Above, we used the property $\mu_\ell = 0$ for $\ell < 0$.
We conclude that the form $a^e(\cdot, \cdot)$ is $\mathtt{T}^e$-coercive. □

The electromagnetic wave equation is well-posed when $\omega^2 \notin \{\mu_\ell\}_{\ell \geq 0}$, according to Theorem 1.

14

4.3. Discretization of the electromagnetic wave equation

We assume from now on that Ω is a Lipschitz polyhedron. To define finite dimensional subspaces $(\boldsymbol{V}_h)_h$ of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, we consider a family of tetrahedral meshes of Ω (of meshsize h), and we choose the so-called Nédélec's first family of edge finite elements [19, 17]. The construction is detailed in the Appendix. The discrete electromagnetic wave equation writes

$$\begin{cases} \textit{Find } \boldsymbol{e}_h \in \boldsymbol{V}_h \textit{ such that} \\ \displaystyle\int_{\Omega_h} \nu\,\mathbf{curl}\,\boldsymbol{e}_h \cdot \mathbf{curl}\,\boldsymbol{v}_h\,d\Omega \\ \qquad\qquad -\omega^2 \displaystyle\int_{\Omega_h} \varepsilon\boldsymbol{e}_h \cdot \boldsymbol{v}_h\,d\Omega = \int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{v}_h\,d\Omega,\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{23}$$

Again, $\int_{\Omega_h} \cdot\,d\Omega$ stands for integrals possibly computed numerically. We define the discrete forms $a_h^e(\boldsymbol{v}_h, \boldsymbol{w}_h) := \int_{\Omega_h} \nu\,\mathbf{curl}\,\boldsymbol{v}_h \cdot \mathbf{curl}\,\boldsymbol{w}_h\,d\Omega - \omega^2 \int_{\Omega_h} \varepsilon\boldsymbol{v}_h \cdot \boldsymbol{w}_h\,d\Omega$. We shall prove as before convergence of the finite element discretization using Theorem 2 and its Corollary 1. We assume that all consistency terms and $|||a_h^e - a^e|||$ go to 0 when h goes to 0. On the other hand, we focus on the uniform T-coercivity of the discrete forms and, for that, we define suitable discrete $\mathtt{T}_h^e$ operators. The process here is more involved than in §3.2, because we need to take care, not only of the projection of the discrete fields on the discrete counterpart of the eigenspace $\boldsymbol{V}^-$ ($0 \leq \ell \leq \ell_{max}$), but also of their gradient part ($\ell < 0$).

Hence, let us consider splittings of discrete fields: the exact one, like in Proposition 2, and then a discrete one. To begin with, given $\boldsymbol{v}_h \in \boldsymbol{V}_h$, we know that there exists one, and only one $(\varphi, \boldsymbol{w}) \in H_0^1(\Omega) \times \boldsymbol{W}_\varepsilon$ such that

$$\boldsymbol{v}_h = \nabla\varphi + \boldsymbol{w}, \tag{24}$$

and by construction $\mathbf{curl}\,\boldsymbol{v}_h = \mathbf{curl}\,\boldsymbol{w}$. This is the continuous, or exact, splitting of the discrete field $\boldsymbol{v}_h$. As the sum is orthogonal in Proposition 2, it follows that this splitting is stable, i.e. $\|\nabla\varphi\|_{\mathbf{curl}} \leq \|\boldsymbol{v}_h\|_{\mathbf{curl}}$ and $\|\boldsymbol{w}\|_{\mathbf{curl}} \leq \|\boldsymbol{v}_h\|_{\mathbf{curl}}$. Below, we propose a discrete splitting of $\boldsymbol{v}_h$, in the same spirit as (24), and moreover we establish some bounds on the 'distance' between the two splittings. To obtain this result, we recall a regular-singular splitting of elements of $\boldsymbol{W}_\varepsilon$, and more generally of elements of

$$\boldsymbol{X}_\varepsilon := \{\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)\ :\ \mathrm{div}(\varepsilon\boldsymbol{w}) \in L^2(\Omega)\},\ \text{with graph norm } \|\cdot\|_{\boldsymbol{X}_\varepsilon}.$$

The stability([5]) of the discrete splitting is proved under assumptions on ε similar to those of §3.3, which we make from now on. In the present case, we denote by

---

[5]Because the functional space $\boldsymbol{G}$ is infinite dimensional, we need a *uniform* estimate on the approximation of that part of the field. This is the reason why assumptions on the coefficient ε are required. Whereas in §3, the assumptions on the coefficient are needed only to derive convergence rates.

$\mathcal{P} := \mathcal{P}(\varepsilon)$ the partition of Ω, and by $\boldsymbol{PH}^t(\Omega, \mathcal{P})$ the Sobolev space of vector, piecewise-$\boldsymbol{H}^t$ fields (for $t > 0$).

Theorem 4. *Let $\boldsymbol{w} \in \boldsymbol{X}_\varepsilon$.*
Then one can split $\boldsymbol{w}$ as

$$\boldsymbol{w} = \boldsymbol{w}_R + \nabla\psi, \ \ with \ \begin{cases} \boldsymbol{w}_R \in \boldsymbol{X}_\varepsilon \cap \boldsymbol{PH}^1(\Omega, \mathcal{P}) \\ \psi \in \Psi(\varepsilon) \end{cases}. \tag{25}$$

Furthermore,

$$\|\boldsymbol{w}_R\|_{\boldsymbol{X}_\varepsilon} + \|\boldsymbol{w}_R\|_{\boldsymbol{PH}^1(\Omega,\mathcal{P})} + \|\psi\|_{H^1(\Omega)} + \|\operatorname{div}\varepsilon\nabla\psi\|_{L^2(\Omega)} \leq C \, \|\boldsymbol{w}\|_{\boldsymbol{X}_\varepsilon}, \tag{26}$$

with $C := C(\Omega, \varepsilon) > 0$ independent of $\boldsymbol{w}$.

The result above has been proven in [3, Theorem 3.1] in the case of a constant coefficient ε, and in [11, Theorem 3.5] in the case of a piecewise constant ε.

Proposition 5. *Consider a discrete field $\boldsymbol{v}_h \in \boldsymbol{V}_h$, whose exact splitting is given by (24). Then, there exist $\varphi_h \in V_h$ and $\boldsymbol{w}_h \in \boldsymbol{V}_h$ such that*

$$\boldsymbol{v}_h = \nabla\varphi_h + \boldsymbol{w}_h, \tag{27}$$
$$\|\nabla(\varphi - \varphi_h)\|_{\mathbf{curl}} = \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathbf{curl}} \leq C_r \, h^{\min(1,\mathsf{s})} \, \|\boldsymbol{v}_h\|_{\mathbf{curl}}, \tag{28}$$

with $\mathsf{s} := \mathsf{s}(\Omega, \varepsilon) > 0$ defined as in §3.3, $C_r > 0$ independent of $\boldsymbol{v}_h$.

PROOF. Let us start from the exact splitting (24) of $\boldsymbol{v}_h$: $\boldsymbol{v}_h = \boldsymbol{w} + \nabla\varphi$, $\boldsymbol{w} \in \boldsymbol{W}_\varepsilon$, $\varphi \in H_0^1(\Omega)$. Then, we split $\boldsymbol{w}$ as in (25), namely $\boldsymbol{w} = \boldsymbol{w}_R + \nabla\psi$, $\boldsymbol{w}_R \in \boldsymbol{X}_\varepsilon \cap \boldsymbol{PH}^1(\Omega, \mathcal{P})$, $\psi \in \Psi(\varepsilon)$, which yields

$$\boldsymbol{v}_h = \boldsymbol{w}_R + \nabla(\varphi + \psi), \ \ with \ \ \mathbf{curl}\,\boldsymbol{w}_R = \mathbf{curl}\,\boldsymbol{v}_h.$$

In any tetrahedron K, one has $(\boldsymbol{w}_R)_{|K} \in \boldsymbol{H}^1(K)$, whereas $(\mathbf{curl}\,\boldsymbol{w}_R)_{|K} = (\mathbf{curl}\,\boldsymbol{v}_h)_{|K}$ is constant (hence smooth), so the local interpolant $\Pi_K \boldsymbol{w}_R$ exists according to Proposition 6. Furthermore, according to Proposition 8, one has

$$\|\boldsymbol{w}_R - \Pi_K \boldsymbol{w}_R\|_{\boldsymbol{H}(\mathbf{curl},K)} \leq C_1 \left(\|\boldsymbol{w}_R\|_{\boldsymbol{H}^1(K)} + \|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(K)}\right) h_K$$

with C_1 independent of K, $\boldsymbol{w}_R$ and $\boldsymbol{v}_h$ (as $(\mathbf{curl}\,\boldsymbol{v}_h)_{|K}$ is constant, one has $\|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{H}^1(K)} = \|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(K)}$). In addition, $\boldsymbol{w}_R \in \boldsymbol{H}(\mathbf{curl};\Omega)$, so one can apply the global interpolation operator Π_h to it. Summing up over all tetrahedra yields

$$\|\boldsymbol{w}_R - \Pi_h \boldsymbol{w}_R\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq \sqrt{2} C_1 \left(\|\boldsymbol{w}_R\|_{\boldsymbol{PH}^1(\Omega,\mathcal{P})} + \|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}\right) h. \tag{29}$$

Hence, according to Proposition 7, there exists $z_h \in V_h$ such that $\Pi_h(\nabla(\varphi + \psi)) = \nabla z_h$ and moreover

$$\boldsymbol{v}_h = \Pi_h \boldsymbol{v}_h = \Pi_h \boldsymbol{w}_R + \nabla z_h. \tag{30}$$

On the other hand, one has $\psi \in \Psi(\varepsilon)$, which is continuously embedded in $PH^{1+\mathsf{s}}(\Omega, \mathcal{P})$, with $\mathsf{s} = \mathsf{s}(\Omega, \varepsilon)$. Using the (modified) Clément, or the Scott-Zhang, interpolation operators with values in V_h, we know that there exists $\psi_h \in V_h$ such that

$$\|\psi - \psi_h\|_{H^1(\Omega)} \le C_2 \left(\|\psi\|_{H^1(\Omega)} + \|\operatorname{div}\varepsilon\nabla\psi\|_{L^2(\Omega)} \right) h^{\mathsf{s}}, \qquad (31)$$

with C_2 independent of ψ.

Then, we define $\varphi_h := z_h - \psi_h$ and $\boldsymbol{w}_h := \Pi_h \boldsymbol{w}_R + \nabla\psi_h$. By construction, one has $\boldsymbol{w}_h \in \boldsymbol{V}_h$ and $\varphi_h \in V_h$, and moreover

$$\boldsymbol{v}_h \overset{(30)}{=} \boldsymbol{w}_h - \nabla\psi_h + \nabla z_h = \boldsymbol{w}_h + \nabla\varphi_h, \text{i.e. } (27), \text{ and}$$

$$\boldsymbol{w} - \boldsymbol{w}_h \overset{(24)}{=} (\boldsymbol{v}_h - \nabla\varphi) - (\Pi_h \boldsymbol{w}_R + \nabla\psi_h) \overset{(30)}{=} \nabla z_h - \nabla\varphi - \nabla\psi_h = \nabla(\varphi_h - \varphi).$$

To obtain the estimate (28), we write

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathbf{curl}} \le \|\boldsymbol{w}_R - \Pi_h \boldsymbol{w}_R\|_{\mathbf{curl}} + \|\varepsilon\|^{1/2}_{L^\infty(\Omega)} \|\psi - \psi_h\|_{H^1(\Omega)}.$$

We then use (29), (31) and (26), recalling finally that one has

$$\|\boldsymbol{w}\|_{\boldsymbol{X}_\varepsilon} = \|\boldsymbol{w}\|_{\mathbf{curl}} \le \|\boldsymbol{v}_h\|_{\mathbf{curl}},$$

as the stability of the continuous splitting of $\boldsymbol{v}_h$ yields the last inequality. $\qquad\square$

Remark 11. Because the two splittings are sufficiently 'close' one to the other when h is small enough, we have that the discrete splitting (27) is stable, i.e. $\|\nabla\varphi_h\|_{\mathbf{curl}} \le C_{split} \|\boldsymbol{v}_h\|_{\mathbf{curl}}$ and $\|\boldsymbol{w}_h\|_{\mathbf{curl}} \le C_{split} \|\boldsymbol{v}_h\|_{\mathbf{curl}}$, with $C_{split} > 0$ independent of h and $\boldsymbol{v}_h$.

So, we can tackle the gradient part of $\boldsymbol{v}_h$ by transforming $\nabla\varphi_h$ into $-\nabla\varphi_h$.
To address the part of the discrete field $\boldsymbol{v}_h$ which is 'close' to $\boldsymbol{V}^-$, we then follow §3.2, applying the same procedure to $\boldsymbol{w}_h$. According to the basic approximability property for the edge finite element, we can find, for all h and for $0 \le \ell \le \ell_{max}$, $\boldsymbol{e}_{\ell,h} \in \boldsymbol{V}_h$ such that $\|\boldsymbol{e}_\ell - \boldsymbol{e}_{\ell,h}\|_{\mathbf{curl}} \le \delta(h)$, with δ depending only on ℓ_{max} and $\lim_{h\to 0} \delta(h) = 0$. The finite element space $\boldsymbol{V}_h^- := \operatorname{span}_{0\le\ell\le\ell_{max}}(\boldsymbol{e}_{\ell,h})$ is of dimension $\ell_{max} + 1$ when h is small enough. Moreover, $(\boldsymbol{e}_{\ell,h})_{0\le\ell\le\ell_{max}}$ can be chosen to be orthonormal and, defining the orthogonal projection operator P_h^- from $\boldsymbol{V}_h$ to $\boldsymbol{V}_h^-$, one has

$$|||\mathsf{P}^- - \mathsf{P}_h^-||| \le \delta(h), \ \lim_{h\to 0} \delta(h) = 0. \qquad (32)$$

Finally, we can define the discrete operator T_h^e in the vector case. Given $\boldsymbol{v}_h \in \boldsymbol{V}_h$, we split it as in (24-28): in particular, $\boldsymbol{v}_h = \nabla\varphi_h + \boldsymbol{w}_h$ and we set

$$\mathsf{T}_h^e(\boldsymbol{v}_h) := -\nabla\varphi_h + (\mathsf{I}_{\boldsymbol{V}_h} - 2\mathsf{P}_h^-)(\boldsymbol{w}_h).$$

In this case, due to the stability of the discrete splitting (27), we have obviously that $\mathsf{T}_h^e \in \mathcal{L}(\boldsymbol{V}_h)$ and there remains only to prove Corollary 1 in the electromagnetics case.

Theorem 5. *The discrete solution e_h converges to the exact solution e of the electromagnetics problem, with a convergence rate that is governed by (11).*

PROOF. Given $v_h \in V_h$, we compute

$$
\begin{aligned}
(\mathrm{T}^e - \mathrm{T}_h^e)v_h &= -\nabla\varphi + w - 2\mathrm{P}^- w + \nabla\varphi_h - w_h + 2\mathrm{P}_h^- w_h \\
&= \nabla(\varphi_h - \varphi) + (w - w_h) + 2(\mathrm{P}_h^- w_h - \mathrm{P}^- w) \\
&= 2(w - w_h) + 2\mathrm{P}_h^-(w_h - w) + 2(\mathrm{P}_h^- - \mathrm{P}^-)w.
\end{aligned}
$$

To obtain the last line, we used the equality $w - w_h = \nabla(\varphi_h - \varphi)$ (see the proof of Proposition 5). Hence, according to both (28) and (32), one has $\lim_{h\to 0} |||\mathrm{T}^e - \mathrm{T}_h^e||| = 0$. We conclude as in the proof of Theorem 3. $\square$

4.4. Discussions on the convergence rate for the electromagnetic wave equation

We assume that Ω is a Lipschitz polyhedron. Following §4.3, we retain the assumptions on the coefficient ε, with a partition $\mathcal{P} := \mathcal{P}(\varepsilon)$, etc. and we focus again on bounding from above the quantity $\inf_{v_h \in V_h} \|e - v_h\|_{\mathbf{curl}}$ in (11). To that aim, we decompose the solution e as $e = w_e + \nabla\varphi_e$, $w_e \in W_\varepsilon$, $\varphi_e \in H_0^1(\Omega)$ (cf. Proposition 2).
First, we remark that

$$
\operatorname{div}\varepsilon\nabla\varphi_e = \operatorname{div}\varepsilon e \overset{(19)}{=} -\frac{1}{\omega^2}\operatorname{div} f \text{ in } H^{-1}(\Omega).
$$

Hence, we can provide a bound for the curl-free, or electrostatic, part of the solution exactly as in §3.3, assuming that the data f belongs to $\boldsymbol{H}(\operatorname{div};\Omega)$. Indeed, for all $v_h \in V_h$, one has $\nabla v_h \in \boldsymbol{V}_h$ and also

$$
\|\nabla\varphi_e - \nabla v_h\|_{\mathbf{curl}} \leq \|\varepsilon\|_{L^\infty(\Omega)}^{1/2} \|\varphi_e - v_h\|_{H^1(\Omega)},
$$

so all the discussions and results of §3.3 carry over (replacing σ there by ε here). For instance, one derives estimates like (18), $\|\operatorname{div} f\|_{L^2(\Omega)}$ replacing $\|f\|_{L^2(\Omega)}$. About the divergence-ε-free part of the solution w_e, we note that $\mathbf{curl}\, w_e = \mathbf{curl}\, e$. In other words, the situation is 'close' to the one we addressed in Proposition 5 (replacing w there by w_e here), the only difference being the *a priori* smoothness of $(\mathbf{curl}\, w_e)_{|K}$. Let us investigate the consequences of this fact. We write, cf. (25), $w_e = w_{R,e} + \nabla\psi_e$, $w_{R,e} \in \boldsymbol{X}_\varepsilon \cap \boldsymbol{PH}^1(\Omega,\mathcal{P})$, $\psi_e \in \Psi(\varepsilon)$. In particular, the gradient part $\nabla\psi_e$ can be handled as the electrostatic part (without any assumption on f other than $f \in \boldsymbol{L}^2(\Omega)$), which leads again to estimates similar to (18), with $\|f\|_{\boldsymbol{L}^2(\Omega)}$ now replacing $\|f\|_{L^2(\Omega)}$. Last, about the piecewise smooth part $w_{R,e}$ of the solution, we remark that

$$
\nu\,\mathbf{curl}\, w_{R,e} \in \boldsymbol{Y}_{\nu^{-1}} := \{w \in \boldsymbol{H}(\mathbf{curl};\Omega) \ : \ \operatorname{div}(\nu^{-1}w) = 0, \ \nu^{-1}w \cdot n_{|\partial\Omega} = 0\}.
$$

To obtain error estimates for this last part of the solution, we would like to apply Proposition 8. For that, we need that ν be piecewise constant (or smooth), and that $\boldsymbol{Y}_{\nu^{-1}}$ be continuously embedded in $\boldsymbol{PH}^{t'}(\Omega,\mathcal{P}')$ for some $t' > 1/2$, where the partition here depends on ν, i.e. $\mathcal{P}' := \mathcal{P}'(\nu^{-1})$. On the other hand, we know that $w_{R,e} \in \boldsymbol{PH}^1(\Omega,\mathcal{P})$.

18

Remark 12. To be able to infer local estimates from Proposition 8, we choose compatible meshes with respect to *both* partitions $\mathcal{P}$ and $\mathcal{P}'$.

But, we know from [11, Theorem 3.5] that any element of $\boldsymbol{Y}_{\nu^{-1}}$ can be decomposed similarly to those of $\boldsymbol{X}_\varepsilon$, which leads to expressions like (25)-(26). Hence, handling the piecewise smooth part of $\nu^{-1}\,\mathbf{curl}\,\boldsymbol{w}_{R,\boldsymbol{e}}$ is no difficulty, but we need that

$$\{\theta \in H^1(\Omega) \ : \ \mathrm{div}(\nu^{-1}\nabla\theta) \in L^2(\Omega), \ \nu^{-1}\frac{\partial\theta}{\partial\boldsymbol{n}}_{|\partial\Omega} = 0\}$$

be continuously embedded in $PH^{1+t'}(\Omega, \mathcal{P}')$ for some $t' > 1/2$.
This is the case if ν^{-1} is (globally) smooth, i.e. $\nu^{-1} \in W^{1,\infty}(\Omega)$. More generally, admissible configurations are discussed at length by Costabel *et al.* We refer for instance to [11, Theorem 7.1].
We conclude that, given admissible configurations, one has

$$\hat{\mathrm{E}} \qquad \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{e} - \boldsymbol{v}_h\|_{\mathbf{curl}} \leq C \left(\|\mathrm{div}\,\boldsymbol{f}\|_{L^2(\Omega)}\,h^{\mathsf{s}} \right.$$

$$\left. + \|\boldsymbol{f}\|_{L^2(\Omega)}\,h^{\min(\mathsf{s},\mathsf{s}')} \right), \ C > 0 \text{ indep. of } \boldsymbol{f} \text{ and } \boldsymbol{e}.$$

Above, the estimate holds for $\mathsf{s} < \mathsf{s}_{\max}$ and $\mathsf{s}' < \mathsf{s}'_{\max}$, where we have from the previous analyses $\mathsf{s}_{\max} := \mathsf{s}_{\max}(\Omega, \varepsilon) > 0$ and $\mathsf{s}'_{\max} := \mathsf{s}'_{\max}(\Omega, \nu^{-1}) > 1/2$.

Remark 13. When the coefficient ν^{-1} yields singular behaviors, that is when $\mathsf{s}'_{\max}(\Omega, \nu^{-1}) < 1/2$, one can try and reverse the roles of ε and ν^{-1} by solving the time-harmonic problem expressed in the magnetic field $\boldsymbol{h}$.

References

[1] Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional non-smooth domains. Math. Meth. Appl. Sci. **21**, 823–864 (1998)

[2] Aubin, J.P.: Approximation of elliptic boundary-value problems. Wiley-Interscience, New York, USA (1972)

[3] Birman, M.S., Solomyak, M.Z.: Maxwell operator in regions with nonsmooth boundaries. Sib. Math. J. **28**, 12–24 (1987)

[4] Bonnet-Ben Dhia, A.-S., Ciarlet Jr., P., Zwölf, C.-M.: Time harmonic wave diffraction problems in materials with sign-shifting coefficients. J. Comput. Appl. Math. **234**, 1912–1919, Corrigendum 2616 (2010). *Eighth International Conference on Mathematical and Numerical Aspects of Waves (Waves 2007)*.

[5] Brenner, S., Scott, L.R.: The mathematical theory of finite element methods, 3rd Edition. Springer Verlag (2008)

[6] Buffa, A., Costabel, M., Schwab, C.: Boundary element methods for Maxwell's equations on non-smooth domains. Numer. Math. **92**, 679–710 (2002)

[7] Chung, E.T., Ciarlet, Jr., P.: Scalar transmission problems between dielectrics and metamaterials: T-coercivity for the Discontinuous Galerkin approach. Technical Report, CUHK-2011-01, Chinese University of Hong Kong, Hong Kong (2011)

[8] Ciarlet, P.G.: The finite element method for elliptic problems, *Classics in Applied Mathematics*, vol. 40. SIAM (2002)

[9] Ciarlet, Jr., P., Zou, J.: Fully discrete finite element approaches for time-dependent Maxwell's equations. Numer. Math. **82**, 193–219 (1999)

[10] Costabel, M., Dauge, M.: Singularities of electromagnetic fields in polyhedral domains. Arch. Rational Mech. Anal. **151**, 221–276 (2000)

[11] Costabel, M., Dauge, M., Nicaise, S.: Singularities of Maxwell interface problems. Math. Mod. Num. Anal. **33**, 627–649 (1999)

[12] Demkowicz, L.: Asymptotic convergence in finite and boundary element methods. Part I: theoretical results. Comput. Math. Applic. **27**, 69–84 (1994)

[13] Demkowicz, L.: Asymptotic convergence in finite and boundary element methods. Part II: the LBB constant for rigid and elastic scattering problems. Comput. Math. Applic. **28**, 93–109 (1994)

[14] Ern, A., Guermond, J.-L.: Theory and practice of finite elements. Springer Verlag (2004)

[15] Gross, P., Kotiuga, P.: Electromagnetic theory and computation: a topological approach. MSRI Publications Series. Cambridge University Press (2004)

[16] Jerison, D.S., Kenig, C.E.: The inhomogeneous Dirichlet problem in Lipschitz domains. J. Functional Analysis **130**, 161–219 (1995)

[17] Monk, P.: Finite element methods for Maxwell's equations. Oxford University Press, Oxford, G. B. (2003)

[18] Monk, P., Demkowicz, L.: Discrete compactness and the approximation of Maxwell's equations in $\mathbb{R}^3$. Math. Comp. **70**(234), 507–523 (2001)

[19] Nédélec, J.C.: Mixed finite elements in $\mathbb{R}^3$. Numer. Math. **35**, 315–341 (1980)

[20] Weber, C.: A local compactness theorem for Maxwell's equations. Math. Meth. Appl. Sci. **2**, 12–25 (1980)

Appendix

Here, we recall the construction of edge finite elements. To fix ideas, we consider that Ω is a polyhedron, which is triangulated by a regular family of meshes $(\mathcal{T}_h)_h$, made up of tetrahedra. Denoting by K a tetrahedron, by h_K its diameter and by $h := \max_K h_K$ the meshsize, we introduce Nédélec's $H(\mathbf{curl}, \Omega)$-conforming (first family, first order) finite element spaces

$$\boldsymbol{V}_h^+ := \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathbf{curl}, \Omega) \,:\, \boldsymbol{v}_{h|K} \in \mathcal{R}_1(K), \ \forall K \in \mathcal{T}_h\}, \ \ \boldsymbol{V}_h := \boldsymbol{V}_h^+ \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega),$$

where $\mathcal{R}_1(K)$ is the six-dimensional vector space of polynomials on K defined by

$$\mathcal{R}_1(K) := \{\boldsymbol{v} \in (P_1(K))^3 \; : \; \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}, \; \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}.$$

It is shown in [19, Theorem 1] that any element $\boldsymbol{v}$ in $\mathcal{R}_1(K)$ is uniquely determined by the degrees of freedom in the moment set $M_E(\boldsymbol{v})$:

$$M_E(\boldsymbol{v}) := \left(\int_e \boldsymbol{v} \cdot \tau \, dl\right)_{e \in A_K}.$$

Above, A_K is the set of edges of K, and τ is a unit vector along the edge e.
To define a suitable interpolation operator Π_h^+ on $\boldsymbol{V}_h^+$ (resp. Π_h on $\boldsymbol{V}_h$), we recall first that moments in $M_E(\boldsymbol{v})$ have a meaning provided that $\boldsymbol{v}$ belongs to

$$\boldsymbol{X}_p(K) := \{\boldsymbol{v} \in \boldsymbol{L}^p(K) \; : \; \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(K), \; \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{L}^p(\partial K)\}, \text{ for some } p > 2.$$

This result is proved in [1, Lemma 4.7]. Due to classical Sobolev embedding theorems, one can show that if $\boldsymbol{v} \in \boldsymbol{H}^t(K)$ for some $t > 1/2$, then there exists $p := p(t) > 2$ such that $\boldsymbol{v} \in \boldsymbol{L}^p(K)$, $\boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{L}^p(\partial K)$.

Proposition 6. *Assume that $\boldsymbol{v}$ and $\mathbf{curl}\,\boldsymbol{v}$ belong to $\boldsymbol{H}^t(K)$ for some $t > 1/2$, then its moments $M_E(\boldsymbol{v})$ are well-defined.*

One introduces the local interpolation operator

$$\Pi_K \; : \; \boldsymbol{X}_p(K) \to \mathcal{R}_1(K),$$

where, given $\boldsymbol{v} \in \boldsymbol{X}_p(K)$, $\Pi_K \boldsymbol{v}$ is by definition the only element of $\mathcal{R}_1(K)$ with moments equal to $M_E(\boldsymbol{v})$. Then, one defines the global interpolation operator Π_h^+ with values in $\boldsymbol{V}_h^+$ (resp. Π_h with values in $\boldsymbol{V}_h$), for all elements $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ (resp. $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$) such that $\boldsymbol{v}_{|K} \in \boldsymbol{X}_p(K)$ for all $K \in \mathcal{T}_h$, by

$$(\Pi_h^{(+)}\boldsymbol{v})_{|K} := \Pi_K \boldsymbol{v}, \; \forall K \in \mathcal{T}_h.$$

Below, we consider specifically scalar finite element spaces V_h^+ and V_h defined *via* P_1 Lagrange finite elements over $\mathcal{T}_h$. By construction, $\nabla V_h^{(+)} \subset \boldsymbol{V}_h^{(+)}$.
The next result is proved in [19, Lemma 3].

Proposition 7. *Given $\varphi \in H^1(K)$, if $\Pi_K(\nabla \varphi)$ is well-defined, then there exists $\varphi_K \in P^1(K)$ such that $\Pi_K(\nabla \varphi) = \nabla \varphi_K$.*

Last, one has the following approximability result, cf. [9, Lemmas 3.2 & 3.3].

Proposition 8. *Let $t \in\,]1/2, 1]$. There exists $C := C(t) > 0$ independent of K such that, for all $\boldsymbol{v} \in \{\boldsymbol{v}' \in \boldsymbol{H}^t(K) \; : \; \mathbf{curl}\,\boldsymbol{v}' \in \boldsymbol{H}^t(K)\}$, $\Pi_K \boldsymbol{v}$ exists and*

$$\|\boldsymbol{v} - \Pi_K \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, K)} \leq C \left(\|\boldsymbol{v}\|_{\boldsymbol{H}^t(K)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{H}^t(K)}\right) h_K^t.$$