



HAL
open science

A vector bundle proof of Poncelet's closure theorem

Jean Vallès

► **To cite this version:**

Jean Vallès. A vector bundle proof of Poncelet's closure theorem. *Expositiones Mathematicae*, 2012, 30, pp.399-405. hal-00664308v1

HAL Id: hal-00664308

<https://hal.science/hal-00664308v1>

Submitted on 31 Jan 2012 (v1), last revised 23 Jan 2019 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A vector bundle proof of Poncelet theorem

Jean Vallès *

January 31, 2012

Abstract

In the town of Saratov where he was prisoner, Poncelet, continuing the work of Euler and Steiner on polygons simultaneously inscribed in a circle and circumscribed around another circle, proved the following generalization.

Theorem *Let C and D be two smooth conics in $\mathbb{P}^2(\mathbb{C})$. If D passes through the $\binom{n}{2}$ vertices of a complete polygon with n sides tangent to C then D passes through the vertices of infinitely many such polygons.*

According to Berger [Be] this theorem is the nicest result about the geometry of conics. Even if it is, there are few proofs of it. To my knowledge there are only three. The first proof, published in 1822 and based on infinitesimal deformations, is due to Poncelet ([Po]). Later, Jacobi proposed a new proof based on finite order points on elliptic curves; his proof, certainly the most famous, is explained in a modern way and in detail by Griffiths and Harris ([GH]). In 1870 Weyr proved a Poncelet theorem in space (more precisely for two quadrics) that implies the one above when one quadric is a cone; this proof is explained by Barth and Bauer ([BB]).

Our aim in this short note is to involve vector bundles techniques to propose a new proof of this celebrated result. Poncelet did not appreciate Jacobi's for the reason that it was too far from the geometric intuition. I guess that he would not appreciate our proof either for the same reason.

1 Preliminaries

In all this text the ground field is \mathbb{C} . A set of n vertices connected by n distinct lines form a n -gon, when a set consisting of n distinct lines with their $\binom{n}{2}$ vertices form a complete n -gon.

We say that a n -gon (respectively a complete n -gon) is inscribed in a given curve if this curve passes through the n -vertices (respectively the $\binom{n}{2}$ vertices). We say that a n -gon, or a complete n -gon, is circumscribed around a smooth conic C if the sides of the polygon, i.e. the n lines, are tangent to the C .

*Author partially supported by ANR-09-JCJC-0097-0 INTERLOW and ANR GEOLMI

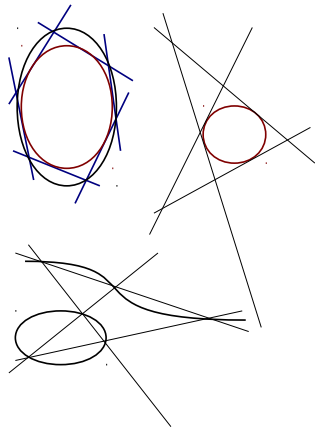


Figure 1: A complete 5-gon inscribed in a cubic, a complete 4-gon circumscribed around a conic and a pentagon simultaneously inscribed in a conic and circumscribed around another

2 Schwarzenberger bundles

First of all let us introduce a vector bundle $E_{n,C}$ naturally associated to any set of n lines tangent to a fixed smooth conic $C \subset \mathbb{P}^2$ (they were defined by Schwarzenberger in [S]). A set of n lines tangent to C corresponds by projective duality to a degree n divisor on the dual conic $C^\vee \subset \mathbb{P}^{2\vee}$. According to the isomorphism $C^\vee \simeq \mathbb{P}^1$ we can consider the subvariety $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ defined by the equation $x_0u^2 + x_1uv + x_2v^2 = 0$ ((x_0, x_1, x_2) are the homogeneous coordinates on \mathbb{P}^2 and (u, v) the homogeneous coordinates on \mathbb{P}^1) and the projections p and q respectively on \mathbb{P}^2 and \mathbb{P}^1 .

$$\begin{array}{ccc} X & \xrightarrow{q} & \mathbb{P}^1 \\ p \downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

The variety X is a double cover of \mathbb{P}^2 ramified along C . If $x \in \mathbb{P}^1$ then $p(q^{-1}(x))$ is a line in \mathbb{P}^2 tangent to C . If, instead of considering a point, we are considering a line bundle on \mathbb{P}^1 we will find a vector bundle of rank two on \mathbb{P}^2 by taking the direct image of its inverse image. Moreover, following Schwarzenberger, we know a very explicit resolution of this bundle. Indeed, tensorizing the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-1, -2) \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

is surjective, there exists a non-zero section $t \in H^0(E_{n,C})$ (i.e. another n -gon) such that the determinant

$$\mathcal{O}_{\mathbb{P}^2}^2 \xrightarrow{(s,t)} E_{n,C}$$

is the equation of S . This proves the theorem since any linear combination of s and t vanish along the vertices of a complete n -gon. \square

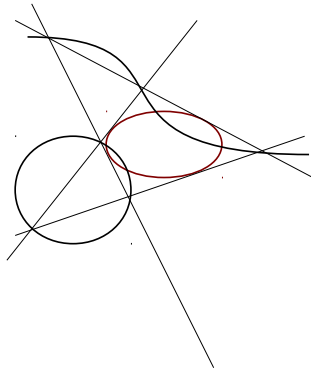


Figure 2: A complete circumscribed 4 gon and a cubic Poncelet curve

These curves described by Darboux are called *Poncelet curves*. When $n = 5$ they are the so-called Luröth quartics (see [OS]).

4 Poncelet theorem

Let us now consider n -gons that are simultaneously inscribed in a smooth conic and circumscribed around a other one. For these configurations Poncelet proved ([Po], page 362) :

Theorem 4.1. *Let $C \subset \mathbb{P}^2$ and $D \subset \mathbb{P}^2$ be two smooth conics such that there exist one n -gon inscribed in D and circumscribed around C . Then there are infinitely many of such n -gons.*

Proof. Let us consider one such n -gon. It is the union of n lines l_1, \dots, l_n with the $\binom{n}{2}$ vertices $l_i \cap l_j$ for $1 \leq i, j \leq n$ and $i \neq j$. There is a non-zero section $s \in H^0(E_{n,C})$ vanishing along the vertices of these lines. We denote by $Z(s)$ the set of these vertices.

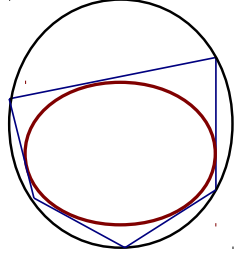


Figure 3: pentagon inscribed and circumscribed

Let us tensorize the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow E_{n,C} \longrightarrow \mathcal{I}_{Z(s)}(n-1) \longrightarrow 0$$

by \mathcal{O}_D . Since $D \cap Z(s)$ consists in n -points, it induces the following decomposition of $E_{n,C}$ along D :

$$E_{n,C} \otimes \mathcal{O}_D = \mathcal{O}_D\left(\frac{n-2}{2}\right) \oplus \mathcal{O}_D\left(\frac{n}{2}\right).$$

According to this decomposition, we consider the following exact sequence :

$$0 \longrightarrow F \longrightarrow E_{n,C} \longrightarrow \mathcal{O}_D\left(\frac{n-2}{2}\right) \longrightarrow 0$$

where F is a rank two vector bundle over \mathbb{P}^2 . Taking the cohomology long exact sequence we verify immediately that $h^0(F) \geq 2$. Then, let us consider a pencil of sections of F and also the pencil of sections of $E_{n,C}$ induced by it. We obtain a commutative diagram:

$$\begin{array}{ccccccc} \mathcal{O}_{\mathbb{P}^2}^2 & \xlongequal{\quad} & \mathcal{O}_{\mathbb{P}^2}^2 & & & & \\ \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & F & \longrightarrow & E_{n,C} & \longrightarrow & \mathcal{O}_D\left(\frac{n-2}{2}\right) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{L}_2 & \longrightarrow & \mathcal{O}_D\left(\frac{n-2}{2}\right) \longrightarrow 0. \end{array}$$

The sheaf \mathcal{L}_2 is supported by a curve Γ_2 of degree $(n-1)$ that is the determinant of a pencil of sections of $E_{n,C}$. This curve Γ_2 is a Poncelet curve. Then a general

point on Γ_2 is a vertex of a complete n -gon inscribed in Γ_2 and circumscribed around C . Moreover any intersection point of the n lines forming the n -gon with Γ_2 is a vertex of this n -gon (it is clear by Bézout theorem since $n(n-1) = 2 \times \binom{n}{2}$). Let Γ_1 be the curve supporting the sheaf \mathcal{L}_1 . We have of course $\Gamma_2 = D \cup \Gamma_1$. Then D is an irreducible component of a Poncelet curve and by the way any general point on D is the vertex of complete n -gon inscribed in Γ_2 . Then this configuration meets the conic D in at least (because there are n lines) and at most (because they are vertices and the decomposition of the bundle along D is fixed) n -points, so exactly n -points, each counting doubly. □

References

- [BB] Barth, W., Bauer, Th. : Poncelet theorems. Expo. Math. 14-2 (1996), 125–144.
- [Be] Berger, M. Géométrie vivante ou l'échelle de Jacob. Cassini (2009).
- [BKOR] Bos, H.J.M., Kers, C., Oort, F., Raven, D.W. : Poncelet's closure theorem, its history, its modern formulation, a comparison of its modern proof with those by Poncelet and Jacobi, and some mathematical remarks inspired by these early proofs. Expo. Math. 5 (1987), 289–364.
- [Da] Darboux, G. : Principes de géométrie analytique. Paris, Gauthier-Villars, 1917.
- [GH] Griffiths, P., Harris, J. : On Cayley's explicit solution to Poncelet's porism. Enseign. Math. (2) 24 (1978), no. 1-2, 31–40.
- [OS] Ottaviani, G., Sernesi, E. : On the hypersurface of Lüroth quartics. Michigan Math. J. 59 (2010), no. 2, 365–394.
- [Po] Poncelet, J. V. : Traité des propriétés projectives des figures, Bachelier Libraire. Edition de 1822.
- [S] R. L. E. Schwarzenberger, Vector bundles on the projective plane, Proc. London Math. Soc. 11, (1961), 623–640.
- [Va] Vallès, J. : Fibrés de Schwarzenberger et coniques de droites sauteuses, Bull. Soc. Math. France 128, (2000), 433-449.