Static Analysis of Sandwich Plates by Hybrid Finite Elements

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August 31, 1996

Abstract

As sandwich plates become more and more an important structural component, it would be essential to develop analysis tools taking their specificities into acount.
The present work concerns the development of hybrid sandwich finite elements modelling the mechanical behaviour of sandwich plates in shear in a more realistic way, and respecting physical phenomena at the interfaces between the skins and the core which have in general very heterogeneous mechanical properties.
These elements are initially composed of three sub-elements in the same manner as the sandwich plate. The hybrid formulation has been retained because it permits to define, in a simple way, a modelling corresponding to all required specifications with the minimum number of variables. In the direction of the thickness, stress and displacement fields are both linear in each layer. They both are quadratic in the others directions to take flexural effects into account.
We present in this paper several results and comparisons of static linear problems.

Keywords : Finite Elements Method, Hybrid Finite Element, Sandwich Plates, Composite Materials, Static Analysis

Notations

$\{\ \}$	vector (column)
$\langle\ \rangle$	transposed vector (row)
$[\ ]$	matrix
$\mathbb{O}$	zero matrix or zero vector
Ω	element's interior (volume or surface)
Γ	element's boundary (surface or curve) $\Gamma = \partial\Omega = \Gamma_u \cup \Gamma_\sigma$; $\Gamma_u \cap \Gamma_\sigma = \emptyset$
Γ_u	part of Γ where displacements are prescribed
Γ_σ	part of Γ where forces are prescribed
u	(small) displacements
σ	Cauchy's stresses
ε	(small) strains
H_{ijkl}	Hooke's matrix $\{\sigma\} = [H]\{\varepsilon\}$
$\overline{f}_\Omega$	value of prescribed forces in Ω
$\overline{T}$	value of prescribed forces on Γ_σ
$\overline{U}$	value of prescribed displacements on Γ_u
Γ_I	interface
$[\mathcal{L}]$	differential operator $\{\varepsilon\} = [\mathcal{L}]\{U\}$

nodal data :

q	displacement
L	Lagrange multiplier (stress)
f_u	equivalent force

1 Introduction

Sandwich plates have become more and more an important structural component in the field of transport, especially for the TGV Duplex and automobiles. With a view to the better use of this plate taking full advantage of its mechanical propreties, above all its stiffness and lightness in relatively sophisticated structures, it is therefore indispensable to make use of a powerful and efficient analysis tool. The subject of the present work concerns the development of hybrid sandwich finite elements in order to present better the mechanical behaviour in shear of sandwich plates, respecting physical phenomena at the interfaces between the skins and the core which have in general very heterogeneous properties.

These hybrid sandwich finite elements permit at the same time the balancing of the shear stress distribution through the thickness in a more realistic manner and present more correctly the phenomena of delamination or of cracks at the interfaces between the skins and the core in case of damage. They ensure the continuity of the displacement field and the equilibrium state at the interfaces at the same time.

A sandwich material is a laminated material with 3 layers, these layers having very different properties.

From a numerical point of view, taking only the stress distribution through the thickness, it does not exist any significant difference between laminated and sandwich plates. On the contrary, taking into account the specificity of the nature of the core, numerous physical differences appear :

- crack modes : shear rupture of the core;
- instability modes : crimping, wrinkling, dimpling;
- behaviour under local loading : local denting; :
- viscoelastic behaviour of the core : creep, damping properties.

Essential points related to laminated plates and/or particular to sandwich plates are the following :

- transverse shear effects must be taken into account : it is experimentally shown that sandwich plates often fails in delamination, so under transverse shear effects;
- continuity of displacements and equilibrium state of stresses at each interface of layers : these conditions stand for the mechanical cohesion at interfaces;
- the way each layer works due do the large difference between geometrical and mechanical propreties of layers;
- relatives magnitudes of stresses : the plane stress assumption used in the theory of laminated plates appears generally to be insufficient for sandwich plates.

The experience (and also experiments) shows that the normal component of stresses in the direction of thickness is not equal to zero, and that, even if its magnitude is still small related to the other components, this values is sufficiently important for the core whose crushing strength is not very large. Hence it is important to include this component in our study of sandwich plates.

In order to take into account the above-mentioned points, we develop a new familly of hybrid elements called hybrid sandwich elements.

2 Finite Elements Modelling

In a finite elements modelling, different choices can be done :

- displacements elements (will be referred to as classical elements);
- mixed elements;
- hybrid elements.

Classical elements only involve the displacement field, leading to good results for displacements. On the contrary, stresses computation implies a numerical derivative which is prejudicial to the numerical accuracy and the equilibrium equations at the interfaces are not satisfied and none of the stresses components is continuous at the interfaces.

The accurate computation of stresses is also important : mixed models in which both displacement and stress fields are independently approximated, can be seen as good candidates for the modelization. Nevertheless, they present some inconvenients :

- the number of unknowns becomes rapidly very high;
- some of the stress variables must be removed, not only on external faces, but also at interfaces because otherwise all stress components would be continuous at the interfaces;
- finally, obtained stiffness matrices are no more definite-positive, contrary to the classical method.

The most simple method to build a "minimal model", i.e. a model having the minimal number of required variables in displacement and stress is the hybride method : in this method, the stress field is only approximated along interfaces. As for the mixed method, obtained stiffness matrices are not definite-positive, but we do not face the critical problem of removing variables. For these reasons, this method has been retained.

The hybrid sandwich element developed here is initially composed of three sub-elements in the thickness, exactly as a sandwich material : the two extremes represents the skins and the one at the center does the core. In the thickness, stress and displacement fields both linear in each layer. They both are quadratic in the others directions to take flexural effects into account.

The considered hybrid sandwich element, divided in three sub-elements, is shown in a dispatched way, for its 2D version, in Figure (1).

This element is built on the principle of virtual work with extra interfaces relations introduced through Langrange multipliers $\{\lambda\}$.

We also work with the following veriationnal principle [2] :

$$\begin{aligned}\Pi \quad = \quad & \sum_{i=1}^{3}\left(\frac{1}{2}\int_{\Omega^i}\langle\varepsilon^i\rangle\left\{\sigma^i\right\}\,d\Omega^i - \int_{\Omega^i}\langle u^i\rangle\left\{\overline{f}^i_\Omega\right\}\,d\Omega^i - \int_{\Gamma^i_\sigma}\langle u^i\rangle\left\{\overline{T}^i\right\}\,d\Gamma^i_\sigma\right)\\ & + \int_{\Gamma^1_I}\langle\lambda^1\rangle\left(\left\{u^1\right\}-\left\{u^2\right\}\right)\,d\Gamma^1_I + \int_{\Gamma^2_I}\langle\lambda^2\rangle\left(\left\{u^2\right\}-\left\{u^3\right\}\right)\,d\Gamma^2_I\end{aligned}$$

expressed only with displacements :

$$\delta\langle\varepsilon\rangle\left\{\sigma\right\} = \delta\langle[\mathcal{L}]^T\{u\}\rangle\,[H]\,\{[\mathcal{L}]\{u\}\}$$

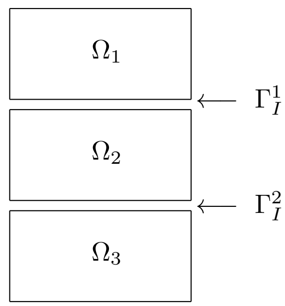


Figure 1: Sub-domains

and where boundary conditions for displacements are assumed to be satisfied explicitly by the choice of fonctions.

The variation $\delta\Pi$ yields :

$$\begin{aligned}\delta\Pi &= ...\\ &+\int_{\Gamma_I^1}(\sigma_{ij}^1 n_j^1-\lambda_i^1)\delta u_i^1+(\sigma_{ij}^2 n_j^2-\lambda_i^1)\delta u_i^2-(u_i^1-u_i^2)\delta\lambda_i\; d\Gamma_I^1\\ &+...\end{aligned}$$

where n_j^k are the components of the exterior normal to Ω^k on Γ_I^1. This proves that Langrage multipliers $\{\lambda\}$ are stresses. This method is known as hybrid displacement model.

Finally, the previous relation leads, for the 2D case which will be considered in the following for simplicity, to :

$$\begin{array}{c}\sigma_{11}n_x+\sigma_{12}n_y\\ \sigma_{21}n_x+\sigma_{22}n_y\end{array}$$

and, the normal being so that $n_x=0$ and $n_y=1$, we obtain, beyond the continuity of displacements at the interfaces, the continuity of the components σ_{12} and σ_{22} of stresses.

As for the component σ_{11}, it is computed in each sub-element using components σ_{12} and σ_{22} and mechanical properties of the sub-element. This component is also, *a priori* discountinuous at interfaces.

Using the following shape functions to approximate the different fields :

$$\begin{aligned}\{U^i\} &= [N_{u^i}]\{q^i\}\\ \{\lambda^i\} &= [N_{\lambda^i}]\{L^i\}\end{aligned}$$

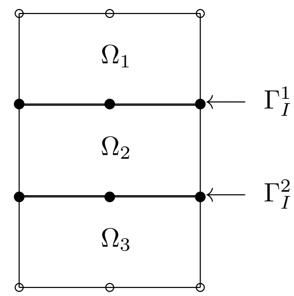
we reach the element shown in Figure (2).

Finally, the matricial system obtained is :

$$[\mathbb{K}]\{\mathbb{Q}\}=\{\mathbb{F}\} \tag{1}$$

with :

$$[\mathbb{K}]=\begin{bmatrix}[K_1] & [Q_{11}] & \mathbb{O} & \mathbb{O} & \mathbb{O}\\ [Q_{11}]^T & \mathbb{O} & [Q_{21}]^T & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & [Q_{21}] & [2K_2] & [Q_{22}] & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & [Q_{22}]^T & \mathbb{O} & [Q_{32}]^T\\ \mathbb{O} & \mathbb{O} & \mathbb{O} & [Q_{32}] & [K_3]\end{bmatrix}$$



∘ node in displacements
• node in displacements and stresses

Figure 2: Hybrid sandwich finite element

$$\{\mathbb{Q}\} = \left\{ \begin{array}{c} \{q^1\} \\ \{L^1\} \\ \{q^2\} \\ \{L^2\} \\ \{q^3\} \end{array} \right\}$$

$$\{\mathbb{F}\} = \left\{ \begin{array}{c} \{f_{u^1}\} \\ \mathbb{O} \\ \{f_{u^2}\} \\ \mathbb{O} \\ \{f_{u^3}\} \end{array} \right\}$$

and :

$$[K_i] = \int_{\Omega_i} \left([\mathcal{L}]\,[N_{u_i}]\right)^T \left[H^i\right] \left([\mathcal{L}]\,[N_{u^i}]\right)\; d\Omega_i \tag{2}$$

$$\{f_{u^i}\} = \int_{\Omega^i} [N_{u^i}]^T \left\{\overline{f}^i_\Omega\right\}\; d\Omega^i + \int_{\Gamma_{\sigma^i}} [N_{u^i}]^T \left\{\overline{T}\right\}\; d\Gamma_{\sigma^i} \tag{3}$$

$$[Q_{ij}] = (-1)^{1+\delta_{ij}} \int_{\Gamma_I^j} [N_{u^i}]^T \left[N_{\lambda^j}\right]\; d\Gamma_I^j \tag{4}$$

Terms (2) and (3) are the same as in the classical formulation, whereas terms of equation (4) relate displacements to stresses (hybrid terms).

3 Results

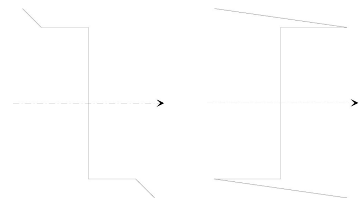
In this section, we will expose the kinds of results which can be obtained using hybrid sandwich finite elements.

Figure (3) shows different shapes of stress distribution through the thickness of a sandwhich which can be obtained with the classical method.

Figure (4) presents the same thing but obtained using hybrid sandwich elements.

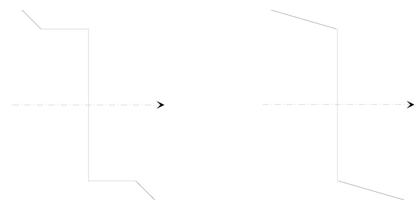
Hence it is clear that the classical method cannot ensure the continuity of any component σ_y nor σ_{xy} of stresses at interfaces.

It can be noticed that the use of hybrid sandwich elements oblige to impose boundary conditions for stresses : i.e. it is also possible to specify that stresses must equal zero



σ_x, σ_y or σ_{xy} all discontinuous

Figure 3: Results obtained with displacements elements



σ_x discontinuous σ_y or σ_{xy} continuous

Figure 4: Results obtained with hybrid sandwich elements

at a free end for example. A classical calculation would show stresses vanishing rapidly near this free end, but never equal zero.

We present a study in order to illustrate the efficiency of developed hybride sandwich finite elements : a contilever beam given in Figure (5).

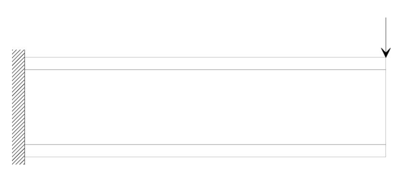
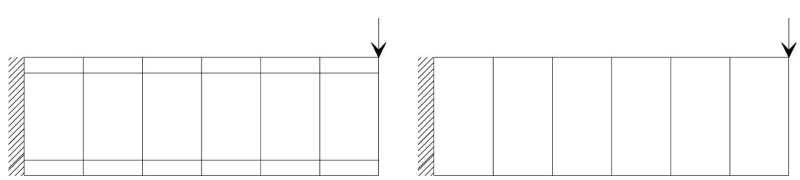


Figure 5: Cantilever beam



Displacement elements Hybrid sandwich elements

Figure 6: Meshing

Let n be the number of hybrid sandwich elements used as defined in Figure (6), then the number of degrees of freedom is :

displacements	:	$28 + 18(n-1)$
sandwich	:	$40 + 26(n-1)$ from which :
		$28 + 18(n-1)$ displacements
		$12 + 8(n-1)$ stresses

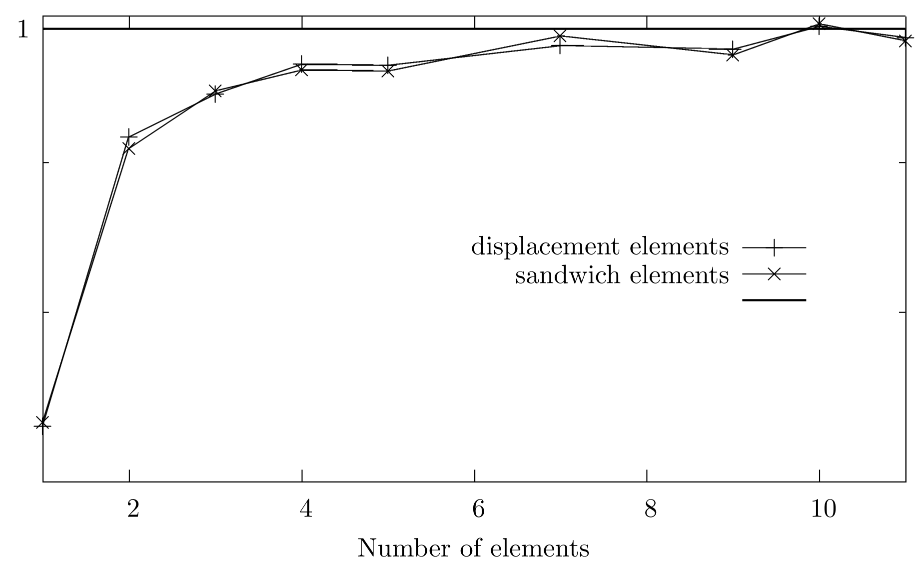


Figure 7: Convergence test

Figure (7) shows the convergence rate of displacements ralated to the theoretical solution. Both methods are seen to converge in a very similar way : this is normal, because matrices $[K_i]$ and vectors $\{f_{u^i}\}$ given in equations (2) and (3) are similar in both formulations.

4 Conclusion- Perspectives

The formulation of such a hybrid sandwich finite element takes into account the specificity of these very-heterogeneous materials. Its stiffness matrix is symetric and definite. But we can mention that this type of element (and also structures modelled by it) is compatible with the majority of used solvers.

The stresses at the interfaces can be determined, and their components σ_y and σ_{xy} are continuous. They correspond to the stresses verifying the equilibrium state at the interfaces.

The determination of these stresses at the interfaces can be of particular importance when introducing stress-based rupture criteria.

Moreover, it is possible to increase the number of nodal variables by doubling the displacement variables at the interfaces nodes : it permits, when introducing crack criteria, to simulate in a simple way the eventual disbond of different layers of the sandwich as illustrated Figure (8)

As it is said in [3], the sandwich plate is inclined to fail under local buckling : i.e. a form of buckling involving local delamination.The possibility of introducing, just by

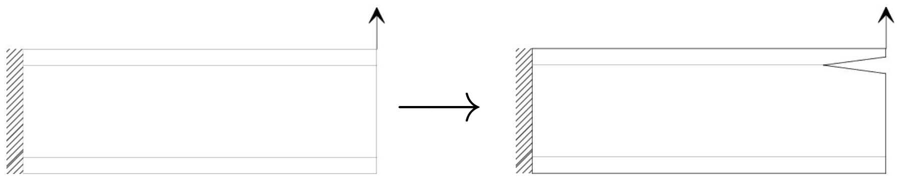


Figure 8: Delamination of layers

modifying a simple parameter, this phenomenon in our sandwich element, allows the computation of the foresee situations that could be omitted in more classical calculations.

References

[1] Lerooy, J.-F., "Calcul des contraintes de cisaillement transversales dans les structures modérément épaisses", Ph.D Thesis, Institut National Polytechnique de Lorraine, 1983.

[2] Washizu, K., "Variational methods in elasticity and plasticity", Third Edition, Pergamon Press, 1992.

[3] Teti, R. et Caprino, G., "Mechanical behavior of structural sandwiches", Sandwich Construction, pp. 53-67, 1989.