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# A Poincaré cone condition in the Poincaré group

Tardif Camille \*

## Abstract

In [8], Ben Arous and Gradinaru described the singularity of the Green function of a general sub-elliptic diffusion. In this article we first adapt their proof to the more general context of a hypoelliptic diffusion. In a second time, we deduce a Wiener criterion and a Poincaré cone condition for Dudley's diffusion.

Key words: Green function. Wiener test. Poincaré cone condition. Relativistic diffusion. Hypoelliptic operator.

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## 1 Introduction

In [14], Dudley introduced a relativistic Brownian motion whose trajectories are time-like and which is invariant in law under Lorentz transformations. The space of states of this relativistic Brownian motion is the unit tangent bundle of the Minkowski spacetime, and the diffusion consists in a Brownian motion in  $\mathbb{H}^d$  (the hyperboloid model of the hyperbolic space) and its time integral in Minkowski spacetime  $\mathbb{R}^{1,d}$ . In the spirit of the Eells-Elworthy construction of the Brownian motion on  $\mathbb{H}^d$ , Dudley's diffusion can be obtained by projecting a diffusion  $(g_t, \xi_t)$  in the orthonormal frame bundle of  $\mathbb{R}^{1,d}$ . The point  $\xi_t$  belongs to  $\mathbb{R}^{1,d}$ , while  $g_t$  is an orthonormal frame at point  $\xi_t$  for Minkowski metric. It is fortunate that this bundle has a group structure and the latter diffusion can be chosen to be left invariant in that group, with a generator of the form

$$\mathcal{L} = \frac{\sigma^2}{2} \sum_{i=1}^d V_i^2 + H_0, \tag{1}$$

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satisfying the weak Hörmander hypoellipticity condition (this means, in particular, that the drift  $H_0$  is needed to generate a Lie algebra of maximal rank).

Intuitively,  $\{(g_t, \xi_t)\}_{t \geq 0}$  describes the timelike trajectory of a small rigid object in Minkowski spacetime and consists in a stochastic perturbation of a geodesic trajectory by random perturbation of its velocity. This Lorentzian analogue to the Euclidian Brownian motion, was studied by Bailleul in [5] where he determined its Poisson boundary by providing a comprehensive description of asymptotic behaviour. This study was completed in [25] where the Lyapunov spectrum of its flow was described. In the present work we are interested in describing, by means of a (local) Wiener criterion, the thinness of sets with respect to this diffusion. This kind of result is based on the knowledge of the geometry of the level sets of the Green function, and more generally on the knowledge of its behaviour on the diagonal.

For the case of a generic sub-elliptic diffusion with Hörmander type generator

$$\frac{1}{2} \sum_{i=1}^m X_i^2$$

Ben Arous and Gradinaru ([8]) showed, following Nagel, Stein and Wainger ([24]) but using stochastic tools, that the Green function is equivalent on the diagonal to the Green function of a Brownian motion on a free nilpotent Lie group. The estimates of the Green function they obtained allowed them to write a general Wiener criterion for such diffusions.

The diffusion studied in this article, generated by (1), is of parabolic type. The Green function is not explicit and we don't have estimates which allow us to describe the geometry of its level sets. Contrary to the previous sub-elliptic situation, results about estimates of general parabolic Green functions are not known. We find in the literature some studies of particular examples. In [26], Uchiyama obtained a probabilistic proof of a Wiener criterion for the heat operator based on a fine study of the level sets of the Green function (which has an explicit expression in that case). More general results have been obtained by Garofalo & al. ([17], [19], [18]) in different parabolic cases (heat operator with variable coefficients in  $\mathbb{R}^n$ , heat operator in the Heisenberg group and Kolmogorov parabolic operator with constant coefficients). The proofs are based on either explicit expression of Green functions or strong Gaussian estimates. Recently, Menozzi & al. ([13], [21], [23]) obtained, by a parametrix method, some Gaussian estimates of the Green function for general Kolmogorov parabolic type operators. Unfortunately their results cannot be used directly for Dudley's diffusion.

In this article, we provide a weak Wiener criterion for  $\{(g_t, \xi_t)\}_{t \geq 0}$ , which concerns sets into some homogeneous cones and we deduce a Poincaré cone sufficient condition for thinness. For this we first refine the result of Ben Arous and Gradinaru [8] to the case of generic diffusions with Hörmander type generator

$$\frac{1}{2} \sum_{i=1}^m X_i^2 + X_0$$

under the condition that the vector space spanned by the Lie brackets of at most  $r$  of the vectors  $X_1, \dots, X_m$  and  $X_0$  have full rank everywhere. We prove in Section 2 (Theorem 1) that the Green function of a such hypoelliptic diffusion is equivalent on the diagonal to the Green function a Brownian motion with a drift on a  $r$ -step free nilpotent Lie group. Some natural dilations on that group make it possible to express the Green function of this model situation in simple term. Theorem 1 is far from giving a full description of the behaviour of the Green function on the diagonal for a general hypoelliptic diffusion, nevertheless we use it to obtain in Section 3 a weak Wiener criterion and a Poincaré cone condition for  $\{(g_t, \xi_t)\}_{t \geq 0}$  in the Poincaré group.

## 2 Estimates of the Green function for a general hypoelliptic diffusion

Let  $X_0, X_1, \dots, X_m$  be smooth vector fields on a smooth connected manifold  $\mathcal{M}$  of dimension  $n \geq 3$  such that the Lie algebra generated by  $X_0, X_1, \dots, X_m$  has full rank everywhere:

$$\forall x \in \mathcal{M}, \quad \text{Lie}(X_0, X_1, \dots, X_m)|_x = T_x \mathcal{M}. \quad (\text{H})$$

Under this condition, Hörmander [20] showed the hypoellipticity of the operator

$$\mathcal{L} := \frac{1}{2} \sum_{i=1}^m X_i^2 + X_0,$$

and Bony ([10]) proved the existence of the Green function  $G_U(\cdot, \cdot)$  for smooth bounded domain  $U \subset \mathcal{M}$ . Given  $f \in C^0(U)$ , the unique function  $\phi \in C^0(\bar{U})$  such that

$$\begin{aligned} \mathcal{L}\phi &= f \quad \text{on } U \\ \phi &= 0 \quad \text{on } \partial U, \end{aligned}$$

holds in the sense of distributions is given by

$$\phi(x) = \int_U G_U(x, y) f(y) dy,$$

for any  $x \in U$ . Moreover,  $G_U$  is a fundamental solution for  $\mathcal{L}^* = \frac{1}{2} \sum_{i=1}^m X_i^* X_i^* + X_0^*$ ; ( $\mathcal{L}^* G_U(x, \cdot) = \delta_x$  in the sense of distribution). Thus  $G_U$  is smooth off the diagonal and the singularity at  $(x, x)$  does not depend of the domain  $U$  (indeed, if  $U' \supset U$   $G_{U'}(x, \cdot) - G_U(x, \cdot)$  is  $\mathcal{L}^*$ -harmonic in  $U$  and, by hypoellipticity, smooth at  $x$ ). Fix  $U$ . Let now  $(B^1, \dots, B^m)$  be a  $m$ -dimensional Brownian motion and consider the solution  $(x_t)$  of the Stratonovich equation

$$dx_t = \sum_{i=1}^m X_i(x_t) \circ dB_t^i + X_0(x_t) dt, \quad x_0 = x,$$

killed at the first exit time  $\tau$  from  $U$ ,

$$\tau := \inf\{t > 0, x_t \in U^c\}$$

. The Green function  $G$  is also the density of occupation time measure of  $(x_t)_{0 \leq t \leq \tau}$ : for every test function  $f \in C_c^\infty(U)$ ,

$$\mathbb{E} \left[ \int_0^\tau f(x_t) dt \right] = \int_U f(y) G_U(x, y) dy.$$

Under some assumptions concerning the geometry induced by the Lie brackets of the vector fields we prove the following result, which generalize to a general hypoelliptic diffusion a result of Ben Arous and Gradinaru ([8]) proved under a stronger geometric assumption.

**Theorem 1.** *Given  $x \in \mathcal{M}$  we have*

$$\lim_{\varepsilon \rightarrow 0} \sup_{|y|_x < \varepsilon} \left| G_U(x, y) |y|_x^{Q(x)-2} - \frac{1}{J_x(0)} g^{(x)}(0, \theta_x(y)) \right| = 0.$$

The precise notations are explained later and the proof, which follows the pattern of Ben Arous and Gradinaru's work, is done in the appendix. Nevertheless, let us briefly present the underlying idea. The way how the  $\{X_i\}_{i=0, \dots, m}$  generate  $T_x \mathcal{M}^n$  yields a family of dilatations of  $T_x \mathcal{M}^n$ , denoted by  $T_\lambda$  for  $\lambda > 0$ . Then we consider the rescaled diffusions in  $T_x \mathcal{M}^n \simeq \mathbb{R}^n$  defined by  $v_t^{(x, \varepsilon)} := T_{1/\varepsilon}(x_{\varepsilon^2 t})$ . Then, using the stochastic Taylor formula of Castell [11], we write the Taylor expansion  $v_t^{(x, \varepsilon)} = u_t^{(x)} + \varepsilon R(\varepsilon, t)$  where the remainder term  $R(\varepsilon, t)$  is bounded in probability as  $\varepsilon$  goes to 0. We show that the *tangent process*  $u_t^{(x)}$  which appears in the first term of this Taylor expansion has a smooth density of occupation time measure, say  $g^{(x)}$ . Using estimates obtained by Nagel, Stein and Wainger in [24] we prove the convergence of the Green functions  $G_U^{(x, \varepsilon)}$  of the rescaled diffusions  $v^{(x, \varepsilon)}$  towards  $g^{(x)}$ ; Theorem 1 follows.

**Remark 1.** *In the sub-elliptic situation described in Ben Arous and Gradinaru's work, the Green function  $g^{(x)}$  of the tangent process is strictly positive and, in this case, Theorem 1 provides a precise description of the singularity. In a parabolic situation, where the drift  $X_0$  is needed to generate a Lie algebra of full rank,  $g^{(x)}(0, \cdot)$  may be null in some domain. For a  $y$  which tends to  $x$  such that  $\theta_x(y)$  is in this domain, Theorem 1 do not give the precise behavior of  $G(x, y)$ .*

As we shall exclusively work in a fixed bounded neighbourhood  $U$  of  $x$  we shall write  $G$  for  $G_U$ , and without loss of generality we shall suppose the vector fields  $X_0, X_1, \dots, X_m$  globally bounded with bounded derivatives and there is no explosion of the solutions to the SDE considered below.

Now let introduce the notations of the theorem and the tangent process.

## 2.1 Triangular basis

For every multi-index  $J = (j_1, \dots, j_l) \in \{0, 1, \dots, m\}^l$  we denote by  $X^J$  the vector field

$$X^J := [X_{j_1}, [X_{j_2}, [\dots [X_{j_{l-1}}, X_{j_l}] \dots]],$$

and if  $J = (j)$ ,  $X^J := X_j$ .

Let denote by  $|J|$  the length  $l$  of  $J$  and by  $\|J\|$  the *order* of  $J$ :

$$\|J\| := |J| + \text{number of } 0 \text{ in } J.$$

For  $x \in \mathcal{M}$  set

$$C_i(x) := \text{Vect} \{X^J(x); \|J\| \leq i\}.$$

By assumption (H) we can consider the smallest integer  $r(x)$  such that  $C_{r(x)} = T_x\mathcal{M}$ . We denote by  $Q(x)$  the *graded dimension* of  $\text{Lie}(X_0, \dots, X_m)$  at  $x$ :

$$Q(x) := \sum_{i=1}^{r(x)} i \times (\dim C_i(x) - \dim C_{i-1}(x)).$$

**Assumption 1.** *We assume that the geometry of the Lie brackets is constant, this means that the  $\dim C_i(x)$  are constant, thus  $r$  and  $Q$  are also constant.*

Let  $\mathcal{B} = (J_1, \dots, J_n)$  be a family of multi-indexes such that  $(X^J(x))_{J \in \mathcal{B}}$  is a *triangular basis* of  $T_x\mathcal{M}^n$ , that is, for  $j \leq n$ ,  $\{X^J(x); J \in \mathcal{B}, \|J\| \leq j\}$  is a basis of  $C_j(x)$ . By the previous assumption we have also  $(X^J(y))_{J \in \mathcal{B}}$  is a triangular basis of  $T_y\mathcal{M}$  for  $y$  in a neighbourhood of  $x$ . We suppose that the domain  $U$  is in a such neighbourhood.

For any multi-index  $L$  there exist smooth realvalued functions  $a_J^L$  on  $U$  such that we have

$$X^L = \sum_{J \in \mathcal{B}} a_J^L X^J.$$

Since  $(X^J)_{J \in \mathcal{B}}$  is a triangular basis, if  $\|J\| > \|L\|$  then  $a_J^L = 0$  in  $U$ .

**Remark 2.** *We have chosen one particular  $n$ -tuple  $\mathcal{B}$  and this choice seems to be not intrinsic. Nevertheless, the following section we show that under the assumption 1 this choice does not affect the results.*

## 2.2 Homogeneous norm and a priori estimates

Given a vector field  $X$  on  $\mathcal{M}$ ,  $\exp(X)(x)$  denote the solution at time 1 of the differential equation:

$$\begin{cases} \frac{du}{ds} = X(u(s)) \\ u(0) = x. \end{cases}$$

By a well-known result about the dependence of the solution of a differential equation with respect to its parameters, the map

$$\varphi_x : u \in \mathbb{R}^n \mapsto \exp \left( \sum_{i=1}^n u_i X^{J_i} \right) (x) \in \mathcal{M}^n$$

is smooth, and since  $(X^J(x))_{J \in \mathcal{B}}$  is a basis we can find a neighbourhood  $W$  of 0 in  $\mathbb{R}^n$  such that  $\varphi_x$  is a smooth diffeomorphism from  $W$  onto  $\varphi_x(W)$ . There is no loss of generality in supposing that  $U \subset \varphi_x(W)$ . We define the homogeneous norm of  $y = \varphi_x(u) \in U$  setting

$$|y|_x := \left[ \sum_{k=1}^r \left( \sum_{i, \|J_i\|=k} u_i^2 \right)^{\frac{Q}{2k}} \right]^{\frac{1}{Q}}.$$

During the proof of Theorem 1 we will use homogenous norm at different points near of the reference point  $x$ . For this we need assumption 1 and, since  $\varphi_x$  depends smoothly on  $x$  we can take  $U$  small enough so that every map  $\varphi_y^{-1} : U \rightarrow \varphi_y^{-1}(U)$ , with  $y \in U$ , is a diffeomorphism (see Corollary 4.1 of [24]). Thus,  $|z|_y$  is well defined for  $z, y \in U$ .

Also, for  $u \in \mathbb{R}^n$ , set  $|u|_n := \left[ \sum_{k=1}^r \left( \sum_{i, \|J_i\|=k} u_i^2 \right)^{\frac{2}{2k}} \right]^{\frac{1}{2}}$ , and  $\|u\|$  is the Euclidean norm of  $u$ .

Now we introduce the *dilatations* on  $\mathbb{R}^n$  associated to the family of multi-index  $\mathcal{B}$ . For  $\varepsilon > 0$ , set

$$\begin{aligned} T_\varepsilon : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ u &\longmapsto (\varepsilon^{\|J_i\|} u_i)_{i=1, \dots, n} \end{aligned}$$

The norm  $|\cdot|_n$  is homogeneous with respect to the dilatations  $T_\varepsilon$ ; as  $|T_\varepsilon u|_n = \varepsilon |u|_n$ .

For  $y \in U$ , denote by  $\theta_x(y)$  the *angular variable* ( $|\theta_x(y)|_n = 1$ )

$$\theta_x(y) := T_{1/|y|_x} \circ \varphi_x^{-1}(y).$$

**Remark 3.** We can suppose  $\mathcal{B}$  indexed in such a way that  $(\|J_i\|)_{i=1, \dots, n}$  increase. If  $\mathcal{B}' = (J'_i)_{i=1, \dots, n}$  is an other family with this property, then  $\|J_i\| = \|J'_i\|$  and the definition of the dilatations do not depend on the particular family  $\mathcal{B}$  chosen.

Let now introduce the results obtained by Nagel, Stein and Wainger in [24] giving estimates on the Green function in terms of a geometric pseudo-metric  $\rho$  associated with the vector fields  $X_0, X_1, \dots, X_m$ .

For  $\delta > 0$  denote by  $C(\delta)$  the class of absolutely continuous maps  $\phi : [0, 1] \rightarrow U$  such that we can write

$$\phi'(t) = \sum_{\|J\| \leq r} a_J(t) X^J(\phi(t))$$

for almost all  $t \in [0, 1]$  with functions  $a_J : [0, 1] \rightarrow U$  such that  $|a_J(t)| < \delta^{\|J\|}$ .

We define a pseudo-metric  $\rho$  on  $U$  setting

$$\rho(y, z) = \inf \{ \delta > 0 \mid \exists \phi \in C(\delta) \text{ such that } \phi(0) = y, \phi(1) = z \}$$

for all  $y, z \in U$ . By Corollary p 114 of [24], provided  $U$  is small enough, there exists a constant  $C > 0$  such that

$$|G(y, z)| \leq C \frac{\rho^2(y, z)}{\text{Vol}(B(y, \rho(y, z)))} \quad (2)$$

$$|X_{j_1} \cdots X_{j_k} G(y, z)| \leq \frac{\rho^{2-\|J\|}(y, z)}{\text{Vol}(B(y, \rho(y, z)))} \quad (3)$$

holds for any  $y, z \in U$ , and for any multi-index  $J = (j_1, \dots, j_k)$ .

**Remark 4.** In the case where  $X_0 = 0$ , Theorem 4 of [24] ensures that  $\rho$  is locally equivalent to the sub-Riemannian distance associated to  $\{X_i\}_{i=1, \dots, m}$ .

In the following proposition we show that the homogeneous norm and the pseudo-metric are locally equivalent. The proof of this result, which is a sort of “ball-box theorem” and requires the assumption 1, is given in appendix.

**Proposition 1.** Provided  $U$  is small enough, there exist positive constants  $C_1, C_2$  such that

$$\forall y, z \in U, \quad C_1 \rho(y, z) \leq |z|_y \leq C_2 \rho(y, z). \quad (4)$$

**Remark 5.** This result shows in particular that if  $|\cdot|'$  is the family of homogeneous norm associated to a  $n$ -tuple  $\mathcal{B}'$  such that  $(X^L)_{L \in \mathcal{B}'}$  is a triangular basis then  $|\cdot|'$  is locally equivalent to  $|\cdot|$ . Thus, the particular choice of basis  $\mathcal{B}$  does not matter.

We deduce from the previous proposition and estimates (2) and (3) of [24] the following *a priori* estimates. The proof is given in appendix.

**Proposition 2.** *There is a constant  $C > 0$  such that*

$$\forall y \neq z \in U, \quad |G(y, z)| \leq \frac{C}{|z|_y^{Q-2}}.$$

Moreover for any integer  $k$ , for any multi-index  $J = (j_1, \dots, j_k) \in \{0, \dots, m\}^k$ , and for any  $y \neq z$  in  $U$

$$|X_{j_1} \cdots X_{j_k} G(y, z)| \leq \frac{C}{|z|_y^{Q-2+\|J\|}}.$$

Another consequence of the local equivalence between  $\rho$  and  $|\cdot|$  is the following result needed in the proof of Theorem 1. It is a direct consequence of Proposition 1 and Proposition (1.1) p 107 and (iii') p 109 in [24].

**Proposition 3** (Triangular inequality and comparison with a Euclidian norm). *For any small enough compact set  $U$ , we can find a positive constant  $c_0$  such that for any  $w, y, z \in U$ :*

$$|y|_w \leq c_0 (|z|_w + |z|_y). \quad (5)$$

Moreover, there exist positive constants  $c', c''$  such that for any  $y, z \in U$

$$c' \|\varphi_y^{-1}(z)\| \leq |z|_y \leq c'' \|\varphi_y^{-1}(z)\|^{1/r}. \quad (6)$$

### 2.3 Taylor expansion and tangent process

The underlying idea in Theorem 1 is to use the dilations  $T_\varepsilon$  to zoom on the trajectories of the diffusion  $x_t$ . For this we introduce the *rescaled diffusions*

$$v_t^{(x, \varepsilon)} := T_{1/\varepsilon} \circ \varphi_x^{-1}(x_t^\varepsilon),$$

where  $x_t^\varepsilon$  is a diffusion solution of

$$dx_t^\varepsilon = \varepsilon \sum_{i=1}^m X_i(x_t^\varepsilon) \circ dB_t^i + \varepsilon^2 X_0(x_t^\varepsilon) dt, \quad x_0^\varepsilon = x,$$

and has the same law as  $x_{\varepsilon^2 t}$ . These are diffusions with values in the neighbourhood  $\tilde{U}^\varepsilon := T_{1/\varepsilon} \circ \varphi_x^{-1}(U)$  of  $0 \in \mathbb{R}^n$ , defined up to the exit time  $\tau_\varepsilon := \inf\{t > 0, v_t^{(x, \varepsilon)} \notin \tilde{U}^\varepsilon\}$  of  $\tilde{U}^\varepsilon$ . Note that  $\tau_\varepsilon$  has the same law as  $\tau/\varepsilon^2$  where  $\tau$  is the exit time for  $x_t$  from  $U$ . In order to write a Taylor expansion of  $v_t^{(x, \varepsilon)}$  for  $\varepsilon$  near 0, we introduce notations taken from [11]. For a multi-index  $J = (j_1, \dots, j_l) \in \{0, \dots, m\}^l$  we denote by  $B_t^J$  the Stratonovitch iterated integral

$$B_t^J := \int_{\Delta_t^l} \circ dB_{t_1}^{j_1} \cdots \circ dB_{t_l}^{j_l},$$

where  $\Delta_t^l = \{(t_1, \dots, t_l); 0 < t_1 < \dots < t_l < t\}$  and  $B_t^0 = t$ . For  $\sigma$  a permutation of  $\{1, \dots, l\}$ , we set  $J \circ \sigma = (j_{\sigma(1)}, \dots, j_{\sigma(l)})$  and denote by  $e(\sigma) = \text{Card}\{j \in \{1, \dots, l-1\}; \sigma(j) > \sigma(j+1)\}$  the number of errors in ordering  $\sigma$ . Then, we denote by  $c_t^J$  the linear combination of Stratonovitch iterated integrals

$$c_t^J := \sum_{\sigma \in \mathfrak{S}_{|J|}} \frac{(-1)^{e(\sigma)}}{|J|^2 C^{|J|-1}} B_t^{J \circ \sigma^{-1}}.$$

Now let introduce the *tangent process*  $u_t^{(x)}$ , which is a  $\mathbb{R}^n$ -valued process defined in terms of the Stratonovitch integrals by:

$$u_t^{(x)} := \left( \sum_{L, \|L\|=\|J_i\|} a_{J_i}^L(x) c_t^L \right)_{i=1, \dots, n}.$$

This process is not intrinsic and depends on the choice of the  $n$ -tuple  $\mathcal{B}$ . In general it is not a diffusion but we will show that it is the projection in  $\mathbb{R}^n$  of a diffusion in some bigger space. It is the first term in the Taylor expansion of  $v_t^{(x, \varepsilon)}$  as  $\varepsilon$  goes to 0. More precisely, using the results of Castell [11] we obtain the following proposition proved in appendix.

**Proposition 4.** *Set*

$$T_U^\varepsilon := \inf \left\{ t > 0, \quad \exp \left( \sum_{L, \|L\| \leq r} \varepsilon^{\|L\|} c_t^L X^L \right) (x) \notin U \right\}.$$

For all  $\varepsilon > 0$  and  $t \leq \tau_\varepsilon \wedge T_U^\varepsilon$  we define  $R^{(x)}(\varepsilon, t)$  by the formula

$$v_t^{(x, \varepsilon)} = u_t^{(x)} + \varepsilon R^{(x)}(\varepsilon, t).$$

Then  $R^{(x)}(\varepsilon, t)$  is bounded in probability; more precisely for  $T > 0$  fixed, there exist positive constants  $\alpha$  and  $c$  such that

$$\forall R > c, \quad \lim_{\varepsilon \rightarrow 0} \mathbb{P} \left( \sup_{0 \leq s \leq T} \|R^{(x)}(\varepsilon, s)\| > R, \quad T < \tau_\varepsilon \wedge T_U^\varepsilon \right) \leq \exp \left( -\frac{R^\alpha}{cT} \right).$$

We prove in proposition 5 below that  $u_t^{(x)}$  admits smooth Green function  $g^{(x)}$ ; the first step to do that is to show that it is the projection of a diffusion taking values on a universal Lie group.

- Denote by  $\mathcal{G}_{(m,r)}$  the formal  $r$ -step free Lie algebra generated by  $(Y_0, Y_1, \dots, Y_m)$  such that if  $J$  is a multi-index with  $\|J\| > r$  then  $Y^J = 0$ . This  $r$ -step free algebra can be obtained by taking the quotient of the free Lie algebra generated by  $(Y_0, \dots, Y_m)$  by the ideal  $\text{Vect}\{Y^J, \|J\| \geq r+1\}$ . We denote by  $G_{(m,r)}$  the nilpotent, connected and simply connected, Lie group with Lie algebra  $\mathcal{G}_{(m,r)}$ . Denoting by  $\tilde{Y}_i$  the left-invariant vectors fields associated to the  $Y_i$ , we consider the  $G_{(m,r)}$ -valued diffusion  $\tilde{x}_t$  solution, starting at  $e$ , of the stochastic differential equation

$$d\tilde{x}_t = \sum_{i=0}^d \tilde{Y}_i(\tilde{x}_t) \circ dB_t^i,$$

with  $B_t^0 = t$ .

In [11] Castell shows the Chen-Strichartz formula

$$\tilde{x}_t = \exp \left( \sum_{k=1}^r \sum_{\|L\|=k} c_t^L \tilde{Y}^L \right) (e). \quad (7)$$

Setting  $V_k = \text{Vect}\{Y^J, \|J\| = k\}$  we have the direct sum decomposition

$$\mathcal{G}_{(m,r)} = V_1 \oplus \dots \oplus V_r.$$

- We can complete  $\mathcal{B}$  in  $\mathcal{A}$  such that for  $k = 1, \dots, r$ ,  $(Y^K)_{K \in \mathcal{A}, \|K\|=k}$  is a basis of  $V_k$ . There are constants  $(b_K^L)$  such that for any multi-index  $L$

$$Y^L = \sum_{K \in \mathcal{A}} b_K^L Y^K,$$

with  $b_K^L = 0$  if  $\|K\| \neq \|L\|$ . The Lie algebra  $\mathcal{G}_{(m,r)}$  is free up to order  $r$  so for multi-indices  $L$  of order smaller than  $r$  these previous equalities are universal and hold for any family  $(Y_i)_{i=0 \dots m}$  in any Lie algebra. In particular for  $L$  with  $\|L\| \leq r$  we have

$$X^L = \sum_{K \in \mathcal{A}} b_K^L X^K = \sum_{K \in \mathcal{A}} b_K^L \left( \sum_{J \in \mathcal{B}} a_J^K X^J \right) = \sum_{J \in \mathcal{B}} \left( \sum_{K \in \mathcal{A}} b_K^L a_J^K \right) X^J,$$

and so  $a_J^L = \sum_{K \in \mathcal{A}} b_K^L a_J^K$  and if  $\|L\| = \|J\|$  then

$$a_J^L = \sum_{\substack{K \in \mathcal{A} \\ \|K\| = \|J\|}} b_K^L a_J^K. \quad (8)$$



We rewrite (7)

$$\tilde{x}_t = \exp \left( \sum_{K \in \mathcal{A}} \left( \sum_{L, \|L\|=\|K\|} b_K^L c_t^L \right) \tilde{Y}^K \right) (e)$$

and we denote by  $\tilde{u}_t^{(e)}$  the process

$$\tilde{u}_t^{(e)} := \left( \sum_{K \in \mathcal{A}} \sum_{L, \|L\|=\|K\|} b_K^L c_t^L \right).$$

Set  $d := \text{Card}\mathcal{A} - \text{Card}\mathcal{B} = \dim \mathcal{G}_{(m,r)} - n$ , and introduce the projection  $p_x : \mathbb{R}^{d+n} \rightarrow \mathbb{R}^n$  defined by:

$$p_x(\tilde{u}) := \left( \sum_{\substack{K \in \mathcal{A} \\ \|K\|=\|J\|}} a_J^K(x) \tilde{u}_K \right)_{J \in \mathcal{B}}.$$

We obtain

$$p_x(\tilde{u}_t^{(e)}) = \left( \sum_{\substack{K \in \mathcal{A} \\ \|K\|=\|J\|}} a_J^K(x) \left( \sum_{L, \|L\|=\|K\|} c_t^L b_K^L \right) \right)_{J \in \mathcal{B}} = \left( \sum_{L, \|L\|=\|J\|} c_t^L \underbrace{\left( \sum_{\substack{K \in \mathcal{A} \\ \|K\|=\|J\|}} a_J^K(x) b_K^L \right)}_{=a_J^L \text{ by (8)}} \right)_{J \in \mathcal{B}} = u_t^{(x)}.$$

Now we prove

**Proposition 5.** *The tangent process  $u_t^{(x)}$  admits a smooth Green function, denoted by  $g^{(x)}(0, \cdot)$ .*

*Proof.* Since  $G_{(m,r)}$  is nilpotent, connected and simply connected, the map

$$\tilde{\psi}_e : \tilde{u} \mapsto \tilde{\psi}_e(\tilde{u}) = \exp \left( \sum_{K \in \mathcal{A}} \tilde{u}_K \tilde{Y}^K \right) (e)$$

is a diffeomorphism of  $\mathcal{G}_{(m,r)}$  onto  $G_{(m,r)}$ . Thus the process  $u_t^{(e)} = \tilde{\psi}_e^{-1}(\tilde{x}_t)$  is a hypoelliptic  $\mathcal{G}_{(m,r)}$ -valued diffusion and we denote by  $\tilde{g}$  its Green function. We identify  $\mathcal{G}_{(m,r)}$  to  $\mathbb{R}^{n+d}$  via the basis  $(X^K)_{K \in \mathcal{A}}$ .

For  $\tilde{u} \in \mathbb{R}^{n+d}$  we write  $\tilde{u} = (u, v)$  where  $u := (\tilde{u}_K)_{K \in \mathcal{B}} \in \mathbb{R}^n$  and  $v := (\tilde{u}_K)_{K \in \mathcal{A} \setminus \mathcal{B}} \in \mathbb{R}^d$  and we note that

$$p_x(\tilde{u}) = p_x((u, v)) = u + M_x(v),$$

where  $M_x$  is the  $n \times d$ -matrix defined by

$$M_x(v) = \left( \sum_{\substack{K \in \mathcal{A} \setminus \mathcal{B} \\ \|K\|=\|J\|}} a_J^K v_K \right)_{J \in \mathcal{B}}.$$

Now for a test function  $\varphi \in C_c^\infty(\mathbb{R}^n)$  we obtain

$$\begin{aligned} \mathbb{E} \left[ \int_0^{+\infty} \varphi(u_t^{(x)}) dt \right] &= \mathbb{E} \left[ \int_0^{+\infty} \varphi(p_x(\tilde{u}_t^{(e)})) dt \right] = \int_{\mathbb{R}^n \times \mathbb{R}^d} \varphi(u + M_x(v)) \tilde{g}(0, (u, v)) du dv \\ &= \int_{\mathbb{R}^n} \varphi(u) \left( \int_{\mathbb{R}^d} \tilde{g}(0, (u - M_x(v), v)) dv \right) du. \end{aligned}$$

Then  $u_t^{(x)}$  admits a Green function denoted by  $g^{(x)}$ , which verifies:

$$g^{(x)}(0, u) = \int_{\mathbb{R}^d} \tilde{g}(0, (u - M_x(v), v)) dv. \quad (9)$$

In the proof of Fact 2 of Proposition 6 in appendix we dominate the integrand by a integrable term. Thus  $g^{(x)}(0, u)$  is finite for  $u \neq 0$  and the dominated convergence theorem ensures that it is smooth as a function of  $u$ .  $\square$

Since  $u_{\varepsilon^2 t}^{(x)}$  has the same law as  $T_\varepsilon(u_t^{(x)})$  we deduce that (recall that  $v = T_\varepsilon(u) \Rightarrow dv = \varepsilon^Q du$ )

$$g^{(x)}(0, T_{1/\varepsilon}(u)) = \varepsilon^{Q-2} g^{(x)}(0, u). \quad (10)$$

For  $\varepsilon = 1/|y|_x$  we obtain

$$g^{(x)}(0, \varphi_x^{-1}(y)) = \frac{1}{|y|_x^{Q-2}} g^{(x)}(0, \theta_x(y)). \quad (11)$$

Denote by  $G^{(x,\varepsilon)}$  the Green function of the *rescaled diffusions*  $v_t^{(x,\varepsilon)}$  on  $\tilde{U}^\varepsilon = T_{1/\varepsilon} \circ \varphi_x^{-1}(U)$ . The following computation links  $G^{(x,\varepsilon)}$  to  $G$ .

Denoting by  $J_x := |\text{Jac}(\varphi_x)|$  the Jacobian of  $\varphi_x$  on  $\tilde{U}^\varepsilon$  we obtain for  $u \in \tilde{U}^\varepsilon$  and  $\psi \in C_c^0(\tilde{U}^\varepsilon \setminus \{u\})$

$$\begin{aligned} \int_{\tilde{U}^\varepsilon} \psi(v) G^{(x,\varepsilon)}(u, v) dv &= \mathbb{E}_u \left[ \int_0^{\tau_\varepsilon} \psi(v_t^{(x,\varepsilon)}) dt \right] \\ &= \mathbb{E}_{\varphi_x \circ T_\varepsilon(u)} \left[ \int_0^{\tau_\varepsilon} \psi(T_{1/\varepsilon} \circ \varphi_x^{-1}(x_{\varepsilon^2 t})) dt \right] \\ &= \mathbb{E}_{\varphi_x \circ T_\varepsilon(u)} \left[ \varepsilon^{-2} \int_0^\tau \psi(T_{1/\varepsilon} \circ \varphi_x^{-1}(x_t)) dt \right] \\ &= \varepsilon^{-2} \int_U \psi(T_{1/\varepsilon} \circ \varphi_x^{-1}(y)) G(\varphi_x \circ T_\varepsilon(u), y) dy \\ &= \varepsilon^{Q-2} \int_{\tilde{U}^\varepsilon} \psi(v) G(\varphi_x \circ T_\varepsilon(u), \varphi_x \circ T_\varepsilon(v)) J_x(T_\varepsilon(v)) dv. \end{aligned}$$

Thus

$$G^{(x,\varepsilon)}(u, v) = \varepsilon^{Q-2} J_x(T_\varepsilon(v)) G(\varphi_x \circ T_\varepsilon(u), \varphi_x \circ T_\varepsilon(v)), \quad (12)$$

and in particular, for  $y \in U$

$$G(x, y) = \frac{1}{J_x(\varphi_x^{-1}(y)) \varepsilon^{Q-2}} G^{(x,\varepsilon)}(0, T_{1/\varepsilon} \circ \varphi_x^{-1}(y)). \quad (13)$$

Let us now assume the following assumption about the dimensions of the spaces  $C_i$ .

**Assumption 2.** *We assume*

1.  $r \geq 2$
2.  $\dim C_i - \dim C_{i-1} \geq 1, \forall i = 2, \dots, r$
3.  $\dim C_1 \geq 2$ .

Under these assumptions, and using Taylor expansion of  $v_t^{(x,\varepsilon)}$  as described in Proposition 4, we show the convergence of  $G^{(x,\varepsilon)}$  to  $g^{(x)}$ , uniformly on compact sets of  $\mathbb{R}^n \setminus \{0\}$ .

**Proposition 6.** *For any compact sets  $K \subset \mathbb{R}^n \setminus \{0\}$ , we have:*

$$\sup_{u \in K} \left| G^{(x,\varepsilon)}(0, u) - g^{(x)}(0, u) \right| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Considering (11) and (13), taking for  $K$ , the unit sphere of  $\mathbb{R}^n$  for the homogeneous norm  $|\cdot|_n$ , and  $\varepsilon = |y|_x$ , Theorem 1 follows from Proposition 6.

As an illustration of the interest of that, we apply it in the next section to a diffusion in the Poincaré group constructed as a lift of the relativistic diffusion defined by Dudley in [14]; this leads to a Wiener criterion of thinness and a Poincaré cone condition.

### 3 A Wiener criterion and a Poincaré cone condition in the Poincaré group

#### 3.1 Geometric framework.

Denote by  $(\xi^0, \dots, \xi^d)$  the coordinates of a point  $\xi$  with respect to the canonical basis  $(e_0, e_1, \dots, e_d)$  of  $\mathbb{R}^{d+1}$  and set  $\mathbb{R}^{1,d}$  the space  $\mathbb{R}^{d+1}$  endowed with the Minkowski's quadratic form  $q$

$$q(\xi) = (\xi^0)^2 - \sum_{i=1}^d (\xi^i)^2.$$

Denote by  $SO(1, d)$  the sub-group of  $SL(\mathbb{R}^{d+1})$  made up of direct  $q$ -isometries, and by  $SO_0(1, d)$  the connected component of the identity in  $SO(1, d)$ . The Poincaré group is the group  $G := SO_0(1, d) \times \mathbb{R}^{1,d}$  of affine isometries with group law defined by

$$(g, \xi)(g', \xi') = (gg', \xi + g\xi').$$

We can see  $G$  as the following matrix sub-group of  $SL(d+2)$

$$G = \left\{ \begin{pmatrix} g & \xi \\ 0 & 1 \end{pmatrix}; g \in SO_0(1, d), \xi \in \mathbb{R}^{(1,d)} \right\}.$$

The Lie algebra of  $SO(1, d)$  is

$$so(1, d) = \left\{ \begin{pmatrix} 0 & {}^t(u_i) \\ (u_i) & (u_{ij}) \end{pmatrix}; (u_i) \in \mathbb{R}^d, (u_{ij}) \in \mathbb{R}^{d \times d}; (u_{ji}) = -(u_{ij}) \right\}.$$

Thus,  $SO_0(1, d) = \exp(so(1, d))$  and  $G = \exp(\mathfrak{g})$  where

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix}; (u_i)_{i=1, \dots, d} \in \mathbb{R}^d, (u_{ji}) = -(u_{ij}), (u_0, (u_{i0})_{i=1, \dots, d}) \in \mathbb{R}^{1,d} \right\}.$$

Denote by  $\mathbb{H}^d$  the half-unit sphere of  $\mathbb{R}^{1,d}$

$$\mathbb{H}^d := \{ \xi \in \mathbb{R}^{1,d} \mid q(\xi) = 1 \text{ and } \xi^0 > 0 \}.$$

Endowing  $T\mathbb{H}^d$  with the metric  $-q|_{T\mathbb{H}^d}$  turns  $\mathbb{H}^d$  into a Riemannian manifold of constant negative curvature. This is the hyperboloid model for the hyperbolic space.

#### 3.2 Relativistic diffusion.

We introduce a left invariant diffusion on  $G$  which is a lift of the relativistic diffusion on  $\mathbb{H}^d \times \mathbb{R}^{1,d}$  introduced by Dudley in [14]. The asymptotic behavior of this  $G$ -valued diffusion was studied in [5]. A relativistic diffusion can be seen as a stochastic perturbation of the geodesic flow on a Lorentzian manifold. For more information on this subject see [4].

For  $i = 1 \dots d$ , denote by  $X_i$  the left invariant vector field on  $G$  defined by

$$X_i(g, \xi) = (g, \xi) \underbrace{\begin{pmatrix} 0 & {}^t e_i & 0 \\ e_i & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:= E_i \in \mathfrak{g}}.$$

We denote by  $X_0$  the left invariant vector field on  $G$  defined by

$$X_0(g, \xi) = (g, \xi) \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:= E_0 \in \mathfrak{g}}.$$

We denote by  $\{(g_s, \xi_s)\}_{s \geq 0}$  the diffusion on  $G$  solving

$$d(g_s, \xi_s) = \sum_{i=1}^d X_i(g_s, \xi_s) \circ dB_s^i + X_0(g_s, \xi_s) ds.$$

it has generator

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^d X_i^2 + X_0.$$

Note that  $(g_t(e_0), \xi_t)$  takes values in  $\mathbb{H}^d \times \mathbb{R}^{1,d}$ . We have the following proposition (cf [4], [5], [15])

**Proposition 7.** *The process  $(g_t(e_0), \xi_t)$  in  $\mathbb{H}^d \times \mathbb{R}^{1,d}$  is Dudley's relativistic diffusion and has generator*

$$\frac{1}{2} \Delta_{\mathbb{H}^d} + H_0$$

where the vector field  $H_0$  generates the geodesic flow in  $T^1\mathbb{R}^{1,d} \equiv \mathbb{H}^d \times \mathbb{R}^{1,d}$ . In other words  $g_t(e_0)$  is a Brownian motion in  $\mathbb{H}^d$  and  $\xi_t$  is its time integral:  $\xi_t = \int_0^t g_s(e_0) ds$ .

A simple computation shows that, for  $i, j = 1 \dots d$

$$[X_i, X_j](g, \xi) = (g, \xi) \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_i \otimes e_j - e_j \otimes e_i & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:=E_{ij} \in \mathfrak{g}}$$

$$[X_i, X_0](g, \xi) = (g, \xi) \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{0} & e_i \\ 0 & 0 & 0 \end{pmatrix}}_{:=E_{i0} \in \mathfrak{g}}$$

Thus at every point  $(g, \xi)$  of  $G$  we have

$$\text{Vect}\{X_0, X_i, [X_i, X_j], [X_i, X_0], i, j = 1 \dots d\} = T_{(g, \xi)}G,$$

so  $\mathcal{L}$  is hypoelliptic by Hörmander's Theorem. Moreover we have

$$[X_i, [X_i, X_0]] = X_0, \tag{14}$$

and then,

$$\text{Vect}\{X_i, [X_i, X_j], [X_i, X_0], [X_i, [X_i, X_0]] i, j = 1 \dots d\} = T_{(g, \xi)}G.$$

By Hörmander's Theorem  $g_t$  has a smooth density with respect to the *Haar* measure of  $G$ . With the notation of Section 2 the vectors fields  $X_i$  induce the following graduation of  $T_{(g, \xi)}G = (g, \xi)\mathfrak{g}$ :

$$C_1(g, \xi) = \left\{ (g, \xi) \begin{pmatrix} 0 & {}^t(u_i) & 0 \\ (u_i) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; (u_i) \in \mathbb{R}^d \right\} \tag{15}$$

$$C_2(g, \xi) = \left\{ (g, \xi) \begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & 0 \\ 0 & 0 & 0 \end{pmatrix}; u_0 \in \mathbb{R}, (u_i) \in \mathbb{R}^d, u_{ji} = -u_{ji} \right\} \tag{16}$$

$$C_3(g, \xi) = T_{(g, \xi)}G. \tag{17}$$

Thus we have  $r(g, \xi) = 3$ . We choose the triangular basis  $\mathcal{B} := (X_i, X_0, [X_i, X_j], [X_i, X_0])_{i < j = 1 \dots d}$  for  $T_{(g, \xi)}G = (g, \xi)\mathfrak{g}$ . The graded dimension  $Q$  is constant on  $G$  and is computed explicitly

$$Q(g, \xi) = d + 2(d(d-1)/2 + 1) + 3d = d^2 + 3d + 2.$$

Note that assumptions 1 and 2 are fulfilled.

The family of dilations on  $\mathfrak{g}$  is

$$T_\varepsilon \begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon {}^t(u_i) & \varepsilon^2 u_0 \\ \varepsilon (u_i) & \varepsilon^2 (u_{ij}) & \varepsilon^3 (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix} \quad (18)$$

The homogeneous norm is given at  $\mathbf{e} = (id, 0)$  by the formula

$$\left| \exp \left( \begin{pmatrix} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{pmatrix} \right) \right|_{\mathbf{e}} = \left( \left( \sum_{i=1}^d u_i^2 \right)^{Q/2} + (u_0^2 + \sum_{1=i < j=d} u_{ij}^2)^{Q/4} + \left( \sum_{i=1}^d u_{i0}^2 \right)^{Q/6} \right)^{1/Q} \quad (19)$$

The  $\mathfrak{g}$ -valued tangent process is explicitly given by

$$u_t^{\mathbf{e}} = \begin{pmatrix} 0 & B_t^1 & \dots & B_t^d & t \\ B_t^1 & & & & \frac{1}{2} \left( \int_0^t B_s^1 ds - \int_0^t s \circ dB_s^1 \right) \\ \vdots & \left( \frac{1}{2} \left( \int_0^t B_s^i \circ dB_s^j - \int_0^t B_s^j \circ dB_s^i \right) \right)_{i=1 \dots d}^{j=1 \dots d} & & & \vdots \\ B_t^d & & & & \frac{1}{2} \left( \int_0^t B_s^d ds - \int_0^t s \circ dB_s^d \right) \\ 0 & & 0 & 0 & 0 \end{pmatrix} \quad (20)$$

Proposition 8 of [3] establishes that the support of  $\{(g_t, \xi_t)\}_{t \geq 0}$  coincides with the future half cone at  $(g_0, \xi_0)$ . Thus the Green function is strictly positive on this domain:

$$G(\mathbf{e}, (g, \xi)) > 0 \iff q(\xi) > 0 \text{ and } \xi^0 > 0.$$

By scaling, the support of  $T_{1/\varepsilon} \circ \exp^{-1}(g_{\varepsilon^2 s}, \xi_{\varepsilon^2 s})$  is the half cone

$$\left\{ (g, \xi); (\varepsilon^2 \xi^0)^2 - \|\varepsilon^3 \bar{\xi}\|^2 > 0, \xi^0 > 0 \right\}.$$

When  $\varepsilon$  goes to 0 this half cone becomes eventually the half space  $\{(g, \xi) \in G \mid \xi^0 > 0\}$ . From Proposition 6 we deduce that the support of the tangent process coincides with this half space.

We call a **homogeneous cone** with vertex  $(g_0, \xi_0)$ , a subset  $C_h(g_0, \xi_0)$  of  $G$  which is invariant under the dilations  $T_\varepsilon$  at  $(g_0, \xi_0)$  such that the ‘‘sole’’

$$\{(g, \xi) \in C_h(g_0, \xi_0); |(g, \xi)|_{(g_0, \xi_0)} = 1\}$$

is a compact subset of the half space

$$\{(g, \xi) \in G; (\xi - \xi_0)^0 > 0\}.$$

Since, for such a cone  $C_h(g_0, \xi_0)$ , the sole is compact in a domain where the Green function  $g^{(g_0, \xi_0)}$  of the tangent process is positive, we can find two positive constants  $\alpha, \beta$  such that

$$\alpha \leq g^{(g_0, \xi_0)}(0, \cdot) \leq \beta$$

in the sole. By Theorem 1 we can find a neighbourhood  $U$  of  $(g_0, \xi_0)$  such that for every set  $B \subset U$  in the homogeneous cone  $C_h(g_0, \xi_0)$  we have

$$\forall (g, \xi) \in B, \frac{\alpha}{|(g, \xi)|_{(g_0, \xi_0)}^{Q-2}} \leq G((g_0, \xi_0), (g, \xi)) \leq \frac{\beta}{|(g, \xi)|_{(g_0, \xi_0)}^{Q-2}}. \quad (21)$$

**Recall:**  $Q = d(d+3) + 2$ .

**Remark 6.** The right inequality in (21) remains true even if the sole is not entirely contained in the half space  $\{(g, \xi) \in G; (\xi - \xi_0)^0 > 0\}$ .

### 3.3 A Wiener criterion and a Poincaré cone condition

Recall a point  $(g, \xi)$  is said to be regular with respect to a set  $B$  if  $\mathbb{P}_{(g, \xi)}(T_B = 0) = 1$ , where  $T_B = \inf\{s > 0; (g_s, \xi_s) \in B\}$  is the entrance time in  $B$ .

We denote by  $B^r$  the set of regular points for  $B$ ; by continuity of trajectories we have  $\overset{\circ}{B} \subset B^r \subset \bar{B}$ . It is shown in [3] that  $\mathcal{L}$  admits an adjoint (with respect to the Haar measure of  $G$ ) without zero-order term. Dual processes theory developed in [9] can be applied and we have

**Proposition 8.** *There exists only one measure  $\mu_B$  supported by  $B^r$  such that*

$$\mathbb{P}_{(g, \xi)}[T_B < +\infty] = \int G((g, \xi), (g', \xi')) \mu_B(d(g', \xi')).$$

We call capacity of  $B$ , and denote by  $C(B)$ , the total mass of  $\mu_B$ . We also have

$$C(B) = \sup\{\mu(B); \mu \in \mathcal{M}(B), G\mu \leq 1\}. \quad (22)$$

Here  $\mathcal{M}(B)$  is the set of finite non-negative measures supported in  $B$  and  $G\mu = \int G(\cdot, (g', \xi')) \mu(d(g', \xi'))$ . Fix  $\lambda < 1$ . We denote by

$$B_n := \{(g, \xi) \in B; \lambda^{n+1} \leq |(g, \xi)|_{(g_0, \xi_0)} < \lambda^n\}$$

the homogeneous slices of  $B$ . Using the double inequality (21) and a Borel-Cantelli lemma due to J. Lamperti in [22] we have, as in the elliptic situation ( see [6] ), the following Wiener criterion:

**Proposition 9** (Wiener criterion). *Let  $B$  be a subset of  $G$  which is included in a homogeneous cone  $C_h(g_0, \xi_0)$ . Then  $(g_0, \xi_0)$  is regular for  $B$  if and only if  $\sum_n \lambda^{d(d+3)n} C(B_n) = \infty$ .*

Let  $H$  be the closure of a domain of  $\mathfrak{g}$  included in

$$\left\{ \left( \begin{array}{ccc} 0 & {}^t(u_i) & u_0 \\ (u_i) & (u_{ij}) & (u_{i0}) \\ 0 & 0 & 0 \end{array} \right) \in \mathfrak{g}; u_0 > 0 \right\}.$$

We denote by  $H_\varepsilon := \exp(T_\varepsilon(H))$  a “small” compact set in the future of  $e$ . To study the behavior of  $C(H_\varepsilon)$  when  $\varepsilon \rightarrow 0$  we need the following lemma, where  $u_\varepsilon$  and  $v_\varepsilon$  are two arbitrary points of  $H_\varepsilon$  and  $\theta_{u_\varepsilon}(v_\varepsilon) := T_{1/|v_\varepsilon|u_\varepsilon}(\exp_{u_\varepsilon}^{-1}(v_\varepsilon))$ .

**Lemma 1.** *The following limits exist*

$$\lim_{\varepsilon \rightarrow 0} \frac{|v_\varepsilon|_{u_\varepsilon}}{\varepsilon} := \alpha(u, v) > 0$$

$$\lim_{\varepsilon \rightarrow 0} \theta_{u_\varepsilon}(v_\varepsilon) := \beta(u, v) \in \mathfrak{g} - \{0\}.$$

*Proof.* Let  $u, v \in \mathfrak{g}$  be such that  $u_\varepsilon = \exp(T_\varepsilon(u))$  and  $v_\varepsilon = \exp(T_\varepsilon(v))$ . We want to find  $w \in \mathfrak{g}$  such that  $v_\varepsilon = u_\varepsilon \times \exp(w)$ , i.e,  $\exp(w) = \exp(-T_\varepsilon(u)) \exp(T_\varepsilon(v))$ .

By the Campell-Hausdorff formula we obtain

$$w = T_\varepsilon(v) - T_\varepsilon(u) - \frac{1}{2}[T_\varepsilon(u), T_\varepsilon(v)] \quad (23)$$

$$+ \frac{1}{12}([-T_\varepsilon(u), [-T_\varepsilon(u), T_\varepsilon(v)]] + [T_\varepsilon(v), [T_\varepsilon(v), -T_\varepsilon(u)]] + \dots$$

A simple computation gives

$$[T_\varepsilon(v), T_\varepsilon(u)] = \sum_{i=1}^d o(\varepsilon) X_i + \sum_{1 \leq i < j \leq d} (\varepsilon^2(u_i v_j - u_j v_i) + o(\varepsilon^2)) [X_i, X_j] +$$

$$\sum_{i=1}^d (\varepsilon^3(v_i u_0 - v_0 u_i) + o(\varepsilon^3)) [X_i, X_0] + o(\varepsilon^2) X_0.$$

With (23) we get

$$\begin{aligned} w_i &= \varepsilon(v_i - u_i) + o(\varepsilon), & w_0 &= \varepsilon^2(v_0 - u_0) + o(\varepsilon^2), \\ w_{ij} &= \varepsilon^2(v_{ij} - u_{ij} - \frac{1}{2}(v_i u_j - u_i v_j)) + o(\varepsilon^2), \\ w_{i0} &= \varepsilon^3(v_{i0} - u_{i0} - \frac{1}{2}(u_i v_0 - u_0 v_i)) + o(\varepsilon^3). \end{aligned}$$

By the definition of the homogeneous norm we have  $\frac{|v_\varepsilon|_{u_\varepsilon}}{\varepsilon} = \frac{|w|_e}{\varepsilon}$  and this quantity converges, when  $\varepsilon$  goes to 0, to the homogeneous norm  $\alpha(u, v) \neq 0$  of

$$\begin{pmatrix} 0 & {}^t(v_i - u_i) & (v_0 - u_0) \\ (v_i - u_i) & (v_{ij} - u_{ij} - \frac{1}{2}(v_i u_j - u_i v_j)) & (v_{i0} - u_{i0} - \frac{1}{2}(u_i v_0 - u_0 v_i)) \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Moreover, since by definition  $\theta_{u_\varepsilon}(v_\varepsilon) = T_{1/|w|_e}(w)$  we have

$$\lim_{\varepsilon \rightarrow 0} \theta_{u_\varepsilon}(v_\varepsilon) = \beta(u, v),$$

where

$$\beta(u, v) := \frac{1}{\alpha(u, v)} \begin{pmatrix} 0 & {}^t(v_i - u_i) & (v_0 - u_0) \\ (v_i - u_i) & (v_{ij} - u_{ij} - \frac{1}{2}(v_i u_j - u_i v_j)) & (v_{i0} - u_{i0} - \frac{1}{2}(u_i v_0 - u_0 v_i)) \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

□

Set

$$q(H) := \frac{m(H)}{\max_{u \in \partial H} \int_H \frac{\Phi_\varepsilon(\beta(u, v))}{\alpha(u, v)^{Q-2}} m(dv)}.$$

**Notation:** To simplify notations, set  $\Phi_x(\cdot) := \frac{1}{J_x(0)} g^{(x)}(0, \cdot)$  where  $g^{(x)}$  is the Green function of the tangent process at  $x$ . We denote by  $m$  the image by  $\exp^{-1}$  of the *Haar* measure on  $G$ .

**Proposition 10** (Capacities of small compact sets).

$$\liminf_{\varepsilon \rightarrow 0} \frac{C(H_\varepsilon)}{\varepsilon^{Q-2}} \geq q(H).$$

*Proof.* Denote by  $\nu_\varepsilon := \mathbf{1}_{H_\varepsilon} \text{Haar}$  the image of the measure  $\nu := \mathbf{1}_H m$  by the map  $\exp \circ T_\varepsilon$ .

By (22), to obtain a lower bound of  $C(H_\varepsilon)$  it is sufficient to get an upper bound for  $G\nu_\varepsilon$ . If  $\tau$  denotes the entrance time in  $H_\varepsilon$ , we have, for  $(g, \xi) \in G$ :

$$\begin{aligned} G\nu_\varepsilon(g, \xi) &= \int_{H_\varepsilon} G((g, \xi), (g', \xi')) \text{Haar}(d(g', \xi')) \\ &= \mathbb{E}_{(g, \xi)} \left[ \int_\tau^{+\infty} \mathbf{1}_{H_\varepsilon}(g_t, \xi_t) dt \right] \\ &= \mathbb{E}_{(g, \xi)} \left[ \mathbb{E}_{(g_\tau, \xi_\tau)} \left[ \int_0^{+\infty} \mathbf{1}_{H_\varepsilon}(g_t, \xi_t) dt \right] \right], \end{aligned}$$

hence

$$G\nu_\varepsilon(g, \xi) = \mathbb{E}_{(g, \xi)} [G\nu_\varepsilon(g_\tau, \xi_\tau)]. \quad (24)$$

Thus it is sufficient to find an upper bound for  $G\nu_\varepsilon$  on  $\partial H_\varepsilon$ . For some  $u_\varepsilon \in \partial H_\varepsilon$ , by definition of  $\nu_\varepsilon$ , we have:

$$G\nu_\varepsilon(u_\varepsilon) = \int_H G(u_\varepsilon, v_\varepsilon) m(dv),$$

where  $v_\varepsilon := \exp(T_\varepsilon(v))$ . For any  $\eta > 0$ , Theorem 1 provides some  $\varepsilon_0 > 0$  such that for all  $\varepsilon < \varepsilon_0$

$$\forall u_\varepsilon, v_\varepsilon \in H_\varepsilon, \quad G(u_\varepsilon, v_\varepsilon) \leq \frac{\eta + \Phi_{u_\varepsilon}(\theta_{u_\varepsilon}(v_\varepsilon))}{|v_\varepsilon|_{u_\varepsilon}^{Q-2}}.$$

Moreover, by Lemma 1, we can find  $\varepsilon_1 > 0$  such that for all  $\varepsilon < \varepsilon_1$

$$\forall u_\varepsilon, v_\varepsilon \in H_\varepsilon, \quad \varepsilon^{Q-2} \frac{\eta + \Phi_{u_\varepsilon}(\theta_{u_\varepsilon}(v_\varepsilon))}{|v_\varepsilon|_{u_\varepsilon}^{Q-2}} \leq \frac{\eta + \Phi_\varepsilon(\beta(u, v))}{\alpha(u, v)^{Q-2}} + \eta.$$

Thus,

$$\begin{aligned} \forall u_\varepsilon \in \partial H_\varepsilon \quad \varepsilon^{Q-2} G\nu_\varepsilon(u_\varepsilon) &\leq \int_H \left( \frac{\eta + \Phi_\varepsilon(\beta(u, v))}{\alpha(u, v)^{Q-2}} + \eta \right) m(dv). \\ &\leq \eta \times m(H) + \max_{u \in \partial H} \int_H \frac{\eta + \Phi_\varepsilon(\beta(u, v))}{\alpha(u, v)^{Q-2}} m(dv). \end{aligned}$$

By (24) we have

$$\|G\nu_\varepsilon\|_\infty \leq \frac{1}{\varepsilon^{Q-2}} \left( \eta \times m(H) + \max_{u \in \partial H} \int_H \frac{\eta + \Phi_\varepsilon(\beta(u, v))}{\alpha(u, v)^{Q-2}} m(dv) \right).$$

By (22), the previous upper bound provides the following lower bound for the capacity:

$$C(H_\varepsilon) \geq \frac{\varepsilon^{Q-2} \nu_\varepsilon(H_\varepsilon)}{\eta \times m(H) + \max_{u \in \partial H} \int_H \frac{\eta + \Phi_\varepsilon(\beta(u, v))}{\alpha(u, v)^{Q-2}} m(dv)}.$$

Since  $\nu_\varepsilon(H_\varepsilon) = m(H)$ , letting  $\eta$  go to 0 we finally obtain

$$\liminf_{\varepsilon \rightarrow 0} \frac{C(H_\varepsilon)}{\varepsilon^{Q-2}} \geq q(H) := \frac{m(H)}{\max_{u \in \partial H} \int_H \frac{\Phi_\varepsilon(\beta(u, v))}{\alpha(u, v)^{Q-2}} m(dv)}.$$

□

Propositions 9 and 10 above provide a Poincaré's cone condition of regularity:

**Corollary 1** (Poincaré cone condition of regularity). *If a subset  $B$  contains a homogeneous cone based at  $(g_0, \xi_0)$  then this point is regular for  $B$ .*

*Proof.* It suffices to remark that the vertex of an homogeneous cone is regular for it. Indeed, the slices  $B_n$  are such that  $B_n = T_{\lambda^n}^{(g_0, \xi_0)}(B_0)$  and the capacities  $C(B_n)$  are, by Proposition 10, of order  $\lambda^{n(2-Q)}$ . Thus  $\sum_n \lambda^{(Q-2)n} C(B_n)$  diverges and we conclude by the Wiener criterion of Proposition 9. □

**Remark 7.** • *In our case the Green function is not symmetric and we cannot properly mimic the proof of Theorem 6.1 given by Chaleyat -Maurel and Le Gall in [12] to obtain a precise equivalent of the capacity of small compact sets. As a consequence, it seems to be difficult to obtain more pathwise properties such as limit theorems for the Wiener sausage ( section 7 of [12]).*

- *In [16] Franchi and Le Jan defined relativistic diffusions with values in the unit tangent bundle,  $T^1\mathcal{M}$ , over any Lorentz manifold  $\mathcal{M}$ . Roughly speaking, their diffusions are obtained by “rolling without slipping” the space  $\mathbb{H}^d \times \mathbb{R}^{1,d}$  over  $T^1\mathcal{M}$  along some trajectory of Dudley's diffusion. The asymptotic behavior of such diffusions in Robertson-Walker space-times was studied in [2]. These diffusions are projections of diffusions  $r_t$  in the orthonormal frame bundle  $\mathcal{OM}$  which are solution of the SDE*

$$dr_t = \sigma \sum_{i=1}^d V_i(r_t) \circ dB_t^i + H_0(r_t) dt,$$



where the  $V_i$  are vertical vector fields on  $\mathcal{OM}$  corresponding to the infinitesimal action on the fibres of the infinitesimal boost  $e_i \otimes e_0 + e_0 \otimes e_i \in \mathfrak{so}(1, d)$ . The vector field  $H_0$  stands for the horizontal infinitesimal parallel displacement of the frame along the geodesic started in the direction of the timelike vector of the frame.

It can be shown that the tangent process associated to  $r_t$  is the same that the one associated to  $g_t$  and we obtain a Wiener criterion and a Poincaré cone condition for  $r_t$  as well.

- Since we dispose (see [5]) of a geometric description of the Poisson boundary of Dudley's diffusion, we can expect a Wiener criterion describing the thinness at the boundary. This question is in the spirit of the works of Ancona who provides, for example in [1], an answer in the case of a Brownian motion in a Cartan-Hadamard manifold. Unfortunately the method used in the present article cannot provide such an asymptotic result, as we have looked here only at the infinitesimal behavior.

## 4 Appendix

### 4.1 Proof of Propositions 1 and 2

*Proof of Proposition 1.* Introduce first an intermediate pseudo-distance  $d_{\mathcal{B}}$  which is a slight modification of  $|\cdot|$ . For this denote as in [24] by  $C_3(\delta, \mathcal{B})$  the class of absolutely continuous maps  $\phi : [0, 1] \rightarrow U$  which satisfy the differential equation

$$\phi'(t) = \sum_{J \in \mathcal{B}} a_J X^J(\phi(t))$$

almost everywhere with constants  $a_J$  such that

$$|a_J| < \delta^{\|J\|}.$$

Then define for  $y, z \in U$

$$d_{\mathcal{B}}(y, z) = \inf\{\delta > 0 \mid \exists \phi \in C_3(\delta, \mathcal{B}) \text{ such that } \phi(0) = y, \phi(1) = z\}.$$

Recall that  $U$  has been taken small enough for  $\varphi_y$  be a bijection onto  $U$ . Thus, for  $y, z \in U$  there exists a unique  $(a_J)_{J \in \mathcal{B}}$  such that

$$z = \exp\left(\sum_{J \in \mathcal{B}} a_J X^J\right)(y),$$

$$d_{\mathcal{B}}(y, z) = \max_{J \in \mathcal{B}} |a_J|^{1/\|J\|},$$

and

$$d_{\mathcal{B}}(y, z) \leq |z|_y \leq C d_{\mathcal{B}}(y, z),$$

where

$$C = \left(\sum_{k=1}^r (\dim C_k - \dim C_{k-1})^{Q/k}\right)^{1/Q}.$$

We need to show that  $d_{\mathcal{B}}$  is locally equivalent to  $\rho$ . Since  $C_3(\delta, \mathcal{B}) \subset C(\delta)$  we deduce

$$\forall y, z \in U, \quad \rho(y, z) \leq d_{\mathcal{B}}(y, z).$$

To show that  $d_{\mathcal{B}}$  is locally dominated by  $\rho$  we follow the proof of Lemma 2.16 of [24]; let introduce for that purpose the following notations. For a  $n$ -tuple  $\mathcal{B}'$  of multi-indices of order smaller than  $r$  we denote by  $\|\mathcal{B}'\|$  the sum of the orders of its components

$$\|\mathcal{B}'\| := \sum_{J \in \mathcal{B}'} \|J\|.$$

Denote also by  $\lambda_{\mathcal{B}'}(y)$  the determinant of  $(X^J)_{J \in \mathcal{B}'}$  at  $y$ . Recall that the  $n$ -tuple  $\mathcal{B}$  has been chosen such that  $(X^J(y))_{J \in \mathcal{B}}$  is a triangular basis of  $T_y \mathcal{M}$ . This means that if  $\mathcal{B}'$  is such that  $\lambda_{\mathcal{B}'}(y) \neq 0$  then  $\|\mathcal{B}\| \leq \|\mathcal{B}'\|$ . Moreover

$$M := \sup_{\mathcal{B}', y} |\lambda_{\mathcal{B}'}(y)| < +\infty,$$

where the supremum is taken over all  $y \in U$  and over the finite set of the  $n$ -tuple  $\mathcal{B}'$  of multi-indices of order smaller than  $r$ . We obtain for a  $n$ -tuple  $\mathcal{B}'$  of multi-indices, for  $y \in U$  and  $\delta \in ]0, 1]$

$$|\lambda_{\mathcal{B}'}(y)| \delta^{\|\mathcal{B}'\|} \leq M \delta^{\|\mathcal{B}'\|} \leq M \delta^{\|\mathcal{B}\|} \leq \frac{M}{\inf_{z \in U} |\lambda_{\mathcal{B}}(z)|} |\lambda_{\mathcal{B}}(y)| \delta^{\|\mathcal{B}\|}.$$

Thus, taking

$$t := \frac{\inf_{y \in U} |\lambda_{\mathcal{B}}(y)|}{M} > 0,$$

we have

$$\forall y \in U, \forall \delta \in ]0, 1] \quad |\lambda_{\mathcal{B}}(y)| \delta^{\|\mathcal{B}\|} \geq t \sup_{\mathcal{B}'} |\lambda_{\mathcal{B}'}(y)| \delta^{\|\mathcal{B}'\|}.$$

Now we apply the lemma 2.16 of [24] and find  $\eta > 0$  and  $\varepsilon(t) > 0$  such that

$$\forall y \in U, \forall \delta \in ]0, 1], \quad B(y, \eta \varepsilon(t) \delta) \subset B_{\mathcal{B}}(y, \varepsilon(t) \delta),$$

where  $B$  and  $B_{\mathcal{B}}$  are the balls associated respectively to  $\rho$  and  $d_{\mathcal{B}}$ . Thus if  $\rho(y, z) \leq \eta \varepsilon(t)$  then

$$\rho(y, z) = \inf \{n \varepsilon(t) \delta, z \in B(y, \eta \varepsilon(t) \delta)\} \geq \eta \inf \{\varepsilon(t) \delta, z \in B_{\mathcal{B}}(y, \varepsilon(t) \delta)\} = \eta d_{\mathcal{B}}(y, z). \quad (25)$$

Then  $U$  can be chosen small enough in order that (25) be verified for  $y, z \in U$ .  $\square$

*Proof of Proposition 2.* It is enough to verify that there exists a positive constant  $C$  such that  $\text{Vol}(B(y, \rho(y, z))) \geq C |z|_y^Q$  for  $y, z \in U$ . By Proposition 1, we can find  $C' > 0$  such that  $B(y, \rho(y, z)) \supset B_{\mathcal{B}}(y, C' |z|_y)$ . Then

$$\begin{aligned} \text{Vol}(B(y, \rho(y, z))) &\geq \text{Vol}(B_{\mathcal{B}}(y, C' |z|_y)) = \int_{B_{\mathcal{B}}(y, C' |z|_y)} d\tilde{z} \\ &= \int_{|u|_n \leq C' |z|_y} |\text{Jac } \varphi_y(u)| du. \end{aligned}$$

Since

$$\inf_{y \in U, u \in \varphi_y^{-1}(U)} |\text{Jac } \varphi_y(u)| > 0,$$

there exists  $C'' > 0$  such that

$$\text{Vol}(B(y, \rho(y, z))) \geq C'' \int_{|u|_n < C' |z|_y} du,$$

and remarking that  $d(T_\varepsilon u) = \varepsilon^Q du$  we finally obtain, putting  $C := C' C''$ ,

$$\text{Vol}(B(y, \rho(y, z))) \geq C |z|_y^Q.$$

$\square$

## 4.2 Proof of Proposition 4

*Proof.* According to Theorem 4.1 of [11] (p 234) for  $t \leq T$ , we have

$$x_t^\varepsilon = \exp \left( \sum_{k=1}^r \sum_{L, \|L\|=k} \varepsilon^k c_t^L X^L \right) (x) + \varepsilon^{r+1} \tilde{R}(\varepsilon, t)$$

where  $\tilde{R}(\varepsilon, t)$  is such that there exists  $\alpha, c > 0$  for which

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P} \left( \sup_{0 \leq s \leq T} \|\tilde{R}(\varepsilon, s)\| > R, T < \tau_\varepsilon \wedge T_U^\varepsilon \right) \leq \exp \left( -\frac{R^\alpha}{cT} \right) \quad (26)$$

for every  $R > c$ . Let us consider the surjection  $\psi_x$  defined by:

$$\psi_x((v_L)_{\|L\| \leq r}) = \exp \left( \sum_{L, \|L\| \leq r} v_L X^L \right) (x),$$

so that

$$\begin{aligned} v_t^{(x, \varepsilon)} &= T_{1/\varepsilon} \circ \varphi_x^{-1} \left( \psi_x \left( (\varepsilon^{\|L\|} c_t^L)_{\|L\| \leq r} \right) + \varepsilon^{r+1} \tilde{R}(\varepsilon, t) \right) \\ &= T_{1/\varepsilon} \circ \varphi_x^{-1} \circ \psi_x \left( (\varepsilon^{\|L\|} c_t^L)_{\|L\| \leq r} \right) + \varepsilon \bar{R}(\varepsilon, t) \\ &= \left( \varepsilon^{-\|J_i\|} (\varphi_x^{-1} \circ \psi_x)_i \left( (\varepsilon^{\|L\|} c_t^L)_{\|L\| \leq r} \right) \right)_{i=1, \dots, n} + \varepsilon \bar{R}(\varepsilon, t), \end{aligned}$$

where  $\|\bar{R}(\varepsilon, t)\| \leq \text{Lip}(\varphi_x^{-1}) \|\tilde{R}(\varepsilon, t)\|$ , since  $\|T_{1/\varepsilon}(u)\| \leq \varepsilon^{-r} \|u\|$  for  $\varepsilon < 1$ . We need to prove that, for  $J \in \mathcal{B}$ , the Taylor expansion

$$(\varphi_x^{-1} \circ \psi_x)_J \left( (\varepsilon^{\|L\|} c_t^L)_{\|L\| \leq r} \right) = \varepsilon^{\|J\|} \sum_{L, \|L\| = \|J\|} c_t^L a_J^L(x) + \varepsilon^{\|J\|+1} R_J(\varepsilon, t) \quad (27)$$

is such that  $R_J(\varepsilon, t)$  is bounded in probability as  $\tilde{R}(\varepsilon, t)$ . For this, we use the following lemma.

**Lemma 2.** For  $J \in \mathcal{B}$  and  $L, \|L\| \leq r$

$$\frac{\partial(\varphi_x^{-1} \circ \psi_x)_J}{\partial v_L}(0) = a_J^L(x)$$

which, by triangularity, is 0 if  $\|L\| < \|J\|$ . Under the assumption 1 we obtain moreover, for  $p \geq 2$  and  $L_1 \dots L_p$  with  $\sum_j \|L_j\| \leq \|J\|$ , that

$$\frac{\partial^p(\varphi_x^{-1} \circ \psi_x)_J}{\partial v_{L_1} \dots \partial v_{L_p}}(0) = 0.$$

*Proof.* Following [7] p 93-94 a direct computation shows that

$$\frac{\partial(\varphi_x^{-1} \circ \psi_x)_J}{\partial u_L}(0) = a_J^L(x),$$

$$\frac{\partial^2(\varphi_x^{-1} \circ \psi_x)_J}{\partial u_{L_1} \partial u_{L_2}}(0) = X^{L_1}(a_J^{L_2})(x) + X^{L_2}(a_J^{L_1})(x)$$

and by iteration

$$\frac{\partial^p(\varphi_x^{-1} \circ \psi_x)_J}{\partial u_{L_1} \dots \partial u_{L_p}}(0) = \sum_{\sigma \in \mathfrak{S}_p} X^{L_{\sigma(1)}} \dots X^{L_{\sigma(p-1)}}(a_J^{L_{\sigma(p)}})(x).$$

By Assumption 1, if  $\|L\| < \|J\|$  then  $a_J^L = 0$  on  $U$ , and thus if  $\sum_j \|L_j\| \leq \|J\|$  and  $p \geq 2$  then all the terms in the sum are 0.  $\square$

By this result we write the Taylor expansion up to the order  $\|J\| + 1$ , when  $\varepsilon \rightarrow 0$ ,

$$(\varphi_x^{-1} \circ \psi_x)_J \left( (\varepsilon^{\|L\|} c_t^L)_{\|L\| \leq r} \right) = \sum_{L, \|J\| \leq \|L\| \leq r} \varepsilon^{\|L\|} c_t^L a_J^L + \varepsilon^{\|J\|+1} \hat{R}_J(\varepsilon, t)$$

with  $|\hat{R}_J(\varepsilon, t)| \leq \|(\varepsilon^{\|L\| - 1} c_t^L)\|_{L \leq r} \|J\|^{J+1} \times \|D\|^{J+1} (\varphi_x^{-1} \circ \psi_x)_J\|_\infty$ . Thus the remainder term  $R_J(\varepsilon, t)$  defined by (27) is

$$R_J(\varepsilon, t) = \hat{R}_J(\varepsilon, t) + \sum_{L, \|J\|+1 \leq \|L\| \leq r} \varepsilon^{\|L\| - \|J\| - 1} c_t^L a_J^L,$$

and as in the proof of Lemma 4.1 of [11], by Properties P1 and P2 p 238 of [11] we deduce that there exist  $\alpha_J, c_J > 0$  such that

$$\forall \varepsilon \leq 1, \quad \forall R \geq c_J, \quad \mathbb{P} \left[ \sup_{0 \leq t \leq T} \|R_J(\varepsilon, t)\| \leq R; T < T_U^\varepsilon \right] \leq \exp \left( -\frac{R^{\alpha_J}}{c_J T} \right). \quad (28)$$

Now since  $R^{(x)}(\varepsilon, t) = \bar{R}(\varepsilon, t) + (R_J(\varepsilon, t))_{J \in \mathcal{B}}$  the proposition 4 follows from (26) and (28).  $\square$

### 4.3 Proof of Proposition 6

The proof follows the pattern of Ben Arous and Gradinaru's work [8]. Fix  $K$  a compact set of  $\mathbb{R}^n \setminus \{0\}$ . The following Proposition ensures, using Ascoli's Theorem, that the family  $\{G^{(x, \varepsilon)}(0, \cdot)\}_{\varepsilon > 0}$  is relatively compact for the uniform norm on  $K$ . Thus, to prove Proposition 6 it is enough to show that  $G^{(x, \varepsilon)}(0, \cdot)$  converges weakly to  $g^{(x)}(0, \cdot)$ , which appears to be the unique limit point of  $\{G^{(x, \varepsilon)}(0, \cdot)\}_{\varepsilon > 0}$ .

**Proposition 11.** *We have*

$$\limsup_{\varepsilon > 0} \sup_{u \in K} |G^{(x, \varepsilon)}(0, u)| < +\infty,$$

and for  $i = 1 \dots n$

$$\limsup_{\varepsilon > 0} \sup_{u \in K} \left| \frac{\partial}{\partial u_i} G^{(x, \varepsilon)}(0, u) \right| < +\infty. \quad (29)$$

*Proof.* By (10) for  $u \in K$ ,  $G^{(x, \varepsilon)}(0, u) = \varepsilon^{Q-2} J_x(0) G(x, \varphi_x \circ T_\varepsilon(u))$  so

$$\forall u \in K, \quad |G^{(x, \varepsilon)}(0, u)| \leq C \varepsilon^{Q-2} |G(x, \varphi_x \circ T_\varepsilon(u))| \stackrel{\text{Prop 2}}{\leq} C' \frac{\varepsilon^{Q-2}}{|\varphi_x \circ T_\varepsilon(u)|_x^{Q-2}} = \frac{C'}{|u|_n^{Q-2}}.$$

Note that the derivatives in the second inequality of Proposition 2 are taken with respect to the second variable of  $G$ . To simplify notation set  $G^x(\cdot) = G(x, \cdot)$ ,  $u_\varepsilon^x = \varphi_x \circ T_\varepsilon(u)$  and  $u_\varepsilon = T_\varepsilon(u)$ .

$$\begin{aligned} \left| \frac{\partial}{\partial u_i} G^{(x, \varepsilon)}(0, u) \right| &= |J_x(0) \varepsilon^{Q-2} \frac{\partial}{\partial u_i} (G^x \circ \varphi_x \circ T_\varepsilon(u))| \\ &= \left| J_x(0) \varepsilon^{Q-2+\|J_i\|} d_{u_\varepsilon^x} G^x \circ d_{u_\varepsilon} \varphi_x \left( \frac{\partial}{\partial u_i} \right) \right|. \end{aligned}$$

By the smoothness of  $d_\bullet \varphi_x \left( \frac{\partial}{\partial u_i} \right)$  (see also Lemma 2.12 of [24])

$$d_{u_\varepsilon} \varphi_x \left( \frac{\partial}{\partial u_i} \right) = d_0 \varphi_x \left( \frac{\partial}{\partial u_i} \right) + O(\varepsilon) = X^{J_i} + O(\varepsilon),$$

where  $O(\varepsilon)$  is uniform with respect to  $u \in K$ . Thus  $d_{u_\varepsilon^x} G^x \circ d_{u_\varepsilon} \varphi_x \left( \frac{\partial}{\partial u_i} \right) = X^{J_i} G^x(u_\varepsilon^x) + O(\varepsilon)$  and

$$\begin{aligned} \left| \frac{\partial}{\partial u_i} G^{(x, \varepsilon)}(0, u) \right| &= C' \varepsilon^{Q-2+\|J_i\|} |X^{J_i} G^x(u_\varepsilon^x) + O(\varepsilon)| \\ &\stackrel{\text{Prop 2}}{\leq} C' \frac{\varepsilon^{Q-2+\|J_i\|} (1 + O(\varepsilon))}{|u_\varepsilon^x|_x^{Q-2+\|J_i\|}} \leq \frac{C' (1 + O(\varepsilon))}{|u|_n^{Q-2+\|J_i\|}}. \end{aligned}$$

Taking  $\sup_{u \in K}$  then  $\limsup_{\varepsilon}$  we obtain (29).  $\square$

**Proposition 12.** Let  $K$  be a compact set in  $\mathbb{R}^n \setminus \{0\}$  and  $f$  be a smooth function supported on  $K$ . We have:

$$\lim_{\varepsilon \rightarrow 0} \left| \int f(u) G^{(x, \varepsilon)}(0, u) du - \int f(u) g^{(x)}(0, u) du \right| = 0.$$

*Proof.* By definition of the Green functions we have:

$$\left| \int f(u) G^{(x, \varepsilon)}(0, u) du - \int f(u) g^{(x)}(0, u) du \right| = \left| \mathbb{E} \left[ \int_0^{\tau_\varepsilon} f(v_t^{(x, \varepsilon)}) dt - \int_0^{+\infty} f(u_t^{(x)}) dt \right] \right|.$$

Now fixing  $T > 0$ , we can decompose the term  $\int_0^{\tau_\varepsilon} f(v_t^{(x, \varepsilon)}) dt - \int_0^{+\infty} f(u_t^{(x)}) dt$  into:

$$\begin{aligned} & \mathbf{1}_{T \leq \tau_\varepsilon} \int_0^T \left( f(v_t^{(x, \varepsilon)}) - f(u_t^{(x)}) \right) dt + \mathbf{1}_{T \leq \tau_\varepsilon} \int_T^{\tau_\varepsilon} f(v_t^{(x, \varepsilon)}) dt - \int_T^{+\infty} f(u_t^{(x)}) dt \\ & + \mathbf{1}_{T > \tau_\varepsilon} \int_0^{\tau_\varepsilon} f(v_t^{(x, \varepsilon)}) dt - \mathbf{1}_{T > \tau_\varepsilon} \int_0^T f(u_t^{(x)}) dt. \end{aligned}$$

Thus we have the inequality:

$$\begin{aligned} & \left| \mathbb{E} \left[ \int_0^{\tau_\varepsilon} f(v_t^{(x, \varepsilon)}) dt - \int_0^{+\infty} f(u_t^{(x)}) dt \right] \right| \leq \mathbb{E} \left[ \mathbf{1}_{T \leq \tau_\varepsilon} \int_0^T |f(v_t^{(x, \varepsilon)}) - f(u_t^{(x)})| dt \right] \\ & + \mathbb{E} \left[ \mathbf{1}_{T \leq \tau_\varepsilon} \int_T^{\tau_\varepsilon} |f(v_t^{(x, \varepsilon)})| dt \right] + \mathbb{E} \left[ \int_T^{+\infty} |f(u_t^{(x)})| dt \right] + 2\|f\|_\infty T \mathbb{P}(T \geq \tau_\varepsilon). \end{aligned}$$

Now, taking first the  $\limsup_{\varepsilon \rightarrow 0}$  and secondly the  $\liminf_{T \rightarrow \infty}$ , Proposition 12 follows from the following facts:

**Fact 1.** For  $T > 0$  fixed,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \mathbf{1}_{T \leq \tau_\varepsilon} \int_0^T |f(v_t^{(x, \varepsilon)}) - f(u_t^{(x)})| dt \right] = 0. \quad (30)$$

**Fact 2.**

$$\lim_{T \rightarrow +\infty} \mathbb{E} \left[ \int_T^{+\infty} |f(u_t^{(x)})| dt \right] = 0.$$

**Fact 3.**

$$\liminf_{T \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \mathbf{1}_{T \leq \tau_\varepsilon} \int_T^{\tau_\varepsilon} |f(v_t^{(x, \varepsilon)})| dt \right] = 0.$$

- The proof of Fact 1 is a consequence of the Taylor expansion  $v_t^{(x, \varepsilon)} = u_t^{(x)} + \varepsilon R^{(x)}(\varepsilon, t)$ , where

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P} \left( \sup_{0 \leq s \leq T} \|R^{(x)}(\varepsilon, s)\| > R, T < \tau_\varepsilon \wedge T_U^\varepsilon \right) \leq \exp \left( -\frac{R^\alpha}{cT} \right).$$

Indeed, by Proposition 4, for  $\eta > 0$  we can find  $\varepsilon_0 > 0$  and  $R > 0$  such that for all  $\varepsilon < \varepsilon_0$

$$\mathbb{P} \left( \sup_{0 \leq s \leq T} \|R^{(x)}(\varepsilon, s)\| > R, T < \tau_\varepsilon \wedge T_U^\varepsilon \right) \leq \frac{\eta}{2T\|f\|}.$$

So, decomposing the expectation in (30) depending on whether the event  $\left\{ \sup_{0 \leq s \leq T} \|R^{(x)}(\varepsilon, s)\| > R, T \leq \tau_\varepsilon \wedge T_U^\varepsilon \right\}$  holds or not we obtain

$$\mathbb{E} \left[ \mathbf{1}_{T \leq \tau_\varepsilon} \int_0^T |f(v_t^{(x, \varepsilon)}) - f(u_t^{(x)})| dt \right] \leq \eta + \varepsilon TR \|Df\|_\infty + 2T\|f\|_\infty \mathbb{P}[T > T_U^\varepsilon].$$

As in [11], p. 238, for  $\varepsilon$  sufficiently small

$$\mathbb{P}[T > T_U^\varepsilon] \leq \sum_{L, \|L\| \leq r} \exp\left(-\frac{c_L}{\varepsilon^2 \|L\| T}\right).$$

Hence for all  $\eta > 0$  we have

$$\limsup_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \mathbf{1}_{T \leq \tau_\varepsilon} \int_0^T |f(v_t^{(x, \varepsilon)}) - f(u_t^{(x)})| dt \right] \leq \eta,$$

and Fact 1 is proved.

- To prove Fact 2, it suffices to show that

$$\mathbb{E} \left[ \int_0^{+\infty} \mathbf{1}_{B(0, \rho)}(u_t^{(x)}) dt \right] < \infty, \quad (31)$$

where  $B(0, \rho)$  is the ball of radius  $\rho$  for the homogenous norm  $|\cdot|_n$ . Recall (cf proof of Proposition 5) that  $\tilde{u}_t^{(x)}$  is the image of the diffusion  $u_t^{(e)}$  which lives in a  $r$ -step free nilpotent Lie algebra  $\mathcal{G}_{(m, r)}$  by the projection map  $p_x$ . Denote by  $\tilde{Q} = \sum_{i=1}^r i \times \dim V_i$  the homogeneous dimension of  $\mathcal{G}_{(m, r)}$  and set  $\tilde{T}_\varepsilon(\tilde{u}) = (\varepsilon^{\|K\|} \tilde{u}_K)_{K \in \mathcal{A}}$  the associated dilations. Recalling that  $\tilde{g}$  is the Green function of  $\tilde{u}_t^{(e)}$  we have

$$\mathbb{E} \left[ \int_0^{+\infty} \mathbf{1}_{B(0, \rho)}(u_t^{(x)}) dt \right] = \int_{B(0, \rho)} g^{(x)}(0, u) du \stackrel{\text{by (9)}}{=} \int_{B(0, \rho) \times \mathbb{R}^d} \tilde{g}(0, (u - M_x(v), v)) dudv. \quad (32)$$

The homogeneous character of  $\tilde{g}$  with respect to the dilations reads

$$\tilde{g}(0, \tilde{T}_\varepsilon(\tilde{u})) = \frac{1}{\varepsilon^{\tilde{Q}-2}} \tilde{g}(0, \tilde{u}).$$

Taking  $\varepsilon^{-1} = |\tilde{u}|_{n+d} := \left[ \sum_{k=1}^r \left( \sum_{K \in \mathcal{A}, \|K\|=k} \tilde{u}_K^2 \right)^{\frac{\tilde{Q}}{2k}} \right]^{\frac{1}{\tilde{Q}}}$ , we obtain the following upper bound for  $\tilde{g}$

$$\exists c > 0, \forall \tilde{u} \neq 0, \quad |\tilde{g}(0, \tilde{u})| \leq \frac{c}{|\tilde{u}|_{n+d}^{\tilde{Q}-2}}.$$

Thus with (32) we obtain the upper bound

$$\mathbb{E} \left[ \int_0^{+\infty} \mathbf{1}_{B(0, \rho)}(u_t^{(x)}) dt \right] \leq c \int_{B(0, \rho) \times \mathbb{R}^d} \frac{1}{|(u - M_x(v), v)|_{n+d}^{\tilde{Q}-2}} dudv. \quad (33)$$

For  $k = 1, \dots, r$ , set

$$(u, v)_k := ((u_K)_{K \in \mathcal{B}}, (v_K)_{K \in \mathcal{A} \setminus \mathcal{B}})_{\|K\|=k} \in \mathbb{R}^{\dim V_k}.$$

The following endomorphism of  $\mathbb{R}^{\dim V_k}$

$$(u, v)_k \mapsto \left( \left( u_J - \sum_{\substack{K \in \mathcal{A} \setminus \mathcal{B} \\ \|K\|=k}} a_J^K v_K \right)_{J \in \mathcal{B}, \|J\|=k}, (v_K)_{K \in \mathcal{A} \setminus \mathcal{B}, \|K\|=k} \right)$$

which is the  $k^{\text{th}}$  block of  $(u, v) \mapsto (u - M_x(v), v)$ , is an isomorphism. Therefore there exists a constant  $C_k$  such that

$$\forall (u, v)_k, \quad C_k \left( \sum_{\substack{J \in \mathcal{B} \\ \|J\|=k}} (u_J)^2 + \sum_{\substack{K \in \mathcal{A} \setminus \mathcal{B} \\ \|K\|=k}} (v_K)^2 \right) \leq \sum_{\substack{J \in \mathcal{B} \\ \|J\|=k}} \left( u_J - \sum_{\substack{K \in \mathcal{A} \setminus \mathcal{B} \\ \|K\|=k}} a_J^K v_K \right)^2 + \sum_{\substack{K \in \mathcal{A} \setminus \mathcal{B} \\ \|K\|=k}} (v_K)^2.$$

Thus we can find a constant  $C = \min\{C_k, k = 1, \dots, r\} > 0$  such that

$$\forall (u, v) \in \mathbb{R}^{n+d}, \quad C|(u, v)|_{n+d} \leq |(u - M_x(v), v)|_{n+d}.$$

Moreover, Lemma (A.7) of [8] states exactly that

$$\int_{B(0, \rho) \times \mathbb{R}^d} \frac{1}{|(u, v)|_d} dudv < +\infty,$$

and we deduce, by (33), that

$$\mathbb{E} \left[ \int_0^{+\infty} \mathbf{1}_{B(0, \rho)}(u_t^{(x)}) dt \right] < +\infty.$$

- The proof of Fact 3 is very similar to the proof of Proposition (4.1) of [8], and we prove two lemmas to conclude. Denote by  $\mu_T^{(x, \varepsilon)}$  the measure whose density is  $\mathbf{1}_{T < \tau_\varepsilon}$  with respect to the law of  $v_T^{(x, \varepsilon)}$ . By the Markov property we have:

$$\mathbb{E}_0 \left[ \mathbf{1}_{T < \tau_\varepsilon} \int_T^{\tau_\varepsilon} |f(v_t^{(x, \varepsilon)})| dt \right] = \int_{T_{1/\varepsilon} \circ \varphi_x^{-1}(U)} d\mu_T^{(x, \varepsilon)}(u) \int_{T_{1/\varepsilon} \circ \varphi_x^{-1}(U)} G^{(x, \varepsilon)}(u, v) |f(v)| dv.$$

For  $u, v \in T_{1/\varepsilon} \circ \varphi_x^{-1}(U)$ , we set  $u_\varepsilon^x := \varphi_x \circ T_\varepsilon(u)$  and  $v_\varepsilon^x := \varphi_x \circ T_\varepsilon(v)$ .<sup>1</sup> By (12) we have:

$$G^{(x, \varepsilon)}(u, v) = \varepsilon^{Q-2} J_x(T_\varepsilon(v)) G(u_\varepsilon^x, v_\varepsilon^x),$$

and using Proposition 2 we can find  $C > 0$  such that for all  $\varepsilon > 0$ :

$$\int_{T_{1/\varepsilon} \circ \varphi_x^{-1}(U)} G^{(x, \varepsilon)}(u, v) |f(v)| dv \leq \int_{B(0, \rho)} \frac{C\varepsilon^{Q-2}}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}} dv,$$

where  $\rho$  is large enough, so that the support of  $f$  be included in  $B(0, \rho)$ .

As in [8] we show:

**Lemma 3.** *For all  $R > 0$  there exist  $\varepsilon_0 > 0$  and  $c > 0$  such that for all  $\varepsilon < \varepsilon_0$  and  $u \in \mathbb{R}^n$  such that  $\|u\| \geq R$  we have:*

$$\int_{B(0, \rho)} \frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}} dv \leq c.$$

Moreover,

$$\lim_{\|u\| \rightarrow \infty} \int_{B(0, \rho)} \frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}} dv = 0,$$

uniformly with respect to  $\varepsilon$ .

*Proof of Lemma 3.* For  $u = 0$ , this means  $u_\varepsilon^x = x$ ,  $\frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}} = \frac{1}{|\varphi_x(v)|_x^{Q-2}} = \frac{1}{|v|_n^{Q-2}}$ .

Lemma (A-1) of [8] states, under assumption 2, that  $\int_{B(0, \rho)} \frac{dv}{|v|_d^{Q-2}}$  is bounded by a constant depending only on  $\rho$ . As the family of diffeomorphisms  $\varphi_y^{-1}$  depends smoothly on  $y \in U$ , we can find a neighbourhood  $U_x$  of  $x$ , a constant  $C > 0$  and some  $\varepsilon_0 > 0$  such that for all  $y \in U_x$  and all  $\varepsilon < \varepsilon_0$  we have  $B(0, \varepsilon\rho) \subset \varphi_y^{-1}(U) \cap \varphi_y^{-1}(U)$  and for all  $w \in B(0, \varepsilon\rho)$  we have  $|\varphi_y^{-1} \circ \varphi_x(w)|_n \geq C|w|_n$ . Thus we obtain

$$\begin{aligned} \sup_{y \in U_x} \int_{B(0, \rho)} \frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|_y^{Q-2}} dv &= \sup_{y \in U_x} \int_{B(0, \rho)} \frac{\varepsilon^{Q-2}}{|\varphi_y^{-1}(v_\varepsilon^x)|_n^{Q-2}} dv = \sup_{y \in U_x} \int_{B(0, \rho)} \frac{\varepsilon^{Q-2}}{|\varphi_y^{-1} \circ \varphi_x \circ T_\varepsilon(v)|_n^{Q-2}} dv \\ &\leq \tilde{C} \int_{B(0, \rho)} \frac{\varepsilon^{Q-2}}{|T_\varepsilon(v)|_n^{Q-2}} dv \leq \tilde{C} \int_{B(0, \rho)} \frac{dv}{|v|_n^{Q-2}}. \end{aligned} \quad (34)$$

<sup>1</sup>Be careful:  $u_\varepsilon^x$  is a vector of  $\mathbb{R}^n$  and  $u_\varepsilon^{(x)}$  denotes the tangent process at time  $\varepsilon$ .

Now choose  $\varepsilon_1 > 0$  such that for all  $\varepsilon < \varepsilon_1$  and for all  $u \in \mathbb{R}^n$  with  $\|u\| \leq R$ , we have  $u_\varepsilon^x \in U_x$ . Using (34), there is a constant  $c$  such that for all  $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$  and for all  $u \in \mathbb{R}^n$  with  $\|u\| \leq R$

$$\int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|^{Q-2}} dv \leq c.$$

This is the first point of the lemma. The second point will follow from (5) and (6) of Proposition 3. For  $\|u\|$  large enough such that  $\frac{1}{c_0}|u_\varepsilon^x|_x - |v_\varepsilon^x|_x > 0$  we have

$$\begin{aligned} \int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{|v_\varepsilon^x|^{Q-2}} dv &\leq \int_{B(0,\rho)} \frac{\varepsilon^{Q-2}}{\left(\frac{1}{c_0}|u_\varepsilon^x|_x - |v_\varepsilon^x|_x\right)^{Q-2}} dv \leq \int_{B(0,\rho)} \frac{1}{\left(\frac{1}{c_0}|\varphi^{-1}(u)|_x - |\varphi^{-1}(v)|_x\right)^{Q-2}} dv \\ &\leq \int_{B(0,\rho)} \frac{1}{\left(\frac{c'}{c_0}\|u\| - \rho\right)^{Q-2}} dv. \end{aligned}$$

This inequality ensures that the convergence is uniform in  $\varepsilon$ .  $\square$

Let us return to the proof of Fact 3.

By the previous lemma we obtain

$$\mathbb{E}_0 \left[ \mathbf{1}_{T < \tau_\varepsilon} \int_T^{\tau_\varepsilon} |f(v_t^{(x,\varepsilon)})| dt \right] \leq c\mu_T^{(x,\varepsilon)}(B(0,R)) + \underbrace{\sup_{u, \|u\| \geq R} \int_{B(0,\rho)} \frac{C\varepsilon^{Q-2}}{|v_\varepsilon^x|^{Q-2}} dv}_{\xrightarrow{R \rightarrow +\infty} 0}.$$

To end the proof of Fact 3 it remains to prove the following fact.

**Lemma 4.** *For all  $R > 0$ ,*

$$\liminf_{T \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \mu_T^{(x,\varepsilon)}(B(0,R)) = 0.$$

*Proof of Lemma 4.* By definition of  $\mu_T^{(x,\varepsilon)}$  we have,

$$\mu_T^{(x,\varepsilon)}(B(0,R)) = \mathbb{E}_0 \left[ \mathbf{1}_{T < \tau_\varepsilon} \mathbf{1}_{B(0,R)}(v_T^{(x,\varepsilon)}) \right].$$

Let  $\chi$  be a smooth function which is equals to 1 on  $B(0,R)$  and which is supported on  $B(0,R+1)$ . We have

$$\mu_T^{(x,\varepsilon)}(B(0,R)) \leq \mathbb{E}_0 \left[ \mathbf{1}_{T < \tau_\varepsilon} \chi(v_T^{(x,\varepsilon)}) \right].$$

Using the Taylor expansion of  $v_T^{(x,\varepsilon)}$ , as in the proof of the fact 1, we obtain:

$$\mathbb{E}_0 \left[ \mathbf{1}_{T < \tau_\varepsilon} \chi(v_T^{(x,\varepsilon)}) \right] \xrightarrow{\varepsilon \rightarrow 0} \mathbb{E}_0 \left[ \chi(u_T^{(x)}) \right].$$

Moreover  $\mathbb{E}_0 \left[ \chi(u_T^{(x)}) \right] \leq \mathbb{E}_0 \left[ \mathbf{1}_{B(0,R+1)}(u_T^{(x)}) \right]$  and  $t \mapsto \mathbb{E}_0 \left[ \mathbf{1}_{B(0,R+1)}(u_t^{(x)}) \right]$  is a positive bounded function which is integrable according to (31). Thus we obtain

$$\liminf_{T \rightarrow +\infty} \mathbb{E}_0 \left[ \mathbf{1}_{B(0,R+1)}(u_T^{(x)}) \right] = 0.$$

$\square$

$\square$

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## References

- [1] A. Ancona. *Théorie du potentiel sur les graphes et les variétés*, volume 1427 of *Lecture Notes in Math.* Springer, Berlin, 1990.
- [2] J Angst. *Étude de diffusions à valeurs dans des variétés Lorentziennes*. PhD thesis, Université de Strasbourg, 2009.
- [3] Ismael Bailleul. Poisson boundary of a relativistic diffusion. *Probab. Theory Related Fields*, 141(1-2):283–329, 2008.
- [4] Ismaël Bailleul. A stochastic approach to relativistic diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(3):760–795, 2010.
- [5] Ismael Bailleul and Albert Raugi. Where does randomness lead in spacetime? *ESAIM Probab. Stat.*, 14:16–52, 2010.
- [6] Richard F. Bass. *Probabilistic techniques in analysis*. Probability and its Applications (New York). Springer-Verlag, New York, 1995.
- [7] Gérard Ben Arous. Développement asymptotique du noyau de la chaleur hypoelliptique sur la diagonale. *Ann. Inst. Fourier (Grenoble)*, 39(1):73–99, 1989.
- [8] Gerard Ben Arous and Mihai Gradinaru. Singularities of hypoelliptic Green functions. *Potential Anal.*, 8(3):217–258, 1998.
- [9] R. M. Blumenthal and R. K. Gettoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968.
- [10] Jean-Michel Bony. Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. *Ann. Inst. Fourier (Grenoble)*, 19(fasc. 1):277–304 xii, 1969.
- [11] Fabienne Castell. Asymptotic expansion of stochastic flows. *Probab. Theory Related Fields*, 96(2):225–239, 1993.
- [12] M. Chaleyat-Maurel and J.-F. Le Gall. Green function, capacity and sample path properties for a class of hypoelliptic diffusion processes. *Probab. Theory Related Fields*, 83(1-2):219–264, 1989.
- [13] François Delarue and Stéphane Menozzi. Density estimates for a random noise propagating through a chain of differential equations. *J. Funct. Anal.*, 259(6):1577–1630, 2010.
- [14] R. M. Dudley. Lorentz-invariant Markov processes in relativistic phase space. *Ark. Mat.*, 6:241–268, 1966.
- [15] J Franchi and Y Le Jan. *Hyperbolic Dynamics and Brownian Motion*. Oxford University Press, 2012.
- [16] Jacques Franchi and Yves Le Jan. Relativistic diffusions and Schwarzschild geometry. *Comm. Pure Appl. Math.*, 60(2):187–251, 2007.
- [17] Nicola Garofalo and Ermanno Lanconelli. Wiener’s criterion for parabolic equations with variable coefficients and its consequences. *Trans. Amer. Math. Soc.*, 308(2):811–836, 1988.
- [18] Nicola Garofalo and Ermanno Lanconelli. Level sets of the fundamental solution and Harnack inequality for degenerate equations of Kolmogorov type. *Trans. Amer. Math. Soc.*, 321(2):775–792, 1990.
- [19] Nicola Garofalo and Fausto Segàla. Estimates of the fundamental solution and Wiener’s criterion for the heat equation on the Heisenberg group. *Indiana Univ. Math. J.*, 39(4):1155–1196, 1990.
- [20] Lars Hörmander. *Linear partial differential operators*. Die Grundlehren der mathematischen Wissenschaften, Bd. 116. Academic Press Inc., Publishers, New York, 1963.
- [21] Valentin Konakov, Stéphane Menozzi, and Stanislav Molchanov. Explicit parametrix and local limit theorems for some degenerate diffusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(4):908–923, 2010.
- [22] John Lamperti. Wiener’s test and Markov chains. *J. Math. Anal. Appl.*, 6:58–66, 1963.

- [23] Stéphane Menozzi. Parametrix techniques and martingale problems for some degenerate Kolmogorov equations. *Electron. Commun. Probab.*, 16:234–250, 2011.
- [24] Alexander Nagel, Elias M. Stein, and Stephen Wainger. Balls and metrics defined by vector fields. I. Basic properties. *Acta Math.*, 155(1-2):103–147, 1985.
- [25] C Tardif. *Étude infinitésimale et asymptotique de certains flots stochastiques relativistes*. PhD thesis, Université de Strasbourg, 2012.
- [26] Kōhei Uchiyama. A probabilistic proof and applications of Wiener’s test for the heat operator. *Math. Ann.*, 283(1):65–86, 1989.