Levy multiplicative chaos and star scale invariant random measures

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$\star$-Scale invariant random measures

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Abstract

In this article, we consider the continuous analog of the celebrated Mandelbrot star equation with infinitely divisible weights. Mandelbrot introduced this equation to characterize the law of multiplicative cascades. We show existence and uniqueness of measures satisfying the aforementioned continuous equation. We obtain an explicit characterization of the structure of these measures, which reflects the constraints imposed by the continuous setting. In particular, we show that the continuous equation enjoys some specific properties that do not appear in the discrete star equation. To that purpose, we define a Lévy multiplicative chaos that generalizes the already existing constructions.

1. Introduction

Mandelbrot [16] introduced the so-called random multiplicative cascades to exhibit random processes with nonlinear power-law scalings. This need of constructing random processes with such a power-law scaling goes back to the Kolmogorov theory of fully developed turbulence in the sixties (see [5, 22, 23, 6, 12] and references therein). They render the intermittency effects in turbulence. Random multiplicative cascades are therefore the first mathematical discrete approach of multifractality. Roughly speaking, a (dyadic) multiplicative cascade is a positive random measure M on the unit interval $[0, 1]$ that obeys the following decomposition rule:

$$M(dt) \overset{law}{=} Z^0 \mathbf{1}_{[0,\frac{1}{2}]}(t) M^0(2dt) + Z^1 \mathbf{1}_{[\frac{1}{2},1]}(t) M^1(2dt-1), \tag{1}$$

where M^0, M^1 are two independent copies of M and (Z^0, Z^1) is a random vector with prescribed law and positive components of mean 1 independent from M^0, M^1. Such an equation (and its generalizations to b-adic trees for $b \geqslant 2$), the celebrated star equation introduced by Mandelbrot in [15], uniquely determines the law of the multiplicative cascade. Since the seminal work of Mandelbrot, the star equation (1) has been intensively studied: of particular interest are the founding paper by Kahane and Peyriere [14] and the work by Durrett and Ligget [9]. The following literature on the topic essentially builds on these two works. Let us also mention the article [7] which shows that the free energy of a directed polymer model can be obtained as the limit of the free energy of multiplicative cascade models, thus establishing a link between the two models.

Despite the fact that multiplicative cascades have been widely used as reference models in many applications, they possess many drawbacks related to their discrete scale invariance, mainly they involve a particular scale ratio and they do not possess stationary fluctuations (this comes from the fact that they are constructed on a dyadic tree structure).

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Much effort has been made to develop a continuous parameter theory of suitable stationary multifractal random measures ever since, stemming from the theory of multiplicative chaos introduced by Kahane [13, 3, 22, 2, 19, 20]. Nevertheless, in comparison with the discrete case, the state of the art concerning continuous time models sounds rather empty: laying the foundations like defining a proper continuous star equation is very recent and its solving only concerns the lognormal situation [1]. The main reasons are technical: first, Gaussian processes are very well understood and, second, the analysis of Gaussian multiplicative chaos is much simplified by the use of convexity inequalities for lognormal weights introduced by Kahane (see Kahane's original paper [13] or [1, Lemma 10] for instance).

In this paper, we are concerned with solving the continuous star equation:

$\star$-Scale invariance. *A stationary random measure M on $\mathbb{R}^d$ is said to be $\star$-scale invariant if for all $0<\epsilon\leqslant 1$, M obeys the cascading rule*

$$\big(M(A)\big)_{A\in\mathcal{B}(\mathbb{R}^d)} \stackrel{law}{=} \Big(\int_A e^{\omega_\epsilon(r)}M^\epsilon(dr)\Big)_{A\in\mathcal{B}(\mathbb{R}^d)} \tag{2}$$

where ω_ϵ is a stochastically continuous stationary process and M^ϵ is a random measure independent from ω_ϵ satisfying the relation

$$\big(M^\epsilon(\epsilon A)\big)_{A\in\mathcal{B}(\mathbb{R}^d)} \stackrel{law}{=} \epsilon^d\big(M(A)\big)_{A\in\mathcal{B}(\mathbb{R}^d)}.$$

Intuitively, this relation means that when you zoom in the measure M, you should observe the same behaviour up to an independent factor. Notice that this definition is stated in great generality since no constraint on the law of ω_ϵ is imposed. In the context of discrete multiplicative cascades, given any law for ω_ϵ (up to some integrability conditions), this equation can be solved. However, the continuous case imposes the following constraint on ω_ϵ:

Lemma 1. *We consider a non trivial $\star$-scale invariant measure M on $\mathbb{R}^d$. We suppose that for some x (and hence all x) the family $\epsilon\to\omega_\epsilon(x)$ is continuous in distribution and*

$$\mathbb{E}[(M[0,1]^d)^\gamma]<\infty,\quad \mathbb{E}[e^{(1+\gamma)\omega_\epsilon(x)}]<\infty,\ \forall\epsilon\leqslant 1$$

for some $\gamma>0$. Then, for all ϵ, the process ω_ϵ is infinitely divisible.

Hence, with minimal assumptions on ω_ϵ and the solution M, the process ω_ϵ is infinitely divisible. In view of the above lemma, we can suppose that the process ω_ϵ is infinitely divisible: we will make this assumption in the sequel. As suggested by the Gaussian case [1], this naturally leads to the issue of constructing random measures formally defined by

$$M(dx)=e^{X_x}\,dx,$$

where the process X is infinitely divisible with logarithmic correlations. We carry out this construction in Section 2, which generalizes already existing such attempts [2, 3, 10, 20]. We call such measures Lévy multiplicative chaos. This construction enables us not only to give non trivial solutions to (2) (in Section 3) but also to characterize all the solutions to (2) (up to a few additional technical assumptions). These solutions share the property of a specific structure for the law of the process ω_ϵ. This structure reflects the fact that the continuous star equation is far more restrictive than the discrete one (similarly, Lévy processes are in some sense more restrictive than discrete simple random walks which can be considered with any law for the increments).

1.1 Notations

We will use the following notations throughout the paper. $\mathcal{B}(E)$ stands for the Borelian σ-field of a topological space E. A random measure M is a random variable taking values into the set of positive Radon measures defined on $\mathcal{B}(\mathbb{R}^d)$. We will say that M possesses a moment of order $p > 0$ if $\mathbb{E}[M(K)^p] < +\infty$ for every compact set K. A random measure M is said to be stationary if for all $y \in \mathbb{R}^d$ the random measures $M(\cdot)$ and $M(y + \cdot)$ have the same law. A stochastic process $(X_t)_{t\in\mathbb{R}^d}$ is said to be stochastically continuous if, for each $t \in \mathbb{R}^d$, X_{t+h} converges towards X_t in probability when h goes to 0. We will also use the shortcut ID in place of infinitely divisible. We remind the reader that every stochastically continuous random process admits a measurable version (see [4, Chapter 6]). We will only deal with measurable versions of stochastically continuous process in this paper.

2. Generalized Lévy chaos

This section is devoted to the construction of measures that can formally be written as

$$M(dx) = e^{L_x}\, dx,$$

where L is a stationary ID process with a logarithmic spatial dependency. As in the Gaussian case, such a singularity of the spatial structure imposes to construct these measures through a limiting procedure where the singularity has been "cut off". Hence we will understand these measures as a limit

$$M(dx) = \lim_{\epsilon\to 0} e^{X^\epsilon_x}\, dx,$$

where X^ϵ is a stationary ID process that converges in some sense towards L. The process X^ϵ will basically depend on two parameters: a generator (any stationary ID process) and a rate function. We detail below the construction.

2.1 Generator and rate function

Let $(X_t)_{t\in\mathbb{R}^d}$ be a stochastically continuous stationary ID random process. It follows from [17] that X admits a version given by

$$X_t = b + \int_{\mathbb{R}^d} \cos(t\cdot\lambda)\,\overline{W}(d\lambda) + \int_{\mathbb{R}^d} \sin(t\cdot\lambda)\,\overline{W}'(d\lambda) + \int_S f(T_t(s))\,[\overline{N}(ds) - (1\vee|f(T_t(s))|)^{-1}\theta(ds)] \quad (3)$$

where:

- $b \in \mathbb{R}$,
- $\overline{W}, \overline{W}', \overline{N}$ are independent,
- $\overline{W}, \overline{W}'$ are identically distributed centered Gaussian random measures on $\mathbb{R}^d$ with covariance kernel given by $\mathbb{E}[\overline{W}(A)\overline{W}(B)] = R(A\cap B)$ for some symmetric positive finite measure R on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$,
- $\overline{N}$ is a Poisson random measure on a Borel space S with a σ-finite intensity measure θ,
- $f : S \to \mathbb{R}$ is a measurable deterministic function such that

$$\int_S (|f(s)|^2 \wedge 1)\,\theta(ds) < +\infty,$$

- $(T_x)_x$ is a measure preserving flow on (S, θ).

In what follows, we will say that a stochastically continuous ID process is associated to $(S, \overline{W}, \overline{W}', \overline{N}, \theta, R, f, (T_x)_x)$ if it is given by (3) where all the involved items are defined as described above.

We define the Laplace exponents ψ of X for $p \geqslant 1$ by

$$\mathbb{E}[e^{q_1 X_{t_1}+\cdots+q_p X_{t_p}}] = e^{\psi_{t_1,\dots,t_p}(q_1,\dots,q_p)}$$

for all $(t_1,\dots,t_p) \in (\mathbb{R}^d)^p$ and $q_1,\dots,q_p \in \mathbb{R}$ such that the above expectation makes sense. For the sake of clarity, ψ_0 (i.e. the Laplace exponents of X_0, or equivalently of X_t for any $t \in \mathbb{R}^d$) will be denoted by ψ.

We assume that X possesses a second order exponential moment and we consider the following generalized covariance function:

$$F(x) = \psi_{0,x}(1,1) - 2\psi(1),\ x \in \mathbb{R}^d. \tag{4}$$

Assumption 2. *Let g be a nonnegative function in $L^1_{loc}(\mathbb{R}_+, dy)$ such that*

$$\forall x \in \mathbb{R}^d \setminus \{0\} \text{ and } a \geqslant 1, \qquad \int_a^{+\infty} \frac{|F(g(u)x)|}{u}\, du \leqslant \overline{F} \ln_+ \frac{1}{a|x|} + h(a,x) \tag{5}$$

where h is some bounded continuous function on $\mathbb{R}_+ \times \mathbb{R}^d$ and $\overline{F}$ is some positive constant. The function g will be called rate function.

2.2 Limiting procedure

For any $\epsilon \in]0,1[$, we define a new stochastically continuous ID random process:

$$X_t^\epsilon = b \ln \frac{1}{\epsilon} + \int_1^{\frac{1}{\epsilon}} \int_{\mathbb{R}^d} \cos(tg(y)\cdot\lambda)\, W(d\lambda, dy) + \int_1^{\frac{1}{\epsilon}} \int_{\mathbb{R}^d} \sin(tg(y)\cdot\lambda)\, W'(d\lambda, dy) \tag{6}$$

$$+ \int_1^{\frac{1}{\epsilon}} \int_S f(T_{tg(y)}(s))\, [N(ds,dy) - (1 \vee |f(T_{tg(y)}(s))|)^{-1} \theta(ds) \frac{dy}{y}] \tag{7}$$

where:
-W, W', N are independent,
-W, W' are identically distributed centered Gaussian random measures on $\mathbb{R}^d \times \mathbb{R}_+^*$ with covariance kernel given by $\mathbb{E}[W(A)W(B)] = \int_{A\cap B} R(d\lambda)\frac{dy}{y}$,
-N is a Poisson random measure on the Borel space $S \times \mathbb{R}_+^*$ with intensity measure $\theta(ds) \otimes \frac{dy}{y}$,
-$f : S \to \mathbb{R}$ and $(T_x)_x$ are the same as above.

Clearly, X^ϵ is a stationary ID process. From [17, Theorem 5], it is stochastically continuous. In what follows, we will say that a family $(X^\epsilon)_\epsilon$ of stationary stochastically continuous ID processes is an approximating family associated to $(S, W, W', N, \theta, R, f, (T_x)_x)$ if it is given by (6) where all the involved items are defined as described above. Notice that the whole law of the processes $(X^\epsilon)_{\epsilon \in]0,1]}$ can be recovered from the law of the process X introduced in the previous subsection and the rate function g. For this reasons, the ID process X will be called the **generator** of the approximating sequence $(X^\epsilon)_\epsilon$ and g the **rate function**.

We have

$$\forall q \geqslant 0, \forall x \in \mathbb{R}^d, \quad \mathbb{E}[e^{qX_x^\epsilon}] = \mathbb{E}[e^{qX_0^\epsilon}] = e^{\ln \frac{1}{\epsilon}\psi(q)}. \tag{8}$$

We stress that, in great generality, ψ takes values into $\mathbb{R}_+ \cup \{+\infty\}$ but it is finite at least for $q \in [0,2]$.

For $\epsilon > 0$, we define a random measure

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \widetilde{M}^\epsilon(A) = \int_A e^{X_x^\epsilon - \psi(1)\ln\frac{1}{\epsilon}}\, dx. \tag{9}$$

Clearly, for each fixed A with finite Lebesgue measure, the family $(\widetilde{M}^\epsilon(A))_{\epsilon \in]0,1[}$ is a positive martingale. Thus it converges almost surely. We deduce that the family $(\widetilde{M}^\epsilon)_\epsilon$ almost surely weakly converges towards a limiting random measure M on $\mathcal{B}(\mathbb{R}^d)$. This measure will be called Lévy multiplicative chaos associated to $(S, W, W', N, \theta, R, f, (T_x)_x)$.

2.3 Main properties

By stationarity and the $0-1$ law , we deduce (as in [13, 21])

Proposition 3. *Either of the following events occurs with probability one:*

$$\{M \equiv 0\} \quad \text{or} \quad \{\forall B \text{ non empty ball, } M(B) > 0\}.$$

In the second situation, we will say that the measure M is non degenerate.

The non-degeneracy is expectedly related to the Laplace exponents of the generator:

Theorem 4. *Under Assumption 2, the measure M is non degenerate as soon as $\psi'(1) - \psi(1) < d$.*

Corollary 5. *Under Assumption 2 and provided that $\psi'(1) - \psi(1) < d$, the measure M almost surely does not possess any atom.*

In some particular situations, it can be proved that the condition $\psi'(1) - \psi(1) < d$ is optimal (see [13, 3, 2] for instance). But the situation presented here is far more intricate and it is not optimal in great generality since we only require the correlation structure to be sub-logarithmic (Assumption 2). To illustrate the situation, let us focus on the second order moment. It is well known that, in the particular situations presented in [13, 3, 2], the measure M admits a second order moment if and only if $\psi(2) < d$. In our case, the situation is not that clear. For instance, choose θ equal to the Lebesgue measure on $S = \mathbb{R}^d$, θ any Lévy measure on $\mathbb{R}^d$ and $R = 0$. The flow $(T_t)_t$ is the usual group of translations. Take any positive bounded function f with compact support over $\mathbb{R}^d$ and $g(y) = y^q$ (for $q \geqslant 1$). Notice that the associated function F reduces to 0 for all x such that the supports of f and $T_x f$ are disjoint, say for $|x| \geqslant R$. Then for $a > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$, we have (where $e_x = x/|x|$):

$$\begin{aligned}\int_1^{+\infty} \frac{F(u^q x)}{u}\, du &= \int_{|x|^{1/q}}^{+\infty} \frac{F(u^q e_x)}{u}\, du \\ &\simeq \frac{F(0)}{q} \ln_+ \frac{1}{|x|} = \frac{\psi(2) - 2\psi(1)}{q} \ln_+ \frac{1}{|x|} \qquad \text{as } x \to 0. \end{aligned} \tag{10}$$

Hence it can be proved that M admits a second order moment if and only if $\frac{\psi(2)-2\psi(1)}{q} < d$, which is quite a different condition from [13, 3, 2].

Hence, it appears that the condition $\psi'(1) - \psi(1) < d$ should be optimal when the rate function g is "not far" from the function $g(y) = y$. In that spirit, we claim:

Theorem 6. *If the measure M admits a moment of order $1 + \delta$ for some $\delta > 0$ and if the rate function g satisfies $g(y) \leqslant y$ for $y \geqslant 1$ then*

$$\psi(1 + \delta) - (1 + \delta)\psi(1) \leqslant d\delta.$$

In particular $\psi'(1) - \psi(1) < d$.

3. $\star$-scale invariant random measures

In this section, we explain the connection between $\star$-scale invariant random measures and Lévy multiplicative chaos. On the first hand, we show that every Lévy multiplicative chaos defines a $\star$-scale invariant random measure provided that the rate function is defined by $g(y) = y$ for all $y \geqslant 1$. Then we show that all $\star$-scale invariant random measures with a moment of order strictly greater than 1 are Lévy multiplicative chaos, up to a few additional assumptions.

3.1 Construction

We consider X^ϵ, $\widetilde{M}^\epsilon$ and M as constructed in Section 2 with generator X and rate function g given by $g(y) = y$ for all $y \geqslant 1$. Hence the process X^ϵ is given by

$$X_t^\epsilon = b \ln \frac{1}{\epsilon} + \int_1^{\frac{1}{\epsilon}} \int_{\mathbb{R}^d} \cos(ty \cdot \lambda)\, W(d\lambda, dy) + \int_1^{\frac{1}{\epsilon}} \int_{\mathbb{R}^d} \sin(ty \cdot \lambda)\, W'(d\lambda, dy) \tag{11}$$

$$+ \int_1^{\frac{1}{\epsilon}} \int_S f(T_{ty}(s))\, [N(ds, dy) - (1 \vee |f(T_{ty}(s))|)^{-1} \theta(ds) \frac{dy}{y}]. \tag{12}$$

Let us state a simple criterion to check Assumption 2:

Proposition 7. *Assumption 2 is satisfied if and only if*

$$\sup_{|e|=1} \int_1^{+\infty} \frac{|F(ue)|}{u}\, du < +\infty. \tag{13}$$

Theorem 8. *Assume that Assumption 2 (or equivalently (13)) holds and that $\psi'(1) - \psi(1) < d$. Then M is non trivial and $\star$-scale invariant.*

Hence, the $\star$-scale invariance property only depends on the choice of the rate function. This shows in a way that there are as many $\star$-scale invariant random measures as stochastically continuous ID processes (up to the condition $\psi'(1) - \psi(1) < d$).

The existence of a second order moment is ruled by the following condition, which seems to be more conventional than the counter-example described in (10):

Proposition 9. *The measure M admits a second order moment if and only if $F(0) < d$.*

A straightforward adaptation of our proofs shows that:

Proposition 10. *A $\star$-scale invariant random measure M is multifractal in the sense that:*

$$\lim_{t \downarrow 0} \frac{\ln \mathbb{E}[M([0,t])^q]}{\ln t} = q - \psi(q) + q\psi(1),$$

where ψ is the Laplace exponent of its generator.

3.2 Uniqueness

Conversely, we now want to describe as exhaustively as possible the set of all $\star$-scale invariant random measures. For that purpose, we introduce a few additional assumptions:

Assumption 11. *We will say that a stationary random measure M is a good $\star$-scale invariant random measure if M is $\star$-scale invariant and satisfies:*

1. *the process ω_ϵ admits exponential moments of order 2, that is $\mathbb{E}[e^{2\omega_\epsilon(0)}] < \infty$.*
2. *for $\epsilon < 1$, the generalized covariance kernel associated to the ID process ω_ϵ:*

$$\forall x \in \mathbb{R}^d, \quad F_\epsilon(x) := \log(\mathbb{E}[e^{\omega_\epsilon(x) + \omega_\epsilon(0)}])$$

satisfies

$$\forall x \neq 0, \quad |F_\epsilon(x)| \leqslant C_\epsilon \int_{|x|}^{+\infty} \theta(u)\, du. \tag{14}$$

for some positive constant C_ϵ and some decreasing function $\theta :]0, +\infty[\to \mathbb{R}_+$ such that

$$\int_1^{+\infty} \theta(u) \ln(u)\, du < +\infty. \tag{15}$$

3. there is $\epsilon_0 \in]0,1]$ such that, for each $p \geqslant 1$, $q_1, \dots, q_p \in \mathbb{R}$ and $t_1, \dots, t_p \in \mathbb{R}^p$, the mapping

$$(\epsilon, t_1, \dots, t_p) \mapsto \mathbb{E}[e^{iq_1\omega_\epsilon(t_1)+i\omega_\epsilon(t_p)}]$$

is differentiable w.r.t. ϵ_0 with a derivative continuous w.r.t. $(t_1, \dots, t_p)$.

It turns out that the condition on the exponential moments of order 2 of ω_ϵ is also necessary as soon as the measure M possesses a moment of order 2. Point 2 is a decorrelation property at infinity whereas point 3 is a regularity property. In what follows, we denote by ψ_ϵ the Laplace exponent of ω_ϵ:

$$\psi_\epsilon(q) = \ln \mathbb{E}[e^{q\omega_\epsilon(0)}]$$

for all $q \in \mathbb{R}$ such that the above quantity is finite. Notice that, as soon as the measure M possesses a moment of order 1, the condition $\psi_\epsilon(1) = 0$ is a necessary condition for the solution of (2) to be non trivial.

The main result of this paper is the following:

Theorem 12. *Consider a good $\star$-scale invariant measure M. Assume that M admits a finite moment of order $1+\delta$ for some $\delta > 0$ (i.e. $\mathbb{E}[M(B)^{1+\delta}] < \infty$ for some open ball B). Then there exists a random variable $Y \in L^{1+\delta}$ and a Lévy multiplicative chaos Q (independent from Y and non-degenerate) with associated rate function $g(y) = y$ such that*

$$M(dx) \overset{law}{=} YQ(dx).$$

We conjecture that the same theorem holds if M is a $\star$-scale invariant measure with a finite moment of order $1+\delta$ for some $\delta > 0$. Therefore, we think Assumption 11 is just a technical assumption (which we can not avoid at present) and that our theorem characterizes all $\star$-scale invariant measure with a finite moment of order $1+\delta$ for some $\delta > 0$. The general case of $\star$-scale invariant measures with no finite moment assumption is currently under investigation and requires the introduction of a different set of measures (work in progress).

Remark 13. *When M is a good $\star$-scale invariant random measure, the law of M is entirely characterized by the law of the process ω_ϵ in (2) for some $\epsilon \in]0,1[$. Furthermore, the law of the finite dimensional distributions of the generator X can be recovered from those of ω_ϵ by the following procedure: define the Lévy exponents η^ϵ, η of ω_ϵ and X, that is*

$$\mathbb{E}[e^{iq_1\omega_\epsilon(t_1)+\dots+iq_p\omega_\epsilon(t_p)}] = e^{\eta^\epsilon(q_1,\dots,q_p,t_1,\dots,t_p)}, \quad \mathbb{E}[e^{iq_1X_{t_1}+\dots+iq_pX_{t_p}}] = e^{\eta(q_1,\dots,q_p,t_1,\dots,t_p)}.$$

Then we have

$$\partial_\epsilon \eta^\epsilon(q_1, \dots, q_p, t_1, \dots, t_p) = -\frac{1}{\epsilon^2}\eta(q_1, \dots, q_p, \frac{t_1}{\epsilon}, \dots, \frac{t_p}{\epsilon}).$$

4. Examples

4.1 Lognormal case

The lognormal case, that is when the generator of the $\star$-scale invariant measure is a Gaussian process, has been entirely treated in [1]. Of course, the assumptions are less restrictive concerning good $\star$-scale invariant measures since their generator can be entirely described with its two marginals, that is its covariance function. As a consequence, we do not require Assumption 11 point 3) in the lognormal case.

4.2 Reminder about log-ID independently scattered random measures

The next examples are based on log-ID independently scattered random measures so that we first collect a few well known facts about these measures. The reader is referred to [18] for further details.

We remind the reader that an ID independently scattered random measure μ distributed on a measurable space $(S, \mathcal{B}(S))$ with control measure Γ and kernel K is a collection of random variables $(\mu(A), A \in \mathcal{B}(S))$ such that:

1) For every sequence of disjoint sets $(A_n)_n$ in $\mathcal{B}(S)$, the random variables $(\mu(A_n))_n$ are independent and

$$\mu\big(\bigcup_n A_n\big) = \sum_n \mu(A_n) \text{ a.s.},$$

2) for any measurable set A in $\mathcal{B}(S)$, $\mu(A)$ is an ID random variable whose characteristic function is characterized by

$$\mathbb{E}(e^{iq\mu(A)}) = \mathbb{E}[e^{it\mu(A)}] = \exp\Big(\int_A K(q,s)\Gamma(ds)\Big).$$

The control measure Γ is a positive σ-finite measure on S and the kernel K takes on the form

$$K(q,s) = iqa(s) - \frac{1}{2}q^2\sigma^2(s) + \int_{\mathbb{R}} (e^{iqz} - 1 - iq\tau(z))\varrho(s,dz), \tag{16}$$

where

$$|a(s)| + \sigma^2(s) + \int_{\mathbb{R}} \min(1, z^2)\varrho(s,dz) = 1 \quad \theta\, a.e. \tag{17}$$

Here σ, a belong to $L^\infty(S, \Gamma)$ (σ non-negative) and $\varrho : S \times \mathcal{B}(\mathbb{R}) \to [0, +\infty]$ is such that for each fixed $s \in S$, $\varrho(s, dz)$ is a Lévy measure on $\mathbb{R}$ and for each $B \in \mathcal{B}(\mathbb{R})$ the function $\varrho(\cdot, B)$ is measurable and finite whenever 0 does not belong to the closure of B. The function τ is any truncation function. The random measure μ is characterized by the triple of measures $(a(s)\theta(ds), \sigma^2(s)\theta(ds), \varrho(s,dz)\theta(ds))$. Conversely, to such triple corresponds a unique (in law) ID independently scattered random measure.

4.3 Barral-Mandelbrot's type $\star$-scale invariant MRMs

We consider the situation when the dimension d is equal to 1. We introduce an ID independently scattered random measure μ distributed on $(\mathbb{R} \times \mathbb{R}_+^*, \mathcal{B}(\mathbb{R} \times \mathbb{R}_+^*))$ with control measure

$$\Gamma(dt, dy) = dt\, y^{-2}\, dy$$

and kernel

$$K(q, (t,y)) = \varphi(q) = imq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}^*} (e^{iqx} - 1 - iqx\mathbf{1}_{|x| \leqslant 1})\, \nu(dx)$$

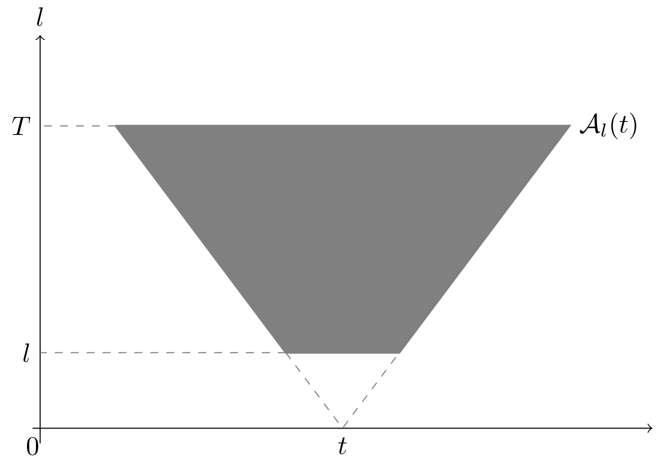
where $\nu(dx)$ is a Lévy measure on $\mathbb{R}$ and $m, \sigma \in \mathbb{R}$. We denote by ψ the Laplace exponent associated to φ, that is $\psi(q) = \varphi(-iq)$ whenever it makes sense to consider such a quantity. We assume that $\psi(1) = 0$.

We can then define the stationary stochastically continuous ID process $(\omega_l(t))_{t\in\mathbb{R}}$ for $l > 0$ by

$$\omega_l(t) = \mu\left(\mathcal{A}_l(t)\right)$$

where $\mathcal{A}_l(t)$ is the triangle like subset $\mathcal{A}_l(t) := \{(s,y) \in \mathbb{R} \times \mathbb{R}_+^* : l \leqslant y \leqslant T, -y/2 \leqslant t-s \leqslant y/2\}$.

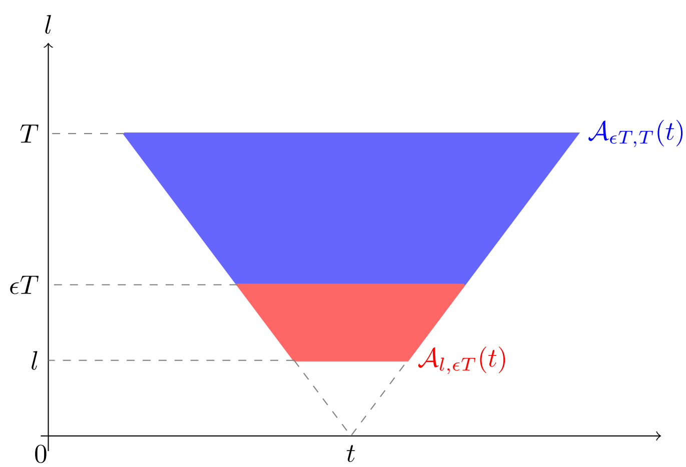
Define now the random measure M_l by $M_l(dt) = e^{\omega_l(t)}dt$. Almost surely, the family of measures $(M_l(dt))_{l>0}$ weakly converges towards a random measure M. When $\psi'(1) - \psi(1) < 1$, this measure is not trivial (see [2, 3]).



Let us check that M is a good $\star$-scale invariant random measure. Fix $\epsilon < 1$ and define the sets $\mathcal{A}_{l,\epsilon T}(t) := \{(s,y) : l \leqslant y \leqslant \epsilon T, -y/2 \leqslant t-s \leqslant y/2\}$ and $\mathcal{A}_{\epsilon T,T}(t) := \{(s,y) : \epsilon T \leqslant y \leqslant T, -y/2 \leqslant t-s \leqslant y/2\}$. Note that $\mathcal{A}_l(t) = \mathcal{A}_{l,\epsilon T}(t) \cup \mathcal{A}_{\epsilon T,T}(t)$ and that those two sets are disjoint. Thus, we can write for every measurable set A

$$M_l(A) = \int_A e^{\omega_{\epsilon T,T}(t)} e^{\omega_{l,\epsilon T}(t)} dt \tag{18}$$

with $\omega_{\epsilon T,T}(t) = \mu(\mathcal{A}_{\epsilon T,T}(t))$ and $\omega_{l,\epsilon T}(t) = \mu(\mathcal{A}_{l,\epsilon T}(t))$.



We then study equation (18) in the limit $l \to 0$; we obtain

$$M(A) = \int_A e^{\omega_{\epsilon T,T}(t)} M^\epsilon(dt) \tag{19}$$

where M^ϵ is the limit when $l \to 0$ of the random measure $M_l^\epsilon(dt) := e^{\omega_{l,\epsilon T}(t)} dt$. We easily verify that $M^\epsilon(\epsilon A) \overset{law}{=} \epsilon M(A)$ writing

$$M_l^\epsilon(\epsilon A) = \epsilon \int_A e^{\omega_{l,\epsilon T}(\epsilon t)} dt \tag{20}$$

and checking that the finite-dimensional marginals of the process $(\omega_{l,\epsilon T}(\epsilon t))_{t\in\mathbb{R}}$ are the same as the one of $(\omega_{l,T}(t))_{t\in\mathbb{R}}$ (see [3]).

By computing the Lévy exponents of the process $\omega_{\epsilon T,T}(t)$:

$$\mathbb{E}[e^{iq_1\omega_{\epsilon T,T}(t_1)+\cdots+iq_p\omega_{\epsilon T,T}(t_p)}] = e^{\psi^\epsilon(q_1,\ldots,q_p,t_1,\ldots,t_p)}, \tag{21}$$

we obtain:

$$\psi^\epsilon(q_1,\ldots,q_p,t_1,\ldots,t_p) = \int_1^{1/\epsilon}\int_{\mathbb{R}} \varphi\Big(\sum_{j=1}^p q_j f(T_{yt_j}(r))\Big)\,dr\,\frac{dy}{y} \tag{22}$$

where $f(r) = \mathbf{1}_{[-\frac{T}{2},\frac{T}{2}]}(r)$ and $T_s : t \in \mathbb{R} \mapsto t-s \in \mathbb{R}$ is the usual shift on $\mathbb{R}$. It is then straightforward to check that M is good provided that $\int_{z>1} e^{2z}\nu(dz) < +\infty$. We stress that the Lévy exponents of the generator, say X, are given by

$$\mathbb{E}[e^{iq_1X(t_1)+\cdots+iq_pX(t_p)}] = \exp\Big(\int_{\mathbb{R}}\varphi\Big(\sum_{j=1}^p q_j f(T_{t_j}(r))\Big)\,dr\Big).$$

In this example, the $\star$-scale invariance property is easily understood via the geometric properties of the process, namely the scaling properties of the cones. Generalizing this example by means of geometric considerations is far from being obvious and has never been done in the literature. On the other hand, in view of the results in this paper, the generalization is straightforward. It suffices to change the function f. To get things simpler, we can, for instance, choose f equal to any measurable function bounded by 1 with compact support.

4.4 Stable Lévy chaos

We focus now on another situations of interest. We consider an infinitely divisible independently scattered random measure μ distributed on $\mathbb{R}$ with the Lebesgue measure ds as control measure and kernel

$$K(q,t) = \overline{\varphi}(q) = imq + \int_0^\infty (e^{-iqx}-1)\frac{dx}{x^{1+\alpha}}$$

for some $\alpha \in]0,1[$. Then the associated Laplace exponent is given by

$$\overline{\psi}(q) = mq - \frac{\Gamma(1-\alpha)}{\alpha}q^\alpha.$$

Let $(T_t)_{t\in\mathbb{R}}$ be the family of usual shifts on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}_+$ be any integrable function with compact support. We define

$$\|f\|_1 = \int_{\mathbb{R}} f(s)\,ds < +\infty, \quad \|f\|_\alpha = \int_{\mathbb{R}} f(s)^\alpha\,ds < +\infty.$$

We consider the stationary ID random process:

$$\forall t \in \mathbb{R}, \quad X_t = \int f(T_t(s))\,\mu(ds).$$

We have

$$\mathbb{E}[e^{qX_t}] = e^{\int_{\mathbb{R}}\psi\left(qf(s)\right)\,ds} = e^{mq\|f\|_1 - \frac{\Gamma(1-\alpha)}{\alpha}\|f\|_\alpha q^\alpha}.$$

So we must set $m = \frac{\Gamma(1-\alpha)\|f\|_\alpha}{\alpha\|f\|_1}$ to ensure the normalizing condition $\psi(1) = 0$. It is obvious to check that X possesses exponential moments of second order. We assume that $\psi'(1) < 1$, that is

$$\|f\|_\alpha < \frac{\alpha}{\Gamma(2-\alpha)}$$

Hence, we can consider the Lévy chaos with generator X and rate function $g(y) = y$. It is a non trivial good $\star$-scale invariant random measure. The scaling factor ω_ϵ appearing in (2) is a stable ID process.

A. Proof of Lemma 1

We first state the following intermediate lemma:

Lemma 14. *Let $(F(x))_{x\in\mathbb{R}^d}$ and $(G(x))_{x\in\mathbb{R}^d}$ be two stationary and non negative stochastically continuous processes. We consider a non trivial stationary random measure η on $\mathbb{R}^d$ independent of F, G. We suppose that there exists $\gamma > 0$ such that $\mathbb{E}[F(x)^{1+\gamma}] < \infty$, $\mathbb{E}[G(x)^{1+\gamma}] < \infty$, and $\mathbb{E}[\eta(K)^\gamma] < \infty$ for all compact set K. If the following equality on measures holds:*

$$F(x)\eta(dx) \overset{law}{=} G(x)\eta(dx)$$

then the two processes F and G have same law.

Proof. We consider the case $d = 1$ (the higher dimensions work the same). Let $\delta > 0$. Notice that $\mathbb{E}[\eta([0,\delta])^\alpha] > 0$ for all $\alpha \in]0,\gamma[$. Indeed, the measure is stationary and non trivial. Choose now $\alpha \in]0, \min(\gamma, 1)[$. Notice that the mapping $x \in \mathbb{R}_+ \mapsto x^\alpha$ is sub-additive. Therefore $|x^\alpha - y^\alpha| \leqslant |x-y|^\alpha$ for any $x, y \geqslant 0$. We deduce the following inequality:

$$\begin{aligned}\Big|\mathbb{E}\left[\left(\int_0^\delta F(x)\eta(dx)\right)^\alpha\right] - \mathbb{E}\left[\left(\int_0^\delta F(0)\eta(dx)\right)^\alpha\right]\Big| &\leqslant \mathbb{E}\left[\Big|\int_0^\delta F(x)\eta(dx) - \int_0^\delta F(0)\eta(dx)\Big|^\alpha\right] \\ &\leqslant \mathbb{E}\left[\Big(\int_0^\delta |F(x) - F(0)|\eta(dx)\Big)^\alpha\right].\end{aligned}$$

The mapping $x \in \mathbb{R}_+ \mapsto x^\alpha$ is concave. So we use Jensen's inequality applied to $\mathbb{E}[.|\eta]$ and we get:

$$\begin{aligned}&\Big|\mathbb{E}\Big[\Big(\int_0^\delta F(x)\eta(dx)\Big)^\alpha\Big] - \mathbb{E}\Big[\Big(\int_0^\delta F(0)\eta(dx)\Big)^\alpha\Big]\Big| \\ &\qquad\leqslant \mathbb{E}\Big[\Big(\int_0^\delta \mathbb{E}[|F(x) - F(0)|]\eta(dx)\Big)^\alpha\Big], \\ &\qquad\leqslant \sup_{x\in[0,\delta]} \mathbb{E}[|F(x) - F(0)|]^\alpha \mathbb{E}\left[\eta[0,\delta]^\alpha\right].\end{aligned}$$

Since $\sup_{x\in[0,\delta]} \mathbb{E}[|F(x) - F(0)|] \underset{\delta\to 0}{\to} 0$, we get that:

$$\frac{\mathbb{E}\Big[\Big(\int_0^\delta F(x)\eta(dx)\Big)^\alpha\Big]}{\mathbb{E}\left[\eta[0,\delta]^\alpha\right]} \underset{\delta\to 0}{\to} \mathbb{E}[F(0)^\alpha].$$

Similarly, we get the above convergence with F replaced by G: this shows that $F(0)$ and $G(0)$ have the same distribution. We show similarly, for all $x_1, \cdots, x_n$, that $(F(x_1), \cdots, F(x_n))$ and $(G(x_1), \cdots, G(x_n))$ have the same distribution. ☐

Now, we can finish the proof of lemma 1:

Proof. By iterating (2) and using the above lemma, the process $(\omega_\epsilon(x))_{x\in\mathbb{R}^d}$ is such that $(\epsilon, \epsilon' < 1)$:

$$(\omega_{\epsilon\epsilon'}(x))_{x\in\mathbb{R}^d} \overset{law}{=} (\omega_\epsilon(x) + \widetilde{\omega}_{\epsilon'}(\frac{x}{\epsilon}))_{x\in\mathbb{R}^d} \tag{23}$$

where ω_ϵ and $\widetilde{\omega}_{\epsilon'}$ are independent copies of ω_ϵ and $\omega_{\epsilon'}$. We fix ϵ and consider $\epsilon_n = \epsilon^{\frac{1}{n}}$. Of course $\epsilon_n^n = \epsilon$. By iterating the cascade rule (23), we get:

$$(\omega_\epsilon(x))_{x\in\mathbb{R}^d} \overset{law}{=} (\sum_{k=0}^{n-1} \omega_{\epsilon_n}^{(k)}(\frac{x}{\epsilon_n^k}))_{x\in\mathbb{R}^d},$$

where the $\omega_{\epsilon_n}^{(k)}$ are independent processes of law ω_{ϵ_n}. Fix $x, y \in \mathbb{R}^d$. We therefore have for all λ, μ:

$$\lambda\omega_\epsilon(x) + \mu\omega_\epsilon(y) \stackrel{law}{=} \sum_{k=0}^{n-1} \mu\omega_{\epsilon_n}^{(k)}(\frac{x}{\epsilon_n^k}) + \lambda\omega_{\epsilon_n}^{(k)}(\frac{y}{\epsilon_n^k})$$

The stochastic continuity of the process ω with respect to ϵ entails, for all $\eta > 0$:

$$\sup_{0 \leqslant k \leqslant n-1} P(|\mu\omega_{\epsilon_n}^{(k)}(\frac{x}{\epsilon_n^k}) + \lambda\omega_{\epsilon_n}^{(k)}(\frac{y}{\epsilon_n^k})| > \eta) \underset{n\to\infty}{\to} 0.$$

By a classical theorem on independent triangular arrays (see chapter XVII in [11]), this shows that the couple $(\omega_\epsilon(x), \omega_\epsilon(y))$ is ID. One proceeds similarly to show that, for all $(x_1, \cdots, x_n)$, the vector $(\omega_\epsilon(x_1), \cdots, \omega_\epsilon(x_n))$ is ID.

□

B. Proof of Theorem 4

We adapt the proofs of [13, 20].

• **The class R_α.**

Let B be a non empty ball of $\mathbb{R}^d$. We introduce the set R_α of Radon measures ν on B satisfying: for any $\varepsilon > 0$, there exist $\delta > 0, D > 0$ and a compact set $K_\varepsilon \subset B$ with $\nu(B \setminus K_\varepsilon) < \varepsilon$ such that the measure $\nu_\varepsilon := \mathbf{1}_{K_\varepsilon}(x)\nu(dx)$ satisfies, for every open set $U \subset B$,

$$\nu_\varepsilon(U) \leqslant D \times diam(U)^{\alpha+\delta}. \tag{24}$$

We further define the set of Radon measures $R_-^\alpha := \cap_{\beta<\alpha} R^\beta$. For a Radon measure ν, we define the quantity

$$C_\alpha(\nu) := \int_{B\times B} \frac{1}{|x-y|^\alpha}\nu(dx)\nu(dy).$$

It is plain to see that

$$C_\alpha(\nu) < \infty \Longrightarrow \nu \in R_-^\alpha.$$

Conversely, a measure obeying (24) satisfies $C_\beta(\nu) < +\infty$ for all $\beta < \alpha + \delta$.

We show the following intermediate result:

Lemma 15. *Consider a Radon measure $\kappa \in R_\alpha$. Let N be the Radon measure defined on B by*

$$N(dx) := \lim_{\varepsilon \searrow 0} e^{X_x^\varepsilon - \psi(1)\ln\left(\frac{1}{\varepsilon}\right)}\kappa(dx) =: \lim_{\varepsilon \searrow 0} N_\varepsilon(dx).$$

If $\overline{F} < \alpha$, then the martingale $(N_\epsilon(B))_\epsilon$ is regular and $N \in R_{\alpha-\psi'(1)+\psi(1)}$.

Proof. We first show that the martingale $(N_\epsilon(B))_\epsilon$ is regular. For this, we use the fact that $F(\cdot)$ verifies Assumption (2) to get (for some positive constant $S = \sup_{\mathbb{R}_+\times\mathbb{R}^d}|h|$):

$$\begin{aligned}
\mathbb{E}[N_\epsilon(B)^2] &= \int_{B\times B} \mathbb{E}\left[e^{X_x^\epsilon + X_y^\epsilon}\right] e^{-2\psi(1)\ln\left(\frac{1}{\epsilon}\right)}\kappa(dx)\kappa(dy) \\
&= \int_{B\times B} e^{\int_1^{1/\varepsilon} F\left(g(u)(x-y)\right)\frac{du}{u}}\kappa(dx)\kappa(dy) \\
&\leqslant \int_{B\times B} e^{\int_1^{\infty} |F\left(g(u)(x-y)\right)|\frac{du}{u}}\kappa(dx)\kappa(dy) \\
&\leqslant \int_{B\times B} e^{\overline{F}\ln_+\frac{1}{|x-y|}+S}\kappa(dx)\kappa(dy) \\
&\leqslant e^S \int_{B\times B} \max\Big(\frac{1}{|x-y|^{\overline{F}}}, 1\Big)\kappa(dx)\kappa(dy)
\end{aligned}$$

and the last integral is finite as soon as $\overline{F} < \alpha$. Hence, the martingale $(N_\epsilon(B))_\epsilon$ is regular.

We consider a compact set $K \subset B$. Even if it means multiplying κ by a positive constant, we assume that $\kappa(K) = 1$. We consider on $\Omega \times K$ the probability measure $\mathbb{Q}$ defined by

$$\int_{\Omega\times K} f(\omega, x) d\mathbb{Q} := \mathbb{E}\left[\int_K f(\omega, x) N(dx)\right]$$

where f is some non negative measurable function.

For $0 < \varepsilon' < \varepsilon < 1$, we define the process $(X_x^{\varepsilon',\varepsilon})_{x\in\mathbb{R}^d}$ by

$$\forall x \in \mathbb{R}^d, \quad X_x^{\varepsilon',\varepsilon} := X_x^{\varepsilon'} - X_x^{\varepsilon} - \psi(1)\ln(\varepsilon/\varepsilon').$$

Because of expression (11), it is straightforward to check that, given $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$, the processes $X^{\varepsilon_1,\varepsilon_2}, X^{\varepsilon_2,\varepsilon_3}, \ldots, X^{\varepsilon_{n-1},\varepsilon_n}$ are $\mathbb{Q}$-independent. Moreover, for $\lambda \geqslant 0$ and because $(N_\epsilon)_\epsilon$ is uniformly integrable, we have

$$\begin{aligned}
&\int e^{\lambda X_x^{\varepsilon',\varepsilon}} d\mathbb{Q} \\
&= \int_K \mathbb{E}\Bigg[e^{\lambda \int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}} \int_{\mathbb{R}^d} \cos(xg(y)\cdot u)\, W(du,dy) + \sin(xg(y)\cdot u)\, W'(du,dy) + \lambda b \ln\frac{\epsilon}{\epsilon'} - \lambda\psi(1)\ln(\varepsilon/\varepsilon')} \\
&\quad \times e^{\lambda \int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}} \int_S f(T_{tg(y)}(s))\, [N(ds,dy) - (1\vee|f(T_{tg(y)}(s))|)^{-1}\theta(ds)\frac{dy}{y}]} e^{X_x^{\epsilon'} - \psi(1)\ln\frac{1}{\epsilon'}} \Bigg] \kappa(dx) \\
&= \int_K \mathbb{E}\Bigg[e^{(\lambda+1) \int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}} \int_{\mathbb{R}^d} \cos(xg(y)\cdot u)\, W(du,dy) + \sin(xg(y)\cdot u)\, W'(du,dy) + (\lambda+1) b \ln\frac{\epsilon}{\epsilon'} - (\lambda+1)\psi(1)\ln(\varepsilon/\varepsilon')} \\
&\quad \times e^{(\lambda+1) \int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}} \int_S f(T_{tg(y)}(s))\, [N(ds,dy) - (1\vee|f(T_{tg(y)}(s))|)^{-1}\theta(ds)\frac{dy}{y}]} \Bigg] \kappa(dx) \\
&= e^{\psi(\lambda+1)\ln(\varepsilon/\varepsilon') - (\lambda+1)\psi(1)\ln(\varepsilon/\varepsilon')}.
\end{aligned}$$

In particular, under $\mathbb{Q}$, the process $u \in \mathbb{R}^+ \mapsto X^{e^{-u},1}$ is an integrable Lévy process. Thus from the strong law of large numbers, we get that $\mathbb{Q}$-almost surely:

$$\frac{X^{e^{-u},1}}{u} \to \psi'(1) - \psi(1)$$

when $u \to \infty$. Consequently, $\mathbb{P}$ almost surely,

$$N \text{ a.s.,} \quad \frac{X_x^{e^{-u}}}{u} \to \psi'(1). \tag{25}$$

In particular, by Egoroff's theorem, there exists a compact set $K_\varepsilon^1 \subset K$ such that $N(K \setminus K_\varepsilon^1) < \varepsilon$ and the convergence (25) is uniform with respect to $x \in K_\varepsilon^1$. Let now $q > 0$, and define $N_q(dy) := \lim_{\epsilon\searrow 0} e^{X_y^{\epsilon,e^{-q}}}\kappa(dy)$ and $P_q(x) := N_q(B_x^q \cap K)$ where B_x^q denotes the ball centered on x and with radius e^{-q}. We finally define the function

$$\theta_q(x,y) := \mathbf{1}_{\{|x-y|\,\leqslant\, e^{-q}\}},$$

in such a way that $P_q(x)=\int_K \theta_q(x,y)N_q(dy)$. Thus we have:

$$\begin{aligned}\int P_q d\mathbb{Q} &= \mathbb{E}\left[\int_{K\times K}\theta_q(x,y)N_q(dx)N(dy)\right]\\ &= \lim_{\epsilon\to 0}\mathbb{E}\left[\int_{K\times K}\theta_q(x,y)e^{X_x^{\epsilon,e^{-q}}+X_y^{\epsilon,e^{-q}}}\kappa(dx)\kappa(dy)\right]\\ &= \int_{K\times K}\theta_q(x,y)e^{\int_{[e^q,\infty]}F(g(u)(y-x))\frac{du}{u}}\kappa(dx)\kappa(dy).\end{aligned}$$

Let $\beta>\overline{F}$ be fixed. By using Assumption 2 and the above relation, we obtain (for some positive constant $S=\sup_{\mathbb{R}_+\times\mathbb{R}^d}|h|$)

$$\begin{aligned}\int\sum_{n\geqslant 1}e^{\beta n}P_n d\mathbb{Q} &= \sum_{n\geqslant 1}\int_{K\times K}\theta_n(x,y)e^{\beta n}e^{\int_{[e^n,\infty]}F(g(u)(y-x))\frac{du}{u}}\kappa(dx)\kappa(dy)\\ &= \int_{K\times K}\sum_{1\leqslant n\leqslant -\ln(|x-y|)}e^{\beta n}e^{\int_{[e^n,\infty]}F(g(u)(y-x))\frac{du}{u}}\kappa(dx)\kappa(dy)\\ &\leqslant e^S\int_{K\times K}\sum_{1\leqslant n\leqslant -\ln(|x-y|)}e^{\beta n}e^{\overline{F}\ln\frac{1}{e^n|x-y|}}\kappa(dx)\kappa(dy)\\ &\leqslant e^S\int_{K\times K}\sum_{1\leqslant n\leqslant -\ln(|x-y|)}e^{(\beta-\overline{F})n}\frac{1}{|x-y|^{\overline{F}}}\kappa(dx)\kappa(dy)\end{aligned}$$

Note that, for some positive constant D,

$$\sum_{1\leqslant n\leqslant -\ln(|x-y|)}e^{(\beta-\overline{F})n}\leqslant D\frac{1}{|x-y|^{\beta-\overline{F}}},$$

in such a way that

$$\int\sum_{n\geqslant 1}e^{\beta n}P_n d\mathbb{Q}\leqslant De^S\int_{K\times K}\frac{1}{|x-y|^{\beta}}\kappa(dx)\kappa(dy)=DD'C_\beta(\kappa).$$

The last term is finite as soon as $\beta<\alpha$. Thus for $\beta\in]\overline{F},\alpha[$, $\mathbb{Q}$ a.s., $e^{\beta n}P_n\to 0$ as $n\to\infty$. In particular, one can find a compact set $K_\varepsilon^2\subset K$ such that $N(K\setminus K_\varepsilon^2)<\varepsilon$ and such that, N almost surely,

$$\limsup_{n\to\infty}\frac{\log(P_n(x))}{n}\leqslant -\beta$$

uniformly for $x\in K_\varepsilon^2$. Setting $\widetilde{K}:=K_\varepsilon^2\cap K_\varepsilon^1$ and $N_{\widetilde{K}}:=\mathbf{1}_{\widetilde{K}}(x)N(dx)$, we get that, uniformly with respect to $x\in\widetilde{K}$:

$$\begin{aligned}\limsup_{n\to\infty}\frac{\log(N_{\widetilde{K}}(B_n^x))}{n} &= \limsup_{n\to\infty}\frac{\log\left(\int_{\widetilde{K}\cap B_n^x}e^{X_u^{e^{-n}}-\psi(1)n}N_n(du)\right)}{n}\\ &\leqslant -\beta+\psi'(1)-\psi(1).\end{aligned}$$

This entails in particular that $M\in R_{\alpha-\psi'(1)+\psi(1)}$. $\square$

Making use of Lemma 15, we now prove Theorem 4.

Proof of Theorem 3. The basic idea is to show that a Lévy multiplicative chaos satisfying $\psi'(1)-\psi(1)<d$ can be decomposed as an iterated Lévy multiplicative chaos.

First, fix an integer n such that

$$\overline{F} < n(d-\psi'(1)+\psi(1)).$$

There exist n independent identically distributed approximating families $(X^{(1),\epsilon},\dots X^{(n),\epsilon})_{\epsilon\in]0,1[}$ respectively associated to $(S, W^{(i)}, W'^{(i)}, N^{(i)}, R/n, \theta/n, f, (T_x)_x)$ where the $(W^{(i)}, W'^{(i)}, N^{(i)})_{1\leqslant i\leqslant n}$ are all independent. We assume that the triples $(W^{(1)}, W'^{(1)}, N^{(1)})$, ..., $(W^{(n)}, W'^{(n)}, N^{(n)})$ are respectively constructed on the probability space $(\Omega_1,\mathbb{P}^1),\dots,(\Omega_n,\mathbb{P}^n)$, and we define $\Omega := \Omega_1\times\dots\times\Omega_n$ equipped with the probability measure $\mathbb{P} := \mathbb{P}^1\otimes\dots\otimes\mathbb{P}^n$.

We define recursively for $1\leqslant k\leqslant n$:

$$M^{(0)}(dx) := dx, \qquad M^{(k)}(dx) := \lim_{\varepsilon\searrow 0} e^{X_x^{(k),\varepsilon}-\frac{\psi(1)}{n}\ln(1/\varepsilon)} M^{(k-1)}(dx) \tag{26}$$

where the limit has to be understood in the sense of weak convergence of Radon measures. For $k\in[1,n-1]$, one has the relation

$$\frac{\overline{F}}{n}\leqslant d-\frac{k}{n}(\psi'(1)-\psi(1)),$$

so that we can apply recursively Lemma 15 to prove that for each $k\leqslant n$,

$$\mathbb{E}[M^{(k)}(B)] = \mathbb{E}[M^{(k-1)}(B)] \text{ and } M^{(k)}\in R_{d-\frac{k}{n}(\psi'(1)-\psi(1))}.$$

In particular, the martingales considered in (26) are uniformly integrable. Then we prove that the measures M and $M^{(n)}$ have the same law. For this, we note that the following equality in law holds:

$$M^{(n)}(dx) = \lim_{\varepsilon\searrow 0} e^{X_x^{(1),\varepsilon}+\dots+X_x^{(n),\varepsilon}-\psi(1)\ln(1/\varepsilon)}dx. \tag{27}$$

Indeed, consider the σ-algebra $\mathcal{G}_\varepsilon$ generated by $\{X_r^{(1),\varepsilon'},\dots,X_r^{(n),\varepsilon'},\varepsilon'>\varepsilon, r\in\mathbb{R}^d\}$. Using the fact that the martingales considered in (26) are uniformly integrable, we compute:

$$\begin{aligned}
\mathbb{E}\left[M^{(n)}(A)\middle|\mathcal{G}_\varepsilon\right] &= \mathbb{E}\left[\mathbb{E}\left[M^{(n)}(A)\middle|(X_r^{(1),\varepsilon'},\dots,X_r^{(n-1),\varepsilon'})_{r\in\mathbb{R}^d,\epsilon'\in]0,1[},(X_r^{(n),\varepsilon'})_{r\in\mathbb{R}^d,\varepsilon'>\varepsilon}\right]\middle|\mathcal{G}_\varepsilon\right]\\
&= \mathbb{E}\left[\mathbb{E}^{(n)}\left[M^{(n)}(A)\middle|(X_r^{(n),\varepsilon'})_{r\in\mathbb{R}^d,\varepsilon'>\varepsilon}\right]\middle|\mathcal{G}_\varepsilon\right]\\
&= \mathbb{E}\left[\int_A e^{X_r^{(n),\varepsilon}-\frac{\psi(1)}{n}\log(1/\varepsilon)}M^{(n-1)}(dr)\middle|\mathcal{G}_\varepsilon\right]\\
&= \dots\\
&= \int_A e^{X_r^{(n),\varepsilon}+\dots+X_r^{(1),\varepsilon}-\psi(1)\log(1/\varepsilon)}dr.
\end{aligned}$$

Since this last quantity has the same law as $M^\epsilon(A)$, (27) follows by passing to the limit as $\epsilon\to 0$. Since $\mathbb{E}[M^{(n)}(A)] = |A|$, we deduce $\mathbb{E}[M(A)] = |A|$. Hence M is not trivial. Furthermore we have proved that $M\in R_{d-\psi'(1)+\psi(1)}$. In particular, M cannot possess any atom. □

C. Proofs of Section 3

C.1 Proof of Proposition 7

We have

$$\int_a^\infty \frac{|F(ux)|}{u}\,du = \int_{a|x|}^\infty \frac{|F(ue_x)|}{u}\,du$$

where $e_x = \frac{x}{|x|}$. For $a|x| \geqslant 1$, this quantity is less than (13). For $a|x| \leqslant 1$, we have the bound:

$$\int_{a|x|}^{\infty} \frac{|F(ue_x)|}{u}\,du = \int_{a|x|}^{1} \frac{|F(ue_x)|}{u}\,du + \int_{1}^{\infty} \frac{F(ue_x)}{u}\,du$$
$$\leqslant F(0)\ln\frac{1}{a|x|} + \int_{1}^{\infty} \frac{|F(ue_x)|}{u}\,du$$

because $|F(x)| \leqslant F(0)$. Actually, because of the continuity of the function F at 0, it turns out that we have $\int_{|x|}^{1} \frac{F(ue_x)}{u}\,du \simeq F(0)\ln\frac{1}{|x|}$ as $|x| \to 0$. We deduce

$$\int_{|x|}^{\infty} \frac{F(ue_x)}{u}\,du \simeq F(0)\ln\frac{1}{|x|} \qquad \text{as } |x| \to 0. \tag{28}$$

□

C.2 Proof of Proposition 9

We just have to compute the second order moment (we use the notation $e_{x-y} = \frac{x-y}{|x-y|}$)

$$\begin{aligned}\mathbb{E}[\widetilde{M}^{\epsilon}(A)^2] &= \int_{A\times A} \mathbb{E}\left[e^{X_x^{\epsilon}+X_y^{\epsilon}}\right] e^{-2\psi(1)\ln(\frac{1}{\epsilon})}\,dxdy\\ &= \int_{A\times A} e^{\int_1^{1/\varepsilon} F\left(u(x-y)\right)\frac{du}{u}}\,dxdy\\ &= \int_{A\times A} e^{\int_{|x-y|}^{|x-y|/\varepsilon} F\left(ue_{x-y}\right)\frac{du}{u}}\,dxdy.\end{aligned}$$

In case M admits a second order moment, we deduce that the quantity

$$\mathbb{E}[M(A)^2] = \int_{A\times A} e^{\int_{|x-y|}^{\infty} F\left(ue_{x-y}\right)\frac{du}{u}}\,dxdy$$

is finite. Because of (28), we necessarily have $F(0) < d$. Conversely, if $F(0) < d$ then $\sup_{\epsilon}\mathbb{E}[\widetilde{M}^{\epsilon}(A)^2]$ is less than the above right-hand side, which is finite. The proof is complete. □

C.3 Proof of Proposition 8

For $0 < \epsilon < 1$, $t_1, \dots, t_p \in (\mathbb{R}^d)^p$ and $q_1, \dots, q_p \in \mathbb{R}$ such that the following expectations make sense, we define the Laplace exponents ψ^{ϵ} of X^{ϵ}:

$$\mathbb{E}[e^{q_1 X_{t_1}^{\epsilon}+\dots+q_p X_{t_p}^{\epsilon}}] = e^{\psi^{\epsilon}_{t_1,\dots,t_p}(q_1,\dots,q_p)}.$$

For $\epsilon' < \epsilon$, we have

$$\begin{aligned}
&\psi^{\epsilon'}_{t_1,\dots,t_p}(q_1,\dots,q_p) = \\
&\quad = b\ln\frac{1}{\epsilon'}\sum_{i=1}^p q_i + \frac{1}{2}\int_1^{\frac{1}{\epsilon'}}\int_{\mathbb{R}^d}\big(\sum_{i=1}^p q_i\cos(yt_iu)\big)^2 R(du)\frac{dy}{y} + \frac{1}{2}\int_1^{\frac{1}{\epsilon'}}\int_{\mathbb{R}^d}\big(\sum_{i=1}^p q_i\sin(yt_iu)\big)^2 R(du)\frac{dy}{y} \\
&\qquad + \int_1^{\frac{1}{\epsilon'}}\int_S\Big(e^{\sum_{i=1}^p q_i f(T_{t_iy}(s))} - 1 - \sum_{i=1}^p q_i\frac{f(T_{t_iy}(s))}{1\vee|f(T_{t_iy}(s))|}\Big)\theta(ds)\frac{dy}{y} \\
&\quad = b\ln\frac{\epsilon}{\epsilon'}\sum_{i=1}^p q_i + \frac{1}{2}\int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}}\int_{\mathbb{R}^d}\big(\sum_{i=1}^p q_i\cos(yt_iu)\big)^2 R(du)\frac{dy}{y} + \frac{1}{2}\int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}}\int_{\mathbb{R}^d}\big(\sum_{i=1}^p q_i\sin(yt_iu)\big)^2 R(du)\frac{dy}{y} \\
&\qquad + \int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}}\int_S\Big(e^{\sum_{i=1}^p q_i f(T_{t_iy}(s))} - 1 - \sum_{i=1}^p q_i\frac{f(T_{t_iy}(s))}{1\vee|f(T_{t_iy}(s))|}\Big)\theta(ds)\frac{dy}{y} + \psi^{\epsilon}_{t_1,\dots,t_p}(q_1,\dots,q_p) \\
&\quad = b\ln\frac{\epsilon}{\epsilon'}\sum_{i=1}^p q_i + \frac{1}{2}\int_1^{\frac{\epsilon}{\epsilon'}}\int_{\mathbb{R}^d}\big(\sum_{i=1}^p q_i\cos(y\frac{t_i}{\epsilon}u)\big)^2 R(du)\frac{dy}{y} + \frac{1}{2}\int_{\frac{1}{\epsilon}}^{\frac{1}{\epsilon'}}\int_{\mathbb{R}^d}\big(\sum_{i=1}^p q_i\sin(y\frac{t_i}{\epsilon}u)\big)^2 R(du)\frac{dy}{y} \\
&\qquad + \int_1^{\frac{\epsilon}{\epsilon'}}\int_S\Big(e^{\sum_{i=1}^p q_i f(T_{\frac{t_i}{\epsilon}y}(s))} - 1 - \sum_{i=1}^p q_i\frac{f(T_{\frac{t_i}{\epsilon}y}(s))}{1\vee|f(T_{\frac{t_i}{\epsilon}y}(s))|}\Big)\theta(ds)\frac{dy}{y} + \psi^{\epsilon}_{t_1,\dots,t_p}(q_1,\dots,q_p) \\
&\quad = \psi^{\frac{\epsilon'}{\epsilon}}_{\frac{t_1}{\epsilon},\dots,\frac{t_p}{\epsilon}}(q_1,\dots,q_p) + \psi^{\epsilon}_{t_1,\dots,t_p}(q_1,\dots,q_p)
\end{aligned}$$

Hence we can write

$$(X^{\epsilon'}_x)_x \overset{\text{law}}{=} (X^{\epsilon}_x + \overline{X}^{\frac{\epsilon'}{\epsilon}}_{\frac{x}{\epsilon}})_x \tag{29}$$

where $\overline{X}^{\frac{\epsilon'}{\epsilon}}$ is independent from X^{ϵ} and has the same law as $X^{\frac{\epsilon'}{\epsilon}}$. It is then plain to deduce that M is $\star$-scale invariant. Indeed, define M^{ϵ} by

$$\forall A\in\mathcal{B}(\mathbb{R}^d),\quad M^{\epsilon}(A) = \lim_{\epsilon'\to 0}\int_A e^{\overline{X}^{\frac{\epsilon'}{\epsilon}}_{\frac{x}{\epsilon}} - \psi(1)\ln\frac{\epsilon}{\epsilon'}}\,dx.$$

A straightforward change of variables shows that

$$M^{\epsilon}(dx) \overset{\text{law}}{=} \epsilon^d M(dx/\epsilon).$$

From (29), we deduce

$$M(dx) = e^{X^{\epsilon}_x - \psi(1)\ln\frac{1}{\epsilon}} M^{\epsilon}(dx).$$

D. Proof of Theorem 12

We carry out the proof in the case when the dimension is equal to 1. This simplifies the notations. In higher dimensions, the proof works the same.

The guiding line is the same as in [1]. But the lack of convexity inequalities, which are specific to the Gaussian case, gives rise to further technical difficulties. So we detail what differs and refer to [1] for the proofs of the results that do not change with respect to the Gaussian case.

D.1 Setting

We consider a non trivial measure satisfying (2) with a moment of order $1+\delta$ for some $\delta > 0$ and a fixed $\epsilon \in]0,1[$. The first step is to prove that the measure M is a Lévy multiplicative chaos. Since

M is not trivial and possesses a moment of order at least 1, we necessarily have

$$\forall x \in \mathbb{R}, \quad \mathbb{E}[e^{\omega_\epsilon(x)}] = 1. \tag{30}$$

Because it is stochastically continuous and ID, the process ω_ϵ admits a version with a representation as in (3) with associated parameters $(S_\epsilon, W_\epsilon, W'_\epsilon, N, \theta_\varepsilon, R_\varepsilon, f_\varepsilon, (T^\epsilon_x)_x)$. The Laplace transform of ω_ϵ is denoted by

$$\psi_\epsilon(q) = \ln \mathbb{E}[e^{q\omega_\epsilon(0)}].$$

It satisfies $\psi_\varepsilon(1) = 0$. We let $(X^n)_n$ denote a sequence of independent stationary stochastically continuous ID processes with common law that of ω_ε. Of course, the law of this sequence depends on ϵ but we remove this dependence from the notations for the sake of clarity. We also define the measure M^N for $N \geqslant 0$ by

$$M^N(A) := \varepsilon^{N+1} M\left(\frac{1}{\varepsilon^{N+1}} A\right). \tag{31}$$

We assume that the sequences $(X_n)_n$ and $(M^N)_N$ are independent. Iterating the relation (2), we get that, for every integer N, the measure $\widetilde{M}^N$ defined by:

$$\widetilde{M}^N(A) = \int_A \exp\left(\sum_{n=0}^{N} X^n_{r/\varepsilon^n}\right) M^N(dr) \tag{32}$$

has the same law as the measure M.

Lemma 16. *(see [1]) Let M be a stationary random measure on $\mathbb{R}$ admitting a moment of order $1+\delta$. There is a nonnegative integrable random variable $Y \in L^{1+\delta}$ such that, for every bounded interval $I \subset \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T} M(TI) = Y|I| \quad \textit{almost surely and in } L^{1+\delta},$$

where $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}$. As a consequence, almost surely the random measure

$$A \in \mathcal{B}(\mathbb{R}) \mapsto \frac{1}{T} M(TA)$$

weakly converges towards $Y|\cdot|$ and $\mathbb{E}_Y[M(A)] = Y|A|$ ($\mathbb{E}_Y[\cdot]$ denotes the conditional expectation with respect to Y).

Thus, in what follows, the random variable Y will be defined as the unique (up to a set of probability 0) random variable such that $\mathbb{E}_Y[M(A)] = Y|A|$ for all Borelian sets A.

For $x \neq 0$, define:

$$S^\varepsilon(x) := \sum_{n=0}^{\infty} F_\varepsilon(x/\varepsilon^n) \tag{33}$$

where $F_\varepsilon(\cdot)$ is the generalized covariance function associated to ω_ε (see Assumption (11)). The uniform convergence of the series on the sets of the type $\{x \in \mathbb{R}; |x| \geqslant \rho\}$ is ensured by (14) (see [1]). Then we can reproduce the proofs if [1, Section 5.2] by replacing K^ϵ by S^ε in the proofs.

D.2 M is a multiplicative Lévy chaos.

Let us define the σ algebra $\mathcal{F}_N := \sigma(X^0, \ldots, X^N, Y)$. For every Borelian subset $A \subset \mathbb{R}$, we define

$$G_N(A) := \mathbb{E}\left[\widetilde{M}^N(A)|\mathcal{F}_N\right]. \tag{34}$$

As in [1], we prove

$$\forall N \geqslant 0, \quad G_N(A) = Y \int_A \exp\left(\sum_{n=0}^{N} X^n_{x/\varepsilon^n}\right) dx. \tag{35}$$

Hence, for each bounded Borelian set A, the sequence $(G_N(A))_N$ is a positive martingale bounded in $L^{1+\delta}$. Being bounded in $L^{1+\delta}$, the martingale $G_N(A)$ converges towards a random variable $Q(A)$ which should be formally thought of as:

$$Q(A) = Y \int_A \exp\left(\sum_{n=0}^{\infty} X^n_{x/\varepsilon^n}\right) dx.$$

The result below is proved in [1] and uses specific properties of Gaussian processes, namely Gaussian concentration inequalities due to Kahane. It turns out that we can carry out the proof while skipping these inequalities:

Lemma 17. *For small enough $\gamma \in]0, \delta[$, there exists $\rho > 0$ such that:*

$$\sup_n n^{1+\rho} \mathbb{E}\left[M\left(\left[0, \frac{1}{n}\right]\right)^{1+\gamma}\right] < \infty. \tag{36}$$

The central lemma for establishing Lemma 17 is the following:

Lemma 18. *The finiteness of a moment of order $1+\delta$ (for some $\delta > 0$) implies*

$$\forall \epsilon < 1, \quad \psi'_\epsilon(1) < \ln \frac{1}{\epsilon}, \tag{37}$$

and $\forall \gamma \in [0, \delta[$

$$\forall \epsilon < 1, \quad \psi_\epsilon(1+\gamma) < \gamma \ln \frac{1}{\epsilon}. \tag{38}$$

Proof. Let us fix $\epsilon < 1$ and define for $q \leqslant 1$:

$$F_\epsilon(q, r) = \ln \mathbb{E}[e^{q\omega_\epsilon(r) + q\omega_\epsilon(0)}].$$

Let us consider $h > 1$ such that $(1+\delta)h = 2$. By concavity of the function $x \mapsto x^{1/h}$, we can make use of Jensen's inequality to get for $N \geqslant 1$:

$$\begin{aligned}
\mathbb{E}\big[M\big([0, \frac{1}{n}]\big)^{1+\delta}\big] &= \mathbb{E}\left[M\big([0, \frac{1}{n}]\big)^{2\times\frac{1}{h}}\right] \\
&= \mathbb{E}\left[\left(\int_0^{1/n}\int_0^{1/n} e^{\sum_{p=0}^{N-1} X^p(\frac{r}{\epsilon^p}) + X^p(\frac{u}{\epsilon^p})} M^{N-1}(dr) M^{N-1}(du)\right)^{1/h}\right] \\
&\geqslant \mathbb{E}\left[\int_0^{1/n}\int_0^{1/n} e^{\frac{1}{h}\sum_{p=0}^{N-1} X^p(\frac{r}{\epsilon^p}) + X^p(\frac{u}{\epsilon^p})} M^{N-1}(dr) M^{N-1}(du) M^{N-1}([0, \frac{1}{n}])^{2/h-2}\right] \\
&\geqslant e^{N \inf_{|r| \leqslant \frac{1}{n\epsilon^N}} F_\epsilon(\frac{1}{h}, r)} \mathbb{E}\left[M^{N-1}([0, \frac{1}{n}])^{1+\delta}\right]
\end{aligned}$$

where we made use of the fact that the sequence $(X^n)_{n \leqslant N-1}$ is independent of the random measure M^{N-1}. Now we choose N such that $\epsilon^N = \frac{1}{\alpha n}$ for some $\alpha > 0$, that is $N = \frac{\ln \alpha + \ln n}{\ln \frac{1}{\epsilon}}$. We obtain:

$$\mathbb{E}\big[M\big([0, \frac{1}{n}]\big)^{1+\delta}\big] \geqslant \frac{1}{n^{1+\delta}} e^{(\ln n + \ln \alpha)\frac{\inf_{|r| \leqslant \alpha} F_\epsilon(\frac{1}{h}, r)}{\ln \frac{1}{\epsilon}}} \frac{1}{\alpha^{1+\delta}} \mathbb{E}\left[M([0, \alpha])^{1+\delta}\right]$$

Now, we use the super-additivity of the function $x \mapsto x^{1+\delta}$ to obtain:

$$\mathbb{E}\big[M\big([0, 1]\big)^{1+\delta}\big] \geqslant \sum_{k=1}^{n} \mathbb{E}\big[M\big([\frac{k-1}{n}, \frac{k}{n}]\big)^{1+\delta}\big] \mathbb{E}\big[M\big([0, \frac{1}{n}]\big)^{1+\delta}\big].$$

By gathering the above inequalities, we deduce:

$$\mathbb{E}\big[M\big([0,1]\big)^{1+\delta}\big] \geqslant n\frac{1}{n^{1+\delta}}e^{(\ln n+\ln\alpha)\frac{\inf_{|r|\,\leqslant\,\alpha}F_\epsilon(\frac{1}{h},r)}{\ln\frac{1}{\epsilon}}}\frac{1}{\alpha^{1+\delta}}\mathbb{E}\left[M([0,\alpha])^{1+\delta}\right].$$

Because the left-hand side is bounded independently of n, we necessarily have:

$$\forall\epsilon>0,\quad \frac{\inf_{|r|\,\leqslant\,\alpha}F_\epsilon(\frac{1}{h},r)}{\ln\frac{1}{\epsilon}}\leqslant\delta. \tag{39}$$

By letting α go to 0 and by continuity of $F_\epsilon(\frac{1}{h},\cdot)$ at 0 (ω_ϵ is stochastically continuous with a moment of order $1+\delta$), we deduce

$$\forall\epsilon>0,\quad \frac{\psi_\epsilon(1+\delta)}{\ln\frac{1}{\epsilon}}\leqslant\delta. \tag{40}$$

By convexity arguments, it is then plain to deduce that

$$\frac{\psi_\epsilon'(1)}{\ln\frac{1}{\epsilon}}<1. \tag{41}$$

Indeed, the (not strict) inequality results from (40). If equality holds, this means that $\psi_\epsilon(1+\gamma)=1$ for all $\gamma\in[0,\delta[$. By analycity arguments, this implies that the law of the process ω_ϵ is that of a constant and the measure M is thus trivial. This is in contradiction with our assumptions. The same type of argument leads to (38). □

Proof of Lemma 17. We consider $\gamma\in]0,\delta[$. As the function $x\mapsto x^{1+\gamma}$ is convex, we make use of Jensen's inequality to get for $N\geqslant 1$:

$$\begin{aligned}
\mathbb{E}\big[M\big([0,\tfrac{1}{n}]\big)^{1+\gamma}\big] &= \mathbb{E}\left[\left(\int_0^{1/n}\frac{e^{\sum_{p=0}^{N-1}X^p(\frac{r}{\epsilon^p})}}{M^{N-1}([0,1/n])}M^{N-1}(dr)\right)^{1+\gamma}M^{N-1}\big([0,\tfrac{1}{n}]\big)^{1+\gamma}\right]\\
&\leqslant \mathbb{E}\left[\int_0^{1/n}e^{(1+\gamma)\sum_{p=0}^{N-1}X^p(\frac{r}{\epsilon^p})}M^{N-1}(dr)M^{N-1}\big([0,\tfrac{1}{n}]\big)^{\gamma}\right]\\
&\leqslant \mathbb{E}\left[e^{(1+\gamma)\sum_{p=0}^{N-1}X^p(0)}\right]\mathbb{E}\left[M^{N-1}\big([0,\tfrac{1}{n}]\big)^{1+\gamma}\right]\\
&= e^{N\psi_\epsilon(1+\gamma)}\mathbb{E}\left[M^{N-1}\big([0,\tfrac{1}{n}]\big)^{1+\gamma}\right]
\end{aligned}$$

where, once again, we made use of the fact that the sequence $(X^n)_{n\,\leqslant\,N-1}$ is independent of the random measure M^{N-1}. We choose $N=\frac{-\ln n}{\ln\epsilon}$ in order to have $\epsilon^N=\frac{1}{n}$. We get that:

$$\mathbb{E}\big[M\big([0,\tfrac{1}{n}]\big)^{1+\gamma}\big]\leqslant\frac{1}{n^{1+\gamma-\frac{\psi_\epsilon(1+\gamma)}{\ln\frac{1}{\epsilon}}}}\mathbb{E}\big[M\big([0,1]\big)^{1+\delta}\big]$$

We are thus left with checking that $\frac{\psi_\epsilon'(1)}{\ln\frac{1}{\epsilon}}<1$. This the the content of Lemma 18. □

Let us stress that, as an immediate consequence of Lemma 17, the measure M does not possess any atom (see [8, Corollary 9.3 VI]). With the above estimation on the function ψ_ϵ, we can prove that Q is a non trivial Lévy multiplicative chaos:

Lemma 19. *The random measure Q is a Lévy multiplicative chaos and it is non trivial.*

Proof. Let us use the decomposition of X^n to write

$$X^n(r) = b_\epsilon+\int_{\mathbb{R}^d} \cos(t\cdot u)\, W^n_\epsilon(du)+\int_{\mathbb{R}^d} \sin(t\cdot u)\, W^{n'}_\epsilon(du)+\int_{S_\epsilon} f_\epsilon(T^\epsilon_t(s))\,[N^n_\epsilon(ds)-(1\vee|f_\epsilon(T^\epsilon_t(s))|)^{-1}\theta_\epsilon(ds)]$$

where the triples $(W^n_\epsilon, W^{n'}_\epsilon, N^n_\epsilon)_n$ are independent. Thus we have

$$\begin{aligned}
Y^N_t :=& \sum_{n=0}^N X^n(\frac{t}{\epsilon^n})\\
=& Nb_\epsilon + \sum_{n=0}^N \int_{\mathbb{R}^d} \cos(\frac{t}{\epsilon^n}\cdot u)\, W^n_\epsilon(du) + \sum_{n=0}^N \int_{\mathbb{R}^d} \sin(\frac{t}{\epsilon^n}\cdot u)\, W^{n'}_\epsilon(du)\\
&+ \sum_{n=0}^N \int_{S_\epsilon} f_\epsilon(T^\epsilon_{\frac{t}{\epsilon^n}}(s))\,[N^n_\epsilon(ds) - (1\vee|f_\epsilon(T^\epsilon_{\frac{t}{\epsilon^n}}(s))|)^{-1}\theta_\epsilon(ds)]
\end{aligned}$$

Let us compute the Lévy exponent of Y^N. For $r_1,\dots,r_p\in\mathbb{R}$ and $\lambda_1,\dots,\lambda_N\in\mathbb{R}$ such that the following expectations make sense, we have:

$$\begin{aligned}
\mathbb{E}[e^{\sum_{i=1}^p \lambda_i Y^N_{r_i}}] =& \exp\Big(Nb_\epsilon + \frac{1}{2}\sum_{n=0}^N \int_{\mathbb{R}^d} \Big(\sum_{i=1}^p \lambda_i \cos(\frac{r_i}{\epsilon^n}\cdot u)\Big)^2 R_\epsilon(du) + \frac{1}{2}\sum_{n=0}^N \int_{\mathbb{R}^d} \Big(\sum_{i=1}^p \lambda_i \sin(\frac{r_i}{\epsilon^n}\cdot u)\Big)^2 R_\epsilon(du)\\
&+ \sum_{n=0}^N \int_{S_\epsilon} \Big(e^{\sum_{i=1}^p \lambda_i f_\epsilon(T^\epsilon_{\frac{r_i}{\epsilon^n}}(s))} - 1 - \sum_{i=1}^p \frac{\lambda_i f_\epsilon(T^\epsilon_{\frac{r_i}{\epsilon^n}}(s))}{1\vee|f_\epsilon(T^\epsilon_{\frac{r_i}{\epsilon^n}}(s))|}\Big)\theta_\epsilon(ds)\Big).
\end{aligned}$$

We point out that the last quantity can be rewritten as

$$\begin{aligned}
\exp\Big(&\int_1^{\frac{1}{\epsilon^N}} \frac{b_\epsilon}{\ln\frac{1}{\epsilon}}\frac{dy}{y} + \frac{1}{2}\int_1^{\frac{1}{\epsilon^N}}\int_{\mathbb{R}^d} \Big(\sum_{i=1}^p \lambda_i \cos(r_i g(y)\cdot u)\Big)^2 \frac{R_\epsilon(du)}{\ln\frac{1}{\epsilon}}\frac{dy}{y}\\
&+ \frac{1}{2}\int_1^{\frac{1}{\epsilon^N}}\int_{\mathbb{R}^d} \Big(\sum_{i=1}^p \lambda_i \sin(r_i g(y)\cdot u)\Big)^2 \frac{R_\epsilon(du)}{\ln\frac{1}{\epsilon}}\frac{dy}{y}\\
&+ \int_1^{\frac{1}{\epsilon^N}}\int_{S_\epsilon} \Big(e^{\sum_{i=1}^p \lambda_i f_\epsilon(T^\epsilon_{r_i g(y)}(s))} - 1 - \sum_{i=1}^p \frac{\lambda_i f_\epsilon(T^\epsilon_{r_i g(y)}(s))}{1\vee|f_\epsilon(T^\epsilon_{r_i g(y)}(s))|}\Big)\frac{\theta_\epsilon(ds)}{\ln\frac{1}{\epsilon}}\frac{dy}{y}\Big)
\end{aligned}$$

where g is defined by $g(y) = \frac{1}{\epsilon^n}$ on the interval $[\frac{1}{\epsilon^n}, \frac{1}{\epsilon^{n+1}}[$. Hence, Q is obviously a Lévy multiplicative chaos. Furthermore, from the relation (34), it is plain to deduce that the martingale $(G_N(A))_N$ is bounded in $L^{1+\delta}$ as $(\widetilde{M}^N)_N$ is. Thus, the martingale $(G_N(A))_N$ converges a.s. and in $L^{1+\delta}$ towards its limit $Q(A)$, which is necessarily non trivial. □

Once we have proved Lemma 17 and 19, we can proceed along the same lines as in [1, Section 5] to have the following description of the set of good $\star$-scale invariant random measures:

Proposition 20. *The random measures $(Q(A))_{A\in\mathcal{B}(\mathbb{R})}$ and $(M(A))_{A\in\mathcal{B}(\mathbb{R})}$ have the same law.*

D.3 Structure of the Lévy chaos

Now, we still have to show that the chaos M can be recovered in the same way as the construction set out in section 3. For $(t_1,\dots,t_p)\in\mathbb{R}^p$, we introduce the Lévy exponent η^ϵ of the random variable $(\omega_\epsilon(t_1),\dots,\omega_\epsilon(t_p))$, namely

$$\mathbb{E}[e^{iq_1\omega_\epsilon(t_1)+\dots+iq_p\omega_\epsilon(t_p)}] = e^{\eta^\epsilon(t_1,\dots,t_p,Q)}$$

where Q stands for the vector $(q_1,\dots,q_p)\in\mathbb{R}^p$. For $T\in\mathbb{R}$, $Q\in\mathbb{R}^p$ and $t\geqslant 0$, we define $G_Q(t,T)=\eta^{e^{-t}}(e^{-T}t_1,\dots,e^{-T}t_p,Q)$. It is the Lévy exponent of the random variable $(\omega_{e^{-t}}(e^{-T}t_1),\dots,\omega_{e^{-t}}(e^{-T}t_p))$. Now, we make use of the cascading equation. We claim that for $\varepsilon,\varepsilon'>0$, the following equality holds:

$$(\omega_r^{\varepsilon\varepsilon'})_{r\in\mathbb{R}}\overset{\text{law}}{=}(\omega_r^{\varepsilon}+\omega_{r/\varepsilon}^{\varepsilon'})_{r\in\mathbb{R}}$$

where the processes ω^ε and $\omega^{\varepsilon'}$ are independent. This is an easy consequence of the cascading equation and Lemma 14. It follows that, for $h>0$,

$$G_Q(t+h,T)=G_Q(t,T)+G_Q(h,T-t),$$

or

$$\frac{G_Q(t+h,T)-G_Q(t,T)}{h}=\frac{G_Q(h,T-t)}{h}.$$

Because of Assumption 11 3), at $t_0=\ln\frac{1}{\epsilon_0}$ and for all $T\in\mathbb{R}$, the left-hand side converges as $h\to 0$, and so does the right-hand side. It follows that G_Q is differentiable w.r.t. t at $t=0$, and then at every $t\geqslant 0$. Furthermore, $\partial_t G(t,T)$ is continuous w.r.t. (t,T) because of Assumption 11 3) again. We deduce that

$$\begin{aligned}G_Q(t,T)-G_Q(s,T)&=\int_s^t H(e^{-T+r}t_1,\dots,e^{-T+r}t_p)\,dr\\&=\int_{e^{s-T}}^{e^{t-T}}H(yt_1,\dots,yt_p)\frac{dy}{y}\end{aligned}$$

where

$$H(t_1,\dots,t_p,Q)=\lim_{\epsilon\to 1}\frac{1}{-\ln\epsilon}\eta^\epsilon(t_1,\dots,t_p,Q).$$

Furthermore, H is a continuous function of $(t_1,\dots,t_p)$. By taking $T,s=0$ and by noticing that $G_Q(s,T)=0$, we deduce:

$$\eta^\epsilon(t_1,\dots,t_p,Q)=\int_1^{\frac{1}{\epsilon}}H(yt_1,\dots,yt_p)\frac{dy}{y}\tag{42}$$

Now we want to prove that H stands for the finite-dimensional distributions of an ID process. For that purpose, observe that (41) and (30), that is $\psi_\epsilon(1)=0$, implies that, for each $t\in\mathbb{R}$, the ID random variable with Lévy exponent

$$q\in\mathbb{R}\mapsto\frac{1}{-\ln\epsilon}\psi^\epsilon(q)$$

is tight. Indeed, both relations imply that its characteristic triple $(b_\epsilon,\sigma_\epsilon,\nu_\epsilon)$ satisfies

$$\sup_\epsilon\Big(\sigma_\epsilon^2+\int_{\mathbb{R}}\min(1,z^2)\nu_\epsilon(dz)\Big)<+\infty\quad\text{and}\quad b_\epsilon+\sigma_\epsilon^2/2+\int_{\mathbb{R}}(e^z-1-z\mathbf{1}_{|z|\leqslant 1})\nu_\epsilon(dz)=0.$$

Hence, for any $(t_1,\dots,t_p)\in\mathbb{R}^p$, the family of ID random variables with Lévy exponents

$$Q\in\mathbb{R}^p\mapsto\frac{1}{-\ln\epsilon}\eta^\epsilon(t_1,\dots,t_p,Q)$$

is tight since its one-dimensional marginals have $\frac{1}{-\ln\epsilon}\psi^\epsilon(q)$ as Lévy exponents. Thus, for any $(t_1,\dots,t_p)\in\mathbb{R}^p$, $Q\mapsto H(t_1,\dots,t_p)$ is necessarily the Lévy exponent of some non trivial $\mathbb{R}^p$-valued ID random variable. Furthermore, H is necessarily associated to a consistent family of random variables since the family $\big(Q \mapsto \frac{1}{-\ln\epsilon}\eta^\epsilon(t_1,\dots,t_p,Q)\big)_{p\geqslant 1,(t_1,\dots,t_p)\in\mathbb{R}^p}$ is. Hence, there exists an ID stationary random process X on $\mathbb{R}$ such that $\forall p\geqslant 1$ and $\forall(t_1,\dots,t_p)\in\mathbb{R}^p$

$$\mathbb{E}[e^{iq_1X_{t_1}+\cdots+iq_pX_{t_p}}]=e^{H(t_1,\dots,t_p,Q)}.$$

The Laplace transform ψ of X_0 (or equivalently of X_t for any $t\in\mathbb{R}$) necessarily satisfies

$$\psi(1)=0 \quad\text{and}\quad \psi(1+\delta)\leqslant\delta.$$

It remains to prove that X is stochastically continuous. Notice that the mapping $t\in\mathbb{R}\mapsto H(0,t,Q)$ is continuous. In particular, we can choose $Q=(q,-q)$ for some $q\in\mathbb{R}$. We deduce $\lim_{t\to 0}\mathbb{E}[e^{iq(X_t-X_0)}]=1$ for all $q\in\mathbb{R}$. In particular, X_t-X_0 converges in law towards 0 as $t\to 0$. Therefore X_t-X_0 converges in probability towards 0 as $t\to 0$. □

E. Proof of Theorem 6

We explain the proof in dimension $d=1$. The generalization to higher dimensions is straightforward. Let us consider $\delta>0$ such that M admits a moment of order $1+\delta$. For any $\epsilon\in]0,1[$ and $n\in\mathbb{N}^*$ with finite Lebesgue measure, we have from Jensen's inequality:

$$\mathbb{E}\big[M([0,\frac{1}{n}])^{1+\delta}\big]\geqslant\mathbb{E}\big[\widetilde{M}^\epsilon([0,\frac{1}{n}])^{1+\delta}\big]$$

We deduce:

$$\mathbb{E}\big[M([0,1])^{1+\delta}\big]\geqslant\sum_{k=1}^n\mathbb{E}\big[M([\frac{k-1}{n},\frac{k}{n}])^{1+\delta}\big]\mathbb{E}\big[M([0,\frac{1}{n}])^{1+\delta}\big]\geqslant n\mathbb{E}\big[\widetilde{M}^\epsilon([0,\frac{1}{n}])^{1+\delta}\big]$$

Let us define for $\epsilon\in]0,1[$ and $q\leqslant 1$

$$F_\epsilon(q,r)=\ln\mathbb{E}[e^{qX_0^\epsilon+qX_r^\epsilon-2q\psi(1)\ln\frac{1}{\epsilon}}]$$

and observe that:

$$F_\epsilon(q,r)=\int_1^{1/\epsilon}[\psi_{0,rg(y)}(q,q)-2q\psi(1)]\frac{dy}{y}$$

Let us consider $h>1$ such that $(1+\delta)h=2$. By using the concavity of the function $x\mapsto x^{1/h}$ is concave and Jensen's inequality, we get:

$$\begin{aligned}
\mathbb{E}\big[\widetilde{M}^\epsilon\big([0,\frac{1}{n}]\big)^{1+\delta}\big]&=\mathbb{E}\left[\widetilde{M}^\epsilon\big([0,\frac{1}{n}]\big)^{2\times\frac{1}{h}}\right]\\
&=\mathbb{E}\left[\left(\int_0^{1/n}\int_0^{1/n}e^{X_r^\epsilon-\psi(1)\ln\frac{1}{\epsilon}+X_u^\epsilon-\psi(1)\ln\frac{1}{\epsilon}}drdu\right)^{1/h}\right]\\
&\geqslant n^{-1-\delta}\mathbb{E}\left[\int_0^{1/n}\int_0^{1/n}e^{\frac{1}{h}X_r^\epsilon+\frac{1}{h}X_u^\epsilon-(1+\delta)\psi(1)\ln\frac{1}{\epsilon}}n^2drdu\right]\\
&\geqslant n^{-1-\delta}e^{\inf_{|r|\leqslant\frac{1}{n}}F_\epsilon(\frac{1}{h},r)}
\end{aligned}$$

Gathering the above inequalities yields:

$$\mathbb{E}\big[M([0,1])^{1+\delta}\big]\geqslant n^{-\delta}e^{\inf_{|r|\leqslant\frac{1}{n}}F_\epsilon(\frac{1}{h},r)}. \tag{43}$$

Now fix $\alpha > 0$. Because the mapping $u \mapsto \psi_{0,u}(\frac{1}{h}, \frac{1}{h})$ is continuous, there exists $\eta > 0$ such that $|\psi_{0,u}(\frac{1}{h}, \frac{1}{h}) - \psi_{0,0}(\frac{1}{h}, \frac{1}{h})| \leqslant \alpha$ for $|u| \leqslant \eta$. We choose $\epsilon = \frac{1}{n\eta}$ and we obtain for $|r| \leqslant \frac{1}{n}$:

$$\begin{aligned}|F_\epsilon(\frac{1}{h}, r) - F_\epsilon(\frac{1}{h}, 0)| &\leqslant \int_1^{n\eta} |\psi_{0,rg(y)}(\frac{1}{h}, \frac{1}{h}) - \psi_{0,0}(\frac{1}{h}, \frac{1}{h})| \frac{dy}{y} \\ &\leqslant \alpha \ln(n\eta).\end{aligned}$$

We deduce

$$\inf_{|r| \leqslant \frac{1}{n}} F_\epsilon(\frac{1}{h}, r) \geqslant F_\epsilon(\frac{1}{h}, 0) - \alpha \ln(n\eta) = \ln(n\eta)\big(\psi(1+\delta) - (1+\delta)\psi(1) - \alpha\big).$$

By plugging this relation into (43), we get:

$$\mathbb{E}\big[M([0,1])^{1+\delta}\big] \geqslant n^{-\delta} e^{\ln(n\eta)\big(\psi(1+\delta)-(1+\delta)\psi(1)-\alpha\big)}.$$

Since this relation must be valid for all n large enough, we necessarily have

$$\psi(1+\delta) - (1+\delta)\psi(1) - \alpha \leqslant \delta.$$

Since $\alpha > 0$ is arbitrary, we deduce

$$\psi(1+\delta) - (1+\delta)\psi(1) \leqslant \delta.$$

In particular, by convexity arguments (as in establishing (41)) we have:

$$\psi'(1) - \psi(1) < 1.$$

□

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