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# Performance analysis of LRT/GLRT-based array receivers for the detection of a known real-valued signal corrupted by noncircular interferences

Abdelkader Oukaci, Jean-Pierre Delmas and Pascal Chevalier

## Abstract

A performance analysis of likelihood ratio test (LRT)-based and generalized likelihood ratio test (GLRT)-based array receivers for the detection of a known real-valued signal corrupted by a potentially noncircular interference is considered in this paper. The distribution of the decision statistics associated with the LRT and GLRT is studied. This allows us to give exact closed-form expressions of the probability of detection ( $P_D$ ) and false alarm ( $P_{FA}$ ) for two LRT-based receivers. Then, asymptotic (with respect to the data length) closed-form expressions are given for  $P_D$  and  $P_{FA}$  for four GLRT-based receivers. Finally, in order to strengthen the obtained results, some illustrative examples are presented.

## Index Terms

Detection, likelihood ratio test (LRT), generalized likelihood ratio test (GLRT), receiver operating characteristics (ROC), noncircular, rectilinear, interference, widely linear.

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## I. INTRODUCTION

These last decades, the problem of the detection of a known signal with unknown parameters in the presence of noise plus interference (called total noise) whose covariance matrix is unknown has been an important problem which has received much attention. This occurs in applications such as radar, satellite localization or time acquisition in radio communications. Most of the proposed detectors available in the literature assume implicitly or explicitly a second order (SO) circular (or proper) total noise. However, in the aforementioned applications,

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SO noncircular (or improper) sources of interference may be potentially omnipresent<sup>1</sup>. Consequently, those detectors become suboptimal.

In SO noncircular context, note that some detectors have been introduced in the literature for this reason. However these detectors have been built under the restrictive condition of either a known signal with known parameters (e.g., [1], [2]) or a random signal [3]. Furthermore, the practical issue consisting to detect a known signal with unknown parameters in the presence of an arbitrary unknown SO noncircular total noise has been investigated to the best of our knowledge only in [4], [5] for completely and partially unknown propagation channels respectively. In these works, no comprehensive theoretical performance analysis of these GLRT detectors has been investigated. For example, only a Monte Carlo simulation exhibiting the non detection probability for a specific false alarm and signal to interference plus noise ratio (SINR) was presented in [5]. Note that most of the GLRT-based array receivers presented in [5] have been patented in [6] and [7] and some results of this paper have been presented in [8].

The purpose of this paper is to complement the list of the GLRT receivers presented in [5] and to present a comprehensive performance analysis of some of these detectors. The paper is organized as follows. The observation model and the statement of the problem are given in Section II. A review of the LRT and some GLRT detectors introduced in [5] and a new GLRT detector are given in Section III. A performance analysis of these detectors is presented and illustrated in Sections IV and V respectively. Note that this performance analysis applies to conventional LRT-based and GLRT-based array receivers for the detection of a known real-valued signal corrupted by a circular interference as well, which has never been reported in the literature.

## II. HYPOTHESES AND PROBLEM FORMULATION

### A. Hypotheses

Let us consider an array of  $N$  narrow-band sensors. Each sensor is assumed to receive a known linearly modulated digital signal<sup>2</sup> composed of  $K$  real-valued<sup>3</sup> known symbols  $a_k$ . This signal is corrupted by a potentially zero-mean noncircular total noise composed of interference and background noise. The pulse shape of this known signal is assumed to be a square root Nyquist filter. Under these assumptions, after matched filtering and sampling at the symbol rate<sup>4</sup> the vector of complex amplitudes of the signals at the output of these

<sup>1</sup>Note that the noncircular assumption is usual in communication systems but not in radar systems. However, with new generation of active digital radar, there is a renewal of waveform generation and processing approaches that includes noncircular waveforms.

<sup>2</sup>This signal may either correspond to a training sequence in a radio communication link, a binary coding signal over the coherent processing interval in radar applications, or a PN code over a symbol period for DS-CDMA networks or GPS systems.

<sup>3</sup>Note that this assumption is not so restrictive since rectilinear signals such as DS-BPSK signals in particular, are currently used in a large domain of practical applications. Extension to complex-valued symbols leads to more involved derivations for some GLRT receivers. Furthermore the performance gain of the optimal LRT and GLRT receivers with respect to the conventional ones are not so attractive compared to real-valued known symbols. But the analysis of Section IV extends straightforwardly.

<sup>4</sup>Note that the samples  $\mathbf{x}_k$  are sufficient statistics for the detection problem when the total noise is whitely Gaussian distributed only.

sensors,  $(\mathbf{x}_k)_{k=1,\dots,K}$  can be written as follows

$$\mathbf{x}_k = \rho_s e^{i\phi_s} a_k \mathbf{s} + \mathbf{n}_k, \quad (1)$$

where  $\mathbf{s}$  is the directional vector of the known signal, such that its first component is equal to one.  $\rho_s$  and  $\phi_s$  control the amplitude and the phase of the known signal on the first sensor respectively, and  $\mathbf{n}_k$  are the samples of the zero-mean total noise at the output of the matched filter.

### B. Second order statistics of the data

The SO statistics of the potentially noncircular data  $\mathbf{x}_k$  are defined by

$$\begin{aligned} \mathbf{R}_x(k) &\stackrel{\text{def}}{=} \mathbb{E}(\mathbf{x}_k \mathbf{x}_k^H) = \pi_s(k) \mathbf{s} \mathbf{s}^H + \mathbf{R}_n(k), \\ \mathbf{C}_x(k) &\stackrel{\text{def}}{=} \mathbb{E}(\mathbf{x}_k \mathbf{x}_k^T) = \pi_s(k) e^{2i\phi_s} \mathbf{s} \mathbf{s}^T + \mathbf{C}_n(k), \end{aligned}$$

where  $\pi_s(k) \stackrel{\text{def}}{=} \rho_s^2 a_k^2$  since  $a_k$  is deterministic and  $\mathbf{R}_n(k) \stackrel{\text{def}}{=} \mathbb{E}(\mathbf{n}_k \mathbf{n}_k^H)$ ,  $\mathbf{C}_n(k) \stackrel{\text{def}}{=} \mathbb{E}(\mathbf{n}_k \mathbf{n}_k^T)$ .

### C. Problem formulation

The problem addressed in this paper is the detection problem with two hypotheses  $H_0$  and  $H_1$ , respectively associated with the presence of total noise  $\mathbf{n}_k$  only and signal plus total noise in the data  $(\mathbf{x}_k)_{k=1,\dots,K}$  based on the LRT/GLRT.

$$\begin{aligned} H_0 : \quad \mathbf{x}_k &= \mathbf{n}_k, & k &= 1, \dots, K \\ H_1 : \quad \mathbf{x}_k &= \rho_s e^{i\phi_s} a_k \mathbf{s} + \mathbf{n}_k, & k &= 1, \dots, K. \end{aligned} \quad (2)$$

To derive GLRT-based receivers, we need the following theoretical assumptions<sup>5</sup> which are not necessarily verified or required in practice.

A.1: the matrices  $\mathbf{R}_n(k)$  and  $\mathbf{C}_n(k)$  do not depend on  $k$

A.2: the samples  $(\mathbf{n}_k)_{k=1,\dots,K}$  are independent zero-mean Gaussian and possibly noncircular.

Under these conditions, the deterministic parameters of the distribution of  $(\mathbf{x}_k)_{k=1,\dots,K}$  are  $(\rho_s, \phi_s)$ ,  $\mathbf{s}$  or  $\psi_s$  (if  $\mathbf{s}$  is totally unknown or parameterized by the direction of arrival (DOA)  $\psi_s$  respectively), and  $(\mathbf{R}_n, \mathbf{C}_n)$ . As each of these parameters may be either known or unknown, depending on the application, different GLRT-based receivers have been derived in [5].

Note that the GLRT receivers introduced and analyzed by Kelly [9] suppose that under the assumption  $H_1$ , free of signal components, called secondary data are available. This is in contrast to our binary hypothesis testing problem (2). So, the GLRT receivers introduced in [5] are not simple extensions of the GLRT receivers

<sup>5</sup>These assumptions are not critical in the sense that the GLRT-based receivers derived under these assumptions still provide good decision performance even if most of the latter are not verified in practice. For example, we will see in Section IV-A that the derived LRT receiver have similar performance with Gaussian and a BPSK interferer with the same second-order statistics.

by Kelly [9] to augmented vectors of observations. Consequently our performance analysis are not simple extensions of those presented in [9] either.

### III. REVIEW OF THE LRT AND GLRT RECEIVERS

#### A. Clairvoyant receivers

We first consider the unrealistic case of completely known parameters. According to the statistical theory of detection [10, Ch. 3], the optimal detector is the LRT receiver that consists in comparing to a threshold, the likelihood ratio  $\text{LR}(\mathbf{x}, K)$  defined by

$$\text{LR}(\mathbf{x}, K) \stackrel{\text{def}}{=} \frac{p[(\mathbf{x}_k)_{k=1,\dots,K}/H_1]}{p[(\mathbf{x}_k)_{k=1,\dots,K}/H_0]}.$$

With assumptions A.1 and A.2, it is straightforward to prove [5], that the LRT receiver decides  $H_1$  if the statistic  $\text{OPT}(\mathbf{x}, K)$  defined by

$$\text{OPT}(\mathbf{x}, K) \stackrel{\text{def}}{=} \tilde{\mathbf{w}}_o^H \hat{\mathbf{r}}_{\mathbf{x},a} \quad (3)$$

is greater than a specific threshold, where  $\tilde{\mathbf{w}}_o \stackrel{\text{def}}{=} \mathbf{R}_{\tilde{n}}^{-1} \tilde{\mathbf{s}}_\phi$  is the so-called widely linear spatial matched filter (SMF) [11],  $\tilde{\mathbf{s}}_\phi \stackrel{\text{def}}{=} [e^{i\phi_s} \mathbf{s}^T, e^{-i\phi_s} \mathbf{s}^H]^T$ ,  $\mathbf{R}_{\tilde{n}} \stackrel{\text{def}}{=} \text{E}[\tilde{\mathbf{n}}_k \tilde{\mathbf{n}}_k^H]$  with  $\tilde{\mathbf{n}}_k \stackrel{\text{def}}{=} [\mathbf{n}_k^T, \mathbf{n}_k^H]^T$  and  $\hat{\mathbf{r}}_{\mathbf{x},a} \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \tilde{\mathbf{x}}_k a_k$  where  $\tilde{\mathbf{x}}_k \stackrel{\text{def}}{=} [\mathbf{x}_k^T, \mathbf{x}_k^H]^T$ . In the particular case of an SO circular total noise ( $\mathbf{C}_n = \mathbf{0}$ ), the statistic  $\text{OPT}(\mathbf{x}, K)$  (3) reduces to the conventional one defined by

$$\text{CONV}(\mathbf{x}, K) \stackrel{\text{def}}{=} 2\Re[\mathbf{w}_c^H \hat{\mathbf{r}}_{\mathbf{x},a}]. \quad (4)$$

with  $\mathbf{w}_c \stackrel{\text{def}}{=} e^{i\phi_s} \mathbf{R}_n^{-1} \mathbf{s}$  and  $\hat{\mathbf{r}}_{\mathbf{x},a} \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k a_k$ .

#### B. GLRT receivers

In most situations of practical interest, the parameters  $(\rho_s, \phi_s)$  and  $(\mathbf{R}_n, \mathbf{C}_n)$  are unknown, while for some applications the directional vector  $\mathbf{s}$  is known, parameterized<sup>6</sup> or totally unknown (see applications given in [5]). Thereby, we resort to GLRT approach where we maximize  $p[(\mathbf{x}_k)_{k=1,\dots,K}; \boldsymbol{\theta}_1]$  and  $p[(\mathbf{x}_k)_{k=1,\dots,K}; \boldsymbol{\theta}_0]$  with respect to the unknown parameters  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_0$  under  $H_1$  and  $H_0$  respectively. We use the resulting LR (denoted  $\text{GLR}(\mathbf{x}, K)$ ) as a decision statistic. Depending on the unknown parameters  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_0$ , different expressions of  $L_G(\mathbf{x}, K) \stackrel{\text{def}}{=} 2 \ln[\text{GLR}(\mathbf{x}, K)]$  have been derived in [5] for  $\mathbf{s}$  known or totally unknown.

For  $\mathbf{s}$  known,  $(\rho_s, \phi_s)$  unknown only (case  $C_1$ ),  $\boldsymbol{\theta}_1 = (\rho_s, \phi_s)$  and  $\boldsymbol{\theta}_0 = \emptyset$ ,  $L_G(\mathbf{x}, K)$  is given by

$$L_{G_1}(\mathbf{x}, K) = K \hat{\mathbf{r}}_{\mathbf{x},a}^H \mathbf{R}_{\tilde{n}}^{-1} \mathbf{S} (\mathbf{S}^H \mathbf{R}_{\tilde{n}}^{-1} \mathbf{S})^{-1} \mathbf{S}^H \mathbf{R}_{\tilde{n}}^{-1} \hat{\mathbf{r}}_{\mathbf{x},a} \quad (5)$$

<sup>6</sup>In this case, the array is assumed to be perfectly calibrated. We suppose here that the directional vector  $\mathbf{s}$  depends only on a scalar-valued DOA  $\psi_s$ , as the extension to a multidimensional-valued DOA is straightforward.

with  $\mathbf{S} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{s} & \mathbf{0} \\ \mathbf{0} & \mathbf{s}^* \end{pmatrix}$ , whereas for  $(\rho_s, \phi_s)$  and  $(\mathbf{R}_n, \mathbf{C}_n)$  unknown (case  $C_2$ ),  $\boldsymbol{\theta}_1 = (\rho_s, \phi_s, \mathbf{R}_n, \mathbf{C}_n)$  and  $\boldsymbol{\theta}_0 = (\mathbf{R}_n, \mathbf{C}_n)$ , we get

$$L_{G_2}(\mathbf{x}, K) = \frac{K \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}^H \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \mathbf{S} \left( \mathbf{S}^H \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \mathbf{S} \right)^{-1} \mathbf{S}^H \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}}{1 - \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}^H \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}}, \quad (6)$$

with  $\widehat{\mathbf{R}}_{\tilde{\mathbf{x}}} \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \tilde{\mathbf{x}}_k \tilde{\mathbf{x}}_k^H$ .

For  $\mathbf{s}$  totally unknown,  $(\rho_s, \phi_s)$  unknown only (case  $C_3$ ),  $\boldsymbol{\theta}_1 = (\rho_s, \phi_s, \mathbf{s})$  and  $\boldsymbol{\theta}_0 = \emptyset$ , we get

$$L_{G_3}(\mathbf{x}, K) = K \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}^H \mathbf{R}_{\tilde{\mathbf{x}}}^{-1} \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a},$$

whereas for  $(\rho_s, \phi_s)$  and  $(\mathbf{R}_n, \mathbf{C}_n)$  unknown (case  $C_4$ ),  $\boldsymbol{\theta}_1 = (\rho_s, \phi_s, \mathbf{s}, \mathbf{R}_n, \mathbf{C}_n)$  and  $\boldsymbol{\theta}_0 = (\mathbf{R}_n, \mathbf{C}_n)$ ,  $L_G(\mathbf{x}, K)$  is given by

$$L_{G_4}(\mathbf{x}, K) = K \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}^H \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}.$$

Finally in the case where  $\mathbf{s}$  is parameterized by an unknown DOA  $\psi_s$  with arbitrary model of parametrization, it is straightforward to derive the GRLT from GLRT (5) and (6) for  $\mathbf{s}$  totally known. We obtain for  $(\mathbf{R}_n, \mathbf{C}_n)$  known (case  $C_5$ ) or unknown (case  $C_6$ ) respectively,  $\boldsymbol{\theta}_1 = (\rho_s, \phi_s, \psi_s)$  and  $\boldsymbol{\theta}_0 = \emptyset$  or  $\boldsymbol{\theta}_1 = (\rho_s, \phi_s, \psi_s, \mathbf{R}_n, \mathbf{C}_n)$  and  $\boldsymbol{\theta}_0 = (\mathbf{R}_n, \mathbf{C}_n)$ , the following expressions of  $L_G(\mathbf{x}, K)$  by maximization over  $\psi$

$$\begin{aligned} L_{G_5}(\mathbf{x}, K) &= \max_{\psi} \left( K \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}^H \mathbf{R}_{\tilde{\mathbf{x}}}^{-1} \mathbf{S}(\psi) \left( \mathbf{S}^H(\psi) \mathbf{R}_{\tilde{\mathbf{x}}}^{-1} \mathbf{S}(\psi) \right)^{-1} \mathbf{S}^H(\psi) \mathbf{R}_{\tilde{\mathbf{x}}}^{-1} \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a} \right) \\ L_{G_6}(\mathbf{x}, K) &= \frac{\max_{\psi} \left( K \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}^H \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \mathbf{S}(\psi) \left( \mathbf{S}^H(\psi) \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \mathbf{S}(\psi) \right)^{-1} \mathbf{S}^H(\psi) \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a} \right)}{1 - \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}^H \widehat{\mathbf{R}}_{\tilde{\mathbf{x}}}^{-1} \widehat{\mathbf{r}}_{\tilde{\mathbf{x}}, a}}, \end{aligned}$$

with  $\mathbf{S}(\psi) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{s}(\psi) & \mathbf{0} \\ \mathbf{0} & \mathbf{s}^*(\psi) \end{pmatrix}$ .

These six GLRT receivers are summarized in the following table for the reader convenience.

Cases	directional vector $\mathbf{s}$	source parameters	noise parameters
$C_1$	$\mathbf{s}$ known	$\rho_s, \phi_s$ unknown	$\mathbf{R}_n, \mathbf{C}_n$ known
$C_2$	$\mathbf{s}$ known	$\rho_s, \phi_s$ unknown	$\mathbf{R}_n, \mathbf{C}_n$ unknown
$C_3$	$\mathbf{s}$ totally unknown	$\rho_s, \phi_s$ unknown	$\mathbf{R}_n, \mathbf{C}_n$ known
$C_4$	$\mathbf{s}$ totally unknown	$\rho_s, \phi_s$ unknown	$\mathbf{R}_n, \mathbf{C}_n$ unknown
$C_5$	$\mathbf{s}$ parameterized	$\rho_s, \phi_s$ unknown	$\mathbf{R}_n, \mathbf{C}_n$ known
$C_6$	$\mathbf{s}$ parameterized	$\rho_s, \phi_s$ unknown	$\mathbf{R}_n, \mathbf{C}_n$ unknown

Table.1 Six analyzed GLRT receivers.

#### IV. THEORETICAL PERFORMANCE ANALYSIS

We present in this section a theoretical performance analysis of the aforementioned detectors where the domain of validity of our approximations are specified if necessary.

##### A. Clairvoyant receivers

To be able to quantify and to compare the performance of the previous clairvoyant receivers, we assume in this subsection that the total noise  $\mathbf{n}_k$  is composed of a BPSK interference with independent equiprobable symbols  $b_k \in \{-1, +1\}$ , plus background noise  $\mathbf{n}'_k$  uncorrelated with each other. Under these assumptions  $\mathbf{n}_k$  is written as

$$\mathbf{n}_k = \rho_1 e^{i\phi_1} b_k \mathbf{j}_1 + \mathbf{n}'_k,$$

where  $\mathbf{j}_1$  is the steering vector of the interference whose first component is equal to one.  $\rho_1$  and  $\phi_1$  control the amplitude and the phase of the interference on the first sensor, and  $(\mathbf{n}'_k)_{k=1, \dots, K}$  are spatially white zero-mean circularly Gaussian independent distributed random variables (RV) with  $E(\mathbf{n}'_k \mathbf{n}'_k^H) = \eta_2 \mathbf{I}$ .

The probability of detection and false alarm associated with the threshold  $\lambda$  are given respectively by

$$\begin{aligned} P_D &= P[\text{OPT}(\mathbf{x}, K) > \lambda/H_1] = P[\rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \tilde{\mathbf{w}}_o^H \tilde{\mathbf{s}}_\phi + \rho_1 \tilde{\mathbf{w}}_o^H \tilde{\mathbf{j}}_\phi \frac{1}{K} \sum_{k=1}^K a_k b_k + \frac{1}{K} \sum_{k=1}^K a_k \tilde{\mathbf{w}}_o^H \tilde{\mathbf{n}}'_k > \lambda] \\ &= P[\beta + \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k + n'_K > \lambda] \end{aligned} \quad (7)$$

$$\begin{aligned} P_{FA} &= P[\text{OPT}(\mathbf{x}, K) > \lambda/H_0] = P[\rho_1 \tilde{\mathbf{w}}_o^H \tilde{\mathbf{j}}_\phi \frac{1}{K} \sum_{k=1}^K a_k b_k + \frac{1}{K} \sum_{k=1}^K a_k \tilde{\mathbf{w}}_o^H \tilde{\mathbf{n}}'_k > \lambda] \\ &= P[\alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k + n'_K > \lambda] \end{aligned} \quad (8)$$

with  $\tilde{\mathbf{j}}_\phi \stackrel{\text{def}}{=} [e^{i\phi_1} \mathbf{j}_1^T, e^{-i\phi_1} \mathbf{j}_1^T]^T$ ,  $\alpha \stackrel{\text{def}}{=} \frac{\rho_1}{K} (\sum_{k=1}^K a_k^2)^{1/2} \tilde{\mathbf{w}}_o^H \tilde{\mathbf{j}}_\phi$  (real-valued),  $\beta \stackrel{\text{def}}{=} \rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \tilde{\mathbf{w}}_o^H \tilde{\mathbf{s}}_\phi = \rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \tilde{\mathbf{s}}_\phi^H \mathbf{R}_n^{-1} \tilde{\mathbf{s}}_\phi > 0$  and  $n'_K \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K a_k \tilde{\mathbf{w}}_o^H \tilde{\mathbf{n}}'_k$  which is a real-valued zero-mean Gaussian RV with variance  $\sigma^2 = \frac{\eta_2}{K^2} (\sum_{k=1}^K a_k^2) \|\tilde{\mathbf{w}}_o\|^2$ . For known interference symbols  $(b_1, \dots, b_K)$ ,  $P_D$  (7) and  $P_{FA}$  (8) are given by

$$P_D = Q\left(\frac{\lambda - \beta - \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k}{\sigma}\right) \quad \text{and} \quad P_{FA} = Q\left(\frac{\lambda - \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k}{\sigma}\right)$$

where  $Q(x) \stackrel{\text{def}}{=} \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$ . Conditioning these probabilities  $P_D$  and  $P_{FA}$  on the different equiprobable

symbols  $(b_1, \dots, b_K)$ , we obtain by the total probability formula

$$P_D = \frac{1}{2^K} \sum_{i=1}^{2^K} Q \left( \frac{\lambda - \beta - \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k^{(i)}}{\sigma} \right), \quad (9)$$

$$P_{FA} = \frac{1}{2^K} \sum_{i=1}^{2^K} Q \left( \frac{\lambda - \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k^{(i)}}{\sigma} \right), \quad (10)$$

where  $(b_1^{(i)}, \dots, b_K^{(i)})_{i=1, \dots, 2^K}$  denote the  $2^K$  different  $K$ -uplets of binary symbols  $(b_1, \dots, b_K)$ .

Expressions (9) and (10) of  $P_D$  and  $P_{FA}$  are valid for the conventional receiver (4) as well, with now  $\alpha \stackrel{\text{def}}{=} \frac{2\rho_1}{K} (\sum_{k=1}^K a_k^2)^{1/2} \Re(e^{i\phi_1} \mathbf{w}_c^H \mathbf{J}_1)$ ,  $\beta \stackrel{\text{def}}{=} 2\rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \Re(e^{i\phi_s} \mathbf{w}_c^H \mathbf{s}) = 2\rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \mathbf{s}^H \mathbf{R}_n^{-1} \mathbf{s} > 0$  and  $n'_K \stackrel{\text{def}}{=} \frac{2}{K} \Re(\sum_{k=1}^K a_k \mathbf{w}_c^H \mathbf{n}'_k)$  which is a zero-mean Gaussian RV with variance  $\sigma^2 = \frac{2\eta_2}{K^2} (\sum_{k=1}^K a_k^2) \|\mathbf{w}_c\|^2$ .

Numerical computation of (9) and (10) are computationally costly for large values of  $K$ . For these values, note that Chernoff-Stein's Lemma [12, Th. 11.8.1] which gives general expressions of false alarm, miss and error probabilities associated with the LRT as the number  $K$  of observations becomes large cannot be applied because the LRT is derived under the Gaussian distribution but is studied under the true distribution, which is a mixture of Gaussian distributions.

But hopefully the RV  $(\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k$ , where  $a_k$  and  $b_k$  are deterministic and random variables respectively, that appears (7) and (8) is a sum of independent RV. Consequently that allows us to apply central limit theorems. For BPSK symbols  $a_k$ ,  $(\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k = K^{-1/2} \sum_{k=1}^K a_k b_k$  is a sum of independent identically distributed RV and the classical central limit theorem applies. And for arbitrary real-valued symbols  $a_k$ , the RV  $a_k b_k$  are no longer identically distributed. But the sequence  $(\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k$  converges also in distribution to the zero-mean and unit variance Gaussian distribution because Liapounov's theorem [13, Th. 2.7.3] can be applied as the following condition is satisfied<sup>7</sup>

$$\left( \sum_{k=1}^K |a_k|^3 \right)^2 = o \left[ \left( \sum_{k=1}^K a_k^2 \right)^3 \right] \quad \text{when } K \rightarrow \infty. \quad (11)$$

Consequently for large values of  $K$ , the distribution of the RV  $(\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k$  can be approximated by a zero-mean and unit variance Gaussian distribution. In this case, the RV  $\alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k + n'_K$  that appear in (7) and (8) is approximately Gaussian distributed with zero-mean and variance  $\alpha^2 + \sigma^2$  because  $\alpha$  is deterministic and  $n'_K$  is a zero-mean Gaussian RV with variance  $\sigma^2$  independent of the RV  $b_k$ . Consequently

<sup>7</sup>Condition (11) is clearly satisfied if we assume the mild assumption that the limit  $\lim_{k \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K a_k^2$  exists and is finite non-zero and the limit  $\lim_{k \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K |a_k^3|$  exists.



from (7) and (8),  $P_D$  and  $P_{FA}$  become respectively<sup>8</sup>

$$P_D \approx Q\left(\frac{\lambda - \beta}{\sqrt{\alpha^2 + \sigma^2}}\right) \quad (12)$$

$$P_{FA} \approx Q\left(\frac{\lambda}{\sqrt{\alpha^2 + \sigma^2}}\right). \quad (13)$$

Noting that the Q function is monotonically decreasing, Q has an inverse that we denote as  $Q^{-1}$  and we obtain the following closed-form expression of the ROC of clairvoyant receivers (3) and (4) by elimination of the threshold  $\lambda = \sqrt{\alpha^2 + \sigma^2} Q^{-1}(P_{FA})$

$$P_D \approx Q\left(Q^{-1}(P_{FA}) - \sqrt{\text{SINR}}\right) \quad (14)$$

where  $\text{SINR} = \frac{\beta^2}{\alpha^2 + \sigma^2}$  represents the ratio between the square of the known signal part and the expected value of the square of the total noise part of the statistics  $\text{OPT}(\mathbf{x}, K)$  (3) and  $\text{CONV}(\mathbf{x}, K)$  (4) for the optimal and conventional clairvoyant receivers. We note that this SINR is  $K$  times the mean SINR (with respect to  $a_k$ ) given by

$$\frac{\rho_s^2 \left(\frac{1}{K} \sum_{k=1}^K a_k^2\right) |\tilde{\mathbf{w}}_o^H \tilde{\mathbf{s}}_\phi|^2}{\rho_1^2 |\tilde{\mathbf{w}}_o^H \tilde{\mathbf{j}}_\phi|^2 + \eta_2 \|\tilde{\mathbf{w}}_o\|^2} \quad \text{and} \quad \frac{\rho_s^2 \left(\frac{1}{K} \sum_{k=1}^K a_k^2\right) (\Re(e^{i\phi_s} \mathbf{w}_c^H \mathbf{s}))^2}{\rho_1^2 (\Re(e^{i\phi_1} \mathbf{w}_c^H \mathbf{j}_1))^2 + \frac{\eta_2}{2} \|\mathbf{w}_c\|^2}$$

at the output of the widely linear SMF  $\tilde{\mathbf{w}}_o$  and of the linear SMF  $\mathbf{w}_c$ , respectively. Computation and comparison of these SINR are done in [11] for BPSK signal of interest and not reported here for want of space.

## B. GLRT receivers

The exact distribution of  $\text{GLR}(\mathbf{x}, K)$  under  $H_0$  and  $H_1$  for the true distribution of the data appears to be challenging to derive. For example, the derivation of the distribution of simplest statistics (5) (w.r.t. the R.V.  $(\mathbf{x}_k)_{k=1,\dots,K}$ ) after conditioning on the symbols  $(b_1, \dots, b_K)$ , comes down to derive the distribution of the Hermitian form  $\mathbf{z}^H \boldsymbol{\Omega} \mathbf{z}$  where  $\mathbf{z}$  is a zero-mean circular Gaussian RV. Unfortunately, matrix  $\boldsymbol{\Omega}$  is not related to the covariance matrix of  $\mathbf{z}$ . Thus, there is no known closed-form expression<sup>9</sup> of the distribution of  $\mathbf{z}^H \boldsymbol{\Omega} \mathbf{z}$ . But asymptotically with respect to  $K$  and under the assumptions for which the GLRT has been derived, i.e., according to a noncircular Gaussian distribution of the data, some general statistical results can be applied, without having to know the exact form of  $\text{GLR}(\mathbf{x}, K)$ .

Depending on the unknown parameters  $\boldsymbol{\theta}$  among  $(\rho_s, \phi_s)$ ,  $\mathbf{s}$ ,  $\psi_s$  and  $(\mathbf{R}_n, \mathbf{C}_n)$ , six practical cases has been defined in Section III-B, for which we note that the PDF of  $(\mathbf{x}_k)_{k=1,\dots,K}$  under  $H_0$  and  $H_1$  is the same, except

<sup>8</sup>Note that approximation (13) can be questionable because the probabilities of the tails of the distributions are generally ill approximated by central limit theorems. But in practice, approximation (13) is satisfactory as it is shown in section V.

<sup>9</sup>We note that closed-form expressions of the cumulative function of the Hermitian form  $\sum_{k=1}^K a|z_{1,k}|^2 + b|z_{2,k}|^2 + cz_{1,k}z_{2,k}^* + c^*z_{1,k}^*z_{2,k}$  where  $(z_{1,k}, z_{2,k})_{k=1,\dots,K}$  are  $K$  independent couples of correlated circular Gaussian variables  $z_{1,k}$  and  $z_{2,k}$ , have been given in the literature, see e.g., [14]. But our generic Hermitian form  $\mathbf{z}^H \boldsymbol{\Omega} \mathbf{z}$  does not reduce to this one.

that the value of the unknown parameter vector  $\boldsymbol{\theta}$  is different. Decompose this general unknown parameter<sup>10</sup>  $\boldsymbol{\theta}$  in  $\boldsymbol{\theta}_s$  and  $\boldsymbol{\theta}_r$  that collects the unknown parameters among  $(\rho_s, \phi_s, \psi_s, \mathbf{s})$  and  $(\mathbf{R}_n, \mathbf{C}_n)$ , respectively.  $s$  and  $r$  represent the dimensions of the real-valued vectors  $\boldsymbol{\theta}_s$  and  $\boldsymbol{\theta}_r$ . Note that  $(\mathbf{R}_n, \mathbf{C}_n)$  is sometimes referred to as a nuisance parameter. For identifiability reasons, we must use reparameterizations, to get the parameters  $\boldsymbol{\theta}_s$  and  $\boldsymbol{\theta}_r$  that are given for each case by

$$\begin{aligned}\boldsymbol{\theta}_s &= [\rho_s \cos(\phi_s), \rho_s \sin(\phi_s)]^T \text{ with } s = 2 \text{ for } C_1 \text{ and } C_2, \\ \boldsymbol{\theta}_s &= [\Re(\rho_s e^{i\phi_s} \mathbf{s})^T, \Im(\rho_s e^{i\phi_s} \mathbf{s})^T]^T \text{ with } s = 2N \text{ for } C_3 \text{ and } C_4. \\ \boldsymbol{\theta}_s &= [\rho_s \cos(\phi_s), \rho_s \sin(\phi_s), \psi_s]^T \text{ with } s = 3 \text{ for } C_5 \text{ and } C_6,\end{aligned}$$

There is no nuisance parameter for  $C_1, C_3$  or  $C_5$  and

$$\boldsymbol{\theta}_r = [(\mathbf{R}_n)_{i,i}, \Re((\mathbf{R}_n)_{i,j}), \Im((\mathbf{R}_n)_{i,j}) \text{ for } 1 \leq i < j \leq N, \Re((\mathbf{C}_n)_{i,j}), \Im((\mathbf{C}_n)_{i,j}) \text{ for } 1 \leq i \leq j \leq N]^T \text{ with } r = N(2N + 1) \text{ for } C_2, C_4 \text{ and } C_6.$$

In the cases<sup>11</sup>  $C_1, C_2, C_3$  and  $C_4$ , detection problem (2) can be recast as the following composite hypothesis testing problem [15], [10, Ch. 6], which is a parameter test of the PDF  $p[(\mathbf{x}_k)_{k=1,\dots,K}; \boldsymbol{\theta}_s, \boldsymbol{\theta}_r]$ , where we wish to test if  $\boldsymbol{\theta}_s = \mathbf{0}$  as opposed to  $\boldsymbol{\theta}_s \neq \mathbf{0}$

$$\begin{aligned}H_0 &: p[(\mathbf{x}_k)_{k=1,\dots,K}; \boldsymbol{\theta}_s = \mathbf{0}, \boldsymbol{\theta}_r] \\ H_1 &: p[(\mathbf{x}_k)_{k=1,\dots,K}; \boldsymbol{\theta}_s \neq \mathbf{0}, \boldsymbol{\theta}_r].\end{aligned}\tag{15}$$

With these notations, GLR( $\mathbf{x}, K$ ) becomes

$$\text{GLR}(\mathbf{x}, K) = \frac{p[(\mathbf{x}_k)_{k=1,\dots,K}; \hat{\boldsymbol{\theta}}_{s_1}, \hat{\boldsymbol{\theta}}_{r_1}]}{p[(\mathbf{x}_k)_{k=1,\dots,K}; \hat{\boldsymbol{\theta}}_{s_0}, \hat{\boldsymbol{\theta}}_{r_0}]},$$

with  $\boldsymbol{\theta}_{s_0} = \mathbf{0}$  and  $(\hat{\boldsymbol{\theta}}_{s_1}, \hat{\boldsymbol{\theta}}_{r_1})$  and  $\hat{\boldsymbol{\theta}}_{r_0}$  are the maximum likelihood estimates of  $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_r)$  and  $\boldsymbol{\theta}_r$  under  $H_1$  and  $H_0$ , respectively.

Wilk's theorem with nuisance parameters [16, p.132] can be applied without having to know the exact form of GLR( $\mathbf{x}, K$ ). Thus the following convergence in distribution follows:

$$L_{G_i}(\mathbf{x}, K) \xrightarrow{\mathcal{L}} \chi^2(s) \quad \text{under } H_0 \text{ for cases } (C_i)_{i=1,2,3,4}\tag{16}$$

where  $\chi^2(s)$  denotes the chi-squared distribution with  $s$  degrees of freedom associated with  $C_i$ . Under  $H_1$ , the derivation of the asymptotic distribution of  $L_{G_i}(\mathbf{x}, K)$  is much more involved. Using a theoretical result by Stroud [17], Stuart *et al* [18, Ch. 23.7] have stated that when  $\boldsymbol{\theta}_s$  can take values<sup>12</sup> near  $\mathbf{0}$ ,  $L_{G_i}(\mathbf{x}, K)$  is

<sup>10</sup>We use here a vector-valued parameter, with a slight difference under  $H_0$  with respect to the definition given in Subsection III-B.

<sup>11</sup>In the cases  $C_5$  and  $C_6$ , detection problem (2) cannot be recast to the parameter test (15) because of the third component  $\psi_s$  of  $\boldsymbol{\theta}_s$  that could take the value 0 under  $H_1$ .

<sup>12</sup>The following more formal condition is given in [17],  $\boldsymbol{\theta}_s$  is embedded in an adequate sequence indexed by  $K$  that converges to zero at the rate  $K^{-1/2}$  or faster, i.e.,  $\|\boldsymbol{\theta}_s\| = O_p(1/K^{1/2})$ . Note the simplified condition given by Kay [10, A. 6A]:  $\|\boldsymbol{\theta}_s\| = c/\sqrt{K}$  for some constant  $c$ , that is reduced to the rough assumption of weak SINR (i.e.,  $K\rho_s^2/(\rho_1^2 + \eta_2) \ll 1$ ) [10, Section 6.5].

approximately distributed<sup>13</sup> as

$$L_{G_i}(\mathbf{x}, K) \overset{a}{\sim} \chi^2(s, \mu_K) \quad \text{under } H_1 \text{ for cases } (C_i)_{i=1,2,3,4} \quad (17)$$

where  $\chi^2(s, \mu_K)$  denotes the noncentral chi-squared distribution with  $s$  degrees of freedom and noncentrality parameter  $\mu_K$  associated with  $C_i$ . The dependance on  $K$  in  $\mu_K$  is emphasized to specify that (17) is an approximation and not a convergence in distribution. We will see in (19) that  $\mu_K$  is in fact proportional to  $K$ .  $\mu_K$  is a measure of discrimination between the two hypotheses, given by [15] and [10, Ch. 6]

$$\mu_K = \boldsymbol{\theta}_{s_1}^T \left[ \mathbf{I}_{s,s}(\mathbf{0}, \boldsymbol{\theta}_r) - \mathbf{I}_{s,r}(\mathbf{0}, \boldsymbol{\theta}_r) \mathbf{I}_{r,r}^{-1}(\mathbf{0}, \boldsymbol{\theta}_r) \mathbf{I}_{r,s}(\mathbf{0}, \boldsymbol{\theta}_r) \right] \boldsymbol{\theta}_{s_1}, \quad (18)$$

where  $\boldsymbol{\theta}_{s_1}$  is true values of  $\boldsymbol{\theta}_s$  under  $H_1$ ,  $\boldsymbol{\theta}_r$  is the true value of the parameter (which is the same under  $H_0$  and  $H_1$ ), and the terms in the brackets are given by partitioning the Fisher information matrix (FIM)  $\mathbf{I}_K(\boldsymbol{\theta})$  of  $(\mathbf{x}_k)_{k=1,\dots,K}$  for  $\boldsymbol{\theta} = (\boldsymbol{\theta}_s^T, \boldsymbol{\theta}_r^T)^T$  as

$$\mathbf{I}_K(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{I}_{s,s}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_r) & \mathbf{I}_{s,r}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_r) \\ \mathbf{I}_{r,s}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_r) & \mathbf{I}_{r,r}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_r) \end{pmatrix}.$$

The derivation of  $\mu_K$  for each case  $(C_i)_{i=1,\dots,6}$  are given in the Appendix. It is proved that  $\mu_K$  takes the following common value for cases  $C_1, C_2, C_3$  and  $C_4$

$$\mu_K = K \left( \frac{1}{K} \sum_{k=1}^K a_k^2 \right) \rho_s^2 \tilde{\mathbf{s}}_\phi^H \mathbf{R}_{\tilde{n}}^{-1} \tilde{\mathbf{s}}_\phi. \quad (19)$$

This shows that under the validity conditions of our analysis (i.e., weak SINR  $K\rho_s^2/(\rho_1^2 + \eta_2)$  and large data length  $K$ ),

- under both  $H_0$  under  $H_1$ , the asymptotic distributions of  $L_G(\mathbf{x}, K)$  are identical in the cases  $C_1$  and  $C_2$ , and also in the cases  $C_3$  and  $C_4$ . In other word this proves that the knowledge of the nuisance parameters  $(\mathbf{R}_n, \mathbf{C}_n)$  does not improve the performance for "large" data records. This can be interpreted by the accuracy of the maximum likelihood (ML) estimation of the nuisance parameter  $(\mathbf{R}_n, \mathbf{C}_n)$  that is relatively independent of the SINR in contrast to the ML estimation of the parameters  $(\rho_s, \phi_s, \mathbf{s})$ . Naturally, this property does not hold for "small" data records, as it has been shown in [5];
- comparing the cases  $(C_1, C_2)$  with  $(C_3, C_4)$ , the distributions of  $L_G(\mathbf{x}, K)$  under  $H_0$ , and under  $H_1$ , differ only by the degree  $s$  of freedom of the chi-squared distributions, that goes from 2 to  $2N$ . Observing the spreading of these distributions, we see that they more overlap for  $(C_3, C_4)$  cases, than for  $(C_1, C_2)$  cases. Consequently, the performance improves when the steering vector is known;

<sup>13</sup>The accurate formulation is  $\lim_{K \rightarrow \infty} \{P(2 \ln[\text{GLR}(\mathbf{x}, K)] < t) - P(V_K < t)\} = 0 \forall t$ , where  $V_K$  has a noncentral chi-squared distribution with  $s$  degrees of freedom and noncentrality parameter  $\mu_K$  that depends on the data length  $K$ .

- the proportionality of  $\mu_K$  to  $K$  (for fixed mean energy by symbol) implies that the distribution of  $L_G(\mathbf{x}, K)$  under  $H_1$  moves to the right and consequently the performance improves when  $K$  increases.

Naturally, the first two properties hold only under these validity conditions.

Since the asymptotic distribution of  $2 \ln[\text{GLR}(\mathbf{x}, K)]$  under  $H_0$  does not depend on any unknown parameter, the threshold required to maintain a constant  $P_{\text{FA}}$  can be a priori found. These detectors are constant false alarm rate (CFAR) detectors. But in general this CFAR property holds only for large data length ( $K \gg 1$ ). Similarly to the LRT receiver, the following asymptotic ROC is deduced from (15) and (16).

$$P_D \approx Q_{\chi^2_{s, \mu_K}} \left( Q_{\chi^2_{s, 0}}^{-1} (P_{\text{FA}}) \right), \quad (20)$$

where  $Q_{\chi^2_{s, 0}}(\cdot)$  and  $Q_{\chi^2_{s, \mu_K}}(\cdot)$  denote the complementary cumulative distribution functions of the non-central chi-squared distribution with  $s$  degrees of freedom and respectively noncentrality parameters 0 and  $\mu_K$  associated with  $(C_i)_{i=1,2,3,4}$ .

Finally, note that this performance analysis could be applied to conventional GLRT-based array receivers for the detection of a known signal corrupted by a circular interference and used with circular interference. In contrast, our analysis cannot be applied to these receivers used with noncircular interference, because in this latter case, the derived receivers are no longer GLRT-based receivers.

## V. ILLUSTRATIONS

To illustrate the performance analysis of Section IV, we consider a linear array of  $N$  omnidirectional sensors equispaced half a wavelength apart. The phase and the direction of arrival with respect to broadside, of both the BPSK signal of interest and the BPSK interference are assumed constant over a burst of  $K$  symbols. They take the following values:  $\phi_s = 0$ ,  $\theta_s = 0$ ,  $\phi_1 = \pi/4$  and  $\theta_1 = \pi/9$ . The input SNR and interference to noise ratio (INR) are defined by  $\text{SNR} = \rho_s^2/\eta_2$  and  $\text{INR} = \rho_1^2/\eta_2$ , respectively and fixed to satisfy the constraint  $\text{INR} = \text{SNR} + 20\text{dB}$ .

Figs.1 and 2 show the empirical (Monte Carlo), the exact (issued from (9) and (10)), approximate (issued from (12) and (13)) theoretical ROCs of the optimal (3) and conventional (4) clairvoyant receivers for  $\text{SNR} = -15\text{dB}$  and  $\text{INR} = 5\text{dB}$ . In Fig.1, the number of sensors is fixed  $N = 2$  with  $K = 4$  and 64, whereas for Fig.2 the data length is fixed  $K = 16$  with  $N = 1, 2, 4$  and 8.

We see in Fig.1 that the optimal receiver largely outperforms conventional one and that the two theoretical (exact and approximate) ROCs coincide for both values of  $K$ . In fact, in many scenarios of  $\text{INR} = \rho_1^2/\eta_2$ , numerical computations of approximations (12) and (13) of respectively (9) and (10), remain very accurate (to two significant digits) for  $K = 4$ . This is due to the spatial matched filters  $\tilde{\mathbf{w}}_o$  and  $\mathbf{w}_c$  which mitigate the power of the BPSK interference with respect to the power of the Gaussian background noise. Thus at the output of the detector, the ratio  $\alpha^2/\sigma^2$  remains relatively low with respect to 1 (For example,  $\alpha^2/\sigma^2 \approx 0.003$  in case

of Fig.1). The same conclusion can be drawn from Fig.2. Furthermore, we note the poor performance of the conventional receiver for  $N = 1$  due to its incapability to reject the strong interference,. This contrasts with the optimal receiver which exploits the SO noncircularity of both BPSK signal of interest and interference. Furthermore, we see that the gain in performance of the optimal receiver with respect to the conventional one decreases when the number of  $N$  of sensors increases.

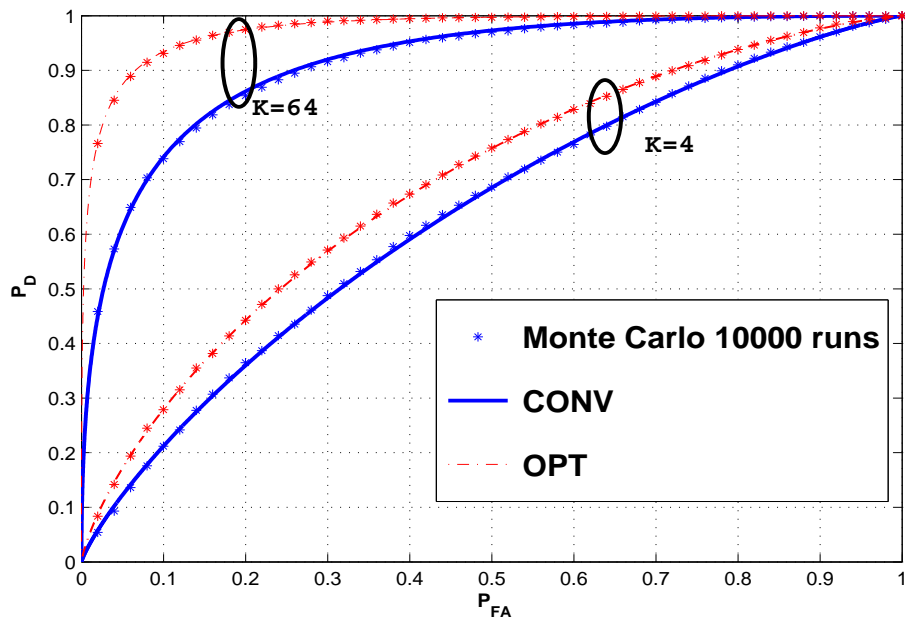


Fig.1 Exact and approximate theoretical, and empirical ROCs for SNR =  $-15$ dB and INR =  $5$ dB with  $N = 2$ .

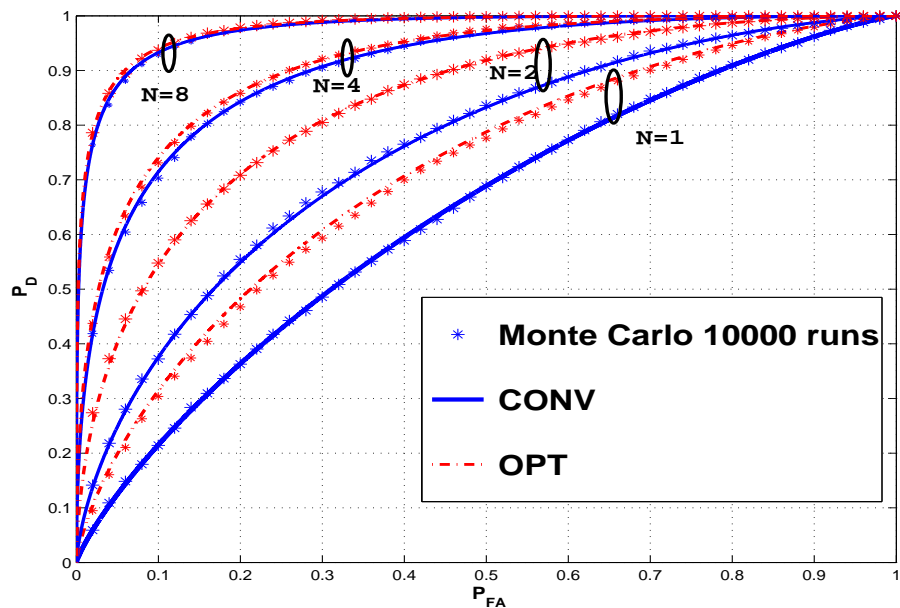


Fig.2 Exact and approximate theoretical, and empirical ROCs for SNR =  $-15$ dB and INR =  $5$ dB with  $K = 16$ .

Considering now the domain of validity of the distribution's approximations. For the different GLRT under  $H_0$  and  $H_1$ , it is much more involved than for the LRT. This is due to the fact that it does not only depend

on  $K$  and  $\text{SINR} = K\rho_s^2/(\rho_1^2 + \eta_2)$ , but also on the concerned domain of  $P_D$  and  $P_{\text{FA}}$ . For example, Figs.3 and 4 show the relative error<sup>14</sup> on probability of detection  $P_D$  given by the GLRT receiver obtained when only  $(\rho_s, \phi_s)$  are unknown ( $C_1$ ) for fixed  $P_{\text{FA}} = 0.001$  with  $N = 2$ , as a function of the data length  $K$  for  $\text{SNR} = -20\text{dB}$ , and as a function of the SNR for the fixed data length  $K = 128$ , respectively. We see that the approximation of  $P_D$  is relatively good<sup>15</sup> for small data length  $K$  and not to low SINR.

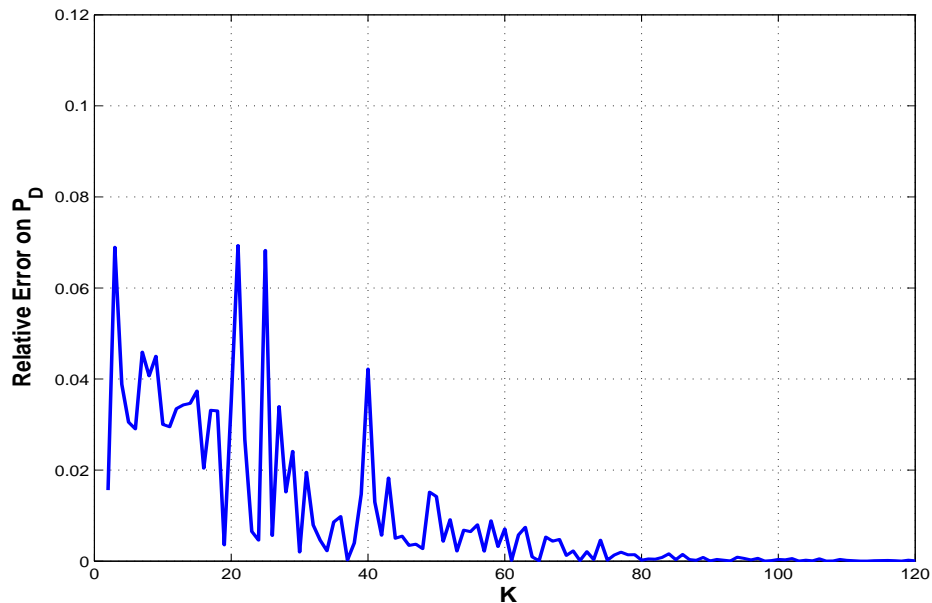


Fig.3 Relative error on probability of detection  $P_D$  for  $P_{\text{FA}} = 0.001$  for  $N = 2$ ,  $\text{SNR} = -20\text{dB}$  and  $\text{INR} = 0\text{dB}$  for the case  $C_1$ .

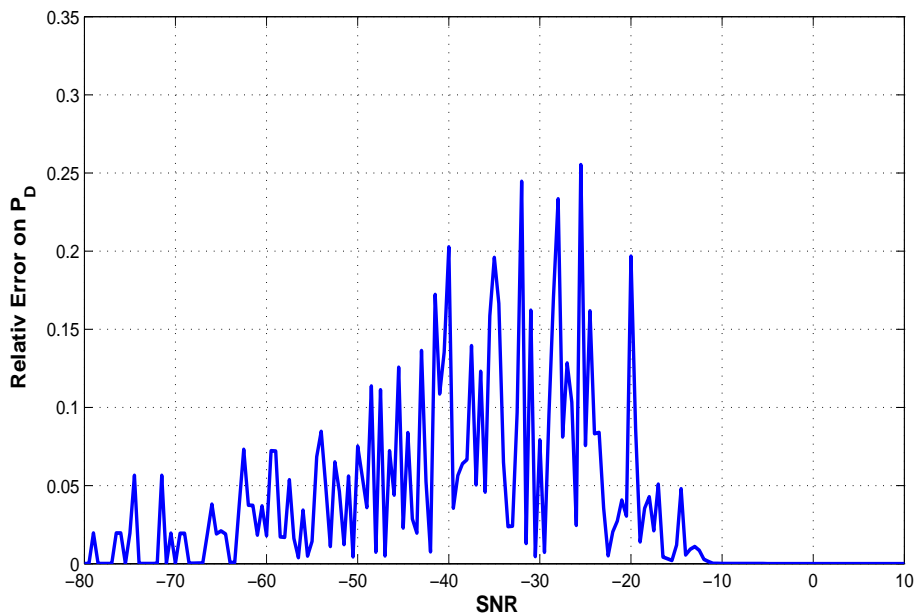


Fig.4 Relative error on probability of detection  $P_D$  for  $P_{\text{FA}} = 0.001$  for  $N = 2$ ,  $K = 128$  and  $\text{INR} = \text{SNR} + 20\text{dB}$  for the case  $C_1$ .

<sup>14</sup>i.e.,  $\frac{P_D^{(2)} - P_D^{(1)}}{P_D^{(2)}}$  where  $P_D^{(2)}$  is deduced from approximative distribution (17) and  $P_D^{(1)}$  is deduced from 10000 Monte Carlo runs

<sup>15</sup>Note that from  $K > 120$  in Fig.3 and for  $\text{SNR} > -10\text{dB}$  in Fig.4,  $P_D^{(1)} \approx P_D^{(2)} \approx 1$  and the detection is perfect.

Fig.5 shows the empirical and the asymptotic theoretical ROCs (20) of the analyzed GLRT receivers compared to the exact theoretical ROC of the optimal clairvoyant receiver described in Section III for  $N = 2$ .

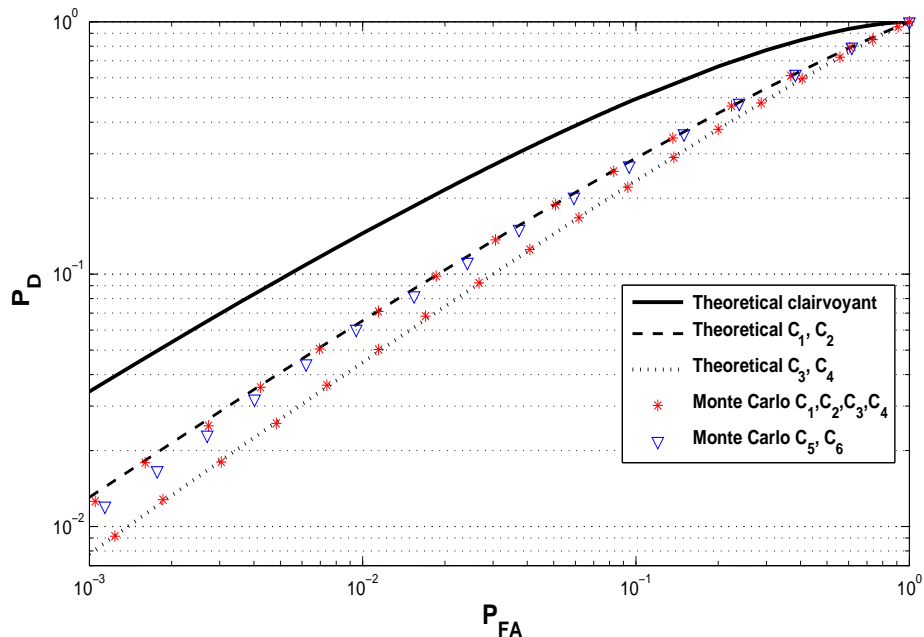


Fig.5 Asymptotic theoretical and empirical ROC for  $\text{SNR} = -25\text{dB}$ ,  $\text{INR} = 0\text{dB}$  and  $K = 128$ .

Comparing these ROCs, we see that the clairvoyant receiver outperforms all the GLRT receivers. The performance improves with the knowledge of the directional vector  $\mathbf{s}$ , i.e., the  $(C_1, C_2)$  GLRT receivers slightly outperform the  $(C_5, C_6)$  GLRT receivers, which outperform the  $(C_3, C_4)$  GLRT receivers. We note that the performances of the  $(C_5, C_6)$  GLRT receivers are very close to those of  $(C_1, C_2)$  GLRT receivers. This was not predicted by our theoretical asymptotic analysis (see footnote 10). For the validity conditions of our analysis (i.e., large size of data and very weak SINR), the knowledge of  $(\mathbf{R}_n, \mathbf{C}_n)$  does not improve the performance in the  $C_1, C_2, C_3$  and  $C_4$  GLRT receivers as predicted by our theoretical asymptotic analysis.

Furthermore, we observe that the empirical ROCs fit the asymptotic theoretical ones for not very weak SINR ( $K\rho_s^2/(\rho_1^2 + \eta_2) = 0.40$ ) and not too large data length ( $K = 128$ ). Finally, note that extensive experiments that are not presented in this paper, show that these conclusions depend on the scenario.

## VI. CONCLUSION

A performance analysis of LRT/GLRT-based array receivers for the detection of a known real-valued signal corrupted by a potentially noncircular interference has been considered in this paper. The exact distributions of the decision statistics associated with two LRT-based array receivers have been given. Then, using central limit theorems, asymptotic (with respect to the data length) Gaussian distributions of these decision statistics allow us to give closed-form expressions of the associated ROC. These expressions prove that taking into account the potential noncircularity of the interference, may dramatically improve the performance of both mono and multi-channels receivers. Concerning the six studied GLRT-based array receivers, asymptotic (for large data

length and weak SINR) distributions of decision statistics associated with four GLRT-based array receivers have been given. They prove that in these conditions, the knowledge of the steering vector dramatically improves the performance in contrast to the knowledge of the nuisance parameters  $(\mathbf{R}_n, \mathbf{C}_n)$ . For parameterized steering vectors with unknown DOA, numerical illustrations show a slight degradation of performance with respect to known steering vectors. Finally, some illustrations show that the empirical ROCs fit the asymptotic theoretical ones for not too large data length and not too weak SINR.

## VII. APPENDIX

### Expression of $\mu_K$

The FIM corresponding to the non-singular and non-circular complex Gaussian distribution of  $(\mathbf{x}_k)_{k=1,\dots,K}$  is deduced from the extended Slepian-Bangs formula [19] that gives elementwise

$$(\mathbf{I}_K(\boldsymbol{\theta}))_{i,j} = \sum_{k=1}^K a_k^2 \left( \frac{\partial \tilde{\mathbf{m}}}{\partial \theta_i} \right)^H \mathbf{R}_{\tilde{n}}^{-1} \frac{\partial \tilde{\mathbf{m}}}{\partial \theta_j} + \frac{K}{2} \text{Tr} \left[ \frac{\partial \mathbf{R}_{\tilde{n}}}{\partial \theta_i} \mathbf{R}_{\tilde{n}}^{-1} \frac{\partial \mathbf{R}_{\tilde{n}}}{\partial \theta_j} \mathbf{R}_{\tilde{n}}^{-1} \right], \quad (21)$$

where  $\tilde{\mathbf{m}} = \rho_s \tilde{\mathbf{s}}_\phi$ . We note that  $\tilde{\mathbf{m}}$  and  $\mathbf{R}_{\tilde{n}}$  depend only on the parameters  $\boldsymbol{\theta}_s$  and  $\boldsymbol{\theta}_r$ , respectively. Consequently these parameters are decoupled in the FIM, i.e.,  $\mathbf{I}_{s,r}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_r) = \mathbf{0}$ , and thus expression (18) of  $\mu_K$  reduces to  $\boldsymbol{\theta}_{s_1}^T \mathbf{I}_{s,s}(\mathbf{0}, \boldsymbol{\theta}_r) \boldsymbol{\theta}_{s_1}$ , that gives from (21)  $\mu_K = \left( \sum_{k=1}^K a_k^2 \right) \boldsymbol{\theta}_{s_1}^T \left( \frac{\partial \tilde{\mathbf{m}}}{\partial \boldsymbol{\theta}_{s_1}} \right)^H \mathbf{R}_{\tilde{n}}^{-1} \frac{\partial \tilde{\mathbf{m}}}{\partial \boldsymbol{\theta}_{s_1}} \boldsymbol{\theta}_{s_1}$ , where

$$\frac{\partial \tilde{\mathbf{m}}}{\partial \boldsymbol{\theta}_{s_1}} = \begin{pmatrix} \mathbf{s} & i\mathbf{s} \\ \mathbf{s}^* & -i\mathbf{s}^* \end{pmatrix} \text{ in cases } C_1 \text{ and } C_2, \quad \left[ \text{resp., } \begin{pmatrix} \mathbf{I} & i\mathbf{I} \\ \mathbf{I} & -i\mathbf{I} \end{pmatrix} \text{ in cases } C_3 \text{ and } C_4 \right]$$

So  $\frac{\partial \tilde{\mathbf{m}}}{\partial \boldsymbol{\theta}_{s_1}} \boldsymbol{\theta}_{s_1} = \tilde{\mathbf{m}} = \rho_s \tilde{\mathbf{s}}_\phi$  in cases  $C_1, C_2, C_3$  and  $C_4$ , which proves (19).

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