Global existence of strong solution for viscous shallow water system with large initial data on the irrotational part

Boris Haspot

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Abstract

We are interested in studying the Cauchy problem for the viscous shallow-water system in dimension $N \geq 2$, we show the existence of global strong solutions with *large* initial data on the irrotational part of the velocity for the scaling of the equations. More precisely our smallness assumption on the initial data is supercritical for the scaling of the equations. It allows us to give a first kind of answer to the problem of the existence of global strong solution with large initial energy data in dimension $N = 2$. To do this, we introduce the notion of *quasi-solutions* which consists in solving the pressureless viscous shallow water system. We can obtain such solutions at least for irrotationnal data which exhibit regularizing effects both on the velocity and also on the density. This smoothing effect is purely non linear and is crucial in order to build solution of the viscous shallow water system as perturbations of the "quasi-solutions". Indeed the pressure term remainder becomes small in high frequencies for the scaling of the equations. To finish we prove the existence of global strong solution with large initial data when $N \geq 2$ for the viscous shallow water system provided that the Mach number is sufficiently large.

1 Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div}u) + \nabla P(\rho) = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$ 0.1

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density and $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ is the strain tensor. We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $\lambda + 2\mu > 0$. Such a condition ensures ellipticity for the momentum equation and is satisfied in the physical cases where $\lambda + \frac{2\mu}{N} > 0$. In the sequel we shall only consider the viscous shallow-water system which corresponds to:

$$\mu(\rho) = \mu\rho \text{ with } \mu > 0 \quad \text{and} \quad \lambda(\rho) = 0.$$

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We supplement the problem with initial condition (ρ_0, u_0). Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$ or to the periodic box $\mathcal{T}_a^N$ with period a_i, in the i-th direction. We restrict ourselves to the case $N \geq 2$.
In this paper we are interested in proving the existence of global strong solution for the system (1.1) with large initial data for the scaling of the equations. Before giving our main result, let us recall some important results concerning the question of the well-posedness for (1.1). For small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved for the first time in [25]. More precisely Matsumura and Nishida obtained the existence of global strong solutions for three-dimensional polytropic ideal fluids and no outer force with initial data chosen small in the following spaces $(\rho_0 - 1, u_0) \in H^3 \times H^3$.
Guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see e.g [27]) we aim at solving (1.1) in the case where the data (ρ_0, u_0) have *critical* regularity. By critical, we mean that we want to solve the system (1.1) in functional spaces with norm invariant by the changes of scales which leave (1.1) invariant. In the case of barotropic fluids, we can observe that the transformations:

$$(\rho(t,x), u(t,x)) \longrightarrow (\rho(l^2 t, lx), lu(l^2 t, lx)), \quad l \in \mathbb{R}, \tag{1.2}$$

have that property, provided that the pressure term has been changed accordingly. Roughly speaking we expect that such spaces are optimal in term of regularity on the initial data in order to prove the well-posedness of the system (1.1).
One of the main difficulty of compressible fluid mechanics is to deal with the vacuum (which corresponds to the state $\rho = 0$), indeed when it occurs, the momentum equation loses its parabolicity. That is why in the sequel we shall work around stable equilibrium in order to be far away the vacuum, we have then the following definition.

Definition 1.1 *Let us note:* $q = \rho - 1$.

The use of critical functional frameworks led to several new well-posedness results for compressible fluids in Besov spaces (see [6, 7, 10, 11, 12, 14, 15, 16, 18, 17, 19, 28]).
We would like to point out that this is in general not sufficient to deal with initial data (ρ_0, u_0) invariant by (1.2) to obtain the existence of global strong solution. Indeed it is important to obtain a control on the L^∞ norm of the density for at least two reasons. First it allows to consider a density which remains far way from the vacuum, it is crucial in order to ensure the parabolic behavior of the velocity u. The second reason is related to the estimations of the non linear term depending on the density as the pressure , indeed in general we need to use composition theorems. Finally it seems necessary to control the velocity in Lipschitz norm in order to estimate the density which is governed by a transport equation.
That is why, we restricted our study to the case where the initial data (q_0, u_0) are in homogeneous Besov spaces with the thord index equal to one:

$$q_0 \in B^{\frac{N}{p}}_{p,1} \text{ and } u_0 \in B^{\frac{N}{p_1}-1}_{p_1,1}$$

with $(p, p_1) \in [1, +\infty[$ suitably chosen. Indeed in this case we recall that $B^{\frac{N}{p}}_{p,1}$ is embedded in L^∞.

Let us mention that in the literature most of the results of existence of global strong solution with small initial data in critical space concern the system (1.1) with constant viscosity coefficient. In particular Danchin in [12] shows for the first time a result of existence of global strong solution with small initial data in critical space for the scaling of the system. More precisely the initial data are chosen as follows $(q_0, u_0) \in (B^{\frac{N}{2}}_{2,1} \cap B^{\frac{N}{2}-1}_{2,1}) \times B^{\frac{N}{2}-1}_{2,1}$. The main difficulty consists in obtaining suitable estimates on the linearized system with convection terms. The crucial point is the proof of damping effect on the density which enables to control the pressure term. This last result has been generalized to the case of Besov space constructed on L^p space by F. Charve and R. Danchin in [6], Chen et al in [10] and the author in [16] by using two different methods. In [6, 10], in order to get estimates on the density and the velocity in Besov space for the linearized system (including the convection terms) associated to (1.1), the authors combine in a very subtle way a paralinearization method and a accurate Fourier study of the linear system. In [16] we introduce a notion of *effective velocity* in high frequencies which allows us to cancel out the coupling between the velocity and the pressure. Indeed the system becomes simply a heat equation with a damped transport equation, as in [12] we obtain then a L^1 decay on q in the high frequency regime. In low frequencies, the first order terms predominate, so that (1.1) has to be treated by means of hyperbolic energy methods (roughly speaking q and the potential part of the velocity verify a wave equation). Let us mention also that a very interesting approach has been proposed by R. Danchin in [14] in order to improve the results of uniqueness inasmuch as initial velocities in critical Besov spaces with negative indices generate a unique local solution. The key tool is the use of Lagrangian coordinates which allows to solve the system by means of the basic contraction mapping theorem.

However the existence of global strong solution with large initial data remains open even in dimension $N = 2$ except for some very specific viscosity coefficients (see [31]). In a remarkable work Vaigant and Kazhikhov prove the existence of global strong solution when the viscosity coefficients are chosen such that $\mu(\rho) = \mu$ and $\lambda(\rho) = \rho^\beta$ with $\beta > 3$. To do this, they use clever energy inequalities which take into account the structure of the viscosity coefficient by introducing the effective flux. The choice $\lambda(\rho) = \rho^\beta$ with $\beta > 3$ is essentially due to technic restrictions when the authors prove some $L^\infty_T(L^p)$ estimates on the density for any $p \in (2, +\infty)$ (we refer also to [18] for a slight extension of the work of [31] to the case $\beta > 2$).

In this paper we would like to improve the previous results by proving the existence of global strong solution with *large* initial data for the scaling of the equations for the viscous shallow water system (the initial data has to be choose irrotational in our case). Indeed at the difference of the previous works that we mention, our smallness assumption on the initial data is supercritical for the scaling of the equations (and in particular *nonlinear*). It implies that it is possible to have initial data (q_0, u_0) which are large in $B^{\frac{N}{2}}_{2,1} \times (B^{\frac{N}{2}-1}_{2,1})^N$ and which generates global strong solution to the system (1.1). It gives in particular a first kind of answer to the problem of the existence of global strong solution with large energy initial data when $N = 2$ for suitable choice on the initial velocity (we refer to the remark 1 for more details).

To do this we are going to work around an irrotational *quasi-solution* of the system (1.1) (we also refer to [18] for a such type of solution in the context of Korteweg system). More

precisely we are interested in solving the pressureless system for an irrotational velocity $u = \nabla\theta$. Due to the strong coupling between the velocity, the density and viscosity tensor, we can check easily that $(\rho^1, -2\mu\nabla \ln \rho^1)$ is a particular irrotational solution of the pressureless system if:

$$\begin{cases} \partial_t \rho^1 - 2\mu\Delta\rho^1 = 0. \\ \rho^1(0,\cdot) = \rho_0^1. \end{cases} \tag{1.3}$$

quasi1

Here we recall that the initial density ρ_0^1 does not admit vacuum, it implies in particular by the maximum principle that $\frac{1}{\rho^1}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ for any $T > 0$. It is then natural to work around this particular *quasi-solution* in order to obtain global strong solution with large initial data for (0.1), to do this we have obviously to take into account the effects of the pressure term $P(\rho_1)$. We can observe that the regularizing effects on the density ρ^1 enable us to consider the pressure term $P(\rho^1)$ as a small term in high frequencies (indeed we get a gain of two derivatives on ρ^1 via the equation (quasi1)). This is essentially due to the fact that the system (0.1) is not completely invariant by scaling because the pressure term.The regularizing effects on ρ^1 is then absolutely crucial and is surprising since the density is governed by a transport equation, it means that this property is purely non linear and is due to the coupling between the density and the velocity (we will discuss on this point in more details in the sequel).
Let us now search solution of the form $\ln\rho = \ln\rho^1 + h^2$ with $\rho_1 = 1 + q^1$, $\rho = \rho^1 e^{h^2}$ and $u = -2\mu\nabla \ln\rho_1 + u^2$, assuming that there is no vacuum, we can rewrite the system (0.1) under the following eulerian form (we recall that $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$):

$$\begin{cases} \partial_t \ln\rho + u\cdot\nabla\ln\rho + \mathrm{div}u = 0, \\ \partial_t u + u\cdot\nabla u - \mu\Delta u - \mu\nabla\mathrm{div}u - 2\mu\nabla\ln\rho\cdot Du + \nabla F(\rho) = 0, \\ (\ln\rho, u)_{/t=0} = (\ln\rho_0, u_0), \end{cases} \tag{1.4}$$

3systeme

with $F'(\rho) = \frac{P'(\rho)}{\rho}$. By using the fact that $(\rho^1, u^1) = (\rho^1, -2\mu\nabla\ln\rho^1)$ is a quasi solution with ρ^1 verifying (quasi1), it gives in particular:

$$\begin{cases} \partial_t \ln\rho^1 + u^1\cdot\nabla\ln\rho^1 + \mathrm{div}u^1 = 0, \\ \partial_t u^1 + u^1\cdot\nabla u^1 - \mu\Delta u^1 - \mu\nabla\mathrm{div}u^1 - 2\mu\nabla\ln\rho^1\cdot Du^1 = 0, \\ (\ln\rho^1, u^1)_{/t=0} = (\ln\rho_0^1, u_0^1). \end{cases} \tag{1.5}$$

3systeme1

We can rewrite the system (3systeme) as follows:

$$\begin{cases} \partial_t h^2 + u\cdot\nabla h^2 + \mathrm{div}u^2 + u^2\cdot\nabla\ln\rho^1 = 0, \\ \partial_t u^2 + u^1\cdot\nabla u^2 + u^2\cdot\nabla u^1 - 2\mu\nabla\ln\rho^1\cdot Du^2 - 2\mu\nabla h^2\cdot Du^1 - \mu\Delta u^2 - \mu\nabla\mathrm{div}u^2 \\ \qquad\qquad + a\nabla h^2 = -a\nabla\ln\rho^1 - u^2\cdot\nabla u^2 + 2\mu\nabla h^2\cdot Du^2, \\ (h^2, u^2)_{/t=0} = (h_0^2, u_0^2). \end{cases} \tag{1.6}$$

0.2

where we have assumed that $P(\rho) = a\rho$ (in order to simplify the notation).
We now are going to prove the existence of global strong solution for the system (0.2). To do this the first step consists in proving estimate in Besov spaces on (h^2, u^2), it will require an accurate study of the linearized system associated to (0.2). In particular we

will have to distinguish the behavior between the low and the high frequencies as in [16]. The last step is to observe that the pressure term $a\nabla \ln \rho^1$ can be considered as a small remainder term in high frequencies. Indeed roughly speaking it must be estimated in $L^1(\mathbb{R}^+, B^{\frac{N}{p}-1}_{p,1})$, but if $\ln \rho_0^1$ is in $B^{\frac{N}{p}}_{p,1}$ then via the heat equation (1.3), $\nabla \ln \rho^1$ belongs to $L^1(\mathbb{R}^+, B^{\frac{N}{p}+1}_{p,1})$. It means that via this gain of regularity on $\nabla \ln \rho^1$ this term is small in high frequencies for the norm $L^1(\mathbb{R}^+, B^{\frac{N}{p}-1}_{p,1})$. It explains why we obtain a result of global strong solution with a nonlinear smallness assumption on ρ_0^1 and this one will be supercritical for the scaling of the equations.

Definition 1.2 *Furthermore we will note $\widetilde{B}^{s_1,s_2}_{(p_1,r_1),(p_2,r_2)}$ the Besov space where the behavior is $B^{s_1}_{p_1,r_1}$ in low frequencies and $B^{s_2}_{p_2,r_2}$ in high frequencies. If $r_1 = r_2$ we will simplify the notation, and we will write $\widetilde{B}^{s_1,s_2}_{p_1,p_2,r_1}$ for $\widetilde{B}^{s_1,s_2}_{(p_1,r_1),(p_2,r_1)}$. For more details on the definition of these spaces we refer to the definition 2.6.*

One can now state our main theorem.

Theorem 1.1 *Let $P(\rho) = a\rho$ and $2 \leq p \leq 4$, $p < 2N$, $q \geq 2$ such that:*

$$\frac{1}{2} \leq \frac{1}{p} + \frac{1}{q}, \frac{1}{N} < \frac{1}{p} + \frac{1}{q}, \frac{1}{p} \leq \frac{1}{N} + \frac{1}{q}.$$

Let $\rho_0 = \rho_0^1 e^{h_0^2}$ with $\rho_0^1 = 1 + q_0^1$, $u_0 = -2\mu\nabla \ln \rho_0^1 + u_0^2$. Furthermore we assume that $\rho_0^1 \geq c > 0$, $q_0^1 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1} \cap \widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}$, $h_0^2 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}$ and $u_0^2 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}$. There exists ϵ_0, ϵ_1, $C > 0$ and two regular function g, g_1 such that if:

$$\begin{aligned} &Cg(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}} \exp\left(Cg_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1}}\right) \leq \epsilon_1, \\ &\|h_2^0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|u_2^0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} \leq \epsilon_0 = g_1(\|\frac{1}{\rho_0^1}\|_{L^\infty}, \|\rho_0^1\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}} \end{aligned} \tag{1.7}$$

crucinitial

then there exists a global solution (ρ, u) of the system (1.1) written under the following form: $\rho = \rho^1 e^{h^2}$ and $u = -2\mu\nabla \ln \rho^1 + u^2$ with:

$$\begin{cases} \partial_t \rho^1 - 2\mu\Delta\rho^1 = 0, \\ \rho^1_{t=0} = \rho_0^1. \end{cases} \tag{1.8}$$

achal

In addition we have:

$$\begin{aligned} h^2 \in \widetilde{C}(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}) \cap L^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}) \\ \text{and } u^2 \in \widetilde{C}(\mathbb{R}^+; \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}) \cap L^1(\mathbb{R}+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}). \end{aligned}$$

Moreover the solution is unique if $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{q}$. We refer to the definition 2.6 for the notion of hybrid Besov spaces.

theo1

r1 **Remark 1** *Let us mention than the main interest of this theorem is to prove the existence of global strong solution with large initial data for the scaling of the equation which is new up to our knowledge. Indeed it suffices to choose* $q_0^1(x) = \varphi(\lambda x)$ *with* $\varphi \in \widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}}$ *(where* $p = q = 2$*) such that* $1 + \varphi \geq c > 0$*, we then verify easily that:*

$$\begin{cases} \|q_0^1\|_{B_{2,1}^{\frac{N}{2}}} = \|\varphi\|_{B_{2,1}^{\frac{N}{2}}}, \\ \|\rho_0^1\|_{L^\infty} = \|1+\varphi\|_{L^\infty}, \\ \|\dfrac{1}{\rho_0^1}\|_{L^\infty} = \|\dfrac{1}{1+\varphi}\|_{L^\infty}, \\ \|q_0^1\|_{B_{2,1}^{\frac{N}{2}-2}} = \dfrac{1}{\lambda^2}\|\varphi\|_{B_{2,1}^{\frac{N}{2}-2}}. \end{cases} \tag{1.9}$$

genial

It implies that q_0^1 *verifies (1.7) by choosing* λ *large enough. In particular it implies that taking* φ *large in* $B_{2,1}^{\frac{N}{2}}$ *our initial density* h_1^0 *is large in the Besov space* $B_{2,1}^{\frac{N}{2}}$ *which is critical for the scaling of the equations. In addition it implies that the* L^∞ *norm of the density* ρ_0 *can be chosen large (let us recall that it is not the case in [6, 10, 16]).*
In particular when $N = 2$*, it is possible to choose* φ *large in* $B_{2,2}^1$ *which shows that there is existence of global strong solution for large initial data in the energy space. Indeed for the viscous shallow water system we recall that the energy data for the BD entropy corresponds to initial data such that* $\nabla\sqrt{\rho_0}$ *and* $\sqrt{\rho_0}u_0$ *belong to* $L^2(\mathbb{R}^N)$ *(see [3, 26] for more details). It gives in particular a first answer to the existence of global strong solution with large initial data when* $N = 2$ *at least for a family of suitable initial data. Indeed our case is not recover by the results of [31] since their viscosity coefficients are completely different. This question remains obviously open for general initial data.*
We could also choose $q_0^1(x) = \ln(\lambda)\varphi(\lambda x)$ *with* $\lambda > 0$ *which improves again the size of the large initial data in* $B_{2,1}^{\frac{N}{2}}$*.*

Remark 2 *We would like to emphasize on the fact that the density* $\ln\rho$ *consists in the sum of a regular function* $\ln\rho^1$ *and of a small perturbation* h^2*. This point is very surprising in the sense that the density is governed by a hyperbolic equation which means that a priori we do notexpect any regularizing effects on the density. It seems that there is a singular behavior around the quasi-solution* $(\rho^1, -\mu\nabla\ln\rho^1)$ *and we note that this effect is strictly non-linear. Indeed it depends on the convection term* $u \cdot \nabla u$*.*
Furthermore this regularizing effect is crucial in order to deal with the term $u^2 \cdot \nabla\ln\rho^1$*, which loses one derivate. Secondly this regularizing effect allows us to consider* $\nabla\ln\rho^1$ *as a small remainder term in high frequencies.*

Remark 3 *We would like to mention that this result is strongly related to the structure of the viscosity coefficients as we are able to construct quasi-solutions. Indeed for constant viscosity coefficient it seems not clear how to construct quasi-solution.*

Remark 4 *Let us point out that that nonlinear condition of smallness as (1.7) have been proved also in some works of Chemin and Gallagher in [8, 9] for incompressible Navier-Stokes equations. Indeed in these works the authors prove the existence of global strong solution for large initial data in* $B_{\infty,\infty}^{-1}$ *which is the largest critical space for the Navier-Stokes equations. However our proof is really different of [8, 9] since our initial data are*

completely irrotational (that is of course not the case for incompressible equation). In addition we work around the quasi-solution which absorb the convection term, it enables to obtain better results in term of smallness assumption (1.7) compared with [8, 9] (indeed in these papers the authors work around the solution of the heat equation, and the process of smallness is related to a smallness assumption on the term of convection). This is obviously due to the fact that our system is compressible which allows us to deal with irrotational data.

Remark 5 *We could weaken the condition on (h^2, u^2) by following also the idea of [7], indeed it may be possible to take u_2 only in $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,2}$ with $\mathrm{div} u_0$ in $\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}$.*

Remark 6 *We think that we could treat also the general case when $\mu(\rho) = \mu\rho^\alpha$ with $\alpha \geq 1 - \frac{1}{N}$ ($\alpha \neq 1$) when μ and λ verify the BD entropy (see [5]). Indeed we have proved in [22, 21] that there exists in this case quasi-solutions of the form $(\rho^1, -\frac{2\mu\alpha}{\alpha-1}\nabla(\rho^1)^{\alpha-1})$ with ρ^1 solution of the fast diffusion equation or the porous medium equation:*

$$\partial_t \rho_1 - 2\mu\Delta(\rho^1)^\alpha = 0.$$

When ρ^1 is far away from the vacuum the previous equation has the same behavior than a heat equation, then we can adapt our proof in a similar way.

We are going to finish by presenting a result of global strong solution with large initial data provided that the Mach number is sufficiently large.

cor3 **Theorem 1.2** *Let $P(\rho) = K\rho$ with $K > 0$ and $2 \leq p \leq 4$, $p < 2N$, $q \geq 2$ such that:*

$$\frac{1}{2} \leq \frac{1}{p} + \frac{1}{q}, \frac{1}{N} < \frac{1}{p} + \frac{1}{q}, \frac{1}{p} \leq \frac{1}{N} + \frac{1}{q}$$

Let $\rho_0 = \rho_0^1 e^{h_0^2}$ with $\rho_0^1 = 1 + q_0^1$, $u_0 = -2\mu\nabla \ln \rho_0^1 + u_0^2$. Furthermore we assume that $\rho_0^1 \geq c > 0$, $q_0^1 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1} \cap \widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}$, $h_0^2 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}$ and $u_0^2 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}$.
There exists $\epsilon_0 > 0$ (depending on h_1^0 and the viscosity coefficient μ) such that for any $K \leq \epsilon_0$, there exits $\epsilon_1 > 0$ such that if

$$\|h_2^0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|u_2^0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} \leq \epsilon_1. \tag{1.10}$$ bcrucinitial

then there exists a global solution (ρ, u) of the system (1.4) with: $u = -2\mu\nabla \ln \rho_1 + u_2$ and $\ln \rho = \ln \rho_1 + h_2$ with $\rho_1 = 1 + q_1$ verifying the following system:

$$\begin{cases} \partial_t \rho_1 - 2\mu\Delta\rho_1 = 0, \\ \rho_1(0, \cdot) = \rho_1^0 = 1 + h_1^0. \end{cases}$$

Furthermore we have:

$$h^2 \in \widetilde{C}(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}) \cap L^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})$$
$$\text{and } u^2 \in \widetilde{C}(\mathbb{R}^+; \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}) \cap L^1(\mathbb{R}+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}).$$

Moreover the solution is unique if $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{q}$.

Remark 7 *The main interest of this theorem is to prove the existence of global strong solution for any large initial data provided that K is sufficiently small with $P(\rho) = K\rho$ (here $K = \frac{1}{\epsilon^2}$ with ϵ the Mach number). In other terms we get global existence and uniqueness for highly compressible fluids (which corresponds to large Mach number ϵ) in any dimension $N \geq 2$. Up to our knowledge it is the first time that we have a result of global strong solution with large initial data in dimension 3 (under a condition of course of high compressibility, it means that K must be sufficiently small in function of the initial data). Roughly speaking K tends to be very small when $\|h_1^0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}}_{2,1}}$ is very large.*

We are now going to consider the viscous shallow water model with friction. This model is also called the Saint-Venant equations and is generally used in oceanography. Indeed it allows to model vertically averaged flows in terms of the horizontal mean velocity field u and the depth variation ρ. In the rotating framework, the model is described by the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu\rho D(u)) + \dfrac{\nabla\rho}{Fr^2} + r\rho u = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.11}$$

10.1

$Fr > 0$ denotes the Froude number. The turbulent regime ($r \geq 0$) is obtained from the friction condition on the bottom (see [29] for more details and [3, 18] for the existence of global weak solutions). We assume now that $r = \frac{1}{2\mu Fr^2}$. With this choice on r, we verify easily that $(\rho^1, -2\mu\nabla \ln \rho 1)$ is an explicit solution of system (1.11) if ρ_1 verifies the heat equation (1.13).

Remark 8 *Let us mention in particular that we could choose initial density ρ_0^1 in $L^1(\mathbb{R}^N)$ with $\rho_0^1 \geq 0$. Let us emphasize that here we can take an initial density which is close from the vacuum, this is generally an open problem for compressible fluids with degenerate viscosity coefficients.*
Indeed since ρ_1 verifies the heat equation, for any $t > 0$ we would have $\rho_1(t,x) > 0$ for all $(t,x) \in]0, +\infty[\times\mathbb{R}^N$ and $\rho_1(t,\cdot) \in C^\infty(\mathbb{R}^N)$ for $t > 0$. In particular $u_1(t,\cdot)$ is well defined for $t > 0$.

Using similar idea than in theorem 1.1 we obtain the following result.

Theorem 1.3 *Let $2 \leq p \leq 4$, $q \geq 2$ such that:*

$$\frac{1}{2} \leq \frac{1}{p} + \frac{1}{q}, \frac{1}{N} < \frac{1}{p} + \frac{1}{q}, \frac{1}{p} \leq \frac{1}{N} + \frac{1}{q}$$

Let $\rho_0 = \rho_0^1 e^{h_0^2}$ with $\rho_0^1 = 1 + q_0^1$, $u_0 = -2\mu\nabla \ln \rho_0^1 + u_0^2$. Furthermore we assume that $\rho_0^1 \geq c > 0$, $q_0^1 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1} \cap \widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}$, $h_0^2 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}$ and $u_0^2 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}$. There exists ϵ_0 depending on ρ_0^1 such that if:

$$\|h_2^0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|u_2^0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} \leq \epsilon_0, \tag{1.12}$$

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then there exists a global solution (ρ, u) *of the system (1.1) written under the following form:* $\rho = \rho^1 e^{h^2}$ *and* $u = -2\mu\nabla \ln \rho^1 + u^2$ *with:*

$$\begin{cases} \partial_t \rho^1 - 2\mu\Delta\rho^1 = 0, \\ \rho^1_{t=0} = \rho^1_0. \end{cases} \tag{1.13}$$

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In addition we have:

$$h^2 \in \widetilde{C}(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}) \cap L^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})$$
$$\text{and } u^2 \in \widetilde{C}(\mathbb{R}^+; \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}) \cap L^1(\mathbb{R}+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}).$$

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Moreover the solution is unique if $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{q}$.

Remark 9 *Compared with the theorem 1.1, we do not need any assumption of smallness on the density* ρ^1_0*, it corresponds to a result of global strong solution for large initial data when* $N \geq 2$*. To do this we simply perturbate the explicit large strong solution* $(\rho^1, -2\mu\nabla \ln \rho^1)$ *with* ρ^1 *verifying the heat equation (1.13). It is the first result up to our knowledge of global strong solution with large initial data for a compressible system when* $N \geq 3$*.*

Our paper is structured as follows. In section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. In section 3, we prove estimates on a linear system with convection terms. In section 4 we prove the theorems 1.1 and the theorems 1.2,1.3.

2 Littlewood-Paley theory and Besov spaces

section2

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$. For all Banach space X, we denote by $C([0,T], X)$ the set of continuous functions on $[0,T]$ with values in X. For $p \in [1, +\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t \to \|f(t)\|_X$ belongs to $L^p(0,T)$. Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any $\varphi \in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and $\chi \in C^\infty(\mathbb{R}^N)$ supported in $B(0, \frac{4}{3})$ such that:

$$\chi(\xi) + \sum_{l \in \mathbb{N}} \varphi(2^{-l}\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^N,$$
$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1 \quad \text{if} \quad \xi \neq 0.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x-y) dy \quad \text{and} \quad S_l u = \chi(2^{-l}u).$$
$$u = \sum_{k \in \mathbb{Z}} \Delta_k u \,.$$

This decomposition is called homogeneous Littlewood-Paley decomposition.

2.1 Homogeneous Besov spaces and first properties

defSh **Definition 2.3** *We denote by $\mathcal{S}_h^{'}$ the space of temperate distribution u such that:*

$$\lim_{j\to-\infty} S_j u = 0 \ \text{ in } \ \mathcal{S}^{'}.$$

Definition 2.4 *For $s\in\mathbb{R},\ p\in[1,+\infty],\ q\in[1,+\infty],$ and $u\in\mathcal{S}^{'}(\mathbb{R}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = (\sum_{l\in\mathbb{Z}}(2^{ls}\|\Delta_l u\|_{L^p})^q)^{\frac{1}{q}}.$$

The homogeneous Besov space $B^s_{p,q}$ is the set of distribution u in $\mathcal{S}_h^{'}$ such that $\|u\|_{B^s_{p,q}}<+\infty$.

Remark 10 *The above definition is a natural generalization of the homogeneous Sobolev and Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the homogeneous Hölder space C^s and that $B^s_{2,2}$ is the homogeneous space H^s.*

interpolation **Proposition 2.1** *The following properties holds:*

1. *there exists a constant universal C such that:*
   $C^{-1}\|u\|_{B^s_{p,r}}\leq\|\nabla u\|_{B^{s-1}_{p,r}}\leq C\|u\|_{B^s_{p,r}}$.
2. *If $p_1<p_2$ and $r_1\leq r_2$ then $B^s_{p_1,r_1}\hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*
3. *$B^{s^{'}}_{p,r_1}\hookrightarrow B^s_{p,r}$ if $s^{'}>s$ or if $s=s^{'}$ and $r_1\leq r$.*
4. *Moreover we have the following interpolation inequalities, it exists $C>0$ such that for any $\theta\in]0,1[$ and $s<\widetilde{s}$ we have:*

$$\|u\|_{B^{\theta s+(1-\theta)\widetilde{s}}_{p,r}}\leq\|u\|^{\theta}_{B^s_{p,r}}\|u\|^{1-\theta}_{B^{\widetilde{s}}_{p,r}},$$
$$\|u\|_{B^{\theta s+(1-\theta)\widetilde{s}}_{p,1}}\leq\frac{C}{\theta(1-\theta)(\widetilde{s}-s)}\|u\|^{\theta}_{B^s_{p,\infty}}\|u\|^{1-\theta}_{B^{\widetilde{s}}_{p,\infty}}.$$

interpolation

Let us now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [2, 1]). BJM,BCD

produit1 **Proposition 2.2** *We have the following laws of product:*

- *For all $s\in\mathbb{R}$, $(p,r)\in[1,+\infty]^2$ we have:*

$$\|uv\|_{B^s_{p,r}}\leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}+\|v\|_{L^\infty}\|u\|_{B^s_{p,r}})\,. \tag{2.14}$$ 2.2

- *Let $(p,p_1,p_2,r,\lambda_1,\lambda_2)\in[1,+\infty]^2$ such that:$\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{p_2}$, $p_1\leq\lambda_2$, $p_2\leq\lambda_1$, $\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{\lambda_1}$ and $\frac{1}{p}\leq\frac{1}{p_2}+\frac{1}{\lambda_2}$. We have then the following inequalities:*
  if $s_1+s_2+N\inf(0,1-\frac{1}{p_1}-\frac{1}{p_2})>0$, $s_1+\frac{N}{\lambda_2}<\frac{N}{p_1}$ and $s_2+\frac{N}{\lambda_1}<\frac{N}{p_2}$ then:

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}}\lesssim\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.15}$$ 2.3

when $s_1+\frac{N}{\lambda_2}=\frac{N}{p_1}$ *(resp* $s_2+\frac{N}{\lambda_1}=\frac{N}{p_2}$*) we replace* $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ *(resp* $\|v\|_{B^{s_2}_{p_2,\infty}}$*) by* $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ *(resp* $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$*), if* $s_1+\frac{N}{\lambda_2}=\frac{N}{p_1}$ *and* $s_2+\frac{N}{\lambda_1}=\frac{N}{p_2}$ *we take* $r=1$.
If $s_1+s_2=0$, $s_1\in(\frac{N}{\lambda_1}-\frac{N}{p_2},\frac{N}{p_1}-\frac{N}{\lambda_2}]$ *and* $\frac{1}{p_1}+\frac{1}{p_2}\le 1$ *then:*

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}}\lesssim\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.16}$$ 2.4

If $|s|<\frac{N}{p}$ *for* $p\ge 2$ *and* $-\frac{N}{p'}<s<\frac{N}{p}$ *else, we have:*

$$\|uv\|_{B^s_{p,r}}\le C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{2.17}$$ 2.5

Remark 11 *In the sequel* p *will be either* p_1 *or* p_2 *and in this case* $\frac{1}{\lambda}=\frac{1}{p_1}-\frac{1}{p_2}$ *if* $p_1\le p_2$, *resp* $\frac{1}{\lambda}=\frac{1}{p_2}-\frac{1}{p_1}$ *if* $p_2\le p_1$.

The study of non stationary PDE's requires space of type $L^\rho(0,T,X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Paley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r=p$). We are now going to define some useful spaces in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.5 *Let* $\rho\in[1,+\infty]$, $T\in[1,+\infty]$ *and* $s_1\in\mathbb{R}$. *We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}=\big(\sum_{l\in\mathbb{Z}}2^{lrs_1}\|\Delta_l u(t)\|^r_{L^\rho(L^p)}\big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *as the set of temperate distribution* u *over* $(0,T)\times\mathbb{R}^N$ *such that* $\lim_{q\to-\infty}S_qu=0$ *in* $\mathcal{S}'((0,T)\times\mathbb{R}^N)$ *and* $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}<+\infty$.

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r})=\widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r})\cap\mathcal{C}([0,T],B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\le\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\ge\rho,\quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\ge\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\le\rho.$$

Remark 12 *It is easy to generalize proposition* produit1 *2.2, to* $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *spaces. The indices* s_1, p, r *behave just as in the stationary case whereas the time exponent* ρ *behaves according to Hölder inequality.*

In the sequel we will need composition lemma in $\widetilde{L}^\rho_T(B^s_{p,r})$ spaces.

composition **Proposition 2.3** *Let* $s>0$, $(p,r)\in[1,+\infty]$ *and* $u\in\widetilde{L}^\rho_T(B^s_{p,r})\cap L^\infty_T(L^\infty)$.

1. *Let* $F\in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ *such that* $F(0)=0$. *Then* $F(u)\in\widetilde{L}^\rho_T(B^s_{p,r})$. *More precisely there exists a function* C *depending only on* s, p, r, N *and* F *such that:*

$$\|F(u)\|_{\widetilde{L}^\rho_T(B^s_{p,r})}\le C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

2. *Let $F \in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u)-F'(0)u \in \widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u)-F'(0)u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|^2_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

Let us give now some estimates on the commutators (see [BCD] chapter 2).

alemme2 **Lemma 1** *Let $1 \leq p_1 \leq p \leq +\infty$ and $\sigma \in (-\min(\frac{N}{p}, \frac{N}{p_1'}), \frac{N}{p}+1)$. There exists a sequence $c_q \in l^1(\mathbb{Z})$ such that $\|c_q\|_{l^1}=1$ and a constant C depending only on N and σ such that:*

$$\forall q \in \mathbb{Z}, \ \ \|[v\cdot\nabla, \Delta_q]a\|_{L^{p_1}} \leq Cc_q 2^{-q\sigma}\|\nabla v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}\|a\|_{B^\sigma_{p_1,1}}. \tag{2.18}$$ 52

In the limit case $\sigma = -\min(\frac{N}{p}, \frac{N}{p_1'})$, we have:

$$\forall q \in \mathbb{Z}, \ \ \|[v\cdot\nabla, \Delta_q]a\|_{L^{p_1}} \leq Cc_q 2^{q\frac{N}{p}}\|\nabla v\|_{B^{\frac{N}{p}}_{p,1}}\|a\|_{B^{-\frac{N}{p_1}}_{p,\infty}}. \tag{2.19}$$ 53

Finally, for all $\sigma > 0$ and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{p}$, there exists a constant C depending only on N and on σ and a sequence $c_q \in l^1(\mathbb{Z})$ with norm 1 such that:

$$\forall q \in \mathbb{Z}, \ \ \|[v\cdot\nabla, \Delta_q]v\|_{L^p} \leq Cc_q 2^{-q\sigma}(\|\nabla v\|_{L^\infty}\|v\|_{B^\sigma_{p_1,1}} + \|\nabla v\|_{L^{p_2}}\|\nabla v\|_{B^{\sigma-1}_{p,1}}). \tag{2.20}$$ 54

2.2 Hybrid Besov spaces

The homogeneous Besov spaces fail to have nice inclusion properties: owing to the low frequencies, the embedding $B^s_{p,1} \hookrightarrow B^t_{p,1}$ does not hold for $s > t$. Still, the functions of $B^s_{p,1}$ are locally more regular than those of $B^t_{p,1}$: for any $\phi \in C^\infty_0$ and $u \in B^s_{p,1}$, the function $\phi u \in B^t_{p,1}$. This motivates the definition of Hybrid Besov spaces introduced by R. Danchin (see [BCD, arma] where the growth conditions satisfied by the dyadic blocks and the coefficient of integrability are not the same for low and high frequencies. Hybrid Besov spaces have been used in [arma] to prove global well-posedness for compressible Navier-Stokes equation. We generalize here a little bit the definition by allowing for different Lebesgue norms in low and high frequencies.

def1.9 **Definition 2.6** *Let $l_0 \in \mathbb{N}$, $s,t, \in \mathbb{R}$, $(r,r_1) \in [1,+\infty]^2$ and $(p,q) \in [1,+\infty]$. We set:*

$$\|u\|_{\widetilde{B}^{s,t}_{p,q,1}} = \sum_{l\leq l_0} 2^{ls}\|\Delta_l u\|_{L^p} + \sum_{l>l_0} 2^{lt}\|\Delta_l u\|_{L^q},$$

and:

$$\|u\|_{\widetilde{B}^{s,t}_{(p,r),(q,r_1)}} = \big(\sum_{l\leq l_0}(2^{ls}\|\Delta_l u\|_{L^p})^r\big)^{\frac{1}{r}} + \big(\sum_{l>l_0}(2^{lt}\|\Delta_l u\|_{L^q})^{r_1}\big)^{\frac{1}{r_1}}.$$

Remark 13 *It will be important in the sequel to chose l_0 big enough.*

Notation 1 *We will often use the following notation:*

$$u_{BF} = \sum_{l\leq l_0}\Delta_l u \quad \text{and} \quad u_{HF} = \sum_{l>l_0}\Delta_l u.$$

Remark 14 *We have the following properties:*

- $\widetilde{B}^{s,s}_{p,p,1} = B^s_{p,1}$.
- *If* $s_1 \geq s_3$ *and* $s_2 \geq s_4$ *then* $\widetilde{B}^{s_3,s_2}_{p,q,1} \hookrightarrow \widetilde{B}^{s_1,s_4}_{p,q,1}$.

Proposition 2.4 *We have:*

$$\|u\|_{\widetilde{B}^{\frac{s}{2}+\frac{\tilde{s}}{2},s'}_{p,q,r}} \leq \|u\|^{\frac{1}{2}}_{\widetilde{B}^{\frac{s}{2},s'}_{p,q,r}} \|u\|^{\frac{1}{2}}_{\widetilde{B}^{\frac{\tilde{s}}{2},s'}_{p,q,r}}.$$

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Proof: For the proof (see [16]). □

We shall also make use of hybrid Besov-spaces.The basic idea of paradifferentiel calculus is that any product of two distributions u and v can be formally decomposed into:

$$uv = T_u v + T_v u + R(u,v) = T_u v + T'_v u$$

where the paraproduct operator is defined by $T_u v = \sum_q S_{q-1} u \Delta_q v$, the remainder operator R, by $R(u,v) = \sum_q \Delta_q u(\Delta_{q-1}v + \Delta_q v + \Delta_{q+1}v)$ and $T'_v u = T_v u + R(u,v)$.
We recall here an important proposition on the paraproduct for hybrid Besov spaces (see [16]).

hybrid **Proposition 2.5** *Let* $p_1, p_2, p_3, p_4 \in [1, +\infty], (s_1, s_2, s_3, s_4) \in \mathbb{R}^4$ *and* $(p,q) \in [1,+\infty]^2$, *we have then the following inequalities:*

- *If* $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda} \leq 1$, $\frac{1}{q} \leq \frac{1}{p_4} + \frac{1}{\lambda'} \leq 1$ *with* $(\lambda, \lambda') \in [1,+\infty]^2$ *and* $p_1 \leq \lambda'$, $p_1 \leq \lambda$, $p_3 \leq \lambda'$ *then:*

$$\|T_u v\|_{\widetilde{B}^{s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2},s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4}}_{p,q,r}} \lesssim \|u\|_{\widetilde{B}^{s_1,s_3}_{p_1,p_3,1}} \|v\|_{\widetilde{B}^{s_2,s_4}_{p_2,p_4,r}}, \tag{2.21}$$ 62

if $s_1 + \frac{N}{\lambda'} \leq \frac{N}{p_1}$, $s_1 + \frac{N}{\lambda} \leq \frac{N}{p_1}$ *and* $s_3 + \frac{N}{\lambda'} \leq \frac{N}{p_3}$.

- *If* $\frac{1}{q} \leq \frac{1}{p_3} + \frac{1}{p_4}$ *and* $s_3 + s_4 + N\inf(0, 1 - \frac{1}{p_3} - \frac{1}{p_4}) > 0$ *then*

$$\sum_{l \geq 4} 2^{l(s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4})} \|\Delta_l R(u,v)\|_{L^q} \lesssim \|u\|_{\widetilde{B}^{s_1,s_3}_{p_1,p_3,1}} \|v\|_{\widetilde{B}^{s_2,s_4}_{p_2,p_4,r}}. \tag{2.22}$$ 63

- *If* $\frac{1}{p} \leq \frac{1}{p_3} + \frac{1}{p_4} \leq 1$, $\frac{1}{p} \leq \frac{1}{p_3} + \frac{1}{p_2} \leq 1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_4} \leq 1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ *and* $s_3 + s_4 > 0$, $s_3 + s_2 > 0$, $s_4 + s_1 > 0$, $s_1 + s_2 > 0$ *then*

$$\sum_{l \leq 4} 2^{l(s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2})} \|\Delta_l R(u,v)\|_{L^p} \lesssim \|u\|_{\widetilde{B}^{s_1,\frac{N}{p_3}-\frac{N}{p_1}+s_1}_{p_1,p_3,1}} \|v\|_{\widetilde{B}^{s_2,\frac{N}{p_4}-\frac{N}{p_2}+s_2}_{p_2,p_4,r}}. \tag{2.23}$$ 63

with $s_3 = \frac{N}{p_3} - \frac{N}{p_1} + s_1$ *and* $s_4 = \frac{N}{p_4} - \frac{N}{p_2} + s_2$.

- *If* $u \in L^\infty$, *we also have:*

$$\|T_u v\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}} \lesssim \|u\|_{L^\infty} \|v\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}}, \tag{2.24}$$ 65

and if $\min(s_1, s_2) > 0$ *then:*

$$\|R(u,v)\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}} \lesssim \|u\|_{L^\infty} \|v\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}}. \tag{2.25}$$ 66

2.3 The heat equation and the mass conservation equation

section3

Let us now give some estimates for the heat equation:

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Proposition 2.6 *Let $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ and $1 \leq \rho_2 \leq \rho_1 \leq +\infty$. Assume that $u_0 \in B^s_{p,r}$ and $f \in \widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})$. Let u be a solution of:*

$$\begin{cases} \partial_t u - \mu\Delta u = f \\ u_{t=0} = u_0\,. \end{cases}$$

Then there exists $C > 0$ depending only on N, μ, ρ_1 and ρ_2 such that:

$$\|u\|_{\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s+2/\rho_1}_{p,r})} \leq C\big(\|u_0\|_{B^s_{p,r}} + \|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})}\big)\,.$$

If in addition r is finite then u belongs to $C([0,T], B^s_{p,r})$.

Let us now give some estimates in Besov spaces for the solution of the transport equation. For more details, the reader is referred to [1].

Proposition 2.7 *Let $1 \leq p_1 \leq p \leq +\infty$, $r \in [1,+\infty]$ and $s \in \mathbb{R}$ be such that:*

$$-N\min(\frac{1}{p_1}, \frac{1}{p'}) < s < 1 + \frac{N}{p_1}.$$

Suppose that $q_0 \in B^s_{p,r}$, $F \in L^1(0,T,B^s_{p,r})$ and that $q \in L^\infty_T(B^s_{p,r}) \cap C([0,T];\mathcal{S}')$ solves the following transport equation:

$$\begin{cases} \partial_t q + u\cdot\nabla q = F, \\ q_{\ t=0} = q_0. \end{cases}$$

There exists a constant C depending only on N, p, p_1, r and s such that , we have for a.e $t \in [0,T]$:

$$\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \leq e^{CU(t)}\big(\|q_0\|_{B^s_{p,r}} + \int_0^t e^{-CU(\tau)}\|F(\tau)\|_{B^s_{p,r}}d\tau\big), \tag{2.26}$$

20

with: $U(t) = \int_0^t \|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty} d\tau$.

transport1

We want to study now the following damped transport equation:

$$(\mathcal{H}) \qquad \begin{cases} \partial_t q + u\cdot\nabla q + \alpha q = F, \\ q_{/t=0} = q_0. \end{cases}$$

Above q is the unknown function. We assume that $F \in \widetilde{L}^1(0,T;B^s_{p,r})$, that v is time dependent vector-fields with coefficients in $L^1(0,T;B^{\frac{N}{p_1}+1}_{p_1,1})$ and $\alpha > 0$ a constant.

Proposition 2.8 *Let $1 \leq p_1 \leq p \leq +\infty$, $r \in [1, +\infty]$ and $s \in \mathbb{R}$ be such that:*

$$-N \min(\frac{1}{p_1}, \frac{1}{p'}) < s < 1 + \frac{N}{p_1}.$$

There exists a constant C depending only on N, p, p_1, r and s such that for all $a \in L^\infty([0,T], B^\sigma_{p,r})$ of $(\mathcal{H})$ with initial data a_0 in $B^s_{p,r}$ and $g \in L^1([0,T], B^s_{p,r})$, we have for a.e $t \in [0,T]$:

$$\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|q\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq e^{CU(t)}\big(\|q_0\|_{B^s_{p,r}} + \|F\|_{\widetilde{L}^1_t(B^s_{p,r})}\big), \tag{2.27}$$ 20

with: $U(t) = \int_0^t \|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} d\tau$.

transport2

Proof: Applying Δ_l to $(\mathcal{H})$ yields:

$$\partial_t \Delta_l q + u \cdot \nabla \Delta_l q + \alpha \Delta_l q = R_l + \Delta_l F,$$

with $R_l = [u \cdot \nabla, \Delta_l] q$. Multiplying by $\Delta_l a |\Delta_l a|^{p-2}$ then performing a time integration, we easily get:

$$\|\Delta_l q(t)\|_{L^p} + \alpha \int_0^t \|\Delta_l q(s)\|_{L^p} ds \leq \|\Delta_l q_0\|_{L^p} + \int_0^t \big(\|R_l\|_{L^p} + \frac{1}{p}\|\mathrm{div} u\|_{L^\infty} \|\Delta_l q\|_{L^p} + \|\Delta_l F\|_{L^p}\big) d\tau. \tag{2.28}$$ a3.7

Next the term $\|R_l\|_{L^p}$may be bounded according to lemma alemme2 1. We get then:

$$\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \alpha \|q\|_{\widetilde{L}^1_t(B^s_{p,r})} ds \leq \|\Delta_l q_0\|_{B^s_{p,r}} + \int_0^t \big(\|F(\tau)\|_{B^s_{p,r}} + CU'(\tau)\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})}\big) d\tau.$$

We end up with Grönwall lemma by letting $X(t) = \|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \alpha \|q\|_{\widetilde{L}^1_t(B^s_{p,r})}$. □

Remark 15 *Let us mention that by using the equality (a3.7 2.28), we show easily that:*

$$\|q_{HF}\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|q_{HF}\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq \|(q_0)_{HF}\|_{B^s_{p,r}} + \|F(\tau)\|_{\widetilde{L}^1(B^s_{p,r})} + \int_0^t \|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} \|q(\tau)\|_{B^s_{p,r}} d\tau, \tag{2.29}$$ 20bis

transportHF

3 A linear model with convection

section4

In this section, we are interested in studying the linear system associated to the system (0.2 1.6) that we can write under the following form with $u = u^1 + u^2$:

$$\begin{cases} \partial_t h^2 + u \cdot \nabla h^2 + \mathrm{div} u^2 + u^2 \cdot \nabla \ln \rho^1 = F, \\ \partial_t u^2 + u^1 \cdot \nabla u^2 + u_2 \cdot \nabla u^1 - 2\mu \nabla \ln \rho^1 \cdot Du^2 - 2\mu \nabla h^2 \cdot Du^1 - \mu \Delta u^2 - \mu \nabla \mathrm{div} u^2 \\ \qquad\qquad + a \nabla h^2 = G, \\ (h^2, u^2)_{/t=0} = (h^2_0, u^2_0). \end{cases} \tag{3.30}$$ 0.2a

with (F,G) some external force with regularity that we shall precise in the proposition (3.11). In our case we recall that $u^1=-2\mu\nabla\ln\rho^1$ with $\rho^1=1+q^1$ verifying the following heat equation:

$$\begin{cases}\partial_t\rho^1-2\mu\Delta\rho^1=0,\\ \rho^1(0,\cdot)=1+q_0^1.\end{cases}\tag{3.31}$$

eqchaleur

According to the maximum principle for the heat equation (3.31), the conditions on ρ^1 in theorem 1.1 and by interpolation for Besov space, it exists $c,M>0$ such that for all $t\geq 0$:

$$0<c\leq\rho^1(t,\cdot)\leq\|\rho_0^1\|_{L^\infty}\leq 1+\|q_0^1\|_{B^0_{\infty,1}},\tag{3.32}$$

Furthermore by the propositions 2.6, 2.3 and the initial condition on q_0^1 in theorem 1.1 ($q_0^1\in\widetilde{B}^{\frac{N}{q_1}-1,\frac{N}{q}}_{q_1,q,\infty}\cap\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}\cap B^0_{\infty,1}$) we have:

$$\begin{aligned}&q^1\in\widetilde{L}^1\big(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+2}_{2,q,\infty}\cap B^2_{\infty,1}\big)\cap\widetilde{L}^\infty\big(\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,\infty}\cap B^0_{\infty,1}\big)\\ &\text{and}\\ &u^1\in\widetilde{L}^1\big(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2},\frac{N}{q}+1}_{2,q,\infty}\cap B^1_{\infty,1}\big)\cap\widetilde{L}^\infty\big(\widetilde{B}^{\frac{N}{2}-3,\frac{N}{p}-3}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2}-2,\frac{N}{q}-1}_{2,q,\infty}\cap B^{-1}_{\infty,1}\big),\end{aligned}\tag{3.33}$$

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with p, q, q_1 verifying the conditions of theorem 1.1. More precisely we have by the propositions 2.6, 2.3 for a regular function C:

$$\begin{aligned}\|q^1\|_{\widetilde{L}^1\left(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+2}_{2,q,\infty}\cap B^2_{\infty,1}\right)}&+\|q^1\|_{\widetilde{L}^\infty\left(\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,\infty}\cap B^0_{\infty,1}\right)}\\ &\lesssim\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,\infty}}+\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}}.\\ \|u^1\|_{\widetilde{L}^1\left(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2},\frac{N}{q}+1}_{2,q,\infty}\cap B^1_{\infty,1}\right)}&+\|u^1\|_{\widetilde{L}^\infty\left(\widetilde{B}^{\frac{N}{2}-3,\frac{N}{p}-3}_{2,p,1}\cap\widetilde{B}^{\frac{N}{2}-2,\frac{N}{q}-1}_{2,q,\infty}\cap B^{-1}_{\infty,1}\right)}\\ &\lesssim C(\|q_0^1\|_{L^\infty},\|\frac{1}{\rho_0^1}\|_{L^\infty})\big(\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,\infty}}+\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}}\big).\end{aligned}\tag{3.34}$$

techcru

Let us start with recalling some estimates in Besov space for the following system:

$$\begin{cases}\partial_t h^2+u\cdot\nabla h^2+\mathrm{div}u^2=F',\\ \partial_t u^2-\mu\Delta u^2-\mu\nabla\mathrm{div}u^2+a\nabla h^2=G',\\ (h^2,u^2)_{/t=0}=(h_0^2,u_0^2),\end{cases}\tag{3.35}$$

0.2b

with (F',G') external forces. More precisely in [16, 12, 10] it has been proved the following proposition by some different methods. In particular in [16], we introduce the notion of *effective velocity* which enables us to diagonalize the system (3.35).

arma,DG,CMZ

arma

Proposition 3.9 *Let $2\leq p\leq 4$, $q\geq 2$ such that:*

$$\frac{1}{2}\leq\frac{1}{p}+\frac{1}{q},\frac{1}{N}<\frac{1}{p}+\frac{1}{q},\frac{1}{p}\leq\frac{1}{N}+\frac{1}{q}.$$

Let (h^2,u^2) the solution of (3.35). There exists a constant C depending only on μ, N, s and s' such that the following estimate holds:

$$\|(h^2,u^2)(t)\|_{\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}+\|(h^2,u^2)\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}\times \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})}$$
$$\leq Ce^{V(T)}\big(\|(h_0^2,u_0^2)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}+\|(F',G')\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\big).$$

with:

$$V(T)=\int_0^t\big(\|\nabla u^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|\nabla u^2(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}+\|u^1(s)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|u^2(s)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\big)ds.$$

propestim

Proof: For the sake of the completeness we are going to prove this proposition by following the same arguments than in [16].

What happens in low frequencies?
Let us start by studying the following system in low frequencies:

$$(LH)'\qquad\begin{cases}\partial_t h^2+\mathrm{div}u^2=F'-u\cdot\nabla h^2,\\ \partial_t u^2-\mu\Delta u^2-\mu\nabla\mathrm{div}u^2+a\nabla h^2=G'.\end{cases}$$

Danchinbasa **Proposition 3.10** *Let (h^2,u^2) a solution of $(LH)'$, let $s\in\mathbb{R}$. The following estimate holds:*

$$\|(h^2,u^2)_{BF}\|_{\widetilde{L}^\infty(B^s_{2,1})}+\|(h^2,u^2)_{BF}\|_{\widetilde{L}^1(B^{s+2}_{2,1})}\leq\|(h_0^2,u_0^2)_{BF}\|_{B^s_{2,1}}+\|(F',G')_{BF}\|_{\widetilde{L}^1(B^s_{2,1})}$$
$$+\|(u\cdot\nabla h2)_{BF}\|_{\widetilde{L}^1(B^s_{2,1})}.$$

Proof: In this case for $j\leq 0$, in terms of Green matrix (see [10]), the solution of $(LH)'$ can be expressed as:

$$\begin{pmatrix}\Delta_j h^2(t)\\ \Delta_j u^2(t)\end{pmatrix}=W(t)\begin{pmatrix}\Delta_j q_0\\ \Delta_j u_0\end{pmatrix}+\int_0^t W(t-s)\begin{pmatrix}\Delta_j F'(s)-\Delta_j(u^1\cdot\nabla h^2)(s)\\ \Delta_j G'(s)\end{pmatrix}ds,$$

with W the Green matrix. From proposition 4.4 in [10] and Young's inequality we obtain the result.

What happens in high frequencies?
To do this, we are going to consider the following *pressure velocity* where v^2 is the solution of the following equation:

$$-\mu\Delta v^2-\mu\nabla\mathrm{div}v^2=a\nabla h^2.$$

Let $\mathcal{E}$ the fundamental solution of the Laplacian operator and we verify that v^2 is as follows:

$$\begin{cases}v^2=-\mathcal{E}*(\dfrac{a}{2\mu}\nabla h^2)=-\dfrac{a}{2\mu}(\Delta)^{-1}\nabla h^2,\\ \mathrm{div}v^2=-\dfrac{a}{2\mu}h^2,\ \mathrm{curl}v^2=0.\end{cases}\tag{3.36}$$

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We set set now $w^2 = u^2 + v^2$, and we observe that (h^2, w^2) verify the following system (we use in particular the fact that $\mathrm{div} v^2 = -\frac{a}{2\mu} h^2$):

$$\begin{cases} \partial_t h^2 + u \cdot \nabla h^2 + \mathrm{div} w^2 + \dfrac{a}{2\mu} h^2 = F', \\ \partial_t w^2 - \mu \Delta w^2 - \mu \nabla \mathrm{div} w^2 = G' - \dfrac{a}{2\mu} \partial_t (\Delta)^{-1} \nabla h^2, \\ (h^2, u^2)_{/t=0} = (h_0^2, u_0^2). \end{cases} \tag{3.37}$$

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We are going now to apply the proposition 2.6 and following the proof of the proposition 2.8 in order to obtain estimates in Besov space on (h^2, w^2) in high frequencies. We have then if $\frac{N}{p} \in]-\min(\frac{N}{q}, \frac{N}{p'}), \frac{N}{q} + 1]$, then:

$$\begin{aligned} \|h^2_{HF}\|_{\widetilde{L}^\infty_t(B^{\frac{N}{p}}_{p,1})} + \|h^2_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}}_{p,1})} &\lesssim \|(h_0^2)_{HF}\|_{B^{\frac{N}{p}}_{p,1}} + \|F'_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}}_{p,1})} + \|\mathrm{div} w^2_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}}_{p,1})} \\ &\quad + \int_0^t (\|\nabla u^1(\tau)\|_{B^{\frac{N}{q}}_{q,1}} + \|\nabla u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}) \|(h^2(\tau))_{HF}\|_{\widetilde{L}^\infty_\tau(B^{\frac{N}{p}}_{p,1})} d\tau \\ \|w^2_{HF}\|_{\widetilde{L}^\infty(B^{\frac{N}{p}-1}_{p,1})} + \|w^2_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}+1}_{p,1})} &\leq C\big(\|(w_0^2)_{HF}\|_{B^{\frac{N}{p}-1}_{p,1}} + \|G'_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})} \\ &\quad + \|(\partial_t(\Delta)^{-1}\nabla h^2)_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})}\big). \end{aligned} \tag{3.38}$$

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We have then:

$$\begin{aligned} &\|h^2_{HF}\|_{\widetilde{L}^\infty_t(B^{\frac{N}{p}}_{p,1})} + \|h^2_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}}_{p,1})} + \|w^2_{HF}\|_{\widetilde{L}^\infty(B^{\frac{N}{p}-1}_{p,1})} + \|w^2_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}+1}_{p,1})} \lesssim \\ &\|(h_0^2)_{HF}\|_{B^{\frac{N}{p}}_{p,1}} + \|F'_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}}_{p,1})} + \int_0^t \|\nabla u^1(\tau)\|_{B^{\frac{N}{q}}_{q,1}} \|(h^2(\tau))_{HF}\|_{\widetilde{L}^\infty_\tau(B^{\frac{N}{p}}_{p,1})} d\tau \\ &+ \int_0^t \|\nabla u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}} \|(h^2(\tau))_{HF}\|_{\widetilde{L}^\infty_\tau(B^{\frac{N}{p}}_{p,1})} d\tau + \|(w_0^2)_{HF}\|_{B^{\frac{N}{p}-1}_{p,1}} + \|G'_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})} \\ &\quad + \|(\partial_t(\Delta)^{-1}\nabla h^2)_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})}. \end{aligned} \tag{3.39}$$

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In low frequencies we are going to work with the unknown (h^2, u^2) and applying the proposition 3.10 to the system (3.35) we have:

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$$\begin{aligned} \|(h^2, u^2)_{BF}\|_{\widetilde{L}^\infty_t(B^{\frac{N}{2}-1}_{2,1})} + \|(h^2, u^2)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}+1}_{2,1})} &\lesssim \|(h_0^2, u_0^2)_{BF}\|_{B^{\frac{N}{2}-1}_{2,1}} \\ &\quad + \|(u \cdot \nabla h^2)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})} + \|(F', G')_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})}. \end{aligned} \tag{3.40}$$

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Since we have $u^2 = w^2 - v^2$ and that $v^2 = -\frac{a}{2\mu}(\Delta)^{-1}\nabla h^2$ it implies by proposition ?? that:

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$$\begin{aligned} \|u^2_{HF}\|_{\widetilde{L}^\infty(B^{\frac{N}{p}-1}_{p,1})} + \|u^2_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}+1}_{p,1})} &\lesssim \|w^2_{HF}\|_{\widetilde{L}^\infty(B^{\frac{N}{p}-1}_{p,1})} + \|w^2_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}+1}_{p,1})} \\ &\quad + \|h^2_{HF}\|_{\widetilde{L}^\infty(B^{\frac{N}{p}-2}_{p,1})} + \|h^2_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}. \end{aligned} \tag{3.41}$$

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Combining (3.39), (3.40) and (3.41) we have:

$$\begin{aligned}\|h^2\|_{\widetilde{L}_t^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}+\|h^2\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})}+\|u^2\|_{\widetilde{L}_t^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{p,1})}+\|u^2\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})}\lesssim\\ \big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}+\|u_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}+\|F^{'}\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}+\|G^{'}\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\\ +\int_0^t\|\nabla u^1(\tau)\|_{B^{\frac{N}{q}}_{q,1}}\|(h^2(\tau))_{HF}\|_{\widetilde{L}^\infty_\tau(B^{\frac{N}{p}}_{p,1})}d\tau+\|(\partial_t(\Delta)^{-1}\nabla h^2)_{HF}\|_{\widetilde{L}_t^1(B^{\frac{N}{p}-1}_{p,1})}\\ +\int_0^t\|\nabla u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\|(h^2(\tau))_{HF}\|_{\widetilde{L}^\infty_\tau(B^{\frac{N}{p}}_{p,1})}d\tau+\|(u\cdot\nabla h^2)_{BF}\|_{\widetilde{L}_t^1(B^{\frac{N}{2}-1}_{2,1})}\big).\end{aligned}\tag{3.42}$$

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In other word we have:

$$\begin{aligned}\|h^2\|_{\widetilde{L}_t^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}+\|h^2\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})}+\|u^2\|_{\widetilde{L}_t^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{p,1})}+\|u^2\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{p,1})}\leq\\ \big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}+\|u_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}+\|F^{'}\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}+\|G^{'}\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\\ +\int_0^t\|\nabla u^1(\tau)\|_{B^{\frac{N}{q}}_{q,1}}\|h^2(\tau)\|_{\widetilde{L}^\infty_\tau(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}d\tau+\|(\partial_t(\Delta)^{-1}\nabla h^2)_{HF}\|_{\widetilde{L}_t^1(B^{\frac{N}{p}-1}_{p,1})}\\ +\int_0^t\|\nabla u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\|h^2(\tau)\|_{\widetilde{L}^\infty_\tau(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}d\tau+\|(u\cdot\nabla h^2)_{BF}\|_{\widetilde{L}_t^1(B^{\frac{N}{2}-1}_{2,1})}\big).\end{aligned}\tag{3.43}$$

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It remains only to treat the terms on the right hand side of (3.43). We verify via the first equation of (3.35) that:

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$$\partial_t(\Delta)^{-1}\nabla h^2=(\Delta)^{-1}\nabla F^{'}-(\Delta)^{-1}\nabla\mathrm{div}u^2-(\Delta)^{-1}\nabla(\mathrm{div}(uh^2)-h^2\mathrm{div}u).$$

We have then:

$$\begin{aligned}\|(\partial_t(\Delta)^{-1}\nabla h^2)_{HF}\|_{\widetilde{L}_t^1(B^{\frac{N}{p}-1}_{p,1})}\lesssim\|F^{'}\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}+\|u^2_{HF}\|_{\widetilde{L}_t^1(B^{\frac{N}{p}-1}_{p,1})}\\ +\|(uh^2)_{HF}\|_{\widetilde{L}_t^1(B^{\frac{N}{p}-1}_{p,1})}+\|(h^2\mathrm{div}u)_{HF}\|_{\widetilde{L}_t^1(B^{\frac{N}{p}-2}_{p,1})}.\end{aligned}\tag{3.44}$$

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Let us deal now with the term $(u^1\cdot\nabla h^2)_{BF}$ in (3.43), similarly by using proposition 2.5, 2.4 and interpolation we have with $\lambda=\lambda'=+\infty$:

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$$\begin{aligned}\|(u^1\cdot\nabla h^2)_{BF}\|_{\widetilde{L}_t^1(B^{\frac{N}{2}-1}_{2,1})}\leq\|T_{\nabla h^2}u^1\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1})}+\|T_{u^1}\nabla h^2\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\\ +\|R(u^1,\nabla h^2)_{BF}\|_{\widetilde{L}_t^1(B^{\frac{N}{2}-1}_{2,1})}\end{aligned}\tag{3.45}$$

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$$\begin{aligned}\|T_{\nabla h^2}u^1\|_{\widetilde{L}_t^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1})}&\lesssim\int_0^t\|\nabla h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-1}_{2,p,1}}\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}}d\tau\\ &\lesssim\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}}d\tau.\end{aligned}$$

Similarly we have by proposition 2.5, 2.4 and Young inequality for any $\epsilon > 0$:

$$\begin{aligned}
\|T_{u^1}\nabla h^2\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})} &\lesssim \int_0^t \|u^1(\tau)\|_{L^\infty}\|\nabla h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}d\tau,\\
&\lesssim \int_0^t \|u^1(\tau)\|_{L^\infty}\|h^2(\tau)\|^{\frac{1}{2}}_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}\|h^2(\tau)\|^{\frac{1}{2}}_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau, \qquad (3.46)\\
&\lesssim C_\epsilon\int_0^t \|u^1(\tau)\|^2_{B^0_{\infty,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau + \epsilon\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau.
\end{aligned}$$

test2

By proposition 2.5 we have with $s_1 = \frac{N}{2}$, $s_3 = \frac{N}{q}$, $s_2 = \frac{N}{2}-1$, $s_4 = \frac{N}{p}-1$, $p_1 = p_2 = 2$, $p_3 = q$ and $p_4 = p$ and since $p \geq 2$, $q \geq 2$, $\frac{1}{2} \leq \frac{1}{p}+\frac{1}{q}$, $\frac{1}{p}+\frac{1}{q} > \frac{1}{N}$:

$$\begin{aligned}
\|R(\nabla h^2, u^1)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})} &\lesssim \int_0^t \|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}\|\nabla h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}d\tau,\\
&\lesssim \int_0^t \|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}d\tau, \qquad (3.47)\\
&\lesssim C_\epsilon\int_0^t \|u^1(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau + \epsilon\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau.
\end{aligned}$$

test3

Finally we have obtained for any $\epsilon > 0$ and $C'_\epsilon > 0$ sufficiently large:

$$\begin{aligned}
\|(u^1\cdot\nabla h^2)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})} &\leq \int_0^t \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}}d\tau\\
&+\epsilon\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau + C'_\epsilon\int_0^t(\|u^1(\tau)\|^2_{B^0_{\infty,1}} + \|u^1(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau.
\end{aligned} \qquad (3.48)$$

convecfi

Similarly we are going to deal with the term $(u^2\cdot\nabla h^2)_{BF}$. We have then:

$$\begin{aligned}
\|(u^2\cdot\nabla h^2)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})} &\leq \|T_{\nabla h^2}u^2\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})} + \|T_{u^2}\nabla h^2\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\\
&\quad + \|R(u^2,\nabla h^2)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})}
\end{aligned} \qquad (3.49)$$

batest1

$$\begin{aligned}
\|T_{\nabla h^2}u^2\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})} &\lesssim \int_0^t \|\nabla h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-1}_{2,p,1}}\|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}}d\tau\\
&\lesssim \int_0^t \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}}d\tau.
\end{aligned}$$

Similarly we have by proposition 2.5, 2.4 and Young inequality for any $\epsilon > 0$:

$$\begin{aligned}
\|T_{u^2}\nabla h^2\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})} &\lesssim \int_0^t \|u^2(\tau)\|_{L^\infty}\|\nabla h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}d\tau,\\
&\lesssim \int_0^t \|u^2(\tau)\|_{L^\infty}\|h^2(\tau)\|^{\frac{1}{2}}_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}\|h^2(\tau)\|^{\frac{1}{2}}_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau, \qquad (3.50)\\
&\lesssim C_\epsilon\int_0^t \|u^2(\tau)\|^2_{B^0_{\infty,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau + \epsilon\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau.
\end{aligned}$$

btest2

It remains now to deal with the remainder $R(u^2,\nabla h^2)$, it gives using the proposition hybrid 2.5 since $2\leq p\leq 4$:

$$\|R(\nabla h^2,u^2)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})}\lesssim \int_0^t \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\|\nabla h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}d\tau,$$

$$\lesssim \int_0^t \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}d\tau, \tag{3.51}$$

btest3

$$\lesssim C_\epsilon\int_0^t \|u^2(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau+\epsilon\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau.$$

It gives then:

$$\|(u^2\cdot\nabla h^2)_{BF}\|_{\widetilde{L}^1_t(B^{\frac{N}{2}-1}_{2,1})}\lesssim\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}\|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}}d\tau$$

$$+\epsilon\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau+C'_\epsilon\int_0^t(\|u^2(\tau)\|^2_{B^0_{\infty,1}}+\|u^2(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau. \tag{3.52}$$

bconvecfi

Let us deal now with the term $(u^1h^2)_{HF}$ and $(u^2h^2)_{HF}$ in the right hand side of (rimp4 3.43), by using proposition hybrid 2.5 we have since $\frac{1}{p}\leq\frac{1}{N}+\frac{1}{q}$, interpolation and Young inequality we have:

$$\|(u^1h^2)_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})}\lesssim\int_0^t\big(\|T_{u^1}h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}+\|T_{h^2}u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}-1}_{2,p,1}}$$

$$+\|(R(u^1,h^2))_{HF}(\tau)\|_{B^{\frac{N}{p}-1}_{p,1}}\big)d\tau$$

$$\int_0^t\big(\|T_{u^1}h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}+\|T_{h^2}u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}-1}_{2,p,1}}d\tau$$

$$\lesssim\int_0^t(\|u^1(\tau)\|_{B^0_{\infty,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}+\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-2}_{2,p,1}}\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}})d\tau$$

$$\lesssim C_\epsilon\int_0^t(\|u^1(\tau)\|^2_{B^0_{\infty,1}}+\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau+\epsilon\int_0^t\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau. \tag{3.53}$$

impfi1

In the same way we have since $\frac{1}{p}+\frac{1}{q}>\frac{1}{N}$:

$$\int_0^t\|(R(u^1,h^2))_{HF}(\tau)\|_{B^{\frac{N}{p}-1}_{p,1}}d\tau\lesssim\int_0^t\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}-1}_{2,q,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau. \tag{3.54}$$

rimpa

Similarly we obtain:

$$\|(u^2h^2)_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})} \lesssim \int_0^t \big(\|T_{u^2}h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}} + \|T_{h^2}u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}-1}_{2,p,1}} + \|(R(u^2,h^2))_{HF}(\tau)\|_{B^{\frac{N}{p}-1}_{p,1}}\big)d\tau.$$

$$\int_0^t \big(\|T_{u^2}h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}} + \|T_{h^2}u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}-1}_{2,p,1}} d\tau$$
$$\lesssim \int_0^t (\|u^2(\tau)\|_{B^0_{\infty,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}} + \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-2}_{2,p,1}}\|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}})d\tau$$
$$\lesssim C_\epsilon \int_0^t (\|u^2(\tau)\|^2_{B^0_{\infty,1}} + \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} d\tau + \epsilon \int_0^t \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}} d\tau. \tag{3.55}$$

impfi1b

In the same way we have since $p \leq 4 < 2N$:

$$\int_0^t \|(R(u^2,h^2))_{HF}(\tau)\|_{B^{\frac{N}{p}-1}_{p,1}} d\tau \lesssim \int_0^t \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}-1}_{2,p,1}}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} d\tau. \tag{3.56}$$

rimpab

Finally combining (impfi1 3.53), (rimpa 3.54), (impfi1 3.53) and (rimpab 3.56) we have for any $\epsilon > 0$ and C'_ϵ sufficiently large:

$$\|(u\,h^2)_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})}$$
$$\lesssim C'_\epsilon \int_0^t (\|u^1(\tau)\|^2_{B^0_{\infty,1}} + \|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}} + \|u^2(\tau)\|^2_{B^0_{\infty,1}} + \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} d\tau$$
$$+ \epsilon \int_0^t \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}} d\tau + \int_0^t (\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}-1}_{2,q,1}} + \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}-1}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} d\tau \tag{3.57}$$

superimo

Let us treat now the terms $(\mathrm{div}u^1h^2)_{HF}$ and $(\mathrm{div}u^2h^2)_{HF}$. We have then using again the proposition hybrid 2.5 since $\frac{1}{p} \leq \frac{1}{N} + \frac{1}{q}$:

$$\|(h^2\mathrm{div}u^1)_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-2}_{p,1})} \lesssim \int_0^t \big(\|T_{\mathrm{div}u^1}h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|T_{h^2}\mathrm{div}u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} + \|(R(\mathrm{div}u^1,h^2))_{HF}(\tau)\|_{B^{\frac{N}{p}}_{p,1}}\big)d\tau$$
$$\lesssim \int_0^t (\|\mathrm{div}u^1(\tau)\|_{L^\infty}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}\|\mathrm{div}u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}})d\tau. \tag{3.58}$$

impfi2

Similarly we have:

$$\|(h^2\mathrm{div}u^2)_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-2}_{p,1})} \lesssim \int_0^t \big(\|T_{\mathrm{div}u^2}h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|T_{h^2}\mathrm{div}u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} + \|(R(\mathrm{div}u^2,h^2))_{HF}(\tau)\|_{B^{\frac{N}{p}}_{p,1}}\big)d\tau$$

$$\lesssim \int_0^t (\|\mathrm{div}u^2(\tau)\|_{L^\infty}\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}\|\mathrm{div}u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}})d\tau. \tag{3.59}$$

cimpfi2

Let us combine (3.43), (3.44), (3.48), (3.52), (3.57), (3.58) and (3.59) we have for $\epsilon >$ and C'_ϵ sufficiently large:

$$\|h^2\|_{\widetilde{L}^\infty_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})} + \|h^2\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})} + \|u^2\|_{\widetilde{L}^\infty_t(\widetilde{B}^{\frac{N}{p}-1,\frac{N}{p}-1}_{p,1})} + \|u^2\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{p}+1}_{p,1})} \leq$$

$$\Big(\|h^2_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|u^2_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} + \|F'\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})} + \|G'\|_{\widetilde{L}^1_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}$$

$$+ \int_0^t (\|\nabla u^1(\tau)\|_{B^{\frac{N}{q}}_{q,1}} + \|\nabla u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{L}^\infty_\tau(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}d\tau + \|u^2_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})}$$

$$+ \int_0^t \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}(\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}} + \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}})d\tau$$

$$+ C'_\epsilon \int_0^t (\|u^1(\tau)\|^2_{B^0_{\infty,1}} + \|u^1(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}} + \|u^2(\tau)\|^2_{B^0_{\infty,1}} + \|u^2(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau \tag{3.60}$$

rimp4cc

$$+ \epsilon \int_0^t \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}}d\tau + C'_\epsilon \int_0^t (\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}} + \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau$$

$$+ \int_0^t (\|u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}-1}_{2,q,1}} + \|u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}-1}_{2,p,1}})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau$$

$$+ \int_0^t ((\|\mathrm{div}u^1(\tau)\|_{L^\infty} + \|\mathrm{div}u^2(\tau)\|_{L^\infty})\|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}d\tau$$

$$+ \int_0^t \|h^2(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}(\|\mathrm{div}u^1(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}} + \|\mathrm{div}u^2(\tau)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}})d\tau.$$

It remains only to deal with the term $\|u^2_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})}$ on the right hand side of (3.60), we recall that:

$$\|u^2_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}-1}_{p,1})} \leq \frac{1}{2^{2l_0}}\|u^2_{HF}\|_{\widetilde{L}^1_t(B^{\frac{N}{p}+1}_{p,1})}. \tag{3.61}$$

hautefre

Combining (3.60), (3.61), taking l_0 large enough in order to apply a bootstrap argument in (3.60) and using Gronwall lemma we obtain the desired result for a $C > 0$:

$$\|(h^2,u^2(t)\|_{\widetilde{L}^\infty_t(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})} + \|(h^2,u^2)\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})}$$

$$\leq Ce^{V(t)}\big(\|(h^2_0,u^2_0)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} + \|(F',G')\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\big).$$

with:

$$V(t)=\int_0^t \big(\|\nabla u^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|\nabla u^2(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}+\|u^1(s)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|u^2(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\big)ds.$$

□

We are going to generalize the proposition propestim 3.9 to the case of the system (0.2a 3.30), more precisely we have the following proposition.

Danchinbas **Proposition 3.11** *Let $2\leq p\leq 4$, $q\geq 2$ such that:*

$$\frac{1}{2}\leq\frac{1}{p}+\frac{1}{q},\frac{1}{N}<\frac{1}{p}+\frac{1}{q},\frac{1}{p}\leq\frac{1}{N}+\frac{1}{q}.$$

Let (h^2,u^2) the solution of (0.2b 3.35). There exists a constant C depending only on μ, N, s and $s^{'}$ such that the following estimate holds:

$$\begin{aligned}&\|(h^2,u^2)(t)\|_{\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}+\|(h^2,u^2)\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}\times\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})}\\&\leq Ce^{V(T)}\big(\|(h_0^2,u_0^2)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}+\|(F',G')\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\big).\end{aligned}$$

with:

$$\begin{aligned}V'(T)=C\int_0^t&\big(\|\nabla u^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|\nabla u^2(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}+\|u^1(s)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|u^2(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\\&+\|\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}}+\|\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}+1}_{2,q,1}}+\|\nabla\ln\rho^1(s)\|^2_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1}}\big)ds.\end{aligned}$$

Proof: By using the proposition propestim 3.9 we obtain that:

$$\begin{aligned}&\|(h^2,u^2)\|_{\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}+\|(h^2,u^2)\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}\times\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})}\\&\leq Ce^{V(T)}\big(\|(h_0^2,u_0^2)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}+\|(F_1,G_1)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\times\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\big).\end{aligned}\tag{3.62}$$

1estimimp

with:

$$\begin{aligned}F_1&=F-u^2\cdot\nabla\ln\rho^1,\\G_1&=G-u^2\cdot\nabla u_1-u_1\cdot\nabla u^2+2\mu\nabla\ln\rho^1\cdot Du^2+2\mu\nabla h^2\cdot Du^1,\end{aligned}$$

with:

$$V(t)=\int_0^t\big(\|\nabla u^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|\nabla u^2(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}+\|u^1(s)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}+\|u^2(\tau)\|^2_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\big)ds.$$

Therefore, it is only a matter of proving appropriate estimates for F_1, G_1 by using properties of continuity on the paraproduct.

We start with the first term $u^2\cdot\nabla\ln\rho^1$, by propositions hybrid, interpolation 2.5, 2.1, composition estimates, Hölder inequality and since $\frac{1}{p}\leq\frac{1}{N}+\frac{1}{q}$, $\frac{1}{2}\leq\frac{1}{p}+\frac{1}{q}$ we have for any $\epsilon>0$:

$$\begin{aligned}
&\int_0^T\|u^2\cdot\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}}ds\lesssim\int_0^T(\|\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1}}\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}}\\
&+\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}\|\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}+1}_{2,q,1}}+\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}\|\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}})ds,\\
&\lesssim\int_0^T\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}(\|\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{q}+1}_{2,q,1}}+\|\nabla\ln\rho^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}+1}_{2,q,1}}\\
&\qquad+C_\epsilon\|\nabla\ln\rho^1(s)\|^2_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1}})+\epsilon\int_0^T\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}}ds,
\end{aligned}\tag{3.63}$$

1estim

with $C_\epsilon>0$ sufficiently large.
We are now going to deal with the terms of G_1, let us start with the term $u^2\cdot\nabla u^1$, we have by proposition hybrid 2.5 and the fact that $\frac{1}{p}\leq\frac{1}{N}+\frac{1}{q}$, $\frac{1}{p}+\frac{1}{q}>\frac{1}{N}$, $\frac{1}{2}\leq\frac{1}{p}+\frac{1}{q}$:

$$\begin{aligned}
&\int_0^T\|u^2\cdot\nabla u^1(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds\lesssim\int_0^T\|u^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}\|\nabla u^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}ds\\
&\int_0^T\|\nabla h^2\cdot Du^1(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds\lesssim\int_0^T\|\nabla h^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}\|\nabla u^1(s)\|_{\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1}}ds.
\end{aligned}\tag{3.64}$$

estim2

Similarly we have (since $\frac{1}{p}\leq\frac{1}{N}+\frac{1}{q}$, $\frac{1}{p}+\frac{1}{q}>\frac{1}{N}$, $\frac{1}{2}\leq\frac{1}{p}+\frac{1}{q}$) for all $\epsilon>0$ there exists $C_\epsilon>0$ sufficiently large such that:

$$\begin{aligned}
&\int_0^T\|\nabla\ln\rho^1\cdot Du^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds\lesssim\int_0^T\|\nabla\ln\rho^1\|_{\widetilde{B}^{\frac{N}{p},\frac{N}{q}}_{2,q,1}}\|Du^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds\\
&\lesssim\epsilon\int_0^T\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds+C_\epsilon\int_0^T\|\nabla\ln\rho^1\|^2_{\widetilde{B}^{\frac{N}{p},\frac{N}{q}}_{2,q,1}}\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds.
\end{aligned}\tag{3.65}$$

aestim2

and:

$$\begin{aligned}
&\int_0^T\|u^1\cdot\nabla u^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}\\
&\lesssim\epsilon\int_0^T\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds+C_\epsilon\int_0^T\|u^1\|^2_{\widetilde{B}^{\frac{N}{p},\frac{N}{q}}_{2,q,1}}\|u^2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}}ds.
\end{aligned}\tag{3.66}$$

estim3

By combining the estimates (1estimimpr, 1estim, estim2, aestim2, estim3 3.62), (3.63), (3.64), (3.65), (3.66) and Gronwall lemma, it achieves the proof of the proposition.

4 The proof of theorem theo1 1.1

section5

4.1 Proof of the existence

We recall here that $(\rho^1,u^1)=(\rho^1,-2\mu\nabla\ln\rho^1)$ is a quasi-solution of the system (0.1 1.1) when:

$$\begin{cases}\partial_t\rho^1-2\mu\Delta\rho^1=0,\\ \rho^1_{t=0}=\rho^1_0.\end{cases}\tag{4.67}$$

It means that (ρ^1, u^1) is an irrotational solution of the pressure less system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu\rho D(u)) = 0, \\ (\rho, u)_{/t=0} = (\rho_0^1, u_0^1). \end{cases} \tag{4.68}$$ 0.1bis

Our goal now consists in solving the system (0.2 1.6) in order to obtain global strong solution of the system (0.1 1.1) under the form $\rho = \rho^1 \epsilon^{h^2}$, $u = u^1 + u^2$ with $\rho \geq c^{'} > 0$. To do this, we use a standard scheme:

1. We smooth out the data and get a sequence of local solutions $(h_n^2, u_n^2)_{n\in\mathbb{N}}$ on $[0, T_n]$ to (0.2 1.6) by using the result of JDE [15].
2. We prove uniform estimates on (h_n^2, u_n^2) on $[0, T_n]$ by using the proposition Danchinbas 3.11 and we deduce that $T_n = +\infty$.
3. We use compactness to prove that the sequence (h_n^2, u_n^2) converges, up to extraction, to a solution of (0.2 1.6).

Construction of approximate solutions

We smooth out the data as follows:

$$(h_0^2)_n = S_n h_0^2 \text{ and } (u_0^2)_n = S_n u_0^2 \quad .$$

Note that we have:

$$\forall l \in \mathbb{Z}, \ \ \|\Delta_l (h_0^2)_n\|_{L^p} \leq \|\Delta_l h_0^2\|_{L^p} \quad \text{and} \quad \|(h_0^2)_n\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} \leq \|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}},$$

and similar properties for $(u_0^2)_n$, a fact which will be used repeatedly during the next steps. Now, according JDE [15], one can solve (0.2 1.6) with the smooth data $((q_0^2)_n, (u_0^2)_n)$. We get a solution (h_n^2, u_n^2) on a non trivial time interval $[0, T_n]$ such that:

$$h_n^2 \in \widetilde{C}([0,T_n], \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}) \ \ u_n^2 \in \widetilde{C}([0,T_n], \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}) \cap \widetilde{L}^1([0,T_n], \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}). \tag{4.69}$$ a26

Uniform bounds

In the sequel we set:

$$\ln \rho_n = \ln \rho^1 + h_n^2 \text{ and } u_n = u^1 + u_n^2.$$

We recall that (h_n^2, u_n^2) satisfies the following system:

$$\begin{cases} \partial_t h_n^2 + u^1 \cdot \nabla h_n^2 + \operatorname{div} u_n^2 + u_n^2 \cdot \nabla \ln \rho^1 + u_n^2 \cdot \nabla h_n^2 = 0, \\ \partial_t u_n^2 + u^1 \cdot \nabla u_n^2 + u_n^2 \cdot \nabla u^1 - 2\mu \nabla \ln \rho^1 \cdot Du_n^2 - 2\mu \nabla h_n^2 \cdot Du^1 - \mu \Delta u_n^2 - \mu \nabla \operatorname{div} u_n^2 \\ \qquad\qquad + a\nabla h_n^2 = -a\nabla \ln \rho^1 - u_n^2 \cdot \nabla u_n^2 + 2\mu \nabla h_n^2 \cdot Du_n^2, \\ (h^2, u^2)_{/t=0} = ((h_0^2)_n, (u_0^2)_n). \end{cases} \tag{4.70}$$ 0.2aa

In this part, we aim at getting uniform estimates on (h_n^2, u_n^2) in the following space F_T with the norm $\|\cdot\|_{F_T}$:

$$F_T = \big(\widetilde{L}_T^\infty(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}}) \cap \widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}+1,\frac{N}{p}})\big) \times \big(\widetilde{L}_T^\infty(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1}) \cap \widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}+1,\frac{N}{p}+1})\big).$$

$$\|(h_n^2, u_n^2)\|_{F_T} = \|h_n^2\|_{\widetilde{L}_T^\infty(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}}) \cap \widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}+1,\frac{N}{p}})} + \|u_n^2\|_{\widetilde{L}_T^\infty(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1}) \cap \widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}+1,\frac{N}{p}+1})}.$$

We can observe that (h_n^2, u_n^2) verifies exactly the system (3.30) with:

$$\begin{aligned} F^n &= 0, \\ G^n &= -a\nabla \ln \rho^1 - u_n^2 \cdot \nabla u_n^2 + 2\mu \nabla h_n^2 \cdot Du_n^2. \end{aligned}$$

By using proposition 3.11, we have the following estimate on (h_n^2, u_n^2):

$$\|(h_n^2, u_n^2)\|_{F_T} \leq Ce^{V_n(T)}\big(\|(h_0^2)_n\|_{\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}}} + \|(u_0^2)_n\|_{\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1}} + \|G^n\|_{\widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1})}\big), \tag{4.71}$$

with:

$$\begin{aligned} V_n(T) = C\int_0^t \big(&\|\nabla u^1(s)\|_{\widetilde{B}_{2,q,1}^{\frac{N}{2},\frac{N}{q}}} + \|\nabla u_n^2(s)\|_{\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}}} + \|u^1(s)\|^2_{\widetilde{B}_{2,q,1}^{\frac{N}{2},\frac{N}{q}}} + \|u_n^2(\tau)\|^2_{\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}}} \\ &+ \|\nabla \ln \rho^1(s)\|_{\widetilde{B}_{2,q,1}^{\frac{N}{2}+1,\frac{N}{q}+1}} + \|\nabla \ln \rho^1(s)\|_{\widetilde{B}_{2,q,1}^{\frac{N}{2},\frac{N}{q}+1}} + \|\nabla \ln \rho^1(s)\|^2_{\widetilde{B}_{2,q,1}^{\frac{N}{2}-1,\frac{N}{q}}}\big)ds. \end{aligned} \tag{4.72}$$

We set in the sequel:

$$\begin{aligned} V_1(T) = C\int_0^t \big(&\|\nabla u^1(s)\|_{\widetilde{B}_{2,q,1}^{\frac{N}{2},\frac{N}{q}}} + \|u^1(s)\|^2_{\widetilde{B}_{2,q,1}^{\frac{N}{2},\frac{N}{q}}} + \|\nabla \ln \rho^1(s)\|_{\widetilde{B}_{2,q,1}^{\frac{N}{2}+1,\frac{N}{q}+1}} \\ &+ \|\nabla \ln \rho^1(s)\|_{\widetilde{B}_{2,q,1}^{\frac{N}{2},\frac{N}{q}+1}} + \|\nabla \ln \rho^1(s)\|^2_{\widetilde{B}_{2,q,1}^{\frac{N}{2}-1,\frac{N}{q}}}\big)ds. \end{aligned}$$

$$V_2^n(T) = C\int_0^t (\|\nabla u_n^2(s)\|_{\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}}} + \|u_n^2(\tau)\|^2_{\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}}})ds.$$

Therefore, it is only a matter of proving appropriate estimates for G^n by using proposition 2.5. Let us deal now with the term G^n. We begin by estimating $\nabla h_n^2 \cdot Du_n^2$ and we obtain by proposition 2.5 and since $p < 2N$:

$$\|\nabla h_n^2 \cdot Du_n^2\|_{\widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1})} \lesssim \|\nabla h_n^2\|_{\widetilde{L}_T^\infty(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1})} \|Du_n^2\|_{\widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}. \tag{4.73}$$

Similarly we have:

$$\|u_n^2 \cdot \nabla u_n^2\|_{\widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1})} \lesssim \|u_n^2\|_{\widetilde{L}_T^\infty(\widetilde{B}_{2,p,1}^{\frac{N}{2}-1,\frac{N}{p}-1})} \|\nabla u_n^2\|_{\widetilde{L}_T^1(\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}. \tag{4.74}$$

It remains to deal with the most important term $a\nabla \ln \rho^1$, we have by using propositions 2.5, 2.3, 2.6 and the maximum principle for ρ^1 (indeed we recall that $\rho^1 = 1 + q^1$ with q^1

verifying a heat equation) that it exists regular functions g_1, g_2 such that:

$$\begin{aligned}\|a\nabla \ln \rho^1\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})} &\lesssim \|\ln \rho^1\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})},\\ &\lesssim g_1(\|\frac{1}{\rho^1}\|_{L^\infty}, \|\rho^1\|_{L^\infty})\|q^1\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}, \qquad (4.75)\\ &\lesssim g_2(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}}\end{aligned}$$

estimb3

Finally by combining the estimates (4.71), (4.73), (4.74) and (4.75) we have:

$$\begin{aligned}\|(h^2_n,u^2_n)\|_{F_T} \leq Ce^{V_1(T)+V_2^n(T)}\big(&\|(h^2_0)_n\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|(u^2_0)_n\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} + \|(h^2_n,u^2_n)\|^2_{F_T}\\ &+ g_1(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{q}-2}_{2,p,1}}\big).\end{aligned} \qquad (4.76)$$

crucial estimb1 estimb2 estimb3 superimp

In particular we have:

$$\|(h^2_n,u^2_n)\|_{F_T} \leq Ce^{V_1(T)+V_2^n(T)}\big(\epsilon_0 + \|(h^2_n,u^2_n)\|^2_{F_T} + g_1(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}}\big). \qquad (4.77)$$

superimp1

We are going to prove by a simple bootstrapping argument that for any $T \in (0, T_n)$:

$$\|(h^2_n,u^2_n)\|_{F_T} \leq 4Ce^{V_1(+\infty)}g_1(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}} = \epsilon_1. \qquad (4.78)$$

uniformestim

We are going to set in the sequel $\alpha = g_1(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}}$.

It is right by continuity on a small interval $(0, T'_n)$ choosing ϵ_0 sufficiently small. Let us set T^1_n (with $(T_1)^n < T_n$) the maximal time such that (4.78) remains true. Using (4.77) we observe that:

$$\|(h^2_n,u^2_n)\|_{F_{T^1_n}} \leq Ce^{V_1(+\infty)}\alpha\, e^{\epsilon_1}\big(1 + \frac{\epsilon_0}{\alpha} + \frac{\epsilon_1^2}{\alpha}\big) \qquad (4.79)$$

superimp2

Now we have to choose ϵ_0 and ϵ_1 such that:

$$e^{\epsilon_1}\big(1 + \frac{\epsilon_0}{\alpha} + \frac{\epsilon_1^2}{\alpha}\big) \leq \frac{24}{7}. \qquad (4.80)$$

superimp3

In particular it suffices to take ϵ_1 and ϵ_0 sufficiently small such that we have:

$$e^{\epsilon_1} \leq \frac{8}{7},\ \epsilon_1^2 \leq \alpha,\ \epsilon_0 \leq \alpha. \qquad (4.81)$$

superimp4

In conclusion there exists ϵ_2 sufficiently small such that if:

$$\begin{aligned}&4Ce^{V_1(+\infty)}g_1(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}} \leq \epsilon_2\\ &16C^2e^{2V_1(+\infty)}g_1(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}} \leq 1 \qquad (4.82)\\ &\epsilon_0 \leq g_1(\|\frac{1}{\rho^1_0}\|_{L^\infty}, \|\rho^1_0\|_{L^\infty})\|q^1_0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}},\end{aligned}$$

superimp5

then by a classical bootstrap argument we can show that $(T_1)^n = T_n$. In particular by using proposition 2.6 and 2.3 we deduce that it exists a regular function K_1 such that:

$$e^{-2V_1(+\infty)} \geq e^{-K_1(\|\frac{1}{\rho_0^1}\|_{L^\infty}, \|q_0^1\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1}}}. \tag{4.83}$$

impes13

Combining (4.82) and (4.83) we have the following condition:

$$16C^2 g_1(\|\frac{1}{\rho_0^1}\|_{L^\infty}, \|\rho_0^1\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{p}-2}_{2,p,1}} \leq e^{-K_1(\|\frac{1}{\rho_0^1}\|_{L^\infty}, \|q_0^1\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{q}}_{2,q,1}}}. \tag{4.84}$$

impes14

This last condition corresponds exactly to the assumption (1.7) of theorem 1.1.
We now are interested in proving a continuation criterion in order to prove that $T_n = +\infty$ when (4.78) is verified. Since (h_n^2, u_n^2) is bounded in F_{T^n}, it implies in particular that $\nabla u_n = \nabla u^1 + \nabla u_n^2$ is bounded in $\widetilde{L}^1_{T_n}(\widetilde{B}^{\frac{N}{2},\frac{N}{q}}_{2,q,1} + B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})$ (in particular u is Lipschitz by Besov injection) then by classical continuation criterion (see [13, 17]) we deduce that for any n, $T_n = +\infty$.

Compactness arguments

We have proved that our sequence $(h_n^2, u_n^2)_{n\in\mathbb{N}}$ of solutions of (1.6) with initial data $((h_0^2)_n, (u_0^2)_n)$ is uniformly bounded in $n \in \mathbb{N}$ in:

$$\big(\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})\big) \times \widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})^N.$$

It remains now to prove that up to a subsequence $(h_n^2, u_n^2)_{n\in\mathbb{N}}$ converges in the sense of distribution to a solution (h^2, u^2) of the system (1.6) with (h^2, u^2) belonging to the space:

$$\big(\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})\big) \times \widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})^N.$$

It suffices to use the Ascoli theorem, this is classic and we refer to [1] for the proof.

The proof of the uniqueness

In the case $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{p_1}$, the uniqueness has been established in [1], see also [14].

4.2 Proof of the theorem 1.2 and 1.3

In the theorem (1.2), the smallness assumption is different since we suppose that K is sufficiently small. Indeed by proceeding similarly as in the proof of the theorem 1.1, we obtain the inequality (4.76) at the difference that we take into account the constant K which gives:

$$\begin{aligned}\|(h_n^2, u_n^2)\|_{F_T} \leq Ce^{V_1(T)+V_2^n(T)}\big(&\|(h_0^2)_n\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|(u_0^2)_n\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}} + \|(h_n^2, u_n^2)\|_{F_T}^2 \\ &+ Kg_1(\|\frac{1}{\rho_0^1}\|_{L^\infty}, \|\rho_0^1\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{q}-2}_{2,p,1}}\big).\end{aligned} \tag{4.85}$$

superimpa

We have now to change slightly our bootstrap argument, indeed we are going to set T_n^1 the maximal time such that:

$$\|(h_n^2,u_n^2)\|_{F_{T_n^1}}\leq 4CKe^{V_1(+\infty)}g_1(\|\frac{1}{\rho_0^1}\|_{L^\infty},\|\rho_0^1\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{q}-2}_{2,p,1}})=\epsilon_1'. \tag{4.86}$$ cruim

We set $\alpha'=Kg_1(\|\frac{1}{\rho_0^1}\|_{L^\infty},\|\rho_0^1\|_{L^\infty})\|q_0^1\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{q}-2}_{2,p,1}}$. We have then:

$$\|(h_n^2,u_n^2)\|_{F_{T_1^n}}\leq\frac{\epsilon_1'}{4}e^{\epsilon_1'}\big(1+\frac{\epsilon_0}{\alpha'}+\frac{(\epsilon_1')^2}{\alpha'}\big) \tag{4.87}$$ asuperimp2

Now we have to choose ϵ_0 and ϵ_1' such that:

$$e^{\epsilon_1'}\big(1+\frac{\epsilon_0}{\alpha'}+\frac{(\epsilon_1')^2}{\alpha'}\big)\leq\frac{24}{7}. \tag{4.88}$$ asuperimp3

In particular it suffices to take ϵ_1' and ϵ_0 sufficiently small such that we have:

$$e^{\epsilon_1'}\leq\frac{8}{7},\ (\epsilon_1')^2\leq\alpha',\ \epsilon_0\leq\alpha'. \tag{4.89}$$ asuperimp4

In other word it suffices to take K sufficiently small compared with the initial data q_0^1. It achieves the proof of the theorem cor3 1.2 since the rest of the proof follows the arguments of the proof of the theorem theo1 1.1.

In order to prove the theorem theo3 1.3, it suffices to use the same argument than in theorem theo1 1.1. In particular since $(\rho^1,-2\mu\nabla\ln\rho^1)$ is an exact solution in this case, we do not have remainder term of the form $a\nabla\ln\rho^1$ which implies that the smallness assumption has the form (crucinitialbb 1.12). The only change corresponds to have good estimate in Besov space for the following linearized system:

$$\begin{cases}\partial_t h^2+u\cdot\nabla h^2+\mathrm{div}u^2+u^2\cdot\nabla\ln\rho^1=F,\\ \partial_t u^2+u^1\cdot\nabla u^2+u_2\cdot\nabla u^1-2\mu\nabla\ln\rho^1\cdot Du^2-2\mu\nabla h^2\cdot Du^1-\mu\Delta u^2-\mu\nabla\mathrm{div}u^2\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+a\nabla h^2+bu^2=G,\\ (h^2,u^2)_{/t=0}=(h_0^2,u_0^2).\end{cases} \tag{4.90}$$ bb0.2a

To obtain such estimates, it suffices to follow exactly the same line as in the theorem theo1 1.1. Indeed the damping term bu^2 gives additional estimate on u^2 of the form $\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})$.

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