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# On the Hausdorff dimension of Julia sets of some real polynomials

Genadi Levin, Michel Zinsmeister

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## Résumé

We show that the supremum for  $c$  real of the Hausdorff dimension of the Julia set of the polynomial  $z \mapsto z^d + c$  ( $d$  is an even natural number) is greater than  $2d/(d+1)$ .

## 1 Introduction and statement of the result

The Julia set of a non-linear polynomial  $P : \mathbf{C} \rightarrow \mathbf{C}$  is the set of points having no neighborhood on which the family of iterates  $(P^n)$  is normal. It is a compact non-empty set which is, except for very special polynomials  $P$ , a fractal set. For  $c \in \mathbf{C}$  we denote by  $f_c$  the polynomial

$$f_c(z) = z^d + c,$$

where  $d \geq 2$  is an even integer number, which we fix. Denote by  $J_c$  the Julia set of  $f_c$ .

In the quadratic case  $d = 2$ , the values of  $c$  for which the Hausdorff dimension (HD) of  $J_c$  is big has attracted a lot of attention. It has been known since the pioneering work of Douady-Hubbard [3] that the Hausdorff dimension of the Julia set is less than 2 for every hyperbolic polynomial. Thus  $HD(J_c) < 2$  outside  $\partial M$  and the (hypothetic) non-hyperbolic components of (the interior)  $M$ . We recall that  $M$  stands for the Mandelbrot set, that is the compact subset of  $c \in \mathbf{C}$  such that  $J_c$  is connected. Shishikura [9] was the first to find quadratic Julia sets with Hausdorff dimension 2. He indeed proved that this property holds on a dense  $\mathcal{G}_\delta$  subset of  $\partial M$  or even on a dense  $\mathcal{G}_\delta$  subset on the boundary of every hyperbolic component of  $M$ .

More recently, Buff and Chéritat [12] have found quadratic Julia sets with positive Lebesgue measure (see also <http://annals.math.princeton.edu/articles/3682>).

Both Shishikura's and Buff-Chéritat's results are based on the phenomenon of parabolic implosion which has been discovered and studied by Douady-Hubbard [3]. It should be pointed out that Buff-Chéritat's result is very involved and that we will make no use of it. There is no doubt that if they exist, values of  $c \in \mathbf{R}$  such that the Julia set  $J_c$  has positive measure must be as hard to find as Buff-Chéritat's ones. The aim of present note is much more modest. Its starting point is the second author re-reading of Shishikura's result [10] : it states that if one implodes a polynomial with a parabolic cycle having  $q$  petals, then the dimension of its Julia set automatically gets bigger than  $2q/(q+1)$ . Shishikura's result follows from this by a Baire argument.

Very little is known about Hausdorff dimension of  $J_c$  for real  $c$ . In particular, it is not known if, for a given degree  $d$ ,

$$\sup\{HD(J_c), c \in \mathbf{R}\} = 2.$$

Possible candidates for high dimension are of course (just look at them!) infinitely renormalizable polynomials but the analysis seems to be very delicate and at least no result concerning dimension 2 has been proven so far (for results in the opposite direction, see [1] though). It is for example unknown if the Julia set of the Feigenbaum polynomial has dimension 2 or not (see [4]- [5] for the Julia set of the Feigenbaum universal map though). The only known result about this set is a very general result of Zdunik [14] : it has dimension bigger than 1.

If one tries to use the same ideas as in [9] for real polynomials  $f_c$ , one immediately faces the problem that if  $f_c$ , for  $c$  real, has a parabolic cycle that may be imploded along the real axis then the number of petals is 1 and this does not imply more than Zdunik's general result. The only trick of this paper is to make use of a virtual doubling of petals when the critical point is mapped to a parabolic point (by Lavaurs map). It was inspired by Douady et al paper [13] and implies the following theorem, which is the main result of this work.

**Theorem 1** *Let  $f_c(z) = z^d + c$ ,  $d$  even. Let  $N$  be the set of parameters  $c \in \mathbf{R}$ , such that  $f_c$  has a parabolic cycle of period at least 2 and multiplier 1. Then there exists an open set  $Y$  of  $\mathbf{R}$  whose closure contains  $N$  such that  $J_c$  is connected and*

$$HD(J_c) > \frac{2d}{d+1},$$

for every  $c \in Y$ .

**Comment 1** *In fact, we prove a stronger statement : hyperbolic dimension [9] of  $J_c$  is bigger than  $2d/(d+1)$ .*

**Comment 2** *By [11] ( [6], [8] for  $d = 2$ ), the set of real  $c$  such that  $f_c$  is hyperbolic is dense in  $\mathbf{R}$ . In particular, hyperbolic parameters  $c$  are dense in  $Y$ .*

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## 2 Proof of the theorem

We fix an even integer  $d \geq 2$  and consider the family  $f_c(x) = x^d + c$ , for real  $c$ . Then the Julia set  $J_c$  is connected if and only if  $c \in [a, b]$ , where  $a < 0$  is such that  $f_a^2(0)$  is a fixed point, and  $b > 0$  is such that  $f_b$  has a fixed point with multiplier 1. It is sufficient to prove that, given  $c_0 \in (a, b)$  such that  $f_{c_0}$  has a neutral cycle of period  $k > 1$  with multiplier 1, that is, with one petal, there is an open set  $Z$  accumulating at  $c_0$  for which  $HD(J_c) > 2d/(d+1)$  for  $c \in Z$ .

We begin with three pictures. The first two ones illustrate how to choose the parameter (for  $d = 2$ ) and how the corresponding Julia set looks like. The third one has been kindly drawn for us by the referee and shows the case  $d = 4$ .

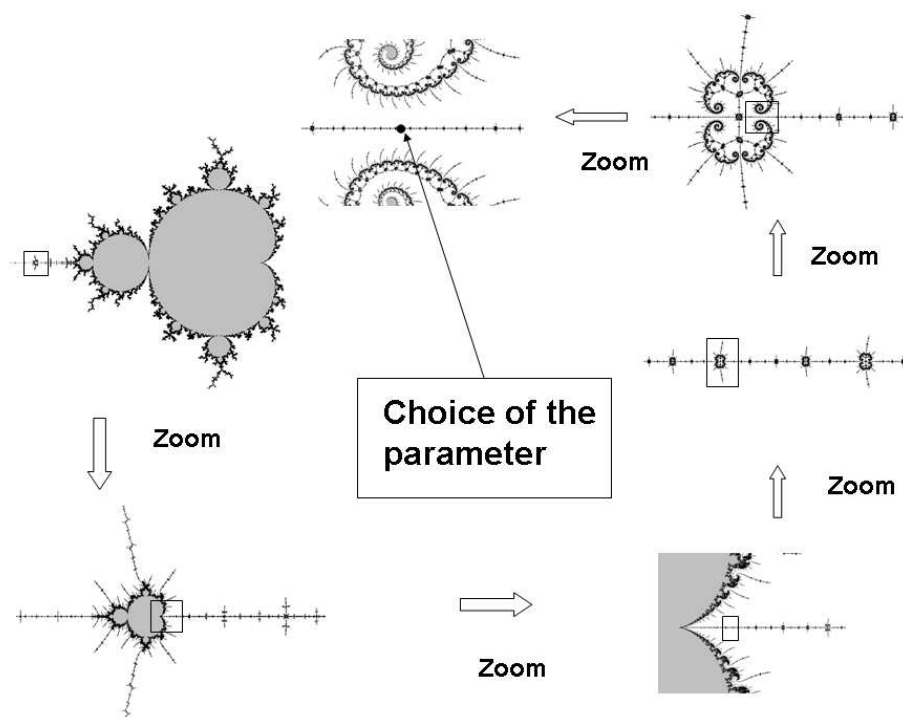


FIGURE 1 – Choice of the parameter

Since  $c_0 \in (a, b)$ , the set  $J_c$  is connected for  $c$  in a small neighborhood  $U$  of  $c_0$ . It is known, that  $f_c$  has an attracting periodic orbit of period  $k$  for  $c \in U$  on the left side of  $c_0$ . Since  $k > 1$ , the corresponding filled-in Julia set  $K_{c_0}$  is such that

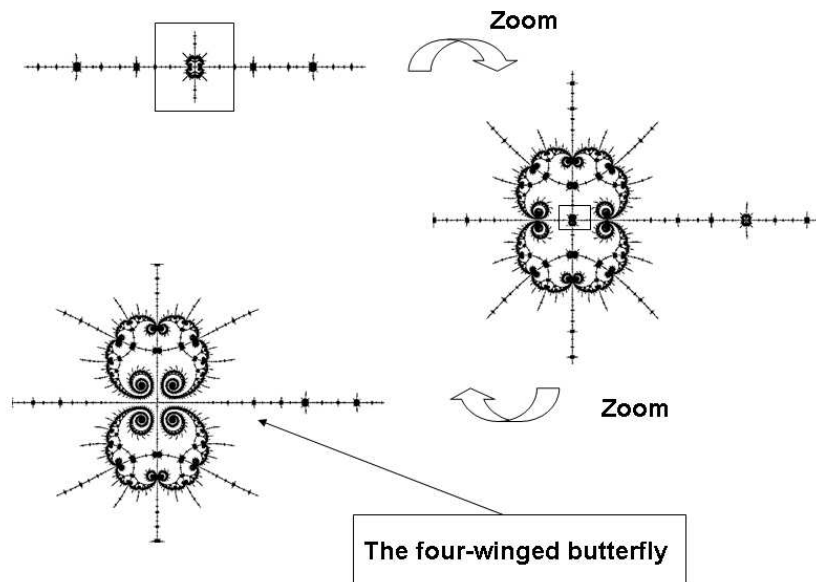


FIGURE 2 – The corresponding Julia set

its interior has a component  $\Delta$  containing 0, and infinitely many preimages of  $\Delta$ . The boundary  $\partial\Delta$  contains a parabolic point  $\alpha$  of period  $k$ . In particular, there is a sequence of preimages of  $\Delta$ , which intersect the real line and accumulate at  $\alpha$ .

For definitivity, one can assume that  $\alpha > 0$ . Then  $F = f_{c_0}^k$  has the following local form :

$$F(z) = z + a(z - \alpha)^2 + b(z - \alpha)^3 + \dots, \quad (1)$$

where  $a > 0$ . This implies that a parabolic implosion phenomenon occurs as  $\epsilon \rightarrow 0$ ,  $\epsilon > 0$ , for the maps  $f_{c_0+\epsilon}$ .

At this point we digress somewhat and describe briefly the theory of parabolic implosion.

Let  $\delta > 0$  be very small and  $D_{\pm}$  be the disks centered at  $\alpha \pm \delta$  with radius  $\delta$ . The map  $F$  sends  $D_-$  into itself while  $D_+ \subset F(D_+)$ . This defines, after identification of  $z$  with  $F(z)$  at the boundary, two cylinders  $U_- = D_- \setminus F(D_-)$  and  $U_+ = F(D_+) \setminus D_+$ . The fact that  $U_{\pm}$  are actual cylinders is best seen in the approximate Fatou coordinate  $I : z \mapsto -1/(a(z - \alpha))$  which sends  $\alpha$  to  $\infty$  and

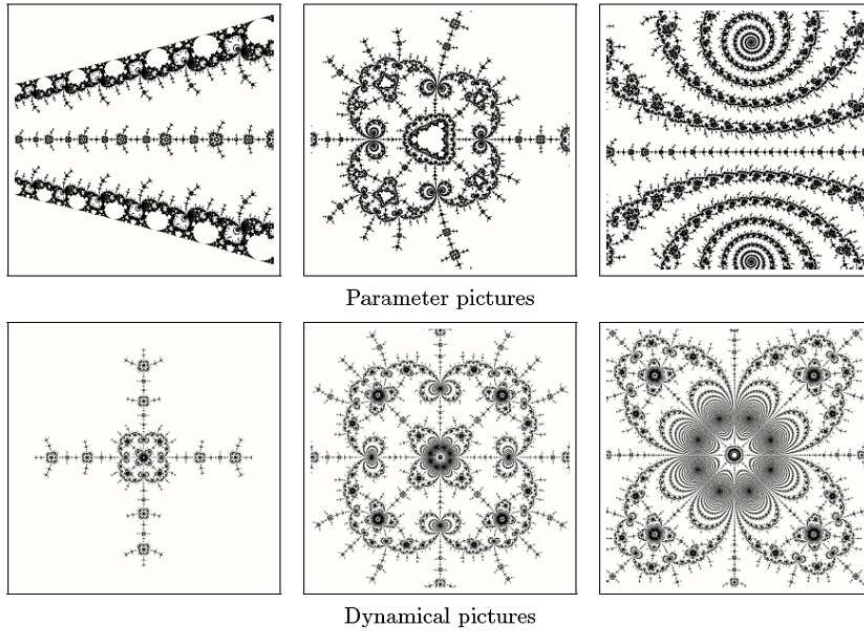


FIGURE 3 – The case  $d=4$

conjugates  $F$  to a map which is asymptotically the translation by 1 at  $\infty$  :

$$F_\infty(w) = I \circ F \circ I^{-1}(w) = w + 1 + \frac{A}{w} + O\left(\frac{1}{|w|^2}\right), \quad (2)$$

where  $A = 1 - b/a^2$ . The real number  $A$  is a conformal invariant. In the case of real polynomial  $F$  which has a parabolic fixed point  $\alpha$  with multiplier 1 and with a single critical point in its immediate basin  $\Delta$ , it is known [9] that  $A > 0$ .

By Riemann mapping theorem these two cylinders may be uniformized by "straight" cylinders. In other words there exists  $\varphi_\pm$  mapping the cylinders  $U_\pm$  to vertical strips  $V_\pm$  of width 1 conjugating  $F$  to the translation by 1. For further use we notice that, by symmetry,  $c_0$  being real, we may assume that  $\varphi_\pm(\bar{z}) = \overline{\varphi_\pm(z)}$ . We also notice that these maps are unique up to post-composition by a real translation. These two maps are called respectively repelling (+) and attracting (-) Fatou coordinates. We normalize them as follows. For every  $\kappa > 0$  and  $R > 0$ , consider two sectors  $\Sigma_-(\kappa, R) = \{w : \operatorname{Re}(w) > R - \kappa|\operatorname{Im}(w)|\}$ ,  $\Sigma_+(\kappa, R) = \{w : \operatorname{Re}(w) < -R + \kappa|\operatorname{Im}(w)|\}$ . Then for any  $\kappa > 0$  there is a big enough  $R(\kappa)$ , such

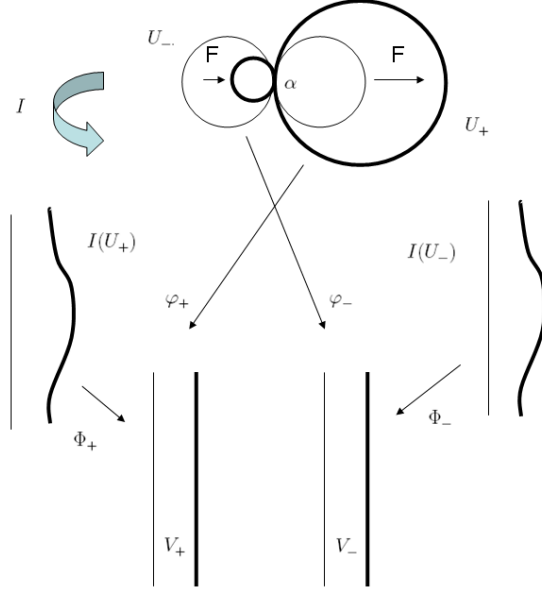


FIGURE 4 – Fatou coordinates

that, if we introduce two sectors  $\Sigma_{\pm}(\kappa) = \Sigma_{\pm}(\kappa, R(\kappa))$ , then  $\varphi_{\pm} = \Phi_{\pm} \circ I$ , where

$$\Phi_{\pm}(w) = w - A \log_{\pm}(w) + C_{\pm} + o(1) \quad (3)$$

as  $w \rightarrow \infty$  within  $\Sigma_{\pm}(\kappa)$  respectively. We specify the constants  $C_{\pm}$  and the log-branches in such a way, that  $\Phi_{\pm}$  are real for real  $w$ . Namely, we choose  $C_- = 0$  and  $\log_-$  to be the standard log-branch in the slit plane  $\mathbf{C} \setminus \{\mathbf{x} \leq \mathbf{0}\}$ . In turn, let  $C_+ = iA\pi$  and  $\log_+$  to be a branch of  $\log$  in  $\mathbf{C} \setminus \{\mathbf{x} \geq \mathbf{0}\}$  so that  $\log_+(w) = \log|w| + i\pi$  for  $w < 0$ .

Now, we extend  $\varphi_{\pm}$  in the following way. Since

$$\varphi_-(F(z)) = T_1(\varphi_-(z)) \quad (4)$$

where  $T_{\sigma}$  denotes the translation by  $\sigma$ , and since every orbit converging to  $\alpha$  passes through  $U_-$  exactly once,  $\varphi_-$  extends uniquely to  $\Delta$  to an holomorphic function still satisfying (4).

It is seen from (4), that the map  $\varphi_- : \Delta \rightarrow \mathbf{C}$  is a branched covering, with the critical points at 0 and all its preimages in  $\Delta$  by  $F^n$ ,  $n > 0$ , and the critical values at the real numbers  $\varphi_-(0) - n$ ,  $n \geq 0$ . In particular, there exists a simply-connected domain  $\Omega_- \subset \Delta \cap \mathbf{H}^+$ , where  $\mathbf{H}^+$  is the upper half-plane, such that  $\varphi_- : \Omega_- \rightarrow \mathbf{H}^+$  is a holomorphic homeomorphism. Moreover, the intersection of the boundary of  $\Omega_-$  with  $\mathbf{R}$  is the interval  $(0, \alpha)$ .

Concerning the repelling Fatou coordinate it is best to consider  $\psi_+ = \varphi_-^{-1} : V_+ \rightarrow U_+$ . The functional relation is now

$$\psi_+(T_1(z)) = F(\psi_+(z)) \quad (5)$$

and we can extend  $\psi_+$  to an entire function by putting, for  $n \in \mathbf{Z}$ ,  $\psi_+(T_n(z)) = F^n(\psi_+(z))$  for  $z \in V_+$ . There exists a simply connected domain  $\Omega_+ \subset \mathbf{H}^+$ , such that  $\psi_+ : \Omega_+ \rightarrow \mathbf{H}^+$  is a homeomorphism.

Let now  $\sigma$  be a real number. We define the Lavaurs map  $g_\sigma$  on the component  $\Delta$  of the interior of the filled-in Julia set of  $f_{c_0}$  by

$$g_\sigma = \psi_+ \circ T_\sigma \circ \varphi_-.$$

The "raison d'être" of this definition is the following theorem due to Douady and Lavaurs ([2]) :

**Theorem 2** : *There exists a sequence of positive  $\epsilon_n$  converging to 0 and a sequence of positive integers  $N_n$  such that*

$$g_\sigma(z) = \lim_{n \rightarrow \infty} f_{c_0 + \epsilon_n}^{kN_n}(z) \quad (6)$$

*uniformly on compact sets of  $\Delta$ .*

Using (3) with the constants  $C_\pm$  and the log-branches specified as above, it is easy to get, that, for every  $\kappa$ , if  $w$  tends to  $\infty$  in  $\Sigma(\kappa) := \Sigma_-(\kappa) \cap \Sigma_+(\kappa) \cap \mathbf{H}^+$ , then  $g_\infty(w) := I \circ g_\sigma \circ I^{-1}(w) = w + (\sigma - iA\pi) + O(\frac{1}{|w|})$ , where  $A$  is real and positive.

Therefore, for every real  $\sigma$  and every  $\kappa > \kappa(\sigma)$ , the inverse map  $g_\infty^{-1}$  leaves the sector  $\Sigma(\kappa)$  invariant and  $g_\infty^{-n} \rightarrow \infty$  as  $n \rightarrow \infty$ . Coming back to the  $z$ -plane, we conclude that the branch  $G = I^{-1} \circ g_\infty^{-1} \circ I$  of  $g_\sigma^{-1}$  leaves the set  $S(\kappa) = I^{-1}(\Sigma(\kappa))$  invariant, and  $G^n(z) \rightarrow \alpha$  as  $n \rightarrow \infty$ , for  $z \in S(\kappa)$  and every  $\kappa > \kappa(\sigma)$ . We have, for  $w \in \Sigma(\kappa)$  :

$$G_\infty(w) := I \circ G \circ I^{-1}(w) = w + (-\sigma + iA\pi) + O(\frac{1}{|w|}), \quad (7)$$

Now, from the definition of  $g_\sigma(z)$  and the global properties of the maps  $\varphi_-$  and  $\psi_+$ , it follows the existence of a simply-connected domain  $\Omega_0 \subset \Omega_-$ , which is mapped by  $g_\sigma$  homeomorphically onto  $\mathbf{H}^+$  and such that  $\alpha \in \bar{\Omega}_0$ . Moreover, from the above description,  $S(\kappa) \subset \Omega_0$ , for every  $\kappa > \kappa(\sigma)$ . Therefore, the branch  $G$  of  $g^{-1}$  which is defined above, extends to a global univalent branch  $G : \mathbf{H}^+ \rightarrow \Omega_0$  of  $g_\sigma^{-1}$ . Since  $\Omega_0 \subset \mathbf{H}^+$ , the iterates  $G^n : \mathbf{H}^+ \rightarrow \Omega_0$ ,  $n > 0$ , converge uniformly on compact sets in  $\mathbf{H}^+$  to a unique fixed point in  $\bar{\Omega}_0$ , which must be  $\alpha$ .

Let us consider the continuous map  $\sigma \mapsto g_\sigma(0) = \psi_+(\varphi_-(0) + \sigma)$  : if  $\sigma$  runs in the interval  $I = \{\varphi_+(x) - \varphi_-(0) : x \in U_+ \cap \mathbf{R}\}$ , then  $g_\sigma(0)$  runs over  $U_+ \cap \mathbf{R}$ . It is thus clear, and this is the key point in the proof, that we can choose  $\sigma$  in such a way that  $g_\sigma(0)$  is a preimage of  $\alpha$  : there is  $j \geq 1$ , such that  $f_{c_0}^j \circ g_\sigma(0) = \alpha$ . Since 0 is a critical point for  $g_\sigma$ , taking the inverse image by  $f_{c_0}^j \circ g_\sigma$  has the same effect as multiplying the number of petals by  $d$  and we may state :



**Lemma 2.1** *There exists an infinite iterated function system defined on a small compact neighborhood  $B_0$  of zero and generated by some holomorphic branches of  $f_{c_0}^{-1}$  and  $g_\sigma^{-1}$  such that its limit set has Hausdorff dimension bigger than  $2d/(d+1)$ .*

**Comment 3** *In fact, the limit set is a subset of a so-called Julia-Lavours set denoted by  $J_{c_0, \sigma}$ . It is defined as follows. The map  $g_\sigma$  can be extended in a natural way from  $\Delta$  to the interior of the filled-in Fatou set of  $f_{c_0}$  : if  $f_{c_0}^k(z) \in \Delta$ , set  $g_\sigma(z) = g_\sigma \circ f_{c_0}^k(z)$ . Then  $J_{c_0, \sigma}$  is simply the closure of the set of points  $z$  for which there exists  $m \in \mathbf{N}$  such that  $g_\sigma^m(z)$  is defined and belongs to  $J(f_0)$ .*

This lemma together with the above discussion implies the theorem. Indeed, by a general property of iterated function systems [7], there exists its finite subsystem with the Hausdorff dimension of its limit set bigger than  $2d/(d+1)$ . On the other hand, the finite iterated function system persists for  $f_{c_0+\epsilon}$  by (6). To be more precise, if  $\{I_j : B_0 \rightarrow X_j, 1 \leq j \leq j_0\}$  is this finite iterated function system, then each  $I_j$  can be extended to a univalent map to a fixed neighborhood  $Y$  of  $B_0$  as  $I_j : Y \rightarrow Y_j$ . Consider the inverse univalent map  $I_j^{-1} : Y_j \rightarrow Y$ . Since the convergence in (6) is uniform on compacts in  $\Delta$ , for every  $\epsilon_n$  small enough there is some integer  $N_j > 0$  and a compact set  $X_{j,n}$ , so that  $f_{c_0+\epsilon_n}^{N_j} : X_{j,n} \rightarrow B_0$  is univalent, too. Now it is clear, that, for every  $\epsilon_n$  small enough, the non-escaping set  $K_n$  of the dynamical system which consists of a finitely many maps  $f_{c_0+\epsilon_n}^{N_j} : X_{j,n} \rightarrow B_0, 1 \leq j \leq j_0$ , has the Hausdorff dimension which is bigger than  $2d/(d+1)$ . On the other hand,  $K_n$  must lie in the Julia set of  $f_{c_0+\epsilon_n}$  because some iterate of the map  $f_{c_0+\epsilon_n}^{N_1 N_2 \dots N_{j_0}}$  leaves the set  $K_n$  invariant and is expanding on it.

**Proof of Lemma 2.1.** As the first step, let us fix a small enough closed ball  $B_0$  around zero, so that it does not contain points of the postcritical set of  $f_{c_0}$ . There exists its preimage  $B'$  by  $F^{-1}$  in  $\Delta \cap \mathbf{H}^+$ . Then we can apply to  $B'$  the maps  $G^n, n > 0$ . By the above,  $B'_n = G^n(B')$  are pairwise disjoint, compactly contained in  $\Delta$ , and  $B'_n \rightarrow \alpha$  as  $n \rightarrow \infty$ . Now, for every  $n \geq n_0$ , so that  $B'_n$  lies in a small enough neighborhood  $U$  of  $\alpha$ , we make “clones” of  $B'_n$  in  $U \cap \Delta$  applying to it  $F^r, r \in \mathbf{Z}$ , where  $F^r$  for  $r < 0$  is a well-defined in  $U \cap \Delta$  branch which fixes  $\alpha$ . We obtain the sets  $B'_{n,r} = F^r(B'_n)$ . On the second step, we consider the map  $f_{c_0}^j \circ g_\sigma$  from a neighborhood  $V$  of 0 onto  $U$ , This map is a ramified cover with the only ramification point at 0 of order  $d$ . Let  $U^* = U \setminus \{x \geq \alpha\}$ , and  $V^* = V \cap \{z : \text{Arg}(z) \in (0, 2\pi/d)\}$ . Denote by  $h$  a branch of  $(f_{c_0}^j \circ g_\sigma)^{-1}$  from  $U^*$  onto  $V^*$ . Let  $B_{n,r} = h(B'_{n,r})$ . We obtain a system of holomorphic maps  $\Psi = \{\psi_{n,r} : B_0 \rightarrow B_{n,r}\}$ , where  $\psi_{n,r} = h \circ F^r \circ G^n \circ F^{-1}, r \in \mathbf{Z}, n \geq n_0$ . If the neighborhood  $U$  is chosen small enough, the maps  $\psi_{n,r}$  extend to univalent maps in a fixed neighborhood  $\tilde{B}$  of  $B_0$  into itself. In particular, the compact sets  $B_{n,r}$  are pairwise disjoint and compactly contained in  $B_0$ . Now, it is quite standard to check that  $\Psi$  form a conformal infinite iterated function system in the sense of [7] (strictly speaking, in the hyperbolic metric of  $\tilde{B}$ , which is equivalent to the Euclidian one

on  $B_0$  though). Let us calculate the parameter  $\theta = \inf\{t : \psi(t) < \infty\}$  of  $\Psi$ , where  $\psi(t) = \sum_{(n,r)} \max_{z \in B_0} |\psi'_{n,r}(z)|^t$ . The map  $\psi_{n,r} = h \circ I^{-1} \circ F_\infty^r \circ G_\infty^n \circ I \circ F^{-1}$ . Here  $F^{-1}$  is a univalent map of a neighborhood  $\tilde{B}$  of  $B_0$  into  $\Delta$ . Now, routine and well-know calculations based on (2)-(7) show (see e.g. [10]), that, for all  $r \in \mathbf{Z}, n \geq n_0$  and some  $C$ , which depends only on a compact set in  $\mathbf{H}^+ \cap \Delta$ , from which  $w$  is taken,  $C^{-1}|r + (-\sigma + i\pi A)n| \leq |F_\infty^r \circ G_\infty^n(w)| \leq C|r + (-\sigma + i\pi A)n|$ , and  $C^{-1} \leq |(F_\infty^r \circ G_\infty^n)'(w)| \leq C$ . On the other hand, the map  $h$  is a composition of a univalent map with an inverse branch of  $z^{1/d}$ . This gives us :  $C_1^{-1}|r + (-\sigma + i\pi A)n|^{-1-1/d} \leq |\psi'_{n,r}(z)| \leq C_1|r + (-\sigma + i\pi A)n|^{-1-1/d}$ , for some  $C_1$  and every  $z \in B_0$ . It follows that the series for  $\psi(t)$  converges if and only if  $t > \theta = 2d/(d+1)$ , and  $\psi(\theta) = \infty$ . Hence [7], the Hausdorff dimension of the limit set of  $\Psi$  is strictly bigger than  $2d/(d+1)$ .

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