# Cubature on $C^1$ space

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**Abstract.** We explore in this paper cubature formulas over the space of functions having a first continuous derivative, i.e., $C^1$. We show that known cubature formulas are not optimal in this case and explain what is the origin of the loss of optimality and how to construct optimal ones; to illustrate we give cubature formulas up to (including) order 9.

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## 1. Introduction

We consider the following controlled ordinary differential equation (ODE)

$$dx(t) = f(x(t), u(t))dt, \quad x(0) = x_0. \tag{1.1}$$

where $f$ is supposed as smooth as required with respect to all variables and $u(t)$ a $C^1$ control that acts on $x(t)$ with $u(0) = u'(0) = 0$. Let $T$ be some final time (which will be set to 1 in all that follows) and denote by $C_0^1([0,T];\mathbb{R})$ the space of $u$. In order to explicitly mark the dependence of $x$ on $u$ we will also write $x_u(t)$ for the solution of (1.1).

We place ourselves in a situation where many $u(t)$ can be chosen and the average (or any aggregate quantity such as higher order moments, etc.) of some functional of $x(T)$ over all such $u(t)$ is to be computed. Typical frameworks where this is relevant is in inverse problems where one can chose several controls $u$, measure the output on the system depending on $x(T)$ and want to identify some parts of the function $f$ by doing this (see [4, 9, 10] for examples).

We need to make precise what average means. Since our primary space for $u(t)$ is $C_0^1([0,T];\mathbb{R})$ a possible way to formalize this average is to consider

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a one dimensional Brownian motion $W_t$ (we write sometimes, as is usual, the time as index instead of $W(t)$ but this means the same thing) and write the following 3-dimensional SDE:

$$dx(t) = f(x(t), u(t))dt, \quad x(0) = x_0 \tag{1.2}$$
$$du(t) = w(t)dt, \quad u(0) = 0 \tag{1.3}$$
$$dw(t) = 1dW_t = 1 \circ dW_t, \quad w(0) = W_0 = 0. \tag{1.4}$$

where the last equality means of course $w(t) = W_t$. The third equality is there only in order to give a formal 3D SDE; the term $\circ dW_t$ signals a Stratonovich formulation (which is the one well adapted to cubature framework because of the Wong-Zakai theorem [15]).

We can now make precise the quantity of interest which is

$$\mathbb{E}F(x(T)), \tag{1.5}$$

where $F$ is some (smooth enough) real function.

The justification of this formal writing is the following: the Brownian motion selects paths on the (Wiener) space of continuous functions null at the origin on $[0,T]$ denoted $C_0^0([0,T];\mathbb{R})$. Any $C_0^1([0,T];\mathbb{R})$ is the definite primitive of a function in the Wiener space. Thus as realizations of $W$ span the Wiener space, $u(t)$ will span the required space.

Following works on infinite dimensional cubature formulas on Wiener space by [7, 8, 11] (see also [12, 14] for an application of cubature to finance and [3] to SPDE; many other works appeared in the literature on these subjects) we want to approximate the mean in (1.5) by a finite sum

$$\mathbb{E}F(x(T)) \simeq \sum_{k=1}^{n} \lambda_k F(x_{u_k}(T)). \tag{1.6}$$

where each $u_k$ corresponds to a given realization $\omega_k$ of the Brownian motion $W$ and the corresponding $u_k(t)$ is given as above

$$u_k(t) = \int_0^t \omega_k(s)ds. \tag{1.7}$$

Such an approximation is called a cubature formula. The question is what weights $\lambda_k$ and paths $\omega_k$ are best for some given $n$ and how good are the approximation properties of such a cubature formula.

A first thought is to use cubature formulas that work on the Wiener space $C_0^0([0,T];\mathbb{R})$ (cf. cited references for the details). As it will be seen in the following this is not necessarily the most efficient choice because of the specific structure of the problem. The purpose of this work is to find optimal cubature formulas for the space $C_0^1([0,T];\mathbb{R})$ up to (including) fourth order.

The plan of the paper is the following: further motivating remarks are the object of Section 2 while a quick introduction to cubature formulas on Wiener space is presented in Section 3. Preliminary computations are given in Section 4 while the actual cubature formulas are given in Section 5.

## 2. Further remarks and motivation

Denote $Y = \begin{pmatrix} x \\ u \\ w \end{pmatrix}$ and note that our equation can be written as

$$dY = \begin{pmatrix} f(x(t), u(t)) \\ w \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \circ dW_t. \tag{2.1}$$

We note that a different circumstance where the term $\mathbb{E}F(x(T))$ appears is in the forward Kolmogorov (or Fokker-Planck) PDE associated to the time evolution of the density of the SDE (2.1). If we denote by $\rho(t, x, u, w)$ the 3D density it satisfies the following degenerate 3-dimensional, time-dependent PDE [6, 13]:

$$\begin{aligned} &\frac{\partial}{\partial_t}\rho(t,x,u,w) + \frac{\partial}{\partial_x}(f(x,u)\rho(t,x,u,w)) + \frac{\partial}{\partial_u}(w\rho(t,x,u,w)) \\ &-\frac{1}{2}\frac{\partial^2}{\partial_w^2}\rho(t,x,u,w) = 0, \qquad (2.2) \\ &\rho(0,x,u,w) = \delta_{x=x_0}. \qquad (2.3) \end{aligned}$$

Then since

$$\mathbb{E}F(x(T)) = \int_{\mathbb{R}^3_+} F(x)\rho(T,x,u,w)dxdudw \tag{2.4}$$

the method presented here also applies to the evaluation of the right hand side of the equation above. An equivalent formulation, that does not require to work with a Dirac mass, involves a degenerate backward (in time) PDE and can be invoked through the Feynman-Kac formula [6, 13]:

$$\mathbb{E}F(x(T)) = \mathcal{F}(0, x_0, 0, 0) \tag{2.5}$$

where

$$\begin{aligned} &\frac{\partial}{\partial_t}\mathcal{F}(t,x,u,w) + f(x,u)\frac{\partial}{\partial_x}\mathcal{F}(t,x,u,w) + w\frac{\partial}{\partial_u}\mathcal{F}(t,x,u,w) \\ &+\frac{1}{2}\frac{\partial^2}{\partial_w^2}\mathcal{F}(t,x,u,w) = 0 \qquad (2.6) \\ &\mathcal{F}(T,x,u,w) = F(x). \qquad (2.7) \end{aligned}$$

Thus the method presented here can be used to solve degenerate PDEs of type (2.6).

## 3. Background on cubature formulas

We follow [7, 11] and introduce below the principle of computing cubature formulas on the Wiener space. Suppose we want to compute $\mathbb{E}g(Z(T))$ with

$g$ a regular function where $Z(t) = (Z_0(t), ..., Z_d(t))^T \in \mathbb{R}^{d+1}$ solves the SDE

$$dZ = \sum_{\ell=0}^{d+1} \zeta_\ell(Z(t)) \circ dB_\ell(t), \tag{3.1}$$

where $B_1(t)$,...$B_d(t)$ are components of a $d$-dimensional Brownian motion, $\zeta_l$ are (generic) smooth functions and we denote $B_0(t) = t$ and set $\zeta_0(\cdot) = 1$ (which ensures $Z_0(t) = t$).

If a path $\omega(t) = (\omega_0(t), ..., \omega_d(t)) \in \mathbb{R}^{d+1}$ with $\omega_0(t) = t$ is given and has some regularity one can define $\xi_\omega(t)$ as the solution of the following ODE

$$d\xi_\omega(t) = \sum_{\ell=0}^{d+1} \zeta_\ell(\xi_\omega(t)) d\omega_\ell(t). \tag{3.2}$$

Use now stochastic Taylor formulas [6, 13] to write

$$\mathbb{E}g(Z(T)) = g(Z(0)) + \sum_j a_j(g, \zeta_0, ..., \zeta_d)\mathbb{E}(P_j) + R \tag{3.3}$$

where $R$ is a remainder of order higher than a predefined order $N$, $a_j(g, \zeta_0, ..., \zeta_d)$ is a real (known) functional depending on $g, \zeta_0, ..., \zeta_d$ and $P_j$ are stochastic polynomials, i.e. integrals of the type

$$\int_0^T \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_{m-1}} \circ dB_{\alpha_m}(t) ... \circ dB_{\alpha_1}(t) \tag{3.4}$$

with $\alpha_p \in \{0, 1, ..., d\}$ for each $p$. The order of a stochastic polynomial is defined adding 1 for each integral involving $\alpha_j > 0$ and 2 for each $\alpha_j = 0$.

If the function $g$ is smooth enough and the remainder $R$ does not contain terms of order $\leq N$ a cubature formula of order $N$

$$\mathbb{E}g(X(T)) \simeq \sum_{k=1}^{n} \lambda_k \xi_{\omega^k}(T) \tag{3.5}$$

is obtained by requiring that cubature paths $\omega^k$ and weights $\lambda_k$ satisfy for each polynomial $P_j$ as in (3.4):

$$\sum_{k=1}^{n} \lambda_k \left( \int_0^T \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_{m-1}} d\omega^k_{\alpha_m}(t) ... d\omega^k_{\alpha_1}(t) \right) \tag{3.6}$$

$$= \mathbb{E}\left( \int_0^T \int_0^{s_1} \int_0^{s_2} ... \int_0^{s_{m-1}} \circ dB_{\alpha_m}(t) ... \circ dB_{\alpha_1}(t) \right). \tag{3.7}$$

*Remark* 3.1. We use here the same naming conventions for the order of the cubature scheme as in [11] which is somehow different from the standard numerical analysis practice. As such, a cubature of order "$N$" will have error of order $O(T^{(N+1)/2})$.

## 4. Stochastic Taylor expansion for averages of deterministic functionals over the class $C_0^1([0,T];\mathbb{R})$

We will use the following convention: for any function $G(\cdot)$ we denote by $\partial_k G$ the partial derivative of function $G$ with respect to its $k$-th argument. We write the stochastic Taylor formula [13] and iterate:

$$\mathbb{E}F(x(T)) = \mathbb{E}F(x(0)) + \mathbb{E}\int_0^T \partial_1 F(x(s_1)) f(x(s_1), u(s_1)) ds_1 \quad (4.1)$$

$$= F(x(0)) + \mathbb{E}\int_0^T \partial_1 F(x(s_1)) f(x(s_1), u(s_1)) ds_1. \quad (4.2)$$

We obtain by iterating :

$$\mathbb{E}F(x(T)) = F(x(0)) + \partial_1 F(x(0)) f(x(0), u(0)) \cdot \mathbb{E}\left(\int_0^T ds_1\right)$$
$$+\mathbb{E}\int_0^T \int_0^{s_1} \partial_1 F(x(s_2))\Big(\partial_1 f(x(s_2), u(s_2)) f(x(s_2), u(s_2)) +$$
$$\partial_2 f(x(s_2), u(s_2)) W(s_2)\Big) + (\partial_1)^2 F(x(s_2)) f(x(s_2), u(s_2)) ds_2 ds_1 \quad (4.3)$$

The first conclusion that can be drawn from this initial computation is that no first order terms appear and the only second order term in $T$ is $\mathbb{E}\left(\int_0^T ds_1\right)$; thus a second order cubature formula (in the sense of the Remark 3.1) has only to satisfy the requirement:

$$\sum_{k=1}^n \lambda_k \int_0^T ds_1 = \mathbb{E}\left(\int_0^T ds_1\right) = T, \quad (4.4)$$

i.e.

$$\sum_{k=1}^n \lambda_k = 1. \quad (4.5)$$

The important remark here is that many terms are missing among which (we only write terms up to order 3 because the others are more cumbersome to

write):

$$\mathbb{E}\left(\int_0^T \circ dW_{s_1}\right), \tag{4.6}$$

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} \circ dW_{s_2} \circ dW_{s_1}\right), \tag{4.7}$$

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} ds_2 \circ dW_{s_1}\right), \tag{4.8}$$

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} \circ dW_{s_2} ds_1\right), \tag{4.9}$$

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} \circ dW_{s_3} \circ dW_{s_2} \circ dW_{s_1}\right), \tag{4.10}$$

terms of order 4 involving Stratonovich integrals (4.11)

... (4.12)

It follows that classical cubature formulas derived for fully general equations on Wiener space lose optimality here. The purpose of this work is to explain what are the constraints that optimal cubature formulas satisfy and give examples of optimal weights and paths up to (including) order 9.

Continuing in the same way the enumeration of orders as they appear iterating the integral form of the stochastic Taylor formula we obtain that the following integrals appear

1. order 2: term $\mathbb{E}\left(\int_0^T ds_1\right)$. The constraint is, as seen above,
$$\sum_{k=1}^n \lambda_k = 1. \tag{4.13}$$
2. (unique) term of order 4: $\mathbb{E}\left(\int_0^T \int_0^{s_1} ds_2 ds_1\right)$. There is no new requirement brought by this term.
3. (unique) term of order 5: $\mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} \circ dW_{s_3} ds_2 ds_1\right)$. The requirement is

$$\sum_{k=1}^n \lambda_k \left(\int_0^T \int_0^{s_1} \int_0^{s_2} d\omega_k(s_3) ds_2 ds_1\right) = \mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} \circ dW_{s_3} ds_2 ds_1\right) = 0. \tag{4.14}$$

We recall that the integral $\int_0^{s_2} d\omega_k(s_3)$ is a Riemann-Stieltjes integral.

4. (unique) term of order 6: $\mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} ds_3 ds_2 ds_1\right)$. There is no new requirement brought by this term.

5. only two terms of order 7:

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \circ dW_{s_4} ds_3 ds_2 ds_1\right) = 0 \tag{4.15}$$

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} ds_4 \circ dW_{s_3} ds_2 ds_1\right) = 0. \tag{4.16}$$

6. order 8 and higher: all the terms beginning by the terms of order 7 and higher.

## 5. Cubature formulas

### 5.1. Cubature formulas or order 6

As seen above cubature formulas up to order 4 (included) are somehow trivial. We thus start our list of cubature formulas from order 5. Note that a formula of order 5 is automatically of order 6 too since terms of order 6 do not bring any new requirement (other that the one implied already by the term at order 2).

There are two equations: (4.13) and (4.14). We will use two paths and thus two weights. A natural choice is to use some path $\omega_1$ and $\omega_2 = -\omega_1$ and $\lambda_1 = \lambda_2 = 1/2$. Then the constraints are both satisfied. We obtain for instance a formula of order 6:

$$\lambda_1 = \lambda_2 = 1/2,\ \omega_1(t) = t, \omega_2(t) = -t. \tag{5.1}$$

Note that this is the same as the third order (dimension one) formula from [11].

### 5.2. Cubature formulas or order 7

There are two new constraints of order 7. But there constraints are again satisfied if one uses $n = 2$ $\lambda_1 = \lambda_2 = 1/2$ and $\omega_2 = -\omega_1$. Thus e.g. formula (5.1) is also of order 7.

### 5.3. Cubature formulas or order 8: a first approach

Two new terms appear that bring new constraints:

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \circ dW_{s_5} \circ dW_{s_4} ds_3 ds_2 ds_1\right) = \frac{T^4}{48} \tag{5.2}$$

$$\mathbb{E}\left(\int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \circ dW_{s_5} ds_4 \circ dW_{s_3} ds_2 ds_1\right) = 0. \tag{5.3}$$

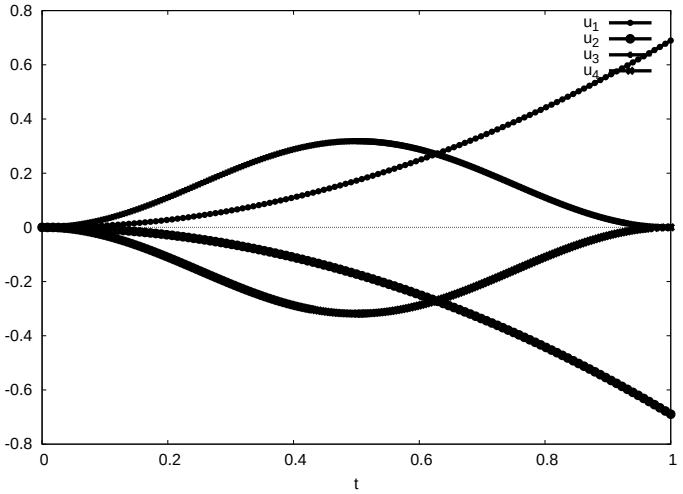


FIGURE 1. The four functions $u_k$ for a order 9 quadrature formula.

We do not enter here into the specifics of the calculation above(see [1]). In terms of the cubature paths and weights the two new constraints read:

$$\sum_{k=1}^{n} \lambda_k \left( \int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} d\omega_k(s_5) d\omega_k(s_4) ds_3 ds_2 ds_1 \right) = \frac{T^4}{48} \quad (5.4)$$

$$\sum_{k=1}^{n} \lambda_k \left( \int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} d\omega_k(s_5) ds_4 d\omega_k(s_3) ds_2 ds_1 \right) = 0. \quad (5.5)$$

Note that the choice $n = 2$, $\lambda_1 = \lambda_2 = 1/2$ and $\omega_2 = -\omega_1 = -t$ does not satisfy these constraints. A first idea is to add two more functions and look for a $n = 4$ cubature formula of order 8. In order to build on conclusions from previous lower order we further choose to set

$$\lambda_2 = \lambda_1, \ \lambda_3 = \lambda_4, \omega_2 = -\omega_1, \ \omega_3 = -\omega_4. \quad (5.6)$$

Denoting

$$\alpha_k = \int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} d\omega_k(s_5) d\omega_k(s_4) ds_3 ds_2 ds_1 \quad (5.7)$$

$$= \int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \omega_k(s_4) d\omega_k(s_4) ds_3 ds_2 ds_1 \quad (5.8)$$

$$= \int_0^T \int_0^{s_1} \int_0^{s_2} \frac{\omega_k^2(s_3)}{2} ds_3 ds_2 ds_1 \quad (5.9)$$

and

$$\beta_k = \int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} d\omega_k(s_5) ds_4 d\omega_k(s_3) ds_2 ds_1 \tag{5.10}$$

$$= \int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \omega_k(s_4) ds_4 d\omega_k(s_3) ds_2 ds_1 \tag{5.11}$$

we obtain that the following requirements are to be satisfied (for $T = 1$):

$$\lambda_1 \alpha_1 + \left(\frac{1}{2} - \lambda_1\right) \alpha_3 = \frac{1}{96} \tag{5.12}$$

$$\lambda_1 \beta_1 + \left(\frac{1}{2} - \lambda_1\right) \beta_3 = 0. \tag{5.13}$$

Let us introduce the parameter $\theta \in \mathbb{R}$ and choose $\omega_1 = \theta t = -\omega_2$; we compute and obtain $\alpha_1 = \frac{\theta^2}{5!} = \beta_1$. It suffices now to choose a family of functions where to look for $\omega_3$ and its opposite $\omega_4$. Instead of piecewise linear functions as in [11] we propose here oscillatory functions $\omega_3(t) = \sin(\frac{2\pi t}{T}) = -\omega_4$. The unknowns are now $\theta$ and $\lambda_1$. Note that $\omega_3$ is such that $\int_0^1 \omega_3(t) dt = 0$.

For this choice of $\omega_3$ we obtain (for $T = 1$)

$$\alpha_3 = \frac{8\,\pi^2 - 3}{192\,\pi^2},\ \beta_3 = -\frac{8\,\pi^2 - 21}{96\,\pi^2}. \tag{5.14}$$

Replacing and solving for $\theta$ and $\lambda_1$ one obtains:

$$\lambda_1 = \frac{5(2\pi^2 - 21)}{6(2\pi^2 - 15)} \simeq 0.39712223492734, \tag{5.15}$$

$$\theta = \frac{\sqrt{8\,\pi^2 - 21}}{\sqrt{4\,\pi^2 - 9}} \simeq 1.378974145172718. \tag{5.16}$$

Note that the natural constraints $\theta^2 > 0$ and $\lambda_1 \in [0, 1/2]$ are satisfied. This is not necessarily the case for other (arbitrary chosen) pairs of functions.

We obtain thus the following integration formula for $C_0^1$ functions (see also Figure 1):

$$\lambda_2 = \lambda_1 = \frac{5(2\pi^2 - 21)}{6(2\pi^2 - 15)},\ \lambda_3 = \lambda_4 = \frac{1}{2} - \lambda_1, \theta = \frac{\sqrt{8\,\pi^2 - 21}}{\sqrt{4\,\pi^2 - 9}}, \tag{5.17}$$

$$u_1(t) = \theta \frac{t^2}{2} = -u_2(t),\ u_3(t) = \frac{1 - \cos(2\pi t)}{2\pi} = -u_4(t). \tag{5.18}$$

**5.4. Minimalistic cubature formulas of order 8 and 9**

Another approach to construct a formula of order 8 is to start with $n = 2$ paths and weights but adapt them to satisfy the constraints. We note that any choice:

$$n = 2,\ \lambda_1 = \lambda_2 = 1/2,\ \omega_2 = -\omega_1 \tag{5.19}$$

(now $\omega_1$ is not necessarily $t$) will automatically satisfy all constraints of odd orders, i.e. involving an odd number of integrations with respect to the paths. The reason is that all such terms have to be zero and are obviously so because are the sum of two contributions, one coming from $\omega_1$ and another, that will

have same modulus but opposite sign, from $\omega_2 = -\omega_1$. In particular, if we find a cubature formula of order 8 with two paths that satisfy (5.19) it will also be of order 9.

Thus all that remains to do is to find a function $\omega_1$ which satisfies equations (5.4) and (5.5) (for $k = 1$). A parametric search as a fractional order polynomial reveals that a suitable solution is:

$$\omega_1(x) = \frac{\sqrt{x}\left(\left(\sqrt{11}+6\right)x - 3\right)}{2}. \tag{5.20}$$

A solution can be also found as a piecewise linear function. The function has two linear parts with slope $a_1$ from 0 to 1/2 and $a_2$ from 1/2 to 1:

$$\omega_1(t) = \frac{(a_2 - a_1)\,|2t-1| + (2a_2 + 2a_1)\,t - a_2 + a_1}{4}, \tag{5.21}$$

$$a_1 = -\frac{\sqrt{\sqrt{161}+17}}{2^{\frac{3}{2}}},\ a_2 = -\frac{\sqrt{\sqrt{7}\,\sqrt{23}+17}\,\left(\sqrt{161}-15\right)}{2^{\frac{5}{2}}}. \tag{5.22}$$

We have thus proved

**Theorem 5.1.** *The following choice is a formula of order* 9 *for* $T = 1$ *(see also Figure* 2*):*

$$\lambda_1 = \lambda_2 = 1/2,\ \omega_1(t) = \frac{\sqrt{t}\left(\left(\sqrt{11}+6\right)t - 3\right)}{2},\ \omega_2 = -\omega_1. \tag{5.23}$$

*or, in terms of the control* $u$*:*

$$\lambda_1 = \lambda_2 = 1/2,\ u_1(t) = \frac{t^{\frac{3}{2}}\left((\sqrt{11}+6)t - 5\right)}{5},\ u_2(t) = -u_1(t). \tag{5.24}$$

*Same holds for:*

$$\omega_1(t) = \frac{(a_2 - a_1)\,|2t-1| + (2a_2 + 2a_1)\,t - a_2 + a_1}{4}, \tag{5.25}$$

$$a_1 = -\frac{\sqrt{\sqrt{161}+17}}{2^{\frac{3}{2}}},\ a_2 = -\frac{\sqrt{\sqrt{7}\,\sqrt{23}+17}\,\left(\sqrt{161}-15\right)}{2^{\frac{5}{2}}}, \tag{5.26}$$

$$\lambda_1 = \lambda_2 = 1/2,\ \omega_2 = -\omega_1. \tag{5.27}$$

*or, in terms of the control* $u$*:*

$$u_1(t) = \frac{\left((2a_2 - 2a_1)\,t - a_2 + a_1\right)|2t-1| + (4a_2 + 4a_1)\,t^2}{16} + \frac{(4a_1 - 4a_2)\,t + a_2 - a_1}{16},\quad \lambda_1 = \lambda_2 = 1/2,\ u_2(t) = -u_1(t). \tag{5.28}$$

*Remark* 5.2. This methodology to find a cubature formula can be extended to the situation of a multi-dimensional state $x(t)$ and even multi-dimensional control $u(t)$ but the cubature formulas will be different.

*Remark* 5.3. The point of the paper is that taking into account the special structure of the equation can help to obtain faster cubature formulas. Up to this point $f$ is not depending explicitly on time; if one needs to work with a non-autonomous version of $f$, the standard treatment is to introduce time

as additional variable, but its SDE is very particular ($dt = 1dt$), this may be combined with the above technique (or not ...) to propose adapted cubature formulas.

We recall that cubature formulas for arbitrary $T$ are simply obtained by rescaling $\omega_k(t)$ to $\sqrt{T}\omega_k(t/T)$ and $u_k(t)$ to $\sqrt{T^3}\omega_k(t/T)$.

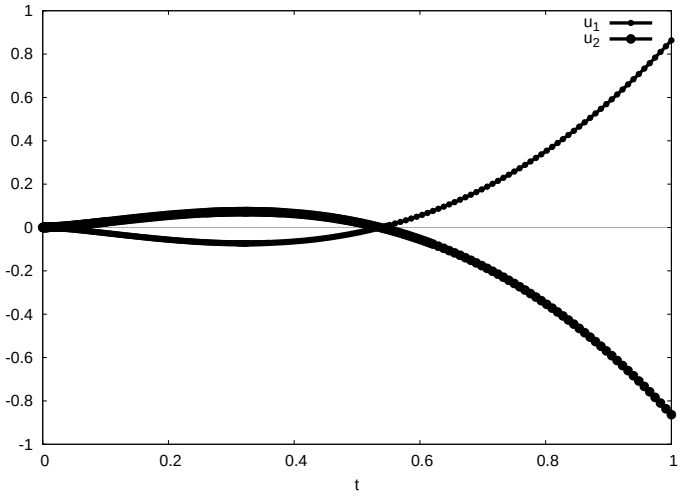


FIGURE 2. The functions $u_k$ for a 9-th order quadrature formula (5.23)-(5.24).

## 6. Numerical results

### 6.1. Linear setting

To test our implementation against trivial errors we considered first

$$f(x,u) = \alpha x + u,\ \alpha \in \mathbb{R},\ x(0) = 0,\ F(x) = x. \tag{6.1}$$

One can show analytically that $\mathbb{E}(F(x(T)) = 0$. The cubature will approximate 0 with $\frac{F(x_{u_1}(T))+F(x_{u_2}(T))}{2}$. But, by linearity $F(x_{u_2}(T)) = -F(x_{u_1}(T))$ so the approximation is in fact (analytically) exact. The numerical implementation for $\alpha = 0$ and $\alpha = 1$ (results not given here) showed indeed that this is the case i.e. $\frac{F(x_{u_1}(T))+F(x_{u_2}(T))}{2}$ was of the order of the roundoff error which in our setting is about $10^{-15}$.

### 6.2. Nonlinear setting

A nonlinear setting was tested next:

$$f(x,u) = x + u^2,\ x(0) = 0,\ F(x) = x. \tag{6.2}$$

The advantage of this example is that using stochastic expansion we know that

$$\mathbb{E}F(x_u(T)) = \frac{T^4}{12} + \frac{T^5}{60} + \frac{T^6}{360} + ... \tag{6.3}$$

We tested the cubature formulas of order 9 for different final times in the range $[10^{-3}, 10^0]$. The range was chosen so that the error is not below the roundoff. The results in Figure 3 confirm the theoretical results i.e. the error behaves as $O(T^5)$ for both order 9 formulas. We also tested the formula (5.1) from the literature that uses also only $n = 2$ functions and found $O(T^4)$. As expected, the formulas (5.23)-(5.24) and (5.25)-(5.28) converge faster.

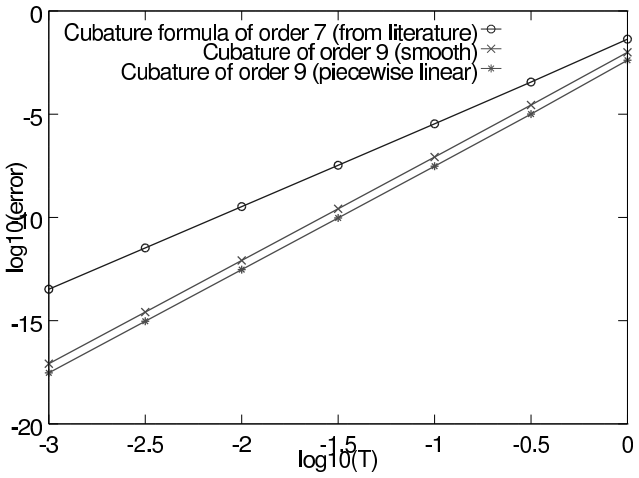


FIGURE 3. The error of the 9-th order cubature formulas (5.23)-(5.24) and (5.25)-(5.28) for test case (6.2) is plotted in log10-log10 axis. The $X$ axis is $log10(T)$ and the $Y$ axis is $log10$ of the error. The resulting plots are lines with slope 5. The 7-th order formula (5.1) is also plotted, it exhibits a slope of about 4.

### 6.3. Nonlinear setting out of the scope of the theoretical result

Finally, we tested a nonlinear setting taken from [5] which is not of the form (1.1). It involves a 2-dimensional SDE $x = (Y, A)$:

$$dY_t = aY_t dt + bY_t \circ dW_t \tag{6.4}$$

$$dA_t = Y_t dt, \ a = 0.1, \ b = 0.2, \tag{6.5}$$

$$F(x) = A^3. \tag{6.6}$$

Here too, we know the explicit solution of this 2-dimensional SDE $Y_t = e^{at+bW_t}, A_t = \int_0^t e^{as+bW_s} ds$. The moment $\mathbb{E}(A(T)^3)$ is not trivial to compute and we will not give here its (cumbersome) formula (see instead [16, 17] and also [2] for a elegant way to express it). We tested the same (three)

cubature formulas for different final times in the range $[10^{-3}, 10^0]$. The results in Figure 4 show that for all cubature formulas the error behaves as $O(T^4)$ which says that, from a numerical perspective, all cubatures are of order 7 for test case (6.4)-(6.6). This hints that formulas (5.23) and (5.25) behave at least as well as (5.1) for situations not covered by the theoretical results; recall that all cubatures have the same number of paths $n = 2$.

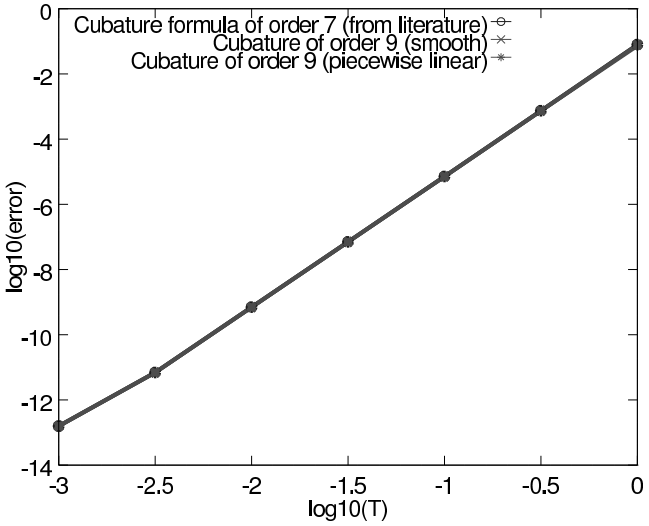


FIGURE 4. The error of the 9-th order cubature formulas (5.23) and (5.25) and the order 7 formula (5.1) for test case (6.4)-(6.6) is plotted in log10-log10 axis. The $X$ axis is $log10(T)$ and the $Y$ axis is $log10$ of the error. The resulting plot is very close to a line of slope 4 for all cubature formulas, the lines coincide graphically (but numerical values are different).

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