A notion of capacity related to elasticity. Applications to homogenization.

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Abstract. We study a notion of capacity related to elasticity which proves convenient to analyze the concentrations of strain energy caused by the rigid displacements of some infinitesimal parts of an elastic body in two or three dimensions. By way of application, we investigate the behavior of solutions to initial boundary value problems describing vibrations of periodic elastic composites with rapidly varying elastic properties. More specifically, we analyze a two-phase medium whereby a set of heavy stiff tiny particles is embedded in a softer matrix. This task is set in the context of linearized elasticity.

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1 Introduction

The study of composites comprising infinitesimal traces of materials with extreme physical properties has attracted a lot of attention over the past few decades [9], [10], [11], [12], [13], [14], [22], [25]. The common feature of this body of work is the emergence of a concentration of energy in a small region of space surrounding the strong components. A similar phenomenon occurs when Dirichlet problems in varying domains are considered [3], [4], [15], [16],[17], [18], [19], [20]. This extra contribution is characterized by a local density of the geometric perturbations in terms of an appropriate capacity depending on the type of equations. We are aiming at complementing this extensive material.

In the spirit of Villaggio [37], we introduce a notion of capacity characterizing the strain energy associated to the displacement of a bounded rigid body T immersed in an elastic space V. More precisely, we consider the family $(c_3((\boldsymbol{v},\boldsymbol{\omega});T;V))_{(\boldsymbol{v},\boldsymbol{\omega})\in(\mathbb{R}^3)^2}$ defined by

$$c_3((\boldsymbol{v},\boldsymbol{\omega});T;V):=\inf\Bigg\{\int_V \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\psi}):\boldsymbol{e}(\boldsymbol{\psi})dx,\quad \boldsymbol{\psi}\in H^1_0(V;\mathbb{R}^3),\\ \boldsymbol{\psi}=\boldsymbol{v}+\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge(\boldsymbol{x}-\boldsymbol{x}_T)\ \text{ in }\ T\Bigg\}, \tag{1}$$

where $\boldsymbol{a}_0$ denotes the elasticity tensor of the medium and $\boldsymbol{x}_T$ stands for the geometrical center of gravity of T. We denote by $\mathbf{Cap}_3(T;V)$ the 6×6 symmetric positive semidefinite matrix associated with the quadratic form $(\boldsymbol{v},\boldsymbol{\omega})\to c_3((\boldsymbol{v},\boldsymbol{\omega});T;V)$ in the canonical basis. In dimension 2, we define in the same way a 3×3 symmetric positive semidefinite matrix $\mathbf{Cap}_2(T;V)$.

The novelty of this notion, compared to what is already available on the subject in the litterature (see [4], [20], [23], [30]), is that the restrictions to T of the minimizers of (1) are helicoidal vector fields instead of constants. This choice is suggested by the small values presumably taken by the symmetrized gradient of the displacement in the parts of the body where the elasticity coefficients are large. The underlying purpose of this tool is to describe, in the setting of homogenization, the concentration of strain energy caused by the rigid displacements of some minuscule parts of a composite. The presence of the parameter $\frac{2}{\mathrm{diam}T}$ in (1) ensures that, given $\boldsymbol{\omega}\neq 0$ and a sequence (T_n) of domains of vanishing size, the norm on ∂T_n of the minimizers of (1) is bounded and not uniformly vanishing. As illustrated in the application developed below, this scaling proves appropriate to study elastic composites comprising small stiff particles homothetical to some fixed bounded domain of $\mathbb{R}^N$ (see Remark 3 (iv)).

For a given bounded smooth open subset Ω of $\mathbb{R}^N$ ($N\in\{2,3\}$), we consider the problem

$$\begin{cases}\rho_\varepsilon\dfrac{\partial^2\boldsymbol{u}_\varepsilon}{\partial t^2}-\mathbf{div}\boldsymbol{\sigma}_\varepsilon=\rho_\varepsilon\boldsymbol{f}\quad \text{in}\quad \Omega\times(0,t_1),\\ \boldsymbol{\sigma}_\varepsilon=\boldsymbol{a}_\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon),\quad \boldsymbol{e}(\boldsymbol{u}_\varepsilon)=\dfrac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon+\boldsymbol{\nabla}^T\boldsymbol{u}_\varepsilon),\\ \boldsymbol{u}_\varepsilon\in C([0,t_1];\,H^1_0(\Omega,\mathbb{R}^N))\cap C^1([0,t_1];\,L^2(\Omega,\mathbb{R}^N)),\\ \boldsymbol{u}_\varepsilon(0)=\boldsymbol{b}_0,\quad \dfrac{\partial\boldsymbol{u}_\varepsilon}{\partial t}(0)=\boldsymbol{c}_0,\\ (\boldsymbol{b}_0,\boldsymbol{c}_0)\in(C(\overline{\Omega};\mathbb{R}^N)\cap H^1_0(\Omega;\mathbb{R}^N))\times C(\overline{\Omega};\mathbb{R}^N),\quad \boldsymbol{f}\in C(\overline{\Omega\times(0,t_1)};\mathbb{R}^N).\end{cases} \tag{2}$$

The elasticity tensor $\boldsymbol{a}_\varepsilon$ and the mass density ρ_ε are supposed to take possibly large values in some subset T_{r_ε} of Ω and constant values $\boldsymbol{a}_0$, ρ_0 in the surrounding matrix (see (32), (34)). The set T_{r_ε} consists of an ε-periodic distribution of tiny grain-like particles of diameter $r_\varepsilon \ll \varepsilon$, homothetical to some bounded connected open subset T of $\mathbb{R}^N$ (see (31)). We show that the homogenized problem associated with (2) depends on the limit $\boldsymbol{C}_N(T)$ of the sum of the images of the connected components of T_{r_ε} under $\mathbf{Cap}_N(.;\Omega)$ per unit volume. The critical case takes place when some eigenvalues of the matrix $\boldsymbol{C}_N(T)$ are positive and finite: this corresponds to particles of diameter of order ε^3 if $N=3$ and, if $N=2$, of diameter r_ε such that $\frac{1}{\varepsilon^2|\log r_\varepsilon|}$ is of order 1. Then, a gap between the mean displacement of the constituent parts of the composite arises, giving rise to the emergence of a concentration of elastic strain energy in a thin zone enveloping the particles (see Remark 3 (iii)). It turns out that the particles behave, at a microscopic scale, like rigid bodies. Their effective displacement is characterized by the limit $(\boldsymbol{v},\boldsymbol{\omega})$ of the sequence $(\tilde{\boldsymbol{v}}_\varepsilon,\tilde{\boldsymbol{\omega}}_\varepsilon)$ defined in terms of the solution $\boldsymbol{u}_\varepsilon$ of (2) by (38), where $\boldsymbol{v}$ describes the effective displacement of their geometrical center of gravity, and $\boldsymbol{\omega}$ their effective rescaled rotation vector (see Remarks 5 and 6). We prove that the effective behavior of the tiny grain-like inclusions is governed by the coupled system of equations in $\Omega\times(0,t_1)$

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} \overline{\rho}\boldsymbol{v}+\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T) \\ \left(\frac{2}{\mathrm{diam}T}\right)^2\boldsymbol{J}^\rho\boldsymbol{\omega}+\frac{2}{\mathrm{diam}T}\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\wedge\boldsymbol{v}\end{pmatrix} = \begin{pmatrix}\overline{\rho}\boldsymbol{f}\\ \overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\wedge\boldsymbol{f}\end{pmatrix} - \boldsymbol{C}_N(T)\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \boldsymbol{\omega}\end{pmatrix}, \tag{3}$$

displaying rigid vibrations, associated with the boundary and initial conditions given in (40), the constants $\overline{\rho}$, $\boldsymbol{J}^\rho$, $\boldsymbol{y}_G$, $\boldsymbol{y}_T$ being defined by (35). The coupling with the effective displacement in the matrix $\boldsymbol{u}$ is marked by the second term of the right hand member of (3), which characterizes the sum of the surface forces applied on the particles by the surrounding medium and their total moment with respect to the center of gravity of the geometric particle. The effective displacement in the matrix is governed by the equation

$$\rho_0\frac{\partial^2\boldsymbol{u}}{\partial t^2}-\mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}))=\rho_0\boldsymbol{f}+\overline{\rho}\left(\boldsymbol{f}-\frac{\partial^2\boldsymbol{v}}{\partial t^2}-\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\omega}}{\partial t^2}\wedge\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\right),$$

where the second term of the right hand member represents the total force per unit volume exerted by the particles on the elastic matrix. We obtain a corrector result (see Section 7.3).

Computing the matrix $\boldsymbol{C}_N(T)$ requires a study of $\mathbf{Cap}_N$ which reveals striking differences depending on N. In the three-dimensional case, we obtain $\boldsymbol{C}_3(T)=\gamma^{(3)}\mathbf{Cap}_3(T;\mathbb{R}^3)$ where $\gamma^{(3)}:=\lim_{\varepsilon\to0}\frac{r_\varepsilon}{\varepsilon^3}$ and, in the critical case, each eigenvalue of $\boldsymbol{C}_3(T)$ is positive and finite. By contrast, only two of the three eigenvalues of $\boldsymbol{C}_2(T)$ are then positive and finite, whereas $(\boldsymbol{C}_2(T))_{33}=+\infty$ regardless of the order of magnitude of r_ε. This means that the effective concentration of strain energy generated by a non vanishing rescaled rotation of the particles is infinite. Accordingly, we find that $\boldsymbol{\omega}=0$ if $N=2$. The second distinctive feature of the two-dimensional case lies in the independence of the upper 2×2 submatrix of $\boldsymbol{C}_2(T)$ with respect to T: we get $(\boldsymbol{C}_2(T))_{\alpha\beta}=\gamma^{(2)}(\boldsymbol{M}_2)_{\alpha\beta}$ where $\gamma^{(2)}=\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2|\log r_\varepsilon|}$ $(\alpha,\beta\in\{1,2\})$ and $\boldsymbol{M}_2$ is defined by (21). The singular behavior of $\mathbf{Cap}_2$ leads to a significant simplification of

the limiting problem (3) in the two-dimensional case: the effective rescaled rotation vector of the particles is then equal to zero and the effective equations, independent of the choice of T, consist of the system

$$\begin{cases} \overline{\rho}\dfrac{\partial^2 \boldsymbol{v}}{\partial t^2} = \overline{\rho}\boldsymbol{f} - \gamma^{(2)}\boldsymbol{M}_2(\boldsymbol{v}-\boldsymbol{u}), \\ \rho_0\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u})) = \rho_0 + \overline{\rho}\left(\boldsymbol{f} - \dfrac{\partial^2}{\partial t^2}\boldsymbol{v}\right), \end{cases}$$

associated with the boundary and initial conditions given in (41). The application $\mathbf{Cap}_2$ is relevant to the study of fibered structures: if T_{r_ε} is made up, instead of particles, of an ε-periodic distribution of parallel cylinders of cross section of size $r_\varepsilon \ll \varepsilon$ homothetical to some bounded connected open subset S of $\mathbb{R}^2$, then the limiting problem depends partly on $\boldsymbol{C}_2(S)$ (see Section 5.4). Accordingly, the rescaled effective angle of torsion of the fibers is equal to zero and no torsion effect take place (see Remark 4).

In the elliptic problem associated with (2), which shows the same general features, the auxiliary variables $\boldsymbol{v},\boldsymbol{\omega}$ can be eliminated from the effective equations, yielding a much simpler homogenized problem which covers the complex behavior of the composite at a microscopic scale (see Section 5.1). This elimination is not possible in the hyperbolic case, where an interesting memory phenomenon arises (see Remark 3 (ii)).

2 Notations

In this paper, $\{\boldsymbol{e}_1,..,\boldsymbol{e}_N\}$ stands for the canonical basis of $\mathbb{R}^N$ ($N \in \{2,3\}$). Points in $\mathbb{R}^N$ or in $\mathbb{Z}^N$ and real-valued functions are represented by symbols beginning by a lightface minuscule (example $x, i, \det \boldsymbol{A}$...) and vectors and vector-valued functions by symbols beginning by a boldface minuscule (examples: $\boldsymbol{x}$, $\boldsymbol{x}_T$, $\boldsymbol{i}$, $\boldsymbol{u}$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\mathbf{div}\boldsymbol{\sigma}_\varepsilon$,...). Matrices and matrix-valued functions are represented by symbols beginning by a boldface majuscule with the following exceptions: $\boldsymbol{\nabla u}$ (displacement gradient), $\boldsymbol{e}(\boldsymbol{u})$ (linearized strain tensor), $\boldsymbol{a}_\varepsilon$ (elasticity tensor field). We denote by u_i or $(\boldsymbol{u})_i$ the components of a vector $\boldsymbol{u}$ and by A_{ij} or $(\boldsymbol{A})_{ij}$ those of a matrix $\boldsymbol{A}$ (that is $\boldsymbol{u} = \sum_{i=1}^N u_i\boldsymbol{e}_i = \sum_{i=1}^N (\boldsymbol{u})_i\boldsymbol{e}_i$; $\boldsymbol{A} = \sum_{i,j=1}^N A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sum_{i,j=1}^N (\boldsymbol{A})_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$). We do not employ the usual repeated index convention for summation. We denote by $\boldsymbol{A}{:}\boldsymbol{B} = \sum_{i,j=1}^N A_{ij}B_{ij}$ the inner product of two matrices, by ε_{ijk} the three-dimensional alternator, by $\boldsymbol{u}\wedge\boldsymbol{v} = \sum_{i,j,k=1}^3 \varepsilon_{ijk}u_jv_k\boldsymbol{e}_i$ the exterior product in $\mathbb{R}^3$, by $\mathbb{S}^M$ ($M \in \mathbb{N}$) the set of all real symmetric matrices of order M, by $\leq$ the partial order relation on $\mathbb{S}^M$ defined by

$$\boldsymbol{A} \leq \boldsymbol{B}, \quad \text{if} \quad \boldsymbol{\xi}.\boldsymbol{A}\boldsymbol{\xi} \leq \boldsymbol{\xi}.\boldsymbol{B}\boldsymbol{\xi} \quad \forall \boldsymbol{\xi} \in \mathbb{R}^M. \tag{4}$$

The symbol $\boldsymbol{I}_M$ represents the $M \times M$ identity matrix. The letter B denotes the open ball of $\mathbb{R}^N$ of center 0 and radius 1. The letter C denotes different constants whose precise values may vary. The symbol $\boldsymbol{a}_0$ stands for a fourth order tensor on $\mathbb{R}^N$ satisfying

$$\begin{aligned} &(\boldsymbol{a}_0)_{ijkh} = (\boldsymbol{a}_0)_{jikh} = (\boldsymbol{a}_0)_{khij} \quad \forall (i,j,k,h) \in \{1,...,N\}^4, \\ &\boldsymbol{a}_0\boldsymbol{M} : \boldsymbol{M} \geq c|\boldsymbol{M}|^2 \quad \forall \boldsymbol{M} \in \mathbb{S}^N, \quad (c>0). \end{aligned} \tag{5}$$

Given an open subset V of $\mathbb{R}^N$ and a bounded connected open subset T of V such that $\overline{T} \subset V$, the symbol $\mathbf{Cap}_N(T;V)$ represents, if $N = 3$, the 6×6 symmetric positive

semidefinite matrix associated to the quadratic form defined by

$$\begin{aligned}
&\boldsymbol{\xi} := \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3 \quad \rightarrow \quad \inf \mathcal{P}_3(T;V;\boldsymbol{\xi}) \qquad (= \boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi}),\\
&\mathcal{P}_3(T;V;\boldsymbol{\xi}) : \inf_{\boldsymbol{\psi}\in H_0^1(V;\mathbb{R}^3)} \left\{ a_V(\boldsymbol{\psi},\boldsymbol{\psi}),\ \boldsymbol{\psi}(x) = \boldsymbol{a} + \frac{2}{\mathrm{diam}T}\boldsymbol{b} \wedge (\boldsymbol{x} - \boldsymbol{x}_T) \text{ in } T \right\},
\end{aligned} \tag{6}$$

where $\boldsymbol{x}_T := \fint_T \boldsymbol{x} dx$ and a_V stands for the bilinear form on $H_0^1(V;\mathbb{R}^3)$ given by

$$a_V(\boldsymbol{\psi},\boldsymbol{\varphi}) := \int_V \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\psi}) : \boldsymbol{e}(\boldsymbol{\varphi}) dx.$$

If $N = 2$, $\mathbf{Cap}_2(T;V)$ denotes the 3×3 symmetric positive semidefinite matrix associated to the quadratic form

$$\begin{aligned}
&\boldsymbol{\xi} := \begin{pmatrix} \boldsymbol{a} \\ b \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R} \quad \rightarrow \quad \inf \mathcal{P}_2(T;V;\boldsymbol{\xi}) \qquad (= \boldsymbol{\xi}.\mathbf{Cap}_2(T;V)\boldsymbol{\xi}),\\
&\mathcal{P}_2(T;V;\boldsymbol{\xi}) : \inf_{\boldsymbol{\psi}\in H_0^1(V;\mathbb{R}^2)} \left\{ a_V(\boldsymbol{\psi},\boldsymbol{\psi}),\ \boldsymbol{\psi}(x) = \boldsymbol{a} + \frac{2}{\mathrm{diam}T} b\boldsymbol{e}_3 \wedge (\boldsymbol{x} - \boldsymbol{x}_T) \text{ in } T \right\}.
\end{aligned} \tag{7}$$

In the context of the heat equation, the same approach leads to the quadratic form $a \in \mathbb{R} \rightarrow \mathrm{cap}_N(T;V)a^2$, where cap_N denotes the harmonic capacity.

3 Study of $\mathbf{Cap}_N$

The main objective of this section is to analyze the behavior of the application $\mathbf{Cap}_N$ with respect to certain small subsets of $\mathbb{R}^N$. In what follows, the letter T denotes a bounded connected Lipschitz open subset of $\mathbb{R}^N$ and V an open subset of $\mathbb{R}^N$ such that $\overline{T} \subset V$.

Lemma 1. *The problems (6) and (7) have minimizing sequences in* $\mathcal{D}(V;\mathbb{R}^N)$.

Proof. Assume that $N = 3$ and fix $\boldsymbol{\xi} := \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$. By (6), there holds

$$\begin{aligned}
&\boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi} = \inf \left\{ a_V(\boldsymbol{\psi},\boldsymbol{\psi}), \quad \boldsymbol{\psi} \in \mathcal{A}_{\boldsymbol{\xi}}(T;V) \right\},\\
&\mathcal{A}_{\boldsymbol{\xi}}(T;V) := \left\{ \boldsymbol{\psi} \in H_0^1(V;\mathbb{R}^3), \quad \boldsymbol{\psi}(x) = \boldsymbol{a} + \boldsymbol{b} \wedge \frac{2}{\mathrm{diam}T}(\boldsymbol{x} - \boldsymbol{x}_T) \ \text{ in } T \right\}.
\end{aligned}$$

As a_V is strongly continuous on $H_0^1(V;\mathbb{R}^3)$, we just have to check that $\mathcal{A}_{\boldsymbol{\xi}}(T;V) \cap \mathcal{D}(V;\mathbb{R}^3)$ is dense in $\mathcal{A}_{\boldsymbol{\xi}}(T;V)$ with respect to the strong topology of $H_0^1(V;\mathbb{R}^3)$. Let us fix $\boldsymbol{\eta} \in \mathcal{A}_{\boldsymbol{\xi}}(T;V)$ and $\boldsymbol{\varphi} \in \mathcal{A}_{\boldsymbol{\xi}}(T;V) \cap \mathcal{D}(V;\mathbb{R}^3)$. There holds $\boldsymbol{\eta} - \boldsymbol{\varphi} = 0$ on ∂T hence, since T is Lipschitz, $\boldsymbol{\eta} - \boldsymbol{\varphi} \in H_0^1(V \setminus T;\mathbb{R}^3)$. Therefore, there exists a sequence $(\boldsymbol{\psi}_n) \subset \mathcal{D}(V \setminus T;\mathbb{R}^3)$ converging strongly to $\boldsymbol{\eta} - \boldsymbol{\varphi}$ in $H_0^1(V \setminus T;\mathbb{R}^3)$. We can extend each $\boldsymbol{\psi}_n$ to V by setting $\boldsymbol{\psi}_n = 0$ in T. Then $\boldsymbol{\psi}_n + \boldsymbol{\varphi} \in \mathcal{A}_{\boldsymbol{\xi}}(T;V) \cap \mathcal{D}(V;\mathbb{R}^3)$ and the sequence $(\boldsymbol{\psi}_n + \boldsymbol{\varphi})$ converges strongly to $\boldsymbol{\eta}$ in $H_0^1(V;\mathbb{R}^3)$. By the same argument, the problem (7) has a minimizing sequence in $\mathcal{D}(V;\mathbb{R}^2)$. □

The next lemma marks a fundamental difference between $\mathbf{Cap}_2$ and $\mathbf{Cap}_3$: the infimum problem $\mathcal{P}_2(T;V;\boldsymbol{\xi})$ is not achieved in general if V is unbounded (see Remark 2 (ii)),

whereas $\mathcal{P}_3(T;V;\boldsymbol{\xi})$ is always achieved provided we substitute for $H_0^1(V;\mathbb{R}^3)$ in (6) the Banach space $K_0(V;\mathbb{R}^3)$ defined by

$$K_0(V;\mathbb{R}^3) := \overline{\mathcal{D}(V;\mathbb{R}^3)}^{|.|_{K_0}}, \quad |\boldsymbol{\psi}|_{K_0} := \left(\int_V |\boldsymbol{\psi}|^6 dx\right)^{\frac{1}{6}} + \left(\int_V |\boldsymbol{\nabla\psi}|^2 dx\right)^{\frac{1}{2}}, \tag{8}$$

where $\overline{\mathcal{D}(V;\mathbb{R}^3)}^{|.|_{K_0}}$ denotes the closure of $\mathcal{D}(V;\mathbb{R}^3)$ with respect to the norm $|.|_{K_0}$. The space $K_0(V;\mathbb{R}^3)$ coincides with $H_0^1(V;\mathbb{R}^3)$ if V is bounded and may be strictly larger otherwise. The discrepancy between the behaviors of $\mathbf{Cap}_2$ and $\mathbf{Cap}_3$ comes in particular from the fact that Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^N} |f|^{p^*} dx \le C \int_{\mathbb{R}^N} |\nabla f|^p dx \quad \forall f \in W^{1,p}(\mathbb{R}^N) \quad (p^* := \frac{Np}{N-p}, \quad p < N), \tag{9}$$

fails to hold for $p = N = 2$.

Lemma 2. *(i) Assume that $N = 3$, and let $\boldsymbol{\xi} := \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$. Then the problem*

$$\mathcal{P}_{K_0}(T;V;\boldsymbol{\xi}) : \inf_{\boldsymbol{\psi}\in K_0(V;\mathbb{R}^3)} \left\{ a_V(\boldsymbol{\psi},\boldsymbol{\psi}),\ \boldsymbol{\psi}(x) = \boldsymbol{a} + \boldsymbol{b} \wedge \frac{2}{\mathrm{diam}T}(\boldsymbol{x} - \boldsymbol{x}_T)x \in T \right\} \tag{10}$$

has a unique solution, the matrix $\mathbf{Cap}_3(T;V)$ is positive definite, and

$$\boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi} = \inf \mathcal{P}_3(T;V;\boldsymbol{\xi}) = \min \mathcal{P}_{K_0}(T;V;\boldsymbol{\xi}).$$

(ii) Assume that $N = 2$, that V is bounded in one direction, and let $\boldsymbol{\xi} := \begin{pmatrix} \boldsymbol{a} \\ b \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}$. Then the problem (7) has a unique solution.

Proof. (i) Setting

$$\mathcal{K}_{\boldsymbol{\xi}}(T;V) := \left\{ \boldsymbol{\psi} \in K_0(V;\mathbb{R}^3), \quad \boldsymbol{\psi} = \boldsymbol{a} + \boldsymbol{b} \wedge \frac{2}{\mathrm{diam}T}(\boldsymbol{x} - \boldsymbol{x}_T) \ \text{ in } T \right\},$$

and repeating the argument of the proof of Lemma 1, we find that $\overline{\mathcal{K}_{\boldsymbol{\xi}}(T;V) \cap \mathcal{D}(V;\mathbb{R}^3)}^{K_0}$ $= \mathcal{K}_{\boldsymbol{\xi}}(T;V)$ and, noticing that a_V is continuous on $K_0(V;\mathbb{R}^3)$, deduce that

$$\boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi} = \inf \left\{ a_V(\boldsymbol{\psi},\boldsymbol{\psi}), \quad \boldsymbol{\psi} \in \mathcal{K}_{\boldsymbol{\xi}}(T;V) \right\}. \tag{11}$$

By Gagliardo-Nirenberg-Sobolev inequality in $H^1(\mathbb{R}^3;\mathbb{R}^3)$ (see (9)), Korn inequality in $H_0^1(V;\mathbb{R}^3)$, and (5), there holds (extending $\boldsymbol{\psi}$ to $\mathbb{R}^3$ by setting $\boldsymbol{\psi} = 0$ in $\mathbb{R}^3 \setminus V$)

$$\begin{aligned} \left(\int_V |\boldsymbol{\psi}|^6 dx\right)^{\frac{1}{6}} = \left(\int_{\mathbb{R}^3} |\boldsymbol{\psi}|^6 dx\right)^{\frac{1}{6}} \le C \left(\int_{\mathbb{R}^3} |\boldsymbol{\nabla\psi}|^2 dx\right)^{\frac{1}{2}} = C \left(\int_V |\boldsymbol{\nabla\psi}|^2 dx\right)^{\frac{1}{2}} \\ \le C \left(\int_V |\boldsymbol{e}(\boldsymbol{\psi})|^2 dx\right)^{\frac{1}{2}} \le C \left(a_V(\boldsymbol{\psi},\boldsymbol{\psi})\right)^{\frac{1}{2}} \le C|\boldsymbol{\psi}|_{K_0} \quad \forall\, \boldsymbol{\psi} \in K_0(V;\mathbb{R}^3), \end{aligned} \tag{12}$$

hence the application $|.|_{a_V} := \sqrt{a_V(.,.)}$ is a norm on $K_0(V;\mathbb{R}^3)$ equivalent to $|.|_{K_0}$. Equiped with this norm, $K_0(V;\mathbb{R}^3)$ is a Hilbert space on which the bilinear form a_V is continuous and coercive. As $\mathcal{K}_{\boldsymbol{\xi}}(T;V)$ is a closed convex subset of $K_0(V;\mathbb{R}^3)$, by

Stampacchia Theorem the infimum (11) is achieved at a unique point $\boldsymbol{\psi} \in \mathcal{K}_{\boldsymbol{\xi}}(T;V)$. If $\boldsymbol{\xi} \neq 0$, then $\boldsymbol{\psi} \neq 0$, hence $\boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi} = a_V(\boldsymbol{\psi},\boldsymbol{\psi}) = |\boldsymbol{\psi}|^2_{a_V} > 0$, therefore the matrix $\mathbf{Cap}_3(T;V)$ is positive definite.
(ii) If V is bounded in one direction, by Poincaré and Korn inequalities in $H^1_0(V;\mathbb{R}^2)$,

$$|\boldsymbol{\psi}|^2_{H^1_0(V;\mathbb{R}^2)} \leq C \int_V |\boldsymbol{e}(\boldsymbol{\psi})|^2 dx \leq C a_V(\boldsymbol{\psi},\boldsymbol{\psi}) \quad \forall \boldsymbol{\psi} \in H^1_0(V;\mathbb{R}^2). \tag{13}$$

Then we repeat the argument of the case $N = 3$, substituting (13) for (12) and $H^1_0(V;\mathbb{R}^2)$ for $K_0(V;\mathbb{R}^3)$. □
The next Lemma, whose proof is straightforward, states that in regard to the order relation (4), the application $(T,V) \to \mathbf{Cap}_N(T;V)$ is decreasing with respect to V and the $N \times N$ upper left submatrix of $\mathbf{Cap}_N(T;V)$ is increasing with respect to T. However, $\mathbf{Cap}_N(T;V)$ is presumably not increasing with respect to T (see Remark 2 (i)).

Lemma 3. *(i) Let V_1 and V_2 be two open subsets of $\mathbb{R}^N$ such that $\overline{T} \subset V_1 \subset V_2$. Then*

$$\mathbf{Cap}_N(T;V_1) \geq \mathbf{Cap}_N(T;V_2).$$

(ii) Let T_1 and T_2 be two bounded connected open subsets of $\mathbb{R}^N$ such that $\overline{T}_1 \subset \overline{T}_2 \subset V$. Then

$$\begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix}.\mathbf{Cap}_N(T_1;V)\begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix} \leq \begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix}.\mathbf{Cap}_N(T_2;V)\begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix} \quad \forall \boldsymbol{a} \in \mathbb{R}^N. \tag{14}$$

In the following lemma, we investigate the continuity properties of $\mathbf{Cap}_N(T,V)$ with respect to V.

Lemma 4. *Let (V_n) be an increasing sequence of open subsets of $\mathbb{R}^N$ such that $\overline{T} \subset V_1$ and $\bigcup_{n=1}^{+\infty} V_n = V$.*
(i) There holds

$$\lim_{n\to+\infty} \mathbf{Cap}_N(T;V_n) = \mathbf{Cap}_N(T;V).$$

(ii) Assume that $N = 3$, and let $\boldsymbol{\psi}_n$ be the solution of $\mathcal{P}_{K_0}(T;V_n;\boldsymbol{\xi})$ (see (10)) extended to V by setting $\boldsymbol{\psi}_n = 0$ in $V \setminus V_n$. Then $(\boldsymbol{\psi}_n)$ converges strongly in $K_0(V;\mathbb{R}^3)$ to the unique solution of $\mathcal{P}_{K_0}(T;V;\boldsymbol{\xi})$.
(iii) Assume that $N = 2$ and V is bounded in one direction, and let $\boldsymbol{\psi}_n$ be the solution of $\mathcal{P}_2(T;V_n;\boldsymbol{\xi})$ (see (7))), extended to V in the same way. Then $(\boldsymbol{\psi}_n)$ converges strongly in $H^1_0(V;\mathbb{R}^2)$ to the unique solution of $\mathcal{P}_2(T;V;\boldsymbol{\xi})$.

Proof. (i) We fix $\boldsymbol{\xi} \in \mathbb{R}^{\frac{N(N+1)}{2}}$, $\alpha > 0$, $\boldsymbol{\psi} \in \mathcal{D}(V;\mathbb{R}^N) \cap \mathcal{A}_{\boldsymbol{\xi}}(T;V)$ such that $a_V(\boldsymbol{\psi},\boldsymbol{\psi}) \leq \boldsymbol{\xi}.\mathbf{Cap}_N(T;V)\boldsymbol{\xi} + \alpha$ (see Lemma 1) and $n_0 \in \mathbb{N}$ such that $\mathrm{Supp}\,\boldsymbol{\psi} \subset V_n \ \forall n \geq n_0$. We have $\boldsymbol{\xi}.\mathbf{Cap}_N(T;V_n)\boldsymbol{\xi} \leq a_{V_n}(\boldsymbol{\psi},\boldsymbol{\psi}) = a_V(\boldsymbol{\psi},\boldsymbol{\psi}) \leq \boldsymbol{\xi}.\mathbf{Cap}_N(T;V)\boldsymbol{\xi} + \alpha \ \forall n \geq n_0$. Applying Lemma 3 (i), we infer

$$\begin{aligned} \boldsymbol{\xi}.\mathbf{Cap}_N(T;V)\boldsymbol{\xi} &\leq \liminf_{n\to+\infty} \boldsymbol{\xi}.\mathbf{Cap}_N(T;V_n)\boldsymbol{\xi} \\ &\leq \limsup_{n\to+\infty} \boldsymbol{\xi}.\mathbf{Cap}_N(T;V_n)\boldsymbol{\xi} \leq \boldsymbol{\xi}.\mathbf{Cap}_N(T;V)\boldsymbol{\xi} + \alpha. \end{aligned}$$

(ii) By (i), we have (see the proof of Lemma 2 (i))

$$\begin{aligned}\lim_{n\to+\infty}|\boldsymbol{\psi}_n|^2_{a_V} &= \lim_{n\to+\infty} a_{V_n}(\boldsymbol{\psi}_n,\boldsymbol{\psi}_n) = \lim_{n\to+\infty}\boldsymbol{\xi}.\mathbf{Cap}_3(T;V_n)\boldsymbol{\xi}\\ &= \boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi} < +\infty,\end{aligned} \tag{15}$$

hence the sequence $(\boldsymbol{\psi}_n)$ is bounded in $K_0(V;\mathbb{R}^3)$ and converges weakly, up to a subsequence, to some $\boldsymbol{\psi}\in K_0(V;\mathbb{R}^3)$. Since each $\boldsymbol{\psi}_n$ belongs to the convex strongly closed (thus weakly closed) subset $\mathcal{K}_{\boldsymbol{\xi}}(T;V)$ of $K_0(V;\mathbb{R}^3)$, there holds $\boldsymbol{\psi}\in\mathcal{K}_{\boldsymbol{\xi}}(T;V)$. The functional $\boldsymbol{g}\to a_V(\boldsymbol{g},\boldsymbol{g})$ is convex strongly continuous on $K_0(V;\mathbb{R}^3)$, hence weakly lower semi-continuous, therefore $a_V(\boldsymbol{\psi},\boldsymbol{\psi})\le \liminf_{n\to+\infty} a_V(\boldsymbol{\psi}_n,\boldsymbol{\psi}_n) = \boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi}$. We deduce that $\boldsymbol{\psi}$ is the unique solution of $\mathcal{P}_{K_0}(T;V;\boldsymbol{\xi})$ and that $|\boldsymbol{\psi}|^2_{a_V} = \boldsymbol{\xi}.\mathbf{Cap}_3(T;V)\boldsymbol{\xi}$. It follows then from (15) that $\lim_{n\to+\infty}|\boldsymbol{\psi}_n|_{a_V} = |\boldsymbol{\psi}|_{a_V}$. As the space $K_0(V;\mathbb{R}^3)$, equiped with $|.|_{a_V}$, is a Hilbert space, it is uniformly convex, hence the weak convergence of $(\boldsymbol{\psi}_n)$ to $\boldsymbol{\psi}$ joined with the convergence of the norms yields the strong convergence of $(\boldsymbol{\psi}_n)$ to $\boldsymbol{\psi}$ in $K_0(V;\mathbb{R}^3)$.
(iii) Same argument as in the case $N=3$. □

The properties stated below are easily deduced from Lemma 4 and from the change of variable formula.

Lemma 5. *There holds, for* $\lambda>0$

$$\begin{aligned}&\mathbf{Cap}_N(\lambda T;V) = \lambda^{N-2}\mathbf{Cap}_N\left(T;\frac{1}{\lambda}V\right) \quad \text{if} \quad \lambda T\subset V,\\ &\lim_{\lambda\to 0}\mathbf{Cap}_N\left(T;\frac{1}{\lambda}V\right) = \mathbf{Cap}_N(T;\mathbb{R}^N) \quad \text{if} \quad 0\in V.\end{aligned} \tag{16}$$

Proof. Let us fix $\boldsymbol{\xi}\in\mathbb{R}^{\frac{N(N+1)}{2}}$ and $\alpha>0$. By Lemma 1, there exists $\boldsymbol{\psi}\in\mathcal{A}_{\boldsymbol{\xi}}(\lambda T;V)\cap\mathcal{D}(V;\mathbb{R}^N)$ such that $\boldsymbol{\xi}.\mathbf{Cap}_N(\lambda T;V)\boldsymbol{\xi}+\alpha\ge a_V(\boldsymbol{\psi},\boldsymbol{\psi}) = \int_V \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\psi}):\boldsymbol{e}(\boldsymbol{\psi})dx$. We set $\boldsymbol{\varphi}(y):=\boldsymbol{\psi}(\lambda y)$. Then $\boldsymbol{\varphi}\in\mathcal{A}_{\boldsymbol{\xi}}(T;\frac{1}{\lambda}V)$, $\boldsymbol{e}(\boldsymbol{\varphi})(y)=\lambda\boldsymbol{e}(\boldsymbol{\psi})(\lambda y)$, and

$$\begin{aligned}\boldsymbol{\xi}.\mathbf{Cap}_N(\lambda T;V)\boldsymbol{\xi}+\alpha &\ge \int_V \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\psi}):\boldsymbol{e}(\boldsymbol{\psi})dx = \lambda^N\int_{\frac{1}{\lambda}V}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\psi}):\boldsymbol{e}(\boldsymbol{\psi})(\lambda y)dy\\ &= \lambda^{N-2}\int_{\frac{1}{\lambda}V}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\varphi}):\boldsymbol{e}(\boldsymbol{\varphi})(y)dy \ge \lambda^{N-2}\boldsymbol{\xi}.\mathbf{Cap}_N\left(T;\frac{1}{\lambda}V\right)\boldsymbol{\xi}.\end{aligned}$$

By the arbitrary choice of α, λ, T, V, $\boldsymbol{\xi}$, the first line of (16) is proved.
If $0\in V$, we can assume without loss of generality that $B\subset V$. By Lemma 4 we have $\lim_{\lambda\to0}\mathbf{Cap}_N(T,\frac{1}{\lambda}B) = \mathbf{Cap}_N(T,\mathbb{R}^N)$. By passing to the limit as $\lambda\to0$ in the first and third terms of the double inequality $\mathbf{Cap}_N(T,\mathbb{R}^N)\le\mathbf{Cap}_N(T,\frac{1}{\lambda}V)\le\mathbf{Cap}_N(T,\frac{1}{\lambda}B)$ (see Lemma 3), we obtain the second line of (16). □

In the next two lemmas, we investigate the asymptotic behavior of $\mathbf{Cap}_N(r_\varepsilon T;R_\varepsilon B)$, being (r_ε), (R_ε) any bounded sequences of positive reals such that $r_\varepsilon << R_\varepsilon$. The study is straightforward in the case $N=3$:

Lemma 6. *Assume that* $N=3$, *let* T *be a bounded connected Lipschitz open subset of* $\mathbb{R}^3$ *such that* $B\subset T$, *and let* (r_ε) *and* (R_ε) *be two sequences of positive reals such that* $r_\varepsilon\ll R_\varepsilon\le C<+\infty$. *Then,*

$$\lim_{\varepsilon\to0}\frac{1}{r_\varepsilon}\mathbf{Cap}_3(r_\varepsilon T;R_\varepsilon B) = \mathbf{Cap}_3(T;\mathbb{R}^3). \tag{17}$$

Moreover, if

$$\boldsymbol{a}_0\boldsymbol{M} := \lambda_0\,\mathrm{tr}(\boldsymbol{M})\boldsymbol{I}_N + 2\mu_0\boldsymbol{M} \quad \forall \boldsymbol{M}\in\mathbb{S}^N, \quad \mu_0>0,\ 3\lambda_0+2\mu_0>0, \tag{18}$$

then

$$\mathbf{Cap}_3(B;\mathbb{R}^3) = \frac{12\pi\mu_0(\lambda_0+2\mu_0)}{(2\lambda_0+5\mu_0)}\begin{pmatrix}\boldsymbol{I}_3 & 0\\ 0 & 0\end{pmatrix} + 8\pi\mu_0\begin{pmatrix}0 & 0\\ 0 & \boldsymbol{I}_3\end{pmatrix}. \tag{19}$$

Proof. By Lemma 4 and Lemma 5 we have

$$\lim_{\varepsilon\to 0}\frac{1}{r_\varepsilon}\mathbf{Cap}_3(r_\varepsilon T;R_\varepsilon B) = \lim_{\varepsilon\to 0}\mathbf{Cap}_3\left(T;\frac{R_\varepsilon}{r_\varepsilon}B\right) = \mathbf{Cap}_3(T;\mathbb{R}^3).$$

The Dirichlet problem $\mathcal{P}_{K_0}(B;\mathbb{R}^3;\boldsymbol{\xi})$ of the spherical cavity in an infinite isotropic elastic body has been studied by a number of authors (see the bibliography of [29]). Its infimum $\boldsymbol{\xi}.\mathbf{Cap}_3(B;\mathbb{R}^3)\boldsymbol{\xi}$ is given for instance in [29, (8.4.3)]. Alternatively, the real $\boldsymbol{\xi}.\mathbf{Cap}_3(T;\mathbb{R}^3)\boldsymbol{\xi}$ is computed explicitely by Villaggio in [37] for $\boldsymbol{\xi}\in\left\{\begin{pmatrix}\boldsymbol{e}_3\\0\end{pmatrix},\begin{pmatrix}0\\\boldsymbol{e}_3\end{pmatrix}\right\}$ when T is an ellipsoid of revolution along the axis parallel to $\boldsymbol{e}_3$. Then, for $T=B$, the coefficients emerging in (19) are deduced simply by substituting $\left(\mu_0,\frac{\lambda_0}{2(\lambda_0+\mu_0)},1\right)$ for (G,σ,q_0) in [37, line 4, p.347]. □

Remark 1. *Under (18), the solution $\boldsymbol{\theta}_\varepsilon^{(p)}$ (resp., $\boldsymbol{\eta}_\varepsilon^{(p)}$) of the Dirichlet problem for the homogeneous isotropic elastic hollow sphere $\mathcal{P}_3(r_\varepsilon B;R_\varepsilon B;(\boldsymbol{e}_p,0))$ (resp., $\mathcal{P}_3(r_\varepsilon B;R_\varepsilon B;(0,\boldsymbol{e}_p))$) were determined by Thomson in [36] after the method developed by Lamé [26] and are given by (see [29, 8.5.30, 8.5.33]):*

$$\begin{aligned}\boldsymbol{\theta}_\varepsilon^{(p)}(x) &= \alpha_\varepsilon(|\boldsymbol{x}|)\boldsymbol{e}_p + \beta_\varepsilon(|\boldsymbol{x}|)x_p\boldsymbol{x} + \varpi_\varepsilon(|\boldsymbol{x}|)\boldsymbol{x},\\ \boldsymbol{\eta}_\varepsilon^{(p)}(x) &= \frac{r_\varepsilon^2}{R_\varepsilon^3-r_\varepsilon^3}\left(\frac{R_\varepsilon^3}{|\boldsymbol{x}|^3}-1\right)\boldsymbol{e}_p\wedge\boldsymbol{x},\end{aligned} \tag{20}$$

where

$$\begin{aligned}\alpha_\varepsilon(r) &:= \frac{r_\varepsilon}{r_\varepsilon-R_\varepsilon}\left(-\frac{R_\varepsilon}{r}+1\right)\frac{r_\varepsilon R_\varepsilon a}{3\delta_\varepsilon(r_\varepsilon-R_\varepsilon)}\left(-\frac{r_\varepsilon^3-R_\varepsilon^3}{r_\varepsilon^5-R_\varepsilon^5}r_\varepsilon^2R_\varepsilon^2+r^2-\frac{r_\varepsilon^2-R_\varepsilon^2}{r_\varepsilon^5-R_\varepsilon^5}r^5\right)\frac{1}{r^3},\\ \beta_\varepsilon(r) &:= \frac{r_\varepsilon R_\varepsilon a}{3\delta_\varepsilon(r_\varepsilon-R_\varepsilon)}\left(-\frac{r_\varepsilon^3-R_\varepsilon^3}{r_\varepsilon^5-R_\varepsilon^5}r_\varepsilon^2R_\varepsilon^2+r^2-\frac{r_\varepsilon^2-R_\varepsilon^2}{r_\varepsilon^5-R_\varepsilon^5}r^5\right)\left(-\frac{3}{r^5}\right),\\ \varpi_\varepsilon(r) &:= \frac{r_\varepsilon R_\varepsilon}{3\delta_\varepsilon(r_\varepsilon-R_\varepsilon)}\left(-\frac{10}{3}\right)\frac{r_\varepsilon^2-R_\varepsilon^2}{r_\varepsilon^5-R_\varepsilon^5}\left(\frac{r_\varepsilon^2-R_\varepsilon^2}{r_\varepsilon-R_\varepsilon}\frac{r_\varepsilon R_\varepsilon}{r}-\frac{r_\varepsilon^3-R_\varepsilon^3}{r_\varepsilon-R_\varepsilon}+r^2\right),\end{aligned}$$

the constants δ_ε, a, b being defined by

$$\delta_\varepsilon := ab-\frac{10}{9}\frac{r_\varepsilon R_\varepsilon(r_\varepsilon^2-R_\varepsilon^2)^2}{(r_\varepsilon-R_\varepsilon)(r_\varepsilon^5-R_\varepsilon^5)}, \quad a := \frac{2}{3}\frac{\lambda_0+4\mu_0}{\lambda_0+\mu_0}, \quad b := \frac{2}{3}\frac{2\lambda_0+5\mu_0}{\lambda_0+\mu_0}.$$

Deriving $\mathbf{Cap}_3(r_\varepsilon B;R_\varepsilon B)$ from (20) and passing to the limit as $\varepsilon\to 0$ thanks to (17), we find again (19). We verify in passing that $\boldsymbol{\varphi}_\varepsilon\in\{\boldsymbol{\theta}_\varepsilon^{(p)},\boldsymbol{\eta}_\varepsilon^{(p)}\}$ satisfies estimates of the type $|\boldsymbol{\varphi}_\varepsilon(x)|\le C\frac{r_\varepsilon}{|\boldsymbol{x}|}$, $|\boldsymbol{\nabla}\boldsymbol{\varphi}_\varepsilon(x)|\le C\frac{r_\varepsilon}{|\boldsymbol{x}|^2}$, similar to (110).

The case $N = 2$ is appreciably more involved:

Lemma 7. *Under (18), for any bounded connected Lipschitz open subset T of $\mathbb{R}^2$ such that $B \subset T$ and any couple $((r_\varepsilon),(R_\varepsilon))$ of sequences of positive reals such that $r_\varepsilon \ll R_\varepsilon \leq C < +\infty$, there holds*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} |\log r_\varepsilon| (\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B))_{\alpha\beta} = (\boldsymbol{M}_2)_{\alpha\beta} \quad \forall \alpha \in \{1,2\},\\
&\boldsymbol{M}_2 := 4\pi\mu_0 \frac{\lambda_0+2\mu_0}{\lambda_0+3\mu_0} \boldsymbol{I}_2,\\
&\lim_{\varepsilon\to 0} \mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B) = \mathbf{Cap}_2(T;\mathbb{R}^2) = (\mathbf{Cap}_2(T;\mathbb{R}^2))_{33} \boldsymbol{e}_3 \otimes \boldsymbol{e}_3,\\
&(\mathbf{Cap}_2(T;\mathbb{R}^2))_{33} > 0, \quad |(\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B))_{\alpha 3}| \leq \frac{C}{\sqrt{|\log r_\varepsilon|}} \quad \forall \alpha \in \{1,2\}.
\end{aligned} \tag{21}$$

Moreover,

$$\mathbf{Cap}_2(B;\mathbb{R}^2) = 4\pi\mu_0 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3. \tag{22}$$

Proof. At first we assume that $T = B$. Let $\boldsymbol{\eta}_\varepsilon$ (resp., $\boldsymbol{\theta}_\varepsilon^{(\alpha)}$) denote the solution of $\mathcal{P}_2(r_\varepsilon B; R_\varepsilon B; (0,1))$ (resp., $\mathcal{P}_2(r_\varepsilon B; R_\varepsilon B;\ (\boldsymbol{e}_\alpha, 0))$, $\alpha \in \{1,2\}$) (see Lemma 2 (ii)). A straightforward computation yields

$$\begin{aligned}
&\boldsymbol{\eta}_\varepsilon(x) = \frac{r_\varepsilon}{R_\varepsilon^2 - r_\varepsilon^2}\left(\frac{R_\varepsilon^2}{|\boldsymbol{x}|^2} - 1\right) \boldsymbol{e}_3 \wedge \boldsymbol{x},\\
&\mathbf{Cap}_2(r_\varepsilon B; R_\varepsilon B)_{33} = \int_{R_\varepsilon B} \boldsymbol{e}(\boldsymbol{\eta}_\varepsilon) : \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\eta}_\varepsilon) dx = 4\pi\mu_0 \frac{R_\varepsilon^2}{R_\varepsilon^2 - r_\varepsilon^2},\\
&\mathbf{Cap}_2(r_\varepsilon B; R_\varepsilon B)_{\alpha 3} = \int_{R_\varepsilon B} \boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}) : \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\eta}_\varepsilon) dx\\
&\qquad\qquad = \boldsymbol{e}_\alpha . \int_{\partial(r_\varepsilon B)} \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\eta}_\varepsilon) \boldsymbol{\nu} d\mathcal{H}^1(x) = 0.
\end{aligned} \tag{23}$$

The following estimate is established in [10, (5.12)]:

$$\begin{aligned}
(\mathbf{Cap}_2(r_\varepsilon B; R_\varepsilon B))_{\alpha\beta} &= \boldsymbol{e}_\beta . \int_{\partial(r_\varepsilon B)} \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}) \boldsymbol{\nu} d\mathcal{H}^1(x)\\
&= 4\pi\mu_0 \frac{\lambda_0+2\mu_0}{\lambda_0+3\mu_0} \frac{1}{|\log r_\varepsilon|} (1 + o(1)) \delta_{\alpha\beta}.
\end{aligned} \tag{24}$$

By Lemma 4 and Lemma 5 we have

$$\lim_{\varepsilon\to 0} \mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B) = \lim_{\varepsilon\to 0} \mathbf{Cap}_2\left(T; \frac{R_\varepsilon}{r_\varepsilon} B\right) = \mathbf{Cap}_2(T;\mathbb{R}^2). \tag{25}$$

Thanks to (23), (24), and (25), the estimates (21), (22) are proved in the case $T = B$ (or $T = \lambda B$, $\lambda > 0$). If $T \neq B$, there holds

$$B \subset T \subset (\mathrm{diam} T) B. \tag{26}$$

Let us fix $\boldsymbol{a} \in \mathbb{R}^2$. By (14) and (26), we have

$$\begin{aligned}
|\log r_\varepsilon| \begin{pmatrix}\boldsymbol{a}\\0\end{pmatrix} . \mathbf{Cap}_2(r_\varepsilon B; R_\varepsilon B) \begin{pmatrix}\boldsymbol{a}\\0\end{pmatrix} &\leq |\log r_\varepsilon| \begin{pmatrix}\boldsymbol{a}\\0\end{pmatrix} . \mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B) \begin{pmatrix}\boldsymbol{a}\\0\end{pmatrix}\\
&\leq |\log r_\varepsilon| \begin{pmatrix}\boldsymbol{a}\\0\end{pmatrix} . \mathbf{Cap}_2(r_\varepsilon (\mathrm{diam} T) B; R_\varepsilon B) \begin{pmatrix}\boldsymbol{a}\\0\end{pmatrix}.
\end{aligned}$$

By passing to the limit as $\varepsilon \to 0$ in the first and third terms of the last inequalities, taking (24) into account, we infer

$$\lim_{\varepsilon\to 0} |\log r_\varepsilon| \begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix} .\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B) \begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix} = 4\pi\mu_0 \frac{\lambda_0+2\mu_0}{\lambda_0+3\mu_0}|\boldsymbol{a}|^2,$$

yielding by the arbitrary choice of $\boldsymbol{a}$

$$\lim_{\varepsilon\to 0} |\log r_\varepsilon| (\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B))_{\alpha\beta} = 4\pi\mu_0 \frac{\lambda_0+2\mu_0}{\lambda_0+3\mu_0}\delta_{\alpha\beta} \quad \forall \alpha,\beta \in \{1,2\}. \tag{27}$$

Denoting now by $\boldsymbol{\theta}_\varepsilon^{(\alpha)}$ ($\alpha \in \{1,2\}$) (resp., $\boldsymbol{\eta}_\varepsilon$) the solution of $\mathcal{P}_2(r_\varepsilon T; R_\varepsilon B; (\boldsymbol{e}_\alpha, 0))$ (resp., $\mathcal{P}_2(r_\varepsilon T; R_\varepsilon B; (0,1))$), by Cauchy-Schwarz inequality, (25) and (27) we have

$$\begin{aligned} |(\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B))_{\alpha 3}| &= |a_{R_\varepsilon B}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}, \boldsymbol{\eta}_\varepsilon)| \le a_{R_\varepsilon B}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}, \boldsymbol{\theta}_\varepsilon^{(\alpha)})^{\frac{1}{2}} a_{R_\varepsilon B}(\boldsymbol{\eta}_\varepsilon, \boldsymbol{\eta}_\varepsilon)^{\frac{1}{2}} \\ &= (\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B))_{\alpha\alpha}^{\frac{1}{2}} (\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B))_{33}^{\frac{1}{2}} \le \frac{C}{\sqrt{|\log r_\varepsilon|}}. \end{aligned} \tag{28}$$

By (7), (26), there holds

$$\begin{aligned} &(\mathbf{Cap}_2(T;\mathbb{R}^2))_{33} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .\mathbf{Cap}_2(T;\mathbb{R}^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\ge \inf_{\boldsymbol{\psi}\in H_0^1(\mathbb{R}^2;\mathbb{R}^2)} \left\{ a_{\mathbb{R}^2}(\boldsymbol{\psi},\boldsymbol{\psi}), \quad \boldsymbol{\psi} = \frac{2}{\mathrm{diam}T}\boldsymbol{e}_3 \wedge (\boldsymbol{x}-\boldsymbol{x}_T) \ \text{ in } B \right\} \\ &= \inf_{\boldsymbol{\psi}\in H_0^1(\mathbb{R}^2;\mathbb{R}^2)} \left\{ a_{\mathbb{R}^2}(\boldsymbol{\psi},\boldsymbol{\psi}), \ \boldsymbol{\psi} = -\frac{2\boldsymbol{e}_3\wedge\boldsymbol{x}_T}{\mathrm{diam}T} + \frac{2}{\mathrm{diam}B}\frac{\mathrm{diam}B}{\mathrm{diam}T}\boldsymbol{e}_3\wedge\boldsymbol{x} \text{ in } B \right\} \\ &= \begin{pmatrix} -\frac{2\boldsymbol{e}_3\wedge\boldsymbol{x}_T}{\mathrm{diam}T} \\ \frac{\mathrm{diam}B}{\mathrm{diam}T} \end{pmatrix} .\mathbf{Cap}_2(B;\mathbb{R}^2) \begin{pmatrix} -\frac{2\boldsymbol{e}_3\wedge\boldsymbol{x}_T}{\mathrm{diam}T} \\ \frac{\mathrm{diam}B}{\mathrm{diam}T} \end{pmatrix} = 4\pi\mu_0 \left(\frac{\mathrm{diam}B}{\mathrm{diam}T}\right)^2 > 0. \end{aligned} \tag{29}$$

Joining (25), (27), (28), and (29), the estimates (21) are proved. □

We are in position to determine the asymptotic behavior of the sequence $(\boldsymbol{C}_{N\varepsilon}(T))$ introduced in Section 1:

Lemma 8. *Let Ω be a bounded open subset of $\mathbb{R}^N$ such that $0 \in \Omega$. Assume (18) if $N = 2$. Then the estimates deduced by substituting Ω for $R_\varepsilon B$ in (17) and (21) are satisfied. In particular, setting $\boldsymbol{C}_{N\varepsilon}(T) := \frac{1}{\varepsilon^N}\mathbf{Cap}_N(r_\varepsilon T; \Omega)$, we have, if $0 < \gamma^{(N)} < +\infty$ (see (39))*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \boldsymbol{C}_{3\varepsilon}(T) = \gamma^{(3)}\mathbf{Cap}_3(T;\mathbb{R}^3), \\ &\lim_{\varepsilon\to 0} (\boldsymbol{C}_{2\varepsilon}(T))_{\alpha\beta} = \gamma^{(2)}(\boldsymbol{M}_2)_{\alpha\beta}, \quad \alpha,\beta \in \{1,2\}, \\ &\lim_{\varepsilon\to 0} (\boldsymbol{C}_{2\varepsilon}(T))_{33} = (\mathbf{Cap}_2(T;\mathbb{R}^3))_{33} > 0, \ (\boldsymbol{C}_{2\varepsilon}(T))_{\alpha 3} \le \frac{C}{\sqrt{|\log r_\varepsilon|}}, \end{aligned} \tag{30}$$

where $\boldsymbol{M}_2$ is given by (21).

Proof. Let $(c,d) \in \mathbb{R}^2$ be such that $cB \subset \Omega \subset dB$. Lemma 8 follows from Lemma 7 (applied with $R_\varepsilon \in \{c,d\}$) and from the double inequality $\mathbf{Cap}_N(r_\varepsilon T; cB) \ge \mathbf{Cap}_N(r_\varepsilon T; \Omega) \ge \mathbf{Cap}_N(r_\varepsilon T; dB)$. □

Remark 2. *(i) The application* $(T,V) \to \mathbf{Cap}_N(T;V)$ *is not increasing with respect to* T*, as illustrated by the following example: assume that* $N=3$ *and set* $T_\varepsilon := B \cup C_\varepsilon$*, where* $C_\varepsilon := (\varepsilon B_{\mathbb{R}^2}) \times (-2,2)$ *(see fig.1). Then, due to the presence of the parameter* $\frac{2}{\mathrm{diam}T}$ *in (1), it is easy to prove that* $\lim_{\varepsilon\to 0} \begin{pmatrix} 0 \\ \boldsymbol{e}_3 \end{pmatrix} .\mathbf{Cap}_3(T_\varepsilon;\mathbb{R}^3) \begin{pmatrix} 0 \\ \boldsymbol{e}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ \boldsymbol{e}_3 \end{pmatrix} .\mathbf{Cap}_3(B;\mathbb{R}^3) \begin{pmatrix} 0 \\ \boldsymbol{e}_3 \end{pmatrix}$*, although* $B \subset T_\varepsilon$*. Hence, given* $\boldsymbol{\xi} \in \mathbb{R}^{\frac{N(N+1)}{2}}$*, the application* $T \to \boldsymbol{\xi}.\mathbf{Cap}_N(T;V)\boldsymbol{\xi}$ *is not the restriction of a Choquet capacity to the connected relatively compact open subsets of* V*, unless* $\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix}$ *for some* $\boldsymbol{a} \in \mathbb{R}^N$ *(see (14)), and the application* $T \to \mathbf{Cap}_N(T;V)$*, defined for relatively compact connected open subsets of* V*, can presumably not be extended to* 2^V*.*

(ii) If $\boldsymbol{a} \in \mathbb{R}^2 \setminus \{0\}$*, then the infimum problem* $\mathcal{P}_2\left(T;\mathbb{R}^2;\begin{pmatrix} \boldsymbol{a} \\ 0 \end{pmatrix}\right)$ *(see (7)) is not achieved. Otherwise, should* $\boldsymbol{\psi} \in H^1_0(\mathbb{R}^2;\mathbb{R}^2)$ *be a minimum, then by Korn inequality in* $H^1_0(\mathbb{R}^2;\mathbb{R}^2)$ *and the second line of (21),* $|\boldsymbol{\nabla\psi}|^2_{L^2(\mathbb{R}^2;\mathbb{R}^2)} \le Ca_{\mathbb{R}^2}(\boldsymbol{\psi},\boldsymbol{\psi}) = \boldsymbol{\xi}.\mathbf{Cap}_2(T;\mathbb{R}^2)\boldsymbol{\xi} = 0$*, hence* $\boldsymbol{\psi} = 0$*, in contradiction with the fact that* $\boldsymbol{\psi} = \boldsymbol{a}$ *in* T*. This lack of solution is similar to Stokes' paradox in fluid Mechanics [34].*

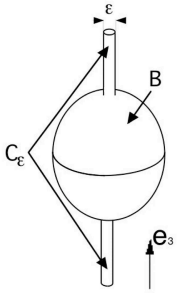

Fig. 1

4 Application to homogenization

Let Ω and T be bounded Lipschitz domains of $\mathbb{R}^N$. Given a sequence of positive reals (r_ε) such that $r_\varepsilon \ll \varepsilon$, we set (see fig. 2)

$$\begin{aligned} &T_{r_\varepsilon} := \bigcup_{i\in I_\varepsilon} T^i_{r_\varepsilon}; \quad T^i_{r_\varepsilon} := \varepsilon\boldsymbol{i} + r_\varepsilon T; \quad I_\varepsilon := \left\{ i \in \mathbb{Z}^N,\ Y^i_\varepsilon \subset \Omega \right\}; \\ &Y^i_\varepsilon := \varepsilon(\{\boldsymbol{i}\} + Y); \quad Y := \left(-\frac{1}{2},\frac{1}{2}\right)^N. \end{aligned} \tag{31}$$

We consider the problem of elastodynamics (2). The elasticity tensor $\boldsymbol{a}_\varepsilon$ and the mass density ρ_ε are supposed to take possibly large values in T_{r_ε} and constant values in the matrix $\Omega \setminus T_{r_\varepsilon}$. More precisely, we assume that

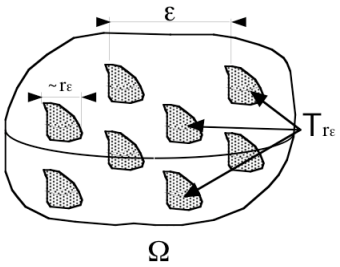

Fig. 2

$$\begin{cases} \rho_\varepsilon(x) = \rho\left(\dfrac{y_\varepsilon(x)}{r_\varepsilon}\right)\dfrac{\varepsilon^N}{r_\varepsilon^N|T|} & \text{if } x \in T_{r_\varepsilon}, \\ \rho_\varepsilon(x) = \rho_0 > 0 & \text{if } x \in \Omega \setminus T_{r_\varepsilon}, \\ \quad \rho \in C(\overline{T}), \quad \rho(y) > c > 0 \quad \forall y \in \overline{T}, \end{cases} \tag{32}$$

where

$$\boldsymbol{y}_\varepsilon(x) := \sum_{i\in\mathbb{Z}^N} 1_{Y_\varepsilon^i}(x)\,(\boldsymbol{x} - \varepsilon\boldsymbol{i})\,, \tag{33}$$

and that

$$\begin{aligned} &(\boldsymbol{a}_\varepsilon)_{ijkh} = (\boldsymbol{a}_\varepsilon)_{jikh} = (\boldsymbol{a}_\varepsilon)_{khij} \quad \forall (i,j,k,h) \in \{1,...,N\}^4, \\ &\boldsymbol{a}_\varepsilon(x)\boldsymbol{M} : \boldsymbol{M} \geq d_\varepsilon(x)|\boldsymbol{M}|^2 \quad \forall \boldsymbol{M} \in \mathbb{S}^N,\ \forall x \in \Omega, \qquad d_\varepsilon(x) > d > 0\ \forall x \in \Omega, \\ &\boldsymbol{a}_\varepsilon(x) = \boldsymbol{a}_0 \ \text{ in } \Omega \setminus T_{r_\varepsilon}, \qquad \lim_{\varepsilon\to 0} c_\varepsilon = +\infty \ \text{ if } N = 3, \qquad c_\varepsilon := \inf_{x\in T_{r_\varepsilon}} d_\varepsilon(x). \end{aligned} \tag{34}$$

We assume also that the elastic material constituting the matrix is isotropic, i. e. that $\boldsymbol{a}_0$ satisfies (18) (see Remark 3 (v)). The scalar $\overline{\rho}$, the vectors $\boldsymbol{y}_T$, $\boldsymbol{y}_G$ and the $N \times N$ symmetric matrix $\boldsymbol{J}^\rho$ defined by

$$\begin{aligned} &\overline{\rho} := \fint_T \rho dy, \quad \boldsymbol{y}_T := \fint_T \boldsymbol{y} dy, \quad \overline{\rho}\boldsymbol{y}_G := \fint_T \rho\boldsymbol{y} dy, \\ &J^\rho_{ij} := -\fint_T \rho(\boldsymbol{y} - \boldsymbol{y}_T)_i(\boldsymbol{y} - \boldsymbol{y}_T)_j dy \ \text{ if } \ i \neq j, \quad J^\rho_{ii} := \sum_{j\neq i} \fint_T \rho|(\boldsymbol{y} - \boldsymbol{y}_T)_j|^2 dy, \end{aligned} \tag{35}$$

characterize respectively the average mass density, the geometrical center of gravity, the center of mass and the mass-inertia matrix of the rescaled particle. We suppose that

$$T \text{ of class } C^3 \text{ if } N = 3, \qquad \int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{b}_0) : \boldsymbol{e}(\boldsymbol{b}_0) dx < C < +\infty, \tag{36}$$

and, without loss of generality, that

$$B \subset T. \tag{37}$$

Denoting by $\boldsymbol{u}_\varepsilon$ the solution of the problem (2), we introduce the auxiliary sequences $(\tilde{\boldsymbol{\omega}}_\varepsilon)$ and $(\tilde{\boldsymbol{v}}_\varepsilon)$ defined by (see Remark 5)

$$\begin{aligned}
\tilde{\boldsymbol{\omega}}_\varepsilon(x,t) &:= \sum_{i\in I_\varepsilon}\left(c(N)\frac{\mathrm{diam}T}{2}\fint_{\partial B^i_{r_\varepsilon}}\left(\frac{\boldsymbol{y}_\varepsilon(s)}{r_\varepsilon}\wedge\boldsymbol{u}_\varepsilon(s,t)\right)d\mathcal{H}^{N-1}(s)\right)1_{Y^i_\varepsilon}(x),\\
\tilde{\boldsymbol{v}}_\varepsilon(x,t) &:= \sum_{i\in I_\varepsilon}\left(\fint_{\partial B^i_{r_\varepsilon}}\boldsymbol{u}_\varepsilon(s,t)d\mathcal{H}^{N-1}(s)\right)1_{Y^i_\varepsilon}(x)-\tilde{\boldsymbol{\omega}}_\varepsilon(x,t)\wedge\boldsymbol{y}_T,
\end{aligned}\tag{38}$$

where $c(2)=1$, $c(3)=\frac{3}{2}$ and $B^i_{r_\varepsilon}$ is obtained by substituting B for T in (31). We show that the limiting problem depends on the parameter $\gamma^{(N)}$ defined by

$$\gamma^{(N)} := \lim_{\varepsilon\to 0}\gamma^{(N)}_\varepsilon \in [0,+\infty],\qquad \gamma^{(2)}_\varepsilon := \frac{1}{\varepsilon^2|\log r_\varepsilon|},\qquad \gamma^{(3)}_\varepsilon := \frac{r_\varepsilon}{\varepsilon^3}.\tag{39}$$

If $0<\gamma^{(N)}<+\infty$, we prove that $(\boldsymbol{u}_\varepsilon,\tilde{\boldsymbol{v}}_\varepsilon,\tilde{\boldsymbol{\omega}}_\varepsilon)$ converges, in the sense defined below, to the unique solution $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega})$ of the problem given, if $N=3$, by

$$\begin{cases}
\rho_0\dfrac{\partial^2\boldsymbol{u}}{\partial t^2}-\mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}))=\\
\quad\rho_0\boldsymbol{f}+\overline{\rho}\left(\boldsymbol{f}-\dfrac{\partial^2\boldsymbol{v}}{\partial t^2}-\dfrac{2}{\mathrm{diam}T}\dfrac{\partial^2\boldsymbol{\omega}}{\partial t^2}\wedge\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\right) & \text{in } \Omega\times(0,t_1),\\
\dfrac{\partial^2}{\partial t^2}\begin{pmatrix}\overline{\rho}\boldsymbol{v}+\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\\ \left(\frac{2}{\mathrm{diam}T}\right)^2\boldsymbol{J}^\rho\boldsymbol{\omega}+\frac{2}{\mathrm{diam}T}\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\wedge\boldsymbol{v}\end{pmatrix}\\
\quad=\begin{pmatrix}\overline{\rho}\boldsymbol{f}\\ \overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\wedge\boldsymbol{f}\end{pmatrix}-\gamma^{(3)}\mathbf{Cap}_3(T;\mathbb{R}^3)\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \boldsymbol{\omega}\end{pmatrix} & \text{in } \Omega\times(0,t_1),\\
(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\in\left(L^\infty(0,t_1;H^1_0(\Omega;\mathbb{R}^3))\times L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^3))^2\right)\\
\qquad\qquad\cap\left(C^1([0,t_1];L^2(\Omega;\mathbb{R}^3))\right)^3,\\
\boldsymbol{u}(0)=\boldsymbol{v}(0)=\boldsymbol{b}_0,\quad \dfrac{\partial\boldsymbol{u}}{\partial t}(0)=\dfrac{\partial\boldsymbol{v}}{\partial t}(0)=\boldsymbol{c}_0,\quad \boldsymbol{\omega}(0)=\dfrac{\partial\boldsymbol{\omega}}{\partial t}(0)=0,
\end{cases}\tag{40}$$

and, if $N=2$, by

$$\begin{cases}
\boldsymbol{\omega}=0 & \text{in } \Omega\times(0,t_1),\\
\rho_0\dfrac{\partial^2\boldsymbol{u}}{\partial t^2}-\mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}))=\rho_0\boldsymbol{f}+\overline{\rho}\left(\boldsymbol{f}-\dfrac{\partial^2\boldsymbol{v}}{\partial t^2}\right) & \text{in } \Omega\times(0,t_1),\\
\overline{\rho}\dfrac{\partial^2\boldsymbol{v}}{\partial t^2}=\overline{\rho}\boldsymbol{f}-\gamma^{(2)}\boldsymbol{M}_2(\boldsymbol{v}-\boldsymbol{u}) & \text{in } \Omega\times(0,t_1),\\
(\boldsymbol{u},\boldsymbol{v})\in\left(L^\infty(0,t_1;H^1_0(\Omega;\mathbb{R}^2))\times L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^2))\right)\\
\qquad\qquad\cap\left(C^1([0,t_1];L^2(\Omega;\mathbb{R}^2))\right)^2,\\
\boldsymbol{u}(0)=\boldsymbol{v}(0)=\boldsymbol{b}_0,\quad \dfrac{\partial\boldsymbol{u}}{\partial t}(0)=\dfrac{\partial\boldsymbol{v}}{\partial t}(0)=\boldsymbol{c}_0.
\end{cases}\tag{41}$$

where $\boldsymbol{M}_2$ is defined by (21). The following result is partly announced in [8].

Theorem 1. *Assume (18), (32)-(36), and* $0 < \gamma^{(N)} < +\infty$. *Let* $\boldsymbol{u}_\varepsilon$ *be the solution of (2) and let* $\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\boldsymbol{\omega}}_\varepsilon$ *be defined by (38). Then* $(\boldsymbol{u}_\varepsilon)$ *converges weak-star in* $L^\infty(0,t_1;H^1_0(\Omega;\mathbb{R}^N))$ *and strongly in* $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$ *to* $\boldsymbol{u}$ *and* $(\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\boldsymbol{\omega}}_\varepsilon)$ *converges weak-star in* $(L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N)))^2$ *to* $(\boldsymbol{v},\boldsymbol{\omega})$. *If* $N=3$, $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega})$ *is the unique solution of (40). If* $N=2$, $\boldsymbol{\omega}=0$ *and* $(\boldsymbol{u},\boldsymbol{v})$ *is the unique solution of (41).*

Remark 3. **(i)** *The conclusions of Theorem 1 can be extended to the cases* $\gamma_N \in \{0,+\infty\}$ *(see Section 7.4):*

- If $\gamma^{(N)} = +\infty$ *and if*

$$r_\varepsilon^2 << \varepsilon^3 \quad \text{if} \quad N=3 \quad \text{and} \quad r_\varepsilon << \varepsilon^2 \quad \text{if} \quad N=2, \tag{42}$$

then $\boldsymbol{u}_\varepsilon$ *converges weak-star in* $L^\infty(0,t_1;H^1_0(\Omega;\mathbb{R}^N))$ *and strongly in* $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$ *to the solution of*

$$\begin{cases} (\rho_0+\overline{\rho})\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u})) = (\rho_0+\overline{\rho})\boldsymbol{f} & \text{in} \quad \Omega\times(0,t_1), \\ \boldsymbol{u}\in L^\infty(0,t_1;H^1_0(\Omega;\mathbb{R}^N))\cap C^1([0,t_1];L^2(\Omega;\mathbb{R}^N)),\ \boldsymbol{u}(0)=\boldsymbol{b}_0,\ \dfrac{\partial \boldsymbol{u}}{\partial t}(0)=\boldsymbol{c}_0. \end{cases} \tag{43}$$

In this case, the sequences $(\tilde{\boldsymbol{v}}_\varepsilon)$ *and* $(\tilde{\boldsymbol{\omega}}_\varepsilon)$ *converge strongly in* $(L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N)))^2$ *respectively to* $\boldsymbol{u}$ *and to* 0.

-If $\gamma^{(N)}=0$, *then* $\boldsymbol{u}_\varepsilon$ *converges weak-star in* $L^\infty(0,t_1;H^1_0(\Omega;\mathbb{R}^N))$ *and strongly in* $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$ *to the solution of*

$$\begin{cases} \rho_0\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u})) = \rho_0\boldsymbol{f} & \text{in} \quad \Omega\times(0,t_1), \\ \boldsymbol{u}\in L^\infty(0,t_1;H^1_0(\Omega;\mathbb{R}^N))\cap C^1([0,t_1];L^2(\Omega;\mathbb{R}^N)),\ \boldsymbol{u}(0)=\boldsymbol{b}_0,\ \dfrac{\partial \boldsymbol{u}}{\partial t}(0)=\boldsymbol{c}_0. \end{cases}$$

(ii) *(Memory effects). Assume for simplicity* $N=3$, $T=B$, $0<\gamma^{(3)}<+\infty$, *and that* ρ *is constant (that is* $\rho=\overline{\rho}$*). Then by (35) there holds* $\boldsymbol{y}_G=\boldsymbol{y}_T=0$. *We deduce from (19) and from the second equation of (40) that*

$$\boldsymbol{J}^\rho\frac{\partial^2\boldsymbol{\omega}}{\partial t^2}+8\pi\mu_0\gamma^{(3)}\boldsymbol{\omega}=0, \quad \text{in} \quad \Omega\times(0,t_1), \qquad \boldsymbol{\omega}(0)=\frac{\partial\boldsymbol{\omega}}{\partial t}(0)=0,$$

therefore $\boldsymbol{\omega}=0$ *and* $\boldsymbol{v}$ *satisfies*

$$\overline{\rho}\frac{\partial^2\boldsymbol{v}}{\partial t^2}+\gamma^{(3)}\chi(\boldsymbol{v}-\boldsymbol{u})=\overline{\rho}\boldsymbol{f} \quad \text{in} \quad \Omega\times(0,t_1), \quad \boldsymbol{v}(0)=\boldsymbol{b}_0, \quad \frac{\partial\boldsymbol{v}}{\partial t}(0)=\boldsymbol{c}_0, \tag{44}$$

where (cf. (19)) $\chi := \dfrac{12\pi\mu_0(\lambda_0+2\mu_0)}{(2\lambda_0+5\mu_0)}$. *Setting* $\delta := \sqrt{\chi\dfrac{\gamma^{(3)}}{\overline{\rho}}}$, *we find*

$$\boldsymbol{v}(x,t)=\int_0^t\frac{\sin\delta(t-\tau)}{\delta}\Big(\boldsymbol{f}(x,\tau)+\delta^2\boldsymbol{u}(x,\tau)\Big)\,d\tau+\boldsymbol{c}_0(x)\frac{\sin\delta t}{\delta}+\boldsymbol{b}_0(x)\cos\delta t. \tag{45}$$

Subtracting (44) from the first equation of (40), we get

$$\rho_0\frac{\partial^2\boldsymbol{u}}{\partial t^2}-\mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}))-\gamma^{(3)}\chi(\boldsymbol{v}-\boldsymbol{u})=\rho_0\boldsymbol{f}. \tag{46}$$

After substitution of (45) in (46), we deduce that $\boldsymbol{u}$ *satisfies the equation*

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}(\boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u})) + \overline{\rho}\delta^2 \left(\boldsymbol{u} - \delta \int_0^t \sin(\delta(t-\tau)) \boldsymbol{u}(\tau) d\tau \right)$$
$$= \rho_0 \boldsymbol{f} + \overline{\rho}\delta \int_0^t \sin(\delta(t-\tau)) \boldsymbol{f}(\tau) d\tau + \overline{\rho}\delta \boldsymbol{c}_0(x) \sin(\delta t) + \overline{\rho}\delta^2 \boldsymbol{b}_0(x) \cos(\delta t),$$

where the memory term "$-\overline{\rho}\delta^3 \int_0^t \sin(\delta(t-\tau)) \boldsymbol{u}(\tau) d\tau$*" emerges.*

(iii) *The total mechanical energy stored in the composite at the instant* τ *is given by (see (96), (117), (118), (144))*

$$e(\tau) = \frac{1}{2} \int_\Omega \rho_0 \left| \frac{\partial \boldsymbol{u}}{\partial t} \right|^2 (\tau) dx$$
$$+ \frac{1}{|T|} \int_{\Omega \times T} \rho \left| \frac{\partial \boldsymbol{v}}{\partial t} + \frac{2}{\mathrm{diam} T} \frac{\partial \boldsymbol{\omega}}{\partial t} \wedge (\boldsymbol{y} - \boldsymbol{y}_T) \right|^2 (\tau) dx dy + \Phi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega}).$$

where, if $N = 3$*,*

$$\Phi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega}) := \frac{1}{2} \int_\Omega \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{u})(\tau) dx$$
$$+ \frac{1}{2} \gamma^{(3)} \int_\Omega \begin{pmatrix} \boldsymbol{v} - \boldsymbol{u} \\ \boldsymbol{\omega} \end{pmatrix} . \mathbf{Cap}_3(T; \mathbb{R}^3) \begin{pmatrix} \boldsymbol{v} - \boldsymbol{u} \\ \boldsymbol{\omega} \end{pmatrix} (\tau) dx, \tag{47}$$

and, if $N = 2$*,*

$$\Phi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega}) := \frac{1}{2} \int_\Omega \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{u}) dx + \frac{1}{2} \gamma^{(2)} \int_\Omega (\boldsymbol{v} - \boldsymbol{u}) . \boldsymbol{M}_2 (\boldsymbol{v} - \boldsymbol{u}) dx \qquad \text{if } \boldsymbol{\omega} = 0,$$
$$\Phi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega}) := +\infty \qquad \text{otherwise.} \tag{48}$$

The second term of Φ *represents the concentration of strain energy, stored in a small zone enveloping the particles, generated by the discrepancy between the effective displacement in the particles and the effective displacement in the matrix.*

(iv) *The choice of the parameter* $\frac{2}{\mathrm{diam} T}$ *in (1) may be inappropriate if the particles have a complicated shape. For instance, in the case of a set* T_ε *consisting of three-dimensional needle-shaped particles parallel to one of the coordinate axes, the variant of* $\mathbf{Cap}_3$ *deduced from (1) by replacing the Dirichlet condition on* T *by*

$$\boldsymbol{\psi} = \boldsymbol{a} + \sum_{i=1,2} \frac{2}{\mathrm{diam} P_i(T)} b_i \boldsymbol{e}_i \wedge (\boldsymbol{x} - \boldsymbol{x}_T),$$

where P_i *denotes the orthogonal projection on the axis* $\mathbb{R}\boldsymbol{e}_i$*, should rather be considered. If* r_ε *denotes the length of the "needles", say* $r_\varepsilon = \mathrm{diam} P_3(T_\varepsilon^i)$ *and* $\alpha_\varepsilon r_\varepsilon$*,* $\beta_\varepsilon r_\varepsilon$ *characterize the size of their cross-sections (*$\alpha_\varepsilon r_\varepsilon = \mathrm{diam} P_1(T_\varepsilon^i)$*,* $\beta_\varepsilon r_\varepsilon = \mathrm{diam} P_2(T_\varepsilon^i)$*), then for each choice of the sequence* (r_ε) *(such that* $c\varepsilon^3 \leq r_\varepsilon < \frac{\varepsilon}{2}$*), there exists presumably several critical sizes of the parameter,* α_ε*,* β_ε *for which some specific microscopic rigid displacements of the particles should induce the emergence of a concentration of strain energy in their neighborhood .*

(v) *The results stated in Theorem 1 are likely to hold true in the anisotropic case: the assumption (18) is used only in Lemma 11 if* $N = 3$ *(resp. in Lemma 7 and in the proof of (146) if* $N = 2$*). However, they may fail to hold if the hypothesis* $\lim_{\varepsilon \to 0} c_\varepsilon = +\infty$ *if* $N = 3$*, stated in (34) and used to prove (93), (138), is not satisfied. In this case, we expect a concentration of strain energy stored inside the particles to emerge in the effective problem.*

5 Variants of the homogenization results

In this section, we comment the elliptic equations associated with (2), the Dirichlet problems in varying domains, the scalar case, the fibered case, and the case of particles distributed on a surface.

5.1 Stationary case

Assuming (31), (32), (34), $\gamma^{(N)} > 0$, we consider the sequence of elliptic problems

$$-\mathbf{div}(\boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon)) = \rho_\varepsilon \boldsymbol{f} \quad \text{in } \Omega,\ \boldsymbol{u}_\varepsilon \in H_0^1(\Omega;\mathbb{R}^N),\ (\boldsymbol{f} \in C(\overline{\Omega};\mathbb{R}^N)). \tag{49}$$

By mimicking the proof of Theorem 1, it is easy to prove that the sequence $(\boldsymbol{u}_\varepsilon)$ of the solution of (49) converges weakly in $H_0^1(\Omega;\mathbb{R}^N)$ to the unique solution $\boldsymbol{u}$ of

$$-\mathbf{div}(\boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u})) = (\rho_0 + \overline{\rho})\boldsymbol{f} \quad \text{in} \quad \Omega, \quad \boldsymbol{u} \in H_0^1(\Omega;\mathbb{R}^N). \tag{50}$$

The seeming simplicity of (50) covers the complex behavior of the displacement in the particles. Indeed, the sequence $(\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\boldsymbol{\omega}}_\varepsilon)$ defined by (38) converges weakly to $(\boldsymbol{v}, \boldsymbol{\omega})$ in $(L^2(\Omega;\mathbb{R}^N))^2$, where $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega})$ is the unique solution in $H_0^1(\Omega;\mathbb{R}^N) \times (L^2(\Omega;\mathbb{R}^N))^2$ of the problem deduced formally from (40), (41), (43), by substituting 0 for the derivatives with respect to t. For instance, if $N = 3$ and $0 < \gamma^{(3)} < +\infty$, we obtain the system of equations

$$\begin{cases} -\mathbf{div}(\boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u})) = (\rho_0 + \overline{\rho})\boldsymbol{f} & \text{in} \quad \Omega, \\ \gamma^{(3)} \mathbf{Cap}_3(T;\mathbb{R}^3) \begin{pmatrix} \boldsymbol{v} - \boldsymbol{u} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \overline{\rho}\boldsymbol{f} \\ \overline{\rho}(\boldsymbol{y}_G - \boldsymbol{y}_T) \wedge \boldsymbol{f} \end{pmatrix} & \text{in} \quad \Omega, \\ (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in H_0^1(\Omega;\mathbb{R}^3) \times (L^2(\Omega;\mathbb{R}^3))^2, \end{cases} \tag{51}$$

associated with the minimization problem

$$\min_{(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega}) \in H_0^1(\Omega;\mathbb{R}^3) \times (L^2(\Omega,\mathbb{R}^3))^2} \Phi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega}) - L(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega}),$$

where Φ is defined by (47) and $L(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega}) := -\int_\Omega \rho_0 \boldsymbol{f}.\boldsymbol{u} + \overline{\rho}\boldsymbol{f}.\boldsymbol{v} + (\overline{\rho}(\boldsymbol{y}_G - \boldsymbol{y}_T) \wedge \boldsymbol{f}).\boldsymbol{\omega} dx$. As the matrix $\mathbf{Cap}_3(T;\mathbb{R}^3)$ is invertible (see Lemma 2), we deduce from (51) that

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u} \\ 0 \end{pmatrix} + \frac{1}{\gamma^{(3)}} (\mathbf{Cap}_3(T;\mathbb{R}^3))^{-1} \begin{pmatrix} \overline{\rho}\boldsymbol{f} \\ \overline{\rho}(\boldsymbol{y}_G - \boldsymbol{y}_T) \wedge \boldsymbol{f} \end{pmatrix}.$$

If $N = 2$, then $\boldsymbol{\omega} = 0$ and the effective problem takes the form

$$\min_{(\boldsymbol{u},\boldsymbol{v}) \in H_0^1(\Omega;\mathbb{R}^2) \times L^2(\Omega,\mathbb{R}^2)} \Phi(\boldsymbol{u}, \boldsymbol{v}, 0) - \int_\Omega \rho_0 \boldsymbol{f}.\boldsymbol{u} + \overline{\rho}\boldsymbol{f}.\boldsymbol{v} dx,$$

being Φ given by (48), yielding $\boldsymbol{v} = \boldsymbol{u} + \frac{1}{\gamma^{(2)}} (\boldsymbol{M}_2)^{-1} (\overline{\rho}\boldsymbol{f})$. The intricate behavior of the composite at a microscopic scale is *only* revealed, in (50), by the presence of the term $\overline{\rho}\boldsymbol{f}$.

5.2 Dirichlet problems in varying domains

We consider the sequence of Dirichlet problems in the perforated domain $\Omega \setminus T_{r_\varepsilon}$,

$$\begin{cases} -\mathbf{div}(\boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon)) = \rho_\varepsilon \boldsymbol{f} & \text{in } \Omega \setminus T_{r_\varepsilon}, \\ \boldsymbol{u}_\varepsilon \in H_0^1(\Omega;\mathbb{R}^N), \quad \boldsymbol{u}_\varepsilon = 0 & \text{in } T_{r_\varepsilon}. \end{cases} \tag{52}$$

Corollary 1. *Under (18), (31)-(36), the sequence* $(\boldsymbol{u}_\varepsilon)$ *of the solution of (52) converges weakly in* $H_0^1(\Omega;\mathbb{R}^N)$ *to the unique solution in* $H_0^1(\Omega;\mathbb{R}^N)$ *of*

$$\begin{aligned} &-\mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u})) + \gamma^{(3)}\boldsymbol{M}_3(T;\mathbb{R}^3)\boldsymbol{u} = \rho_0\boldsymbol{f} && \text{in } \Omega, \text{ if } N=3,\ 0<\gamma^{(3)}<+\infty, \\ &-\mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u})) + \gamma^{(2)}\boldsymbol{M}_2\boldsymbol{u} = \rho_0\boldsymbol{f} && \text{in } \Omega, \text{ if } N=2,\ 0<\gamma^{(2)}<+\infty, \\ &-\mathbf{div}(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u})) = \rho_0\boldsymbol{f} && \text{in } \Omega, \text{ if } \gamma^{(N)}=0, \\ &\boldsymbol{u} = 0 && \text{in } \Omega, \text{ if } \gamma^{(N)}=+\infty, \end{aligned} \tag{53}$$

where $\boldsymbol{M}_3(T;\mathbb{R}^3)$ denotes the upper left 3×3 submatrix of $\mathbf{Cap}_3(T;\mathbb{R}^3)$ and $\boldsymbol{M}_2$ is defined by (21). This result is obtained simply by substituting 0 for $\boldsymbol{\psi}$ and $\boldsymbol{\zeta}$ in the sequence of test functions used in the proof of Theorem 1 (see (119), (139)). The terms $\gamma^{(3)}\boldsymbol{M}_3(T;\mathbb{R}^3)\boldsymbol{u}$ and $\gamma^{(2)}\boldsymbol{M}_2\boldsymbol{u}$ emerging in (53) are analogous to the so-called "strange term" obtained by D. Cioranescu and F. Murat [16] in the context of diffusion equations and to the linear zero-order term for the velocity in the Brinkman's law obtained by G. Allaire in the homogenization of the Stokes and of the Navier-Stokes equations in a domain containing many tiny solid obstacles [3].

5.3 Scalar case

We consider the scalar evolution equation in $L^2(\Omega\times(0,t_1);H_0^1(\Omega))$

$$\begin{aligned} &\rho_\varepsilon \frac{\partial^n u_\varepsilon}{\partial t^n} - \operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = \rho_\varepsilon f \quad \text{in } \Omega\times(0,t_1), \\ &+ \quad \text{initial boundary conditions } (n\in\{1,2\}). \end{aligned}$$

We assume that $a_\varepsilon(x)=1$ in $\Omega\setminus T_{r_\varepsilon}$, $a_\varepsilon(x)>c_\varepsilon>c>0$ in T_{r_ε}, where $\lim_{\varepsilon\to 0} c_\varepsilon = +\infty$ if $N=3$. The results obtained in [6] in the case of three-dimensional spherical particles can be easily extended to the case of particles homothetical to an arbitrary Lipschitz bounded domain T. The effective equations depend then on the parameter $c_N(T) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\operatorname{cap}_N(r_\varepsilon T,\Omega)$, where $\operatorname{cap}_N(T;\Omega)$ denotes the harmonic capacity of T with respect to Ω. We find $c_3(T)=\gamma^{(3)}\operatorname{cap}_3(T;\mathbb{R}^3)$ and $c_2(T)=\gamma^{(2)}2\pi$, being $\gamma^{(N)}$ defined by (39). If $0<\gamma^{(N)}<+\infty$, we obtain an effective system of equations of the type

$$\begin{aligned} &\rho_0\frac{\partial^n u}{\partial t^n} - \Delta u = \rho_0 f + \overline{\rho}\left(f - \frac{\partial^n v}{\partial t^n}\right) && \text{in} \quad \Omega\times(0,t_1), \\ &\overline{\rho}\frac{\partial^n v}{\partial t^n} = \overline{\rho} f - c_N(T)(v-u) && \text{in} \quad \Omega\times(0,t_1), \\ &+ \text{ initial boundary conditions,} \end{aligned}$$

analogous to (40).

5.4 Fibered case

We consider a cylindrical domain $\Omega := \Upsilon \times (0, L)$ of $\mathbb{R}^3$. Given a bounded Lipschitz domain S of $\mathbb{R}^2$, we define the ε-periodic distribution of parallel "fibers" T_{r_ε} of cross section of size r_ε homothetical to S by setting

$$T_{r_\varepsilon} = S_{r_\varepsilon} \times (0, L), \quad S_{r_\varepsilon} = \bigcup_{i \in I_\varepsilon} \varepsilon i + r_\varepsilon S, \quad I_\varepsilon := \{i \in \mathbb{Z}^2,\ Y_\varepsilon^i \subset \Upsilon\}. \tag{54}$$

Under (18), (32)-(36), (54), the problem (2) has been studied by the author with I. Gruais in [10], assuming that the material constituting the fibers is homogeneous and isotropic (the Lamé coefficients taking constant values $\mu_{1\varepsilon}, \lambda_{1\varepsilon}$ in T_{r_ε}) and that the rescaled cross section S is a disk. These results can be extended easily to the case of an arbitrary Lipschitz domain S. The effective problem, which depends partly on the asymptotic behavior of $\left(\frac{1}{\varepsilon^2}\mathbf{Cap}_2(r_\varepsilon S, \Upsilon)\right)$ (see (21), (30)), is then given in the elliptic case by

$$\min_{(\boldsymbol{u},\boldsymbol{v}) \in (L^2(\Omega;\mathbb{R}^3))^2} \overline{\Phi}(\boldsymbol{u}, \boldsymbol{v}) - \int_\Omega \rho_0 \boldsymbol{f}.\boldsymbol{u} + \overline{\rho}\boldsymbol{f}.\boldsymbol{v} dx, \tag{55}$$

the symbol $\overline{\Phi}$ denoting the lower semi-continuous envelop in the strong topology of $(L^2(\Omega;\mathbb{R}^3))^2$ of the functional

$$\begin{aligned}
\Phi(\boldsymbol{u}, \boldsymbol{v}) &:= \frac{1}{2}\int_\Omega \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{u}) dx + \frac{1}{2}\gamma^{(2)} 4\pi\mu_0 \frac{\lambda_0 + 2\mu_0}{\lambda_0 + 3\mu_0} \int_\Omega \sum_{\alpha=1}^{2} |v_\alpha - u_\alpha|^2 dx \\
&\quad + \frac{1}{2}\gamma^{(2)} 2\mu_0 \pi \int_\Omega |v_3 - u_3|^2 dx + \Phi_{fibers}(\boldsymbol{v}), \qquad \text{if } (\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{D}, \\
\Phi(\boldsymbol{u}, \boldsymbol{v}) &:= +\infty, \qquad \text{otherwise,} \\
\mathcal{D} &:= H_0^1(\Omega, \mathbb{R}^3) \times \left\{ \boldsymbol{v} \in L^2(\Omega, H_0^2(0, L; \mathbb{R}^3)),\ \frac{\partial v_1}{\partial x_3} = \frac{\partial v_2}{\partial x_3} = 0 \ \text{ in } \Omega \times \{0, L\} \right\}.
\end{aligned}$$

where the functional Φ_{fibers}, which describes the strain energy stored in the fibers and is the only part of Φ depending on S, is given in terms of the limit $\boldsymbol{v}$ of the sequence $(\tilde{\boldsymbol{v}}_\varepsilon)$ defined by (38) by

$$\begin{aligned}
&\Phi_{fibers}(\boldsymbol{v}) = \frac{1}{2}\frac{3l+2}{2(l+1)} k|S| \int_\Omega \left|\frac{\partial v_3}{\partial x_3}\right|^2 dx + \frac{1}{2}\sum_{\alpha,\beta=1}^{2} \kappa|S| \frac{3l+2}{l+1} J_{\alpha\beta} \int_\Omega \frac{\partial^2 v_\alpha}{\partial x_3^2}\frac{\partial^2 v_\beta}{\partial x_3^2} dx, \\
&J_{\alpha\beta} := \fint_S (\boldsymbol{y} - \boldsymbol{y}_S)_\alpha (\boldsymbol{y} - \boldsymbol{y}_S)_\beta dy, \\
&k := \lim_{\varepsilon\to 0} r_\varepsilon^2 \mu_{1\varepsilon},\ \kappa := \lim_{\varepsilon\to 0} r_\varepsilon^4 \mu_{1\varepsilon},\ l := \lim_{\varepsilon\to 0} \frac{\lambda_{1\varepsilon}}{\mu_{1\varepsilon}}.
\end{aligned}$$

Moreover, the sequence $(\tilde{\boldsymbol{\omega}}_\varepsilon.\boldsymbol{e}_3)$, where $\tilde{\boldsymbol{\omega}}_\varepsilon(., x_3)$ is defined by (38) on each section $x_3 = const$ (setting $N = 2$), converges to zero: this means that the rescaled effective angle of rotation of the fibers is equal to zero.

Remark 4. The lack of torsion effects in the effective problem (55) is a consequence of the specific behavior of $\mathbf{Cap}_2$, whose singularity originates, in particular, in the fact that Gagliardo-Nirenberg-Sobolev inequality fails to hold in $H^1(\mathbb{R}^2)$ (see the paragraph preceding Lemma 2). Therefore torsion effects are likely to take place if the strain

energy density function of the material constituting the matrix has a growth of order $p \in (1,2)$. In linear elasticity, torsion effects are actually obtained in a quite different context, namely, when an ε-periodic distribution of fibers of cross section of size of order ε and of stiffness of order 1, embedded in a very soft matrix of stiffness of order ε^2 is considered (see [7]).

5.5 Three-dimensional particles periodically distributed on a surface

We suppose that the set T_{r_ε} consists of three-dimensional particles of size r_ε, ε-periodically distributed on the portion of hyperplane $\Sigma := \Omega \cap \{x_1 = 0\}$ (see fig. 3). More precisely, setting

$$
\begin{aligned}
&T_{r_\varepsilon} = \bigcup_{i \in J_\varepsilon} T^i_{r_\varepsilon},\ T^i_{r_\varepsilon} = \varepsilon(0, \boldsymbol{i}) + r_\varepsilon T,\ J_\varepsilon = \{i \in \mathbb{Z}^2,\ P^i_\varepsilon \subset \Sigma\},\\
&P^i_\varepsilon = \{0\} \times \varepsilon\left(\boldsymbol{i} + \left(-\frac{1}{2}, \frac{1}{2}\right)^2\right),\ \Omega^- = \Omega \cap \{x_1 < 0\}, \quad \Omega^+ = \Omega \cap \{x_1 > 0\}, \quad (56)\\
&\rho_\varepsilon(x) = \rho_0 1_{\Omega \setminus T_{r_\varepsilon}} + \rho\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right) \frac{\varepsilon^2}{r_\varepsilon^3 |T|} 1_{T_{r_\varepsilon}}(x),
\end{aligned}
$$

we consider the problem (2) under the assumptions (18), (34), (36), (56). The critical case corresponds then to particles of diameter of order ε^2. It is easy to prove, by adapting the argument of the proof of Theorem 1, that if $0 < \gamma := \lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon^2} \leq +\infty$, then the solution $\boldsymbol{u}_\varepsilon$ of (2) converges weak-star in $L^\infty(0, t_1; H^1_0(\Omega; \mathbb{R}^3))$ and strongly in $L^\infty(0, t_1; L^2(\Omega; \mathbb{R}^3))$ to $\boldsymbol{u}$ and the sequence $(\tilde{\boldsymbol{v}}_\varepsilon, \tilde{\boldsymbol{\omega}}_\varepsilon)$ deduced from (38) by substituting $1_{P^i_\varepsilon}(x)$ for $1_{Y^i_\varepsilon}(x)$ (see (56)), converges weak-star in $(L^\infty(0, t_1; L^2(\Sigma; \mathbb{R}^3)))^2$ to $(\boldsymbol{v}, \boldsymbol{\omega})$, where $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega})$ is the unique solution of the system of equations

$$
\begin{cases}
\dfrac{\partial^2}{\partial t^2}\begin{pmatrix} \overline{\rho}\boldsymbol{v} + \frac{2}{\mathrm{diam}T}\boldsymbol{\omega} \wedge \overline{\rho}(\boldsymbol{y}_G - \boldsymbol{y}_T) \\ \left(\frac{2}{\mathrm{diam}T}\right)^2 \boldsymbol{J}^\rho \boldsymbol{\omega} + \frac{2}{\mathrm{diam}T}\overline{\rho}(\boldsymbol{y}_G - \boldsymbol{y}_T) \wedge \boldsymbol{v} \end{pmatrix} \\
\qquad = \begin{pmatrix} \overline{\rho}\boldsymbol{f} \\ \overline{\rho}(\boldsymbol{y}_G - \boldsymbol{y}_T) \wedge \boldsymbol{f} \end{pmatrix} - \gamma \mathbf{Cap}_3(T; \mathbb{R}^3) \begin{pmatrix} \boldsymbol{v} - \boldsymbol{u} \\ \boldsymbol{\omega} \end{pmatrix} \qquad \text{on } \Sigma \times (0, t_1), \\
\rho_0 \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}\boldsymbol{\sigma}_0 = \rho_0 \boldsymbol{f} \qquad \text{in } (\Omega^- \cup \Omega^+) \times (0, t_1), \qquad \boldsymbol{\sigma}_0 = \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u}), \\
(\boldsymbol{\sigma}_0^- - \boldsymbol{\sigma}_0^+)\boldsymbol{\nu} = \overline{\rho}\boldsymbol{f} - \overline{\rho}\dfrac{\partial^2}{\partial t^2}\left(\boldsymbol{v} + \dfrac{2}{\mathrm{diam}T}\boldsymbol{\omega} \wedge (\boldsymbol{y}_G - \boldsymbol{y}_T)\right) \qquad \text{on } \Sigma \times (0, t_1),
\end{cases}
$$

associated to the initial boundary conditions given by

$$
\begin{cases}
(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in L^\infty(0, t_1; H^1_0(\Omega; \mathbb{R}^3)) \times \left(L^\infty(0, t_1; L^2(\Sigma; \mathbb{R}^3))\right)^2 \\
\qquad\qquad \cap\, C^1([0, t_1]; L^2(\Omega; \mathbb{R}^3) \times \left(C^1([0, t_1]; L^2(\Sigma; \mathbb{R}^3))\right)^3, \\
\boldsymbol{u}(0) = \boldsymbol{v}(0) = \boldsymbol{b}_0, \quad \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = \dfrac{\partial \boldsymbol{v}}{\partial t}(0) = \boldsymbol{c}_0, \quad \boldsymbol{\omega}(0) = \dfrac{\partial \boldsymbol{\omega}}{\partial t}(0) = 0,
\end{cases}
$$

being $\boldsymbol{\sigma}_0^-$ (resp. $\boldsymbol{\sigma}_0^+$) the restriction of $\boldsymbol{\sigma}_0 = \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u})$ to Ω^- (resp. Ω^+), and $\boldsymbol{\nu}$ the outward pointing normal to Ω^- (hence $\boldsymbol{\nu} = \boldsymbol{e}_1$ on Σ). The vector field $(\boldsymbol{\sigma}_0^- - \boldsymbol{\sigma}_0^+)\boldsymbol{\nu}$ describes the density of the surface forces exerted by the particles on the matrix.

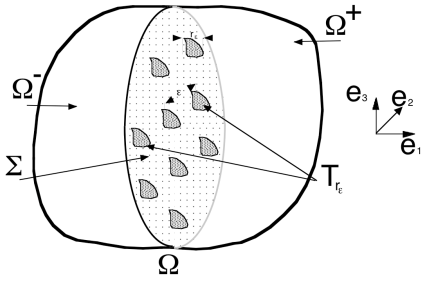

Fig. 3

6 Technical preliminaries and a priori estimates

This section is devoted mainly to the study of the asymptotic behavior of the sequence $(\boldsymbol{u}_\varepsilon)$ of the solutions of (2) and of the auxiliary sequences $(\tilde{\boldsymbol{v}}_\varepsilon)$ and $(\tilde{\boldsymbol{\omega}}_\varepsilon)$ given by (38). As a means to particularize the oscillatory behavior of the displacement in the inclusions, we will use the following variant of the two-scale convergence of Allaire [2] and Nguetseng [31]. Let (m_ε) be the sequence of measures defined by

$$m_\varepsilon := \frac{\varepsilon^N}{r_\varepsilon^N |T|} \mathcal{L}^N \lfloor_{T_{r_\varepsilon}}, \tag{57}$$

where T_{r_ε} is given by (31). A sequence (f_ε) in $L^2(0,t_1;L^2(\Omega))$ will be said to two-scale converge with respect to (m_ε) to $f_0 \in L^2(0,t_1;L^2(\Omega \times T))$ if, for each $\psi \in \mathcal{D}(\Omega \times (0,t_1) \times Y)$, there holds

$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} f_\varepsilon(x,t) \psi\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right) dm_\varepsilon(x) dt = \frac{1}{|T|} \int_{\Omega \times (0,t_1) \times T} f_0 \psi dx dt dy, \tag{58}$$

where $y_\varepsilon(x)$ is given by (33). This convergence will be denoted $f_\varepsilon \overset{m_\varepsilon}{\rightharpoonup} f_0$. We have

Lemma 9. *(i) Let (f_ε) be a sequence in $L^2(0,t_1;L^2(\Omega))$ such that $\sup_{t\in(0,t_1)} \int |f_\varepsilon|^2(t)$ $dm_\varepsilon \le C$. Then (f_ε) two-scale converges with respect to (m_ε), up to a subsequence, to some $f_0 \in L^\infty(0,t_1;L^2(\Omega \times T))$. Furthermore, if $f_\varepsilon \overset{m_\varepsilon}{\rightharpoonup} f_0$, then*

$$\liminf_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} |f_\varepsilon|^2 dm_\varepsilon dt \ge \frac{1}{|T|} \int_{\Omega \times T \times (0,t_1)} |f_0|^2 dx dt dy. \tag{59}$$

(ii) Assume in addition that $f_\varepsilon \overset{m_\varepsilon}{\rightharpoonup} f_0$, $\frac{\partial f_\varepsilon}{\partial t} \in L^2(0,t_1;L^2(\Omega))$ and that $\sup_{t\in(0,t_1)} \int$ $|\frac{\partial f_\varepsilon}{\partial t}|^2(t) dm_\varepsilon \le C$. Then $f_0 \in W^{1,\infty}(0,t_1;L^2(\Omega \times T))$ and $\left(\frac{\partial f_\varepsilon}{\partial t}\right)$ two-scale converges with respect to (m_ε) to $\frac{\partial f_0}{\partial t}$.

Proof. (i) Let ν_ε be the measure on $\overline{\Omega \times (0,t_1) \times T}$ defined by $\int \psi d\nu_\varepsilon := \int_0^{t_1} \int f_\varepsilon(x,t)$ $\psi\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right) dm_\varepsilon(x) dt \ \forall \psi \in C(\overline{\Omega \times (0,t_1) \times T})$. Cauchy-Schwartz inequality yields

$$\begin{aligned} \left| \int \psi d\nu_\varepsilon \right| &\le \int_0^{t_1} dt \left(\int_\Omega |f_\varepsilon(x,t)|^2 \, dm_\varepsilon(x) \right)^{\frac{1}{2}} \left(\int_\Omega \left| \psi\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right) \right|^2 dm_\varepsilon(x) \right)^{\frac{1}{2}} \\ &\le C \int_0^{t_1} dt \left(\int_\Omega \left| \psi\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right) \right|^2 dm_\varepsilon(x) \right)^{\frac{1}{2}} \le C |\psi|_{L^\infty(\Omega \times (0,t_1) \times T)}, \end{aligned} \tag{60}$$

hence the sequence (ν_ε) is bounded in $\mathcal{M}(\overline{\Omega\times(0,t_1)\times T})$ and converges weak-star, up to a subsequence, to some $\nu\in\mathcal{M}(\overline{\Omega\times(0,t_1)\times T})$. By passing to the limit as $\varepsilon\to 0$ in the first and third terms of (60), thanks to the dominated convergence theorem we get

$$\left|\int\psi d\nu\right|\le C|\psi|_{L^1(0,t_1;L^2(\Omega\times T))}\quad\forall\psi\in C(\overline{\Omega\times(0,t_1)\times T}). \tag{61}$$

By (61), the linear form $\psi\in C(\overline{\Omega\times(0,t_1)\times T})\to\int\psi d\nu$ can be extended by density to a continuous linear form on $L^1(0,t_1;L^2(\Omega\times T))$. We deduce that $\nu=\frac{1}{|T|}f_0$ for a suitable $f_0\in L^\infty(0,t_1;L^2(\Omega\times T))$. Then, by the weak-star convergence of (ν_ε) to ν, the sequence (f_ε) two-scale converges with respect to (m_ε) to f_0. Moreover,

$$\begin{aligned}\liminf_{\varepsilon\to0}\int_0^{t_1}\int|f_\varepsilon|^2dm_\varepsilon(x)dt&\ge\lim_{\varepsilon\to0}\int_0^{t_1}\int 2f_\varepsilon\psi\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right)-\left|\psi\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right)\right|^2dm_\varepsilon(x)dt\\&=\frac{1}{|T|}\int_{\Omega\times(0,t_1)\times T}2f_0\psi-\psi^2dxdtdy,\end{aligned}$$

for all $\psi\in C(\overline{\Omega\times(0,t_1)\times T})$. Our sending ψ to f_0 in $L^2(\Omega\times(0,t_1)\times T)$ yields (59).
(ii) By (i) the sequence $(\frac{\partial f_\varepsilon}{\partial t})$ two-sale converges with respect to m_ε, up to a subsequence, to some $\xi_0\in L^\infty(0,t_1;L^2(\Omega\times T))$. For all $\psi\in\mathcal{D}(\Omega\times(0,t_1)\times Y)$, there holds

$$\begin{aligned}\int_{\Omega\times(0,t_1)\times T}\xi_0\psi dxdtdy&=\lim_{\varepsilon\to0}\int_{\Omega\times(0,t_1)}\frac{\partial f_\varepsilon}{\partial t}\psi\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right)dm_\varepsilon dt\\&=-\lim_{\varepsilon\to0}\int_{\Omega\times(0,t_1)}f_\varepsilon\frac{\partial\psi}{\partial t}\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right)dm_\varepsilon dt\\&=-\int_{\Omega\times(0,t_1)\times T}f_0\frac{\partial\psi}{\partial t}dxdtdy,\end{aligned}$$

hence $f_0\in W^{1,\infty}(0,t_1;L^2(\Omega\times T))$, $\frac{\partial f_0}{\partial t}=\xi_0$, and the entire sequence $(\frac{\partial f_\varepsilon}{\partial t})$ two-scale converges with respect to (m_ε) to ξ_0. □

By (39) and (42), we can choose a sequence of positive reals (R_ε) such that

$$\begin{aligned}&r_\varepsilon\ll R_\varepsilon\ll\varepsilon;\quad R_\varepsilon\ll\frac{1}{\gamma_\varepsilon^{(N)}};\\&\varepsilon^3\ll R_\varepsilon\quad\text{if } N=3;\qquad\varepsilon^2|\log R_\varepsilon|\ll1\quad\text{if } N=2.\end{aligned}\tag{62}$$

(Set for instance $R_\varepsilon=\varepsilon^{\frac{3}{2}}$ if $N=3$ and $R_\varepsilon=\varepsilon^2$ if $N=2$). We define (see (31))

$$B^i_{R_\varepsilon}:=\varepsilon\boldsymbol{i}+R_\varepsilon B,\quad H^i_\varepsilon:=B^i_{R_\varepsilon}\setminus T^i_{r_\varepsilon},\quad B_{R_\varepsilon}:=\bigcup_{i\in I_\varepsilon}B^i_{R_\varepsilon},\quad H_\varepsilon:=\bigcup_{i\in I_\varepsilon}H^i_\varepsilon.\tag{63}$$

Given a sequence $(\boldsymbol{u}_\varepsilon)\subset H^1(\Omega;\mathbb{R}^N)$, we consider the sequences $(\tilde{\boldsymbol{v}}_\varepsilon)$ and $(\tilde{\boldsymbol{\omega}}_\varepsilon)$ introduced in (38) and the sequences $(\tilde{\boldsymbol{u}}_\varepsilon)$ and $(\hat{\boldsymbol{v}}_\varepsilon)$ given by (see (70))

$$\begin{aligned}\tilde{\boldsymbol{u}}_\varepsilon(x)&:=\sum_{i\in I_\varepsilon}\left(\fint_{\partial B^i_{R_\varepsilon}}\boldsymbol{u}_\varepsilon(s)d\mathcal{H}^{N-1}(s)\right)1_{Y^i_\varepsilon}(x),\\\hat{\boldsymbol{v}}_\varepsilon(x)&:=\sum_{i\in I_\varepsilon}\left(\fint_{\partial B^i_{r_\varepsilon}}\boldsymbol{u}_\varepsilon(s)d\mathcal{H}^{N-1}(s)\right)1_{Y^i_\varepsilon}(x).\end{aligned}\tag{64}$$

In the next lemma, we establish a series of estimates which will take a crucial part in the proof of Theorem 1.

Lemma 10. *Assume (37), let $\boldsymbol{u}_\varepsilon$ be a sequence in $H^1(\Omega,\mathbb{R}^N)$ and let m_ε, $(\tilde{\boldsymbol{u}}_\varepsilon)$, $(\hat{\boldsymbol{v}}_\varepsilon)$, $(\tilde{\boldsymbol{v}}_\varepsilon)$, $(\tilde{\boldsymbol{\omega}}_\varepsilon)$ be defined by (38), (57), (64). Then the following estimates hold true:*

$$\left.\begin{aligned}&\int_\Omega |\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon|^2dx\le C\frac{\varepsilon^3}{R_\varepsilon}\int_\Omega|\nabla\boldsymbol{u}_\varepsilon|^2dx,\\ &\int_\Omega |\tilde{\boldsymbol{u}}_\varepsilon-\hat{\boldsymbol{v}}_\varepsilon|^2dx\le C\frac{\varepsilon^3}{r_\varepsilon}\int_\Omega|\nabla\boldsymbol{u}_\varepsilon|^2dx,\end{aligned}\right\}\quad \text{if } N=3,$$

$$\left.\begin{aligned}&\int_\Omega |\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon|^2dx\le C\varepsilon^2|\log R_\varepsilon|\int_\Omega|\nabla\boldsymbol{u}_\varepsilon|^2dx,\\ &\int_\Omega |\tilde{\boldsymbol{u}}_\varepsilon-\hat{\boldsymbol{v}}_\varepsilon|^2dx\le C\varepsilon^2|\log r_\varepsilon|\int_\Omega|\nabla\boldsymbol{u}_\varepsilon|^2dx,\end{aligned}\right\}\quad \text{if } N=2, \tag{65}$$

$$\int_\Omega|\boldsymbol{\varphi}_\varepsilon|^2dx=\int|\boldsymbol{\varphi}_\varepsilon|^2dm_\varepsilon\quad\forall\boldsymbol{\varphi}_\varepsilon\in\{\tilde{\boldsymbol{u}}_\varepsilon,\hat{\boldsymbol{v}}_\varepsilon,\tilde{\boldsymbol{v}}_\varepsilon,\tilde{\boldsymbol{\omega}}_\varepsilon\},$$

and

$$\begin{aligned}&\int_{T_{r_\varepsilon}}|\boldsymbol{u}_\varepsilon-\hat{\boldsymbol{v}}_\varepsilon|^2dx\le Cr_\varepsilon^2\int_{T_{r_\varepsilon}}|\nabla\boldsymbol{u}_\varepsilon|^2dx,\\ &\int_{\partial B_{R_\varepsilon}}|\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon|^2d\mathcal{H}^{N-1}(x)\le CR_\varepsilon\int_{B_{R_\varepsilon}}|\nabla\boldsymbol{u}_\varepsilon|^2dx,\\ &\int_{T_{r_\varepsilon}}\left|\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{v}}_\varepsilon-\frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right|^2dx\le Cr_\varepsilon^2\int_{T_{r_\varepsilon}}|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2dx,\\ &\int_{\partial T_{r_\varepsilon}}\left|\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{v}}_\varepsilon-\frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right|^2d\mathcal{H}^{N-1}(x)\le Cr_\varepsilon\int_{T_{r_\varepsilon}}|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2dx,\end{aligned} \tag{66}$$

where T_{r_ε} and B_{R_ε} are given, respectively, by (31) and (63).

Proof. If $N=3$, the estimates (65) are established in [6, Lemma 3.1]. If $N=2$, the estimate $\int_\Omega|\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon|^2dx\le C\varepsilon^2|\log R_\varepsilon|\int_\Omega|\nabla\boldsymbol{u}_\varepsilon|^2dx$ is deduced by freezing the variable x_3 in the estimate obtained in [9, p. 420, l. 7-10]. We get, in the same way, $\int_\Omega|\boldsymbol{u}_\varepsilon-\hat{\boldsymbol{v}}_\varepsilon|^2dx\le C\varepsilon^2|\log r_\varepsilon|\int_\Omega|\nabla\boldsymbol{u}_\varepsilon|^2dx$ and infer $\int_\Omega|\tilde{\boldsymbol{u}}_\varepsilon-\hat{\boldsymbol{v}}_\varepsilon|^2dx\le C\varepsilon^2|\log r_\varepsilon|\int_\Omega|\nabla\boldsymbol{u}_\varepsilon|^2dx$. The last line of (65) can be checked easily. To prove (66), we establish that $\forall\ \boldsymbol{w}\in H^1(T;\mathbb{R}^N)$,

$$\begin{aligned}&\int_T\left|\boldsymbol{w}-\fint_{\partial B}\boldsymbol{w}(s)d\mathcal{H}^{N-1}(s)\right|^2dx\le C\int_T|\nabla\boldsymbol{w}|^2dx,\\ &\int_{\partial T}\left|\boldsymbol{w}-\fint_{\partial B}\boldsymbol{w}(s)d\mathcal{H}^{N-1}(s)\right|^2d\mathcal{H}^{N-1}(x)\le C\int_T|\nabla\boldsymbol{w}|^2dx,\end{aligned} \tag{67}$$

yielding the first two lines of (66) by making suitable changes of variables (set $T=B$ to get the second line of (66)). To that aim, assume that the first line of (67) is false, then there exists a sequence $(\boldsymbol{w}_n)\subset H^1(T;\mathbb{R}^N)$ such that

$$\fint_T\left|\boldsymbol{w}_n-\fint_{\partial B}\boldsymbol{w}_n(s)d\mathcal{H}^{N-1}(s)\right|^2dx=1,\quad\int_T\boldsymbol{w}_ndx=0,\quad\lim_{n\to+\infty}\int_T|\nabla\boldsymbol{w}_n|^2dx=0.$$

Taking Poincaré-Wirtinger inequality $\int_T \left|\boldsymbol{w}_n - \int_T \boldsymbol{w}_n(y)dy\right|^2 dx \leq C\int_T |\boldsymbol{\nabla w}_n|^2\ dx$ into account, we infer that $(\boldsymbol{w}_n)$ converges strongly to 0 in $H^1(T;\mathbb{R}^N)$ and then, by the strong continuity of the trace application from $H^1(T;\mathbb{R}^N)$ to $L^1(\partial B;\mathbb{R}^N)$, that $\lim_{n\to+\infty} \fint_{\partial B} \boldsymbol{w}_n(s)d\mathcal{H}^{N-1}(s) = 0$. By the continuous embedding of $H^1(T;\mathbb{R}^N)$ into $L^2(T;\mathbb{R}^N)$, we get $\lim_{n\to+\infty} \fint_T |\boldsymbol{w}_n - \fint_{\partial B} \boldsymbol{w}_n(s)d\mathcal{H}^{N-1}(s)|^2 dx = 0$, a contradiction. The second line of (67) can be obtained in a similar manner.
The last two lines of (66) remain to be proved. To this purpose, given $\boldsymbol{u} \in H^1(T;\mathbb{R}^N)$, we consider the problem

$$\min_{\boldsymbol{w}\in\mathcal{R}} \left\{ \left(\int_{\partial B} |\boldsymbol{u} - \boldsymbol{w}|^2\, d\mathcal{H}^{N-1} \right)^{\frac{1}{2}} \right\},$$

where $\mathcal{R}$ denotes the space of rigid displacements. This minimum problem is achieved at the point $\boldsymbol{p}(\boldsymbol{u}) \in \mathcal{R}$ defined by

$$\boldsymbol{p}(\boldsymbol{u})(x) := \fint_{\partial B} \boldsymbol{u}(s)d\mathcal{H}^{N-1}(s) + \left(\fint_{\partial B} c(N)(\boldsymbol{s} \wedge \boldsymbol{u}(s))d\mathcal{H}^{N-1}(s) \right) \wedge \boldsymbol{x}, \tag{68}$$

where $c(2) := 1$ and $c(3) := \frac{3}{2}$. The linear subspace of $H^1(T;\mathbb{R}^N)$ defined by $V := \{\boldsymbol{u} \in H^1(T;\mathbb{R}^N),\ \boldsymbol{p}(\boldsymbol{u}) = 0\}$ satisfies $V \cap \mathcal{R} = \{0\}$, hence by Korn inequality there holds

$$|\boldsymbol{u}|_{H^1(T;\mathbb{R}^N)} \leq C|\boldsymbol{e}(\boldsymbol{u})|_{L^2(T;\mathbb{R}^N)} \quad \forall \boldsymbol{u} \in V.$$

Noticing that $\boldsymbol{u} - \boldsymbol{p}(\boldsymbol{u}) \in V$ and $\boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{e}(\boldsymbol{u} - \boldsymbol{p}(\boldsymbol{u}))\ \ \forall \boldsymbol{u} \in H^1(T;\mathbb{R}^N)$, we infer from the continuous embedding of $H^1(T;\mathbb{R}^N)$ into $L^2(T;\mathbb{R}^N)$ and from the continuity of the trace from $H^1(T;\mathbb{R}^N)$ into $L^2(\partial T;\mathbb{R}^N)$ that

$$|\boldsymbol{u} - \boldsymbol{p}(\boldsymbol{u})|_{L^2(T;\mathbb{R}^N)} + |\boldsymbol{u} - \boldsymbol{p}(\boldsymbol{u})|_{L^2(\partial T;\mathbb{R}^N)} \leq C|\boldsymbol{e}(\boldsymbol{u})|_{L^2(T;\mathbb{R}^N)}, \tag{69}$$

for all $\boldsymbol{u} \in H^1(T;\mathbb{R}^N)$. By making appropriate changes of variables in (69), taking (38) and (68) into account (see also (70)), we find the last two lines of (66). □

Remark 5. *Given $\boldsymbol{u}_\varepsilon \in H^1(\Omega;\mathbb{R}^3)$, the field $\tilde{\boldsymbol{v}}_\varepsilon + \frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon \wedge \left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon} - \boldsymbol{y}_T\right)$ defined by (38) coincides in each cell Y_ε^i with the best approximation of $\boldsymbol{u}_\varepsilon$ in $\mathcal{R}$ with respect to the seminorm $\left(\int_{\partial B^i_{r_\varepsilon}} |.|^2\, d\mathcal{H}^{N-1}\right)^{\frac{1}{2}}$. The simplifying assumption $\boldsymbol{y}_T = 0$ would induce a loss of generality, due to the hypothesis (37). Notice that*

$$\hat{\boldsymbol{v}}_\varepsilon + \frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon \wedge \frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon} = \tilde{\boldsymbol{v}}_\varepsilon + \frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon \wedge \left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon} - \boldsymbol{y}_T\right). \tag{70}$$

The main results of Section 6 are stated in the next proposition, where the asymptotic behavior of several sequences associated to the sequence $(\boldsymbol{u}_\varepsilon)$ of the solutions of (2) is specified.

Proposition 1. *Assume (34), let $\boldsymbol{u}_\varepsilon$ be the solution of (2), and let $\tilde{\boldsymbol{v}}_\varepsilon$, $\tilde{\boldsymbol{\omega}}_\varepsilon$, $\tilde{\boldsymbol{u}}_\varepsilon$ be defined by (38), (64). Then*

$$\begin{aligned} &\boldsymbol{u}_\varepsilon \in C([0,t_1], H_0^1(\Omega,\mathbb{R}^N)) \cap C^1([0,t_1], L^2(\Omega,\mathbb{R}^N)), \\ &\frac{\partial^2 \boldsymbol{u}_\varepsilon}{\partial t^2} \in L^2(0,t_1; H^{-1}(\Omega,\mathbb{R}^N)). \end{aligned} \tag{71}$$

Moreover, the following estimates hold ((c_ε), $(\gamma_\varepsilon^{(N)})$ being introduced in (34)), (39))

$$\begin{aligned}
&\int_\Omega \left(|\tilde{\boldsymbol{u}}_\varepsilon|^2 + |\boldsymbol{u}_\varepsilon|^2 + |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 + |\boldsymbol{\nabla} \boldsymbol{u}_\varepsilon|^2 + \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 \right) (\tau) dx \le C, \\
&\int \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 (\tau) dm_\varepsilon \le C, \quad \int_{T_{r_\varepsilon}} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau) dx \le \frac{C}{c_\varepsilon}, \\
&\int |\boldsymbol{u}_\varepsilon|^2 (\tau) dm_\varepsilon \le C \left(1 + \frac{1}{\gamma_\varepsilon^{(N)}} \right),
\end{aligned} \tag{72}$$

and the next convergences take place, up to a subsequence, for some suitable $\boldsymbol{u} \in W^{1,\infty}(0,\, t_1; H_0^1(\Omega;\mathbb{R}^N), L^2(\Omega;\mathbb{R}^N))$, $\boldsymbol{u}_0 \in W^{1,\infty}(0, t_1; L^2(\Omega \times T; \mathbb{R}^N))$ (see (58))

$$\begin{aligned}
&\boldsymbol{u}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{u} && \text{weak-star in } \ L^\infty(0,t_1; H_0^1(\Omega;\mathbb{R}^N)), \\
&\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \overset{\star}{\rightharpoonup} \frac{\partial \boldsymbol{u}}{\partial t} && \text{weak-star in } \ L^\infty(0,t_1; L^2(\Omega;\mathbb{R}^N)), \\
&\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{and} \quad \tilde{\boldsymbol{u}}_\varepsilon \to \boldsymbol{u} && \text{strongly in } \ L^\infty(0,t_1;\, L^2(\Omega;\mathbb{R}^N)), \\
&\boldsymbol{u}_\varepsilon \overset{m_\varepsilon}{\rightharpoonup} \boldsymbol{u}_0 \quad \text{and} \quad \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \overset{m_\varepsilon}{\rightharpoonup} \frac{\partial \boldsymbol{u}_0}{\partial t} && \text{two-scale with respect to } (m_\varepsilon).
\end{aligned} \tag{73}$$

In addition, we have

$$\begin{aligned}
&\int_\Omega |\tilde{\boldsymbol{\omega}}_\varepsilon|^2 (\tau) dx = \int |\tilde{\boldsymbol{\omega}}_\varepsilon|^2 (\tau) dm_\varepsilon(x) \le \frac{\varepsilon^N}{r_\varepsilon^{N-2}}, \\
&\int_\Omega |\tilde{\boldsymbol{v}}_\varepsilon|^2 (\tau) dx = \int |\tilde{\boldsymbol{v}}_\varepsilon|^2 (\tau) dm_\varepsilon(x) \le C \left(1 + \frac{1}{\gamma_\varepsilon^{(N)}} \right),
\end{aligned} \tag{74}$$

and, up to a subsequence,

$$\left.\begin{aligned}
&\tilde{\boldsymbol{\omega}}_\varepsilon \to 0 \quad \text{strongly in } L^\infty(0,t_1; L^2(\Omega;\mathbb{R}^3)) && \text{if } N = 2, \\
&\tilde{\boldsymbol{v}}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{v} \quad \text{weak-star in } L^\infty(0,t_1;\, L^2(\Omega;\mathbb{R}^N)), \\
&\tilde{\boldsymbol{\omega}}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{\omega} \quad \text{weak-star in } L^\infty(0,t_1;\, L^2(\Omega;\mathbb{R}^3)), \\
&\tilde{\boldsymbol{v}}_\varepsilon \overset{m_\varepsilon}{\rightharpoonup} \boldsymbol{v}, \quad \tilde{\boldsymbol{\omega}}_\varepsilon \overset{m_\varepsilon}{\rightharpoonup} \boldsymbol{\omega}, \quad \boldsymbol{u}_0 = \boldsymbol{v} + \frac{2}{\mathrm{diam} T} \boldsymbol{\omega} \wedge (\boldsymbol{y} - \boldsymbol{y}_T), \\
&\frac{\partial \boldsymbol{u}_0}{\partial t} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{2}{\mathrm{diam} T} \frac{\partial \boldsymbol{\omega}}{\partial t} \wedge (\boldsymbol{y} - \boldsymbol{y}_T), && \text{if } \gamma^{(N)} > 0,
\end{aligned}\right\} \tag{75}$$

for a suitable $(\boldsymbol{v}, \boldsymbol{\omega}) \in W^{1,\infty}(0,t_1; L^2(\Omega;\mathbb{R}^N)) \times W^{1,\infty}(0,t_1; L^2(\Omega;\mathbb{R}^3))$. Moreover,

$$\begin{aligned}
&\boldsymbol{\omega} = 0 && \text{if} \quad N = 2, \\
&\boldsymbol{\omega} = 0 \ \text{ and } \ \boldsymbol{v} = \boldsymbol{u} && \text{if} \quad \gamma^{(N)} = +\infty.
\end{aligned} \tag{76}$$

Proof. The assertion (71) follows from the regularity result (95) stated below. Fixing $t \in [0, t_1]$, we multiply (2) by $\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}$ and integrate by parts over Ω. We find

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \int_\Omega \rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(t) \right|^2 dx + \frac{1}{2} \int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon)(t) dx \right) = \int_\Omega \rho_\varepsilon \boldsymbol{f}. \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(t) dx. \tag{77}$$

Then, we fix $\tau \in [0,t_1]$ and integrate (77) with respect to t over $[0,\tau]$. We get

$$\begin{aligned}\frac{1}{2}\left(\int_\Omega \rho_\varepsilon \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(\tau)\right|^2 dx + \int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon)(\tau) dx\right)\\ = \frac{1}{2}\left(\int_\Omega \rho_\varepsilon |\boldsymbol{c}_0|^2 dx + \int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{b}_0) : \boldsymbol{e}(\boldsymbol{b}_0) dx\right) + \int_{\Omega\times(0,\tau)} \rho_\varepsilon \boldsymbol{f} \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dx dt.\end{aligned} \tag{78}$$

By (32) and (57), there holds

$$c_1(\mathcal{L}^N + m_\varepsilon) \le \rho_\varepsilon \mathcal{L}^N \le c_2(\mathcal{L}^N + m_\varepsilon), \tag{79}$$

for some suitable positive constants c_1, c_2, thus (ρ_ε) is bounded in $\mathcal{M}(\overline{\Omega})$. Since $\boldsymbol{c}_0$ is continuous on $\overline{\Omega}$ (see (2)), taking (36) into account we deduce from (78) that

$$\begin{aligned}\frac{1}{2}\left(\int_\Omega \rho_\varepsilon \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(\tau)\right|^2 dx + \int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon)(\tau) dx\right)\\ \le C\left(1 + \sqrt{\int_{\Omega\times(0,t_1)} \rho_\varepsilon \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right|^2 dx dt}\right) \quad \forall\, \tau \in [0,t_1].\end{aligned} \tag{80}$$

By integrating (80) with respect to τ over $(0,t_1)$, we infer that $\int_{\Omega\times(0,t_1)} \rho_\varepsilon \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right|^2 dx dt \le C$, and then deduce from (80) that

$$\int_\Omega \rho_\varepsilon \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(\tau)\right|^2 dx + \int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon)(\tau) dx \le C \quad \forall \tau \in [0,t_1], \tag{81}$$

yielding, by (79),

$$\int_\Omega \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(\tau)\right|^2 dx + \int \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(\tau)\right|^2 dm_\varepsilon \le C \quad \forall \tau \in [0,t_1]. \tag{82}$$

Applying Poincaré and Korn inequalities in $H_0^1(\Omega;\mathbb{R}^N)$, we obtain (see (34))

$$\begin{aligned}\int_\Omega |\boldsymbol{u}_\varepsilon|^2(\tau) dx &\le C\int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^2(\tau) dx \le C\int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2(\tau) dx\\ &\le C\int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2(\tau) dx + C\int_{T_{r_\varepsilon}} c_\varepsilon |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2(\tau) dx\\ &\le C\int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon)(\tau) dx.\end{aligned} \tag{83}$$

By the last line of (65) and the first line of (66) we have (see (57))

$$\begin{aligned}\int |\boldsymbol{u}_\varepsilon|^2(\tau) dm_\varepsilon &\le 2\int |\boldsymbol{u}_\varepsilon - \hat{\boldsymbol{v}}_\varepsilon|^2 dm_\varepsilon + 2\int |\hat{\boldsymbol{v}}_\varepsilon|^2(\tau) dm_\varepsilon\\ &\le C\frac{\varepsilon^N}{r_\varepsilon^{N-2}} \int_{T_{r_\varepsilon}} |\boldsymbol{\nabla u}_\varepsilon|^2(\tau) dx + 2\int_\Omega |\hat{\boldsymbol{v}}_\varepsilon|^2(\tau) dx.\end{aligned} \tag{84}$$

By (39) and (65), there holds

$$\begin{aligned}\int_\Omega |\hat{\boldsymbol{v}}_\varepsilon|^2(\tau) dx &\le C\int_\Omega |\hat{\boldsymbol{v}}_\varepsilon - \tilde{\boldsymbol{u}}_\varepsilon|^2 + |\tilde{\boldsymbol{u}}_\varepsilon - \boldsymbol{u}_\varepsilon|^2 + |\boldsymbol{u}_\varepsilon|^2(\tau) dx\\ &\le \frac{C}{\gamma_\varepsilon^{(N)}} \int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^2(\tau) dx + C\int_\Omega |\boldsymbol{u}_\varepsilon|^2(\tau) dx.\end{aligned} \tag{85}$$

Joining (84) and (85), we infer (see (39))

$$\int |\boldsymbol{u}_\varepsilon|^2(\tau)dm_\varepsilon \leq \frac{C}{\gamma_\varepsilon^{(N)}}\int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^2(\tau)dx + C\int_\Omega |\boldsymbol{u}_\varepsilon|^2(\tau)dx. \tag{86}$$

Collecting (81), (82), (83), (86), taking (34)) into account, we obtain the estimates (72). We infer, up to a subsequence, the convergences stated in two first lines of (73). By (72), the sequence $(\boldsymbol{u}_\varepsilon)$ is bounded in both spaces $W^{1,\infty}(0,t_1;L^2(\Omega;\mathbb{R}^N))$ and $L^\infty(0,t_1;H_0^1(\Omega;\mathbb{R}^N))$, hence by the compact embedding Theorem of Aubin and Simon (see [5], [33]) it is strongly relatively compact in $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$ and in $C(0,t_1;L^2(\Omega;\mathbb{R}^N))$. The first convergence in third line of (73) is proved. By (39) and (65), we have

$$\begin{aligned}
&\int_\Omega |\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{u}}_\varepsilon|^2(\tau)dx \leq C\frac{r_\varepsilon}{R_\varepsilon}\frac{1}{\gamma_\varepsilon^{(3)}}\int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^2(\tau)dx \qquad \text{if } \ N=3,\\
&\int_\Omega |\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{u}}_\varepsilon|^2(\tau)dx \leq C\frac{|\log R_\varepsilon|}{|\log r_\varepsilon|}\frac{1}{\gamma_\varepsilon^{(2)}}\int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^2(\tau)dx \quad \text{if } \ N=2,\\
&\int_\Omega |\hat{\boldsymbol{v}}_\varepsilon - \tilde{\boldsymbol{u}}_\varepsilon|^2(\tau)dx \leq \frac{C}{\gamma_\varepsilon^{(N)}}\int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^2(\tau)dx.
\end{aligned} \tag{87}$$

By (39) and (62) there holds

$$\lim_{\varepsilon\to 0}\frac{r_\varepsilon}{R_\varepsilon}\frac{1}{\gamma_\varepsilon^{(3)}} = 0 \ \text{ if } \ N=3, \qquad \lim_{\varepsilon\to 0}\frac{|\log R_\varepsilon|}{|\log r_\varepsilon|}\frac{1}{\gamma_\varepsilon^{(2)}} = 0 \ \text{ if } \ N=2. \tag{88}$$

The strong convergence of $(\tilde{\boldsymbol{u}}_\varepsilon)$ in $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$ stated in (73) follows from that of $(\boldsymbol{u}_\varepsilon)$, and from (87) and (88). The two scale convergences with respect to (m_ε) stated in the last line of (73) result from (72) and from Lemma 9.
We turn now to the study of the asymptotic behavior of the sequences $(\tilde{\boldsymbol{\omega}}_\varepsilon)$ and $(\tilde{\boldsymbol{v}}_\varepsilon)$ defined by (38). We start from the elementary inequality

$$|\boldsymbol{b}|^2 \leq C\int_T |\boldsymbol{b}\wedge\boldsymbol{y}|^2 d\mathcal{L}^N(y) \quad \forall \boldsymbol{b}\in\mathbb{R}^3, \quad N\in\{2,3\}, \quad T\subset\mathbb{R}^N.$$

By making simple changes of variables, we deduce (see (33))

$$|\boldsymbol{b}|^2 \leq \frac{C}{r_\varepsilon^N}\int_{T_{r_\varepsilon}^i}\left|\boldsymbol{b}\wedge\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}\right|^2 dx \qquad \forall \boldsymbol{b}\in\mathbb{R}^3, \ \forall i\in I_\varepsilon. \tag{89}$$

We can apply (89) for each $i\in I_\varepsilon$ to $\boldsymbol{b}=\tilde{\boldsymbol{\omega}}_\varepsilon(\varepsilon i,\tau)$. Taking the first and third lines of (66), (70) and (72) into account, we infer

$$\begin{aligned}
\int_\Omega |\tilde{\boldsymbol{\omega}}_\varepsilon(\tau)|^2 dx &= \sum_{i\in I_\varepsilon}\varepsilon^N |\tilde{\boldsymbol{\omega}}_\varepsilon(\varepsilon i,\tau)|^2 \leq C\sum_{i\in I_\varepsilon}\frac{\varepsilon^N}{r_\varepsilon^N}\int_{T_{r_\varepsilon}^i}\left|\tilde{\boldsymbol{\omega}}_\varepsilon(\varepsilon i,\tau)\wedge\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}\right|^2 dx\\
&= C\frac{\varepsilon^N}{r_\varepsilon^N}\int_{T_{r_\varepsilon}}\left|\tilde{\boldsymbol{\omega}}_\varepsilon(\tau)\wedge\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}\right|^2 dx\\
&\leq C\frac{\varepsilon^N}{r_\varepsilon^N}\int_{T_{r_\varepsilon}} |\boldsymbol{u}_\varepsilon - \hat{\boldsymbol{v}}_\varepsilon|^2(\tau) + \left|\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{v}}_\varepsilon - \frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon} - \boldsymbol{y}_T\right)\right|^2(\tau)dx\\
&\leq C\frac{\varepsilon^N}{r_\varepsilon^{N-2}}\int_{T_{r_\varepsilon}} |\boldsymbol{\nabla u}_\varepsilon|^2(\tau) + |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2(\tau)dx \leq C\frac{\varepsilon^N}{r_\varepsilon^{N-2}},
\end{aligned} \tag{90}$$

and, noticing that $\frac{\varepsilon^N}{r_\varepsilon^{N-2}} \leq \frac{1}{\gamma_\varepsilon^{(N)}}$ (see (39)), deduce from (66), (72) and (90) that

$$\begin{aligned}\int_\Omega |\tilde{\boldsymbol{v}}_\varepsilon|^2(\tau)dx &= \int_\Omega |\tilde{\boldsymbol{v}}_\varepsilon|^2(\tau)dm_\varepsilon \\ &\leq C\int_\Omega |\boldsymbol{u}_\varepsilon|^2+|\tilde{\boldsymbol{\omega}}_\varepsilon|^2+\left|\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{v}}_\varepsilon-\frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right|^2(\tau)dm_\varepsilon \\ &\leq C\left(1+\frac{1}{\gamma_\varepsilon^{(N)}}\right).\end{aligned} \tag{91}$$

Joining (90) and (91), we obtain (74) and deduce the convergences stated the three first lines of (75). If $\gamma^{(N)}>0$, by Lemma 9 and (74) the sequence $(\tilde{\boldsymbol{v}}_\varepsilon)$ (resp., $(\tilde{\boldsymbol{\omega}}_\varepsilon)$) two-scale converges with respect to (m_ε), up to a subsequence, to some $\boldsymbol{v}_0\in L^\infty(0,t_1;L^2(\Omega\times T;\mathbb{R}^N))$ (resp. $\boldsymbol{\omega}_0\in L^\infty(0,t_1;L^2(\Omega\times T;\mathbb{R}^3))$). We prove below that

$$\begin{aligned}&\boldsymbol{v}_0(x,t,y)=\boldsymbol{v}(x,t) \quad \text{and} \quad \boldsymbol{\omega}_0(x,t,y)=\boldsymbol{\omega}(x,t) \quad \text{in} \quad \Omega\times(0,t_1)\times T,\\ &\boldsymbol{u}_0(x,t,y)=\boldsymbol{v}(x,t)+\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}(x,t)\wedge(\boldsymbol{y}-\boldsymbol{y}_T) \quad \text{in} \quad \Omega\times(0,t_1)\times T.\end{aligned} \tag{92}$$

It follows then from the last line of (73) that $\frac{\partial \boldsymbol{v}}{\partial t}\in L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$, $\frac{\partial\boldsymbol{\omega}}{\partial t}\in L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^3))$ and $\frac{\partial\boldsymbol{u}_0}{\partial t}=\frac{\partial\boldsymbol{v}}{\partial t}+\frac{2}{\mathrm{diam}T}\frac{\partial\boldsymbol{\omega}}{\partial t}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)$ in $\Omega\times(0,t_1)\times T$. Assertion (75) is proved.
If $\gamma^{(N)}=+\infty$, then by (70), (75) and (90) there holds $\boldsymbol{\omega}=0$ and $(\hat{\boldsymbol{v}}_\varepsilon)$ converges weak-star to $\boldsymbol{v}$ in $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$. By (65) and (72),

$$\begin{aligned}\int_{\Omega\times(0,t_1)}|\boldsymbol{u}-\boldsymbol{v}|^2dxdt &\leq \liminf_{\varepsilon\to0}\int_{\Omega\times(0,t_1)}|\tilde{\boldsymbol{u}}_\varepsilon-\hat{\boldsymbol{v}}_\varepsilon|^2dxdt\\ &\leq C\liminf_{\varepsilon\to0}\frac{1}{\gamma_\varepsilon^{(N)}}\int_\Omega|\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon|^2dx=0,\end{aligned}$$

thus $\boldsymbol{u}=\boldsymbol{v}$. Taking (74) into account, Assertion (76) is proved. □

Proof of (92). Since $\tilde{\boldsymbol{v}}_\varepsilon$ is constant in each connected component of T_{r_ε}, for any matrix-valued field $\boldsymbol{\Psi}\in\mathcal{D}(\Omega\times(0,t_1)\times T;\mathbb{R}^N\times\mathbb{R}^N)$ we have $\int_{\Omega\times(0,t_1)}\tilde{\boldsymbol{v}}_\varepsilon.\mathbf{div}_y\boldsymbol{\Psi}\Big(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\Big)dm_\varepsilon dt=0$. By passing to the limit as $\varepsilon\to0$, we get $\frac{1}{|T|}\int_{\Omega\times(0,t_1)\times T}\boldsymbol{v}_0.\mathbf{div}_y\boldsymbol{\Psi}\,dxdtdy=0$, and deduce from the arbitrary choice of $\boldsymbol{\Psi}$ that $\boldsymbol{v}_0$ is independent of y. Let us fix $\boldsymbol{\varphi}\in\mathcal{D}(\Omega\times(0,t_1);\mathbb{R}^N)$ and $\eta\in\mathcal{D}(Y)$ such that $\eta(y)=1\ \forall y\in T$. It is easy to prove that

$$\left|\int_{\Omega\times(0,t_1)}\boldsymbol{\varphi}.\tilde{\boldsymbol{v}}_\varepsilon dxdt-\int_{\Omega\times(0,t_1)}\boldsymbol{\varphi}.\tilde{\boldsymbol{v}}_\varepsilon dm_\varepsilon dt\right|\leq C\varepsilon\left(\int_{\Omega\times(0,t_1)}|\tilde{\boldsymbol{v}}_\varepsilon(\tau)|^2dxdt\right)^{\frac12}\leq C\varepsilon.$$

Taking into account the last inequalities, the weak-star convergence of $(\tilde{\boldsymbol{v}}_\varepsilon)$ to $\boldsymbol{v}$ in $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^N))$ and its two-scale convergence with respect to (m_ε) to $\boldsymbol{v}_0$, we

deduce that

$$\begin{aligned}\int_{\Omega\times(0,t_1)} \boldsymbol{v}.\boldsymbol{\varphi}dxdt &= \lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)} \tilde{\boldsymbol{v}}_\varepsilon.\boldsymbol{\varphi}dxdt = \lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)} \tilde{\boldsymbol{v}}_\varepsilon.\boldsymbol{\varphi}dm_\varepsilon dt\\ &= \lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)} \tilde{\boldsymbol{v}}_\varepsilon(x,t).\left(\boldsymbol{\varphi}(x,t)\eta\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right)\right)dm_\varepsilon dt\\ &= \frac{1}{|T|}\int_{\Omega\times(0,t_1)\times T} \boldsymbol{v}_0\,(x,t)\,.\boldsymbol{\varphi}\,(x,t)\,\eta(y)dxdtdy\\ &= \int_{\Omega\times(0,t_1)} \boldsymbol{v}_0.\boldsymbol{\varphi}dxdt,\end{aligned}$$

thus, by the arbitrariness of $\boldsymbol{\varphi}$, $\boldsymbol{v}_0 = \boldsymbol{v}$. Likewise we find that $\boldsymbol{\omega}_0 = \boldsymbol{\omega}$. Fixing $\boldsymbol{\psi} \in \mathcal{D}(\Omega\times(0,t_1)\times Y;\mathbb{R}^N)$, and testing the two-scale convergence with respect to (m_ε) of $(\tilde{\boldsymbol{\omega}}_\varepsilon)$ to $\boldsymbol{\omega}$ with the test field $(\boldsymbol{y}-\boldsymbol{y}_T)\wedge\boldsymbol{\psi}$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)}\left(\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right).\boldsymbol{\psi}\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right)dm_\varepsilon dt\\ &\quad= \lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)}\tilde{\boldsymbol{\omega}}_\varepsilon.\left(\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\wedge\boldsymbol{\psi}\left(x,t,\frac{y_\varepsilon(x)}{r_\varepsilon}\right)\right)dm_\varepsilon dt\\ &\quad= \frac{1}{|T|}\int_{\Omega\times(0,t_1)\times T}\boldsymbol{\omega}.((\boldsymbol{y}-\boldsymbol{y}_T)\wedge\boldsymbol{\psi})dxdtdy\\ &\quad= \frac{1}{|T|}\int_{\Omega\times(0,t_1)\times T}(\boldsymbol{\omega}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)).\boldsymbol{\psi}dxdtdy.\end{aligned}$$

We infer that $\left(\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right)$ two-scale converges to $(\boldsymbol{\omega}\wedge(\boldsymbol{y}-\boldsymbol{y}_T))$ with respect to (m_ε). It follows that $\left(\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{v}}_\varepsilon-\frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right)$ two-scale converges to $(\boldsymbol{u}_0-\boldsymbol{v}-\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge(\boldsymbol{y}-\boldsymbol{y}_T))$ with respect to (m_ε). Since $\gamma^{(N)}>0$, we deduce from (59), the third line of (66) and (72) that

$$\begin{aligned}&\frac{1}{|T|}\int_{\Omega\times(0,t_1)\times T}|\boldsymbol{u}_0-\boldsymbol{v}-\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)|^2dxdtdy\\ &\quad\le\liminf_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)}\left|\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{v}}_\varepsilon-\frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right|^2dm_\varepsilon dt \qquad (93)\\ &\quad\le\liminf_{\varepsilon\to 0}C\frac{\varepsilon^N}{r_\varepsilon^{N-2}}\frac{1}{c_\varepsilon}=0,\end{aligned}$$

hence $\boldsymbol{u}_0 = \boldsymbol{v}+\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)$. □

We collect in the next theorem some abstract classical results which will be appropriate to check the uniqueness of the solution of the homogenized problem (40). Furthermore, the energy equation (97) is the key to the proof of the corrector result (see Proposition 2). The proof of Theorem 2 can be found in [28, Theorem 8.1 p. 287, Theorem 8.2 and Lemma 8.3 p. 298], [21, Formula (5.20) p. 667, and Theorem 1 p. 670], [27, Remark 1.3 p. 155]. Henceforth, the derivatives in $\mathcal{D}'(0,T;H)$ are identified with the time derivatives in $\mathcal{D}'(\Omega\times(0,T)\times Y)$ and are denoted both by $\frac{\partial\zeta}{\partial t}$ or by ζ'.

Theorem 2. *Let V and H be separable Hilbert spaces such that $V \subset H = H' \subset V'$, with continuous and dense embeddings. Let $||.||_V$, $|.|_H$, $((.,.))_V$, $(.,.)_H$ denote their respective norm and inner product. Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear symmetric form on V. Let $A \in \mathcal{L}(V, V')$ be defined by $a(\xi, \tilde{\xi}) = (A\xi, \tilde{\xi})_{(V',V)}$ $\forall\ (\xi, \tilde{\xi}) \in V^2$. Assume that*

$$\exists (\lambda, \alpha) \in \mathbb{R}_+ \times \mathbb{R}_+^*, \quad a(\xi, \xi) + \lambda |\xi|_H^2 \geq \alpha ||\xi||_V^2 \ \forall\ \xi \in V.$$

Let $h \in L^2(0, t_1; H)$, $\xi_0 \in V$, $\xi_1 \in H$. Then there exists a unique solution ξ of

$$\begin{aligned} &A\xi(t) + \xi''(t) = h(t), \quad \xi \in L^2(0, t_1; V), \\ &\xi' \in L^2(0, t_1; H), \quad \xi(0) = \xi_0, \quad \xi'(0) = \xi_1, \end{aligned} \tag{94}$$

where $\xi' = \frac{\partial \xi}{\partial t}$, $\xi'' = \frac{\partial^2 \xi}{\partial t^2}$. What is more,

$$\xi \in C([0, t_1]; V) \cap C^1([0, t_1]; H),\ \xi' \in L^2(0, t_1; V),\ \xi'' \in L^2(0, t_1; V'). \tag{95}$$

Furthermore, setting

$$e(\tau) := \frac{1}{2} \left[\left(\xi'(\tau), \xi'(\tau)\right)_H + a(\xi(\tau), \xi(\tau)) \right] \ \forall\ \tau \in [0, t_1], \tag{96}$$

there holds

$$e(\tau) = e(0) + \int_0^\tau \left(h, \xi'\right)_H dt \qquad \forall\ \tau \in [0, t_1]. \tag{97}$$

Moreover, (94) is equivalent to

$$\begin{aligned} &\int_0^{t_1} \Big(a(\xi(t), \tilde{\xi})\eta(t) + (\xi(t), \tilde{\xi})_H \eta''(t) \Big) dt + (\xi_0, \tilde{\xi})_H \eta'(0) \\ &\qquad - (\xi_1, \tilde{\xi})_H \eta(0) = \int_0^{t_1} (h, \tilde{\xi})_H \eta(t) dt \\ &\forall\ \tilde{\xi} \in V, \quad \forall\ \eta \in \mathcal{D}(]-\infty, t_1[); \quad \xi \in L^2(0, t_1; V),\ \xi' \in L^2(0, t_1; H). \end{aligned} \tag{98}$$

7 Proof of Theorem 1

Besides the demonstration of Theorem 1, this section contains the statement and the proof of a corrector result (see Section 7.3) and a justification of Remark 3 (i) (see Section 7.4).

Let us briefly outline the proof of Theorem 1. In the spirit of Tartar's method [35], we will multiply (2) by an appropriate sequence of oscillating test fields $(\boldsymbol{\phi}_\varepsilon)$ and, by passing to the limit as $\varepsilon \to 0$ in accordance with the convergences (73) and (75) established in proposition 1, obtain a variational problem of the type (98) satisfied by the triple $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega})$. Theorem 2 will yield then the uniqueness and the regularity of the solution and the initial boundary conditions. We will deduce also that the convergences established in (73) and (75) for subsequences, take place for the entire sequences.

The underlying idea of the construction of $(\boldsymbol{\phi}_\varepsilon)$ is to mimic the asymptotic behavior, studied in Proposition 1, of the sequence $(\boldsymbol{u}_\varepsilon)$ of the solutions of (2). Accordingly, the field $\boldsymbol{\phi}_\varepsilon$ depends on three smooth fields $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\zeta}$ designed to identify, by their arbitrary

nature, the system of equations satisfied by $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega})$ (if $N=2$, we can set $\boldsymbol{\zeta}=0$ because we know by (76) that $\boldsymbol{\omega}=0$). It coincides in each particle with the rigid displacement associated to the rotation vector $\frac{2}{r_\varepsilon \mathrm{diam} T}\boldsymbol{\zeta}_\varepsilon$ and taking the value $\boldsymbol{\psi}_\varepsilon$ at the geometric center of gravity of the particle, being $\boldsymbol{\zeta}_\varepsilon$ and $\boldsymbol{\psi}_\varepsilon$ suitable approximations of $\boldsymbol{\zeta}$ and $\boldsymbol{\psi}$ taking constant values in Y_ε^i (see (101)). The field $\boldsymbol{\phi}_\varepsilon$ coincides with $\boldsymbol{\varphi}$ outside some neighborhood of the particles consisting of an ε-periodic distribution B_{R_ε} of balls of radius R_ε (see (63)), being (R_ε) a sequence of positive reals such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. In the set $B_{R_\varepsilon}\setminus T_{r_\varepsilon}$, the field $\boldsymbol{\phi}_\varepsilon$ takes approximately the value of the displacement minimizing the stored energy corresponding to the elastic state associated to the Lamé coefficients λ_0,μ_0, to vanishing body forces and to the Dirichlet boundary conditions on $\partial(B_{R_\varepsilon}\setminus T_{r_\varepsilon})$ determined by the values taken by $\boldsymbol{\phi}_\varepsilon$ in $\Omega\setminus(B_{R_\varepsilon}\setminus T_{r_\varepsilon})$ as described above. The simplest candidate coping with these conditions is the field given by (139) if $N=2$ and, if $N=3$, by

$$\boldsymbol{\phi}_\varepsilon := \sum_{p=1}^{3}\varphi_p\left(\boldsymbol{e}_p-\boldsymbol{\theta}_\varepsilon^{(p)}\right)+\psi_{\varepsilon p}\boldsymbol{\theta}_\varepsilon^{(p)}+\zeta_{\varepsilon p}\boldsymbol{\eta}_\varepsilon^{(p)},$$

where $\boldsymbol{\theta}_\varepsilon^{(p)}$ (resp., $\boldsymbol{\eta}_\varepsilon^{(p)}$) coincides in each set $B_{R_\varepsilon}^i$ with the solution of the problem $\mathcal{P}_3(T_{r_\varepsilon}^i;B_{R_\varepsilon}^i;(\boldsymbol{e}_p,0))$ (resp., $\mathcal{P}_3(T_{r_\varepsilon}^i;B_{R_\varepsilon}^i;(0,\boldsymbol{e}_p))$, see (6)) and is equal to zero in $\Omega\setminus B_{R_\varepsilon}$. However, this choice of $\boldsymbol{\phi}_\varepsilon$ would lead to technical complications, because at some stage of the proof, to be precise in (138) if $N=3$ and in the proof of (146) if $N=2$, in order to compute the limit of (see (63))

$$\int_{H_\varepsilon\times(0,t_1)}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt, \tag{99}$$

we would have to establish some uniform upper bounds for the norm of the gradients of $\boldsymbol{\theta}_\varepsilon^{(p)}$ and $\boldsymbol{\eta}_\varepsilon^{(p)}$ in the matrix $\Omega\setminus T_{r_\varepsilon}$, like (18) if $N=3$ and like [10, (5.11)] if $N=2$. Such estimates mean that the stress vectors associated to the solution of the problem $\mathcal{P}_N(r_\varepsilon T;R_\varepsilon B;\boldsymbol{\xi})$ defined by (6), (7) do not concentrate on small parts of the connected components of the boundary of $R_\varepsilon B\setminus r_\varepsilon T$ as $\varepsilon\to 0$. They are plausibly satisfied if ∂T is sufficiently smooth and can actually be checked under (18) if T is a ball by means of explicit computations (see Remark 1 and [10, (5.11)]). In the setting of scalar diffusion equations, the corresponding upper bounds can be deduced for a sufficiently smooth set T from the maximum principle which unfortunately fails to hold in linear elasticity. We circumvent this difficulty by assuming that the matrix is homogeneous and isotropic (i.e. that $\boldsymbol{a}_0$ is given by (18)) and by substituting in (99) suitable approximations of the fields $\boldsymbol{\theta}_\varepsilon^{(p)}$ and $\boldsymbol{\eta}_\varepsilon^{(p)}$ for which the last mentioned upper bounds can be proved. A similar approach has been taken by Allaire in [3] in the context of Stokes equations. The choice of these approximations depends on $N\in\{2,3\}$.

-If $N=3$, we substitute the fields $\boldsymbol{\theta}_\varepsilon^{(p)}$ and $\boldsymbol{\eta}_\varepsilon^{(p)}$, in each set $B_{R_\varepsilon}^i\setminus T_{r_\varepsilon}^i$ $(i\in I_\varepsilon)$, for the respective solutions $\mathcal{P}_{K_0}(T_{r_\varepsilon}^i;\mathbb{R}^3;(\boldsymbol{e}_p,0))$ and $\mathcal{P}_{K_0}(T_{r_\varepsilon}^i;\mathbb{R}^3;(0,\boldsymbol{e}_p))$, whose existences are guaranteed by Lemma 2 (i). By using suitable arguments of potential analysis (see Lemma 11), we check that under (18), these approximations of the restrictions of $\boldsymbol{\theta}_\varepsilon^{(p)}$ and $\boldsymbol{\eta}_\varepsilon^{(p)}$ to $B_{R_\varepsilon}\setminus T_{r_\varepsilon}$ satisfy the required estimates provided that ∂T is of class C^3. As they don't vanish on $\partial B_{R_\varepsilon}$, we link them up to zero in the set $B_{2R_\varepsilon}\setminus B_{R_\varepsilon}$ (see (103)) in a way such that the integral $\int_{(B_{2R_\varepsilon}\setminus B_{R_\varepsilon})\times(0,t_1)}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt$ converges to zero as $\varepsilon\to 0$ (see (127), (129)). This method doesn't fit to the case $N=2$ because

the existence of a solution to $\mathcal{P}_2(T^i_{r_\varepsilon};\mathbb{R}^2;(\boldsymbol{a},0))$ in a suitable functional space fails to hold if $\boldsymbol{a}\neq 0$ (see Remark 2 (ii)).

-If $N=2$, we prove that the limit of (99) stays the same if we substitute $\boldsymbol{\theta}^{(p)}_{B\varepsilon}$ for $\boldsymbol{\theta}^{(p)}_{\varepsilon}$ in (139), where, setting $B':=(\mathrm{diam}T)B$ (by (37), $T\subset B'$), $\boldsymbol{\theta}^{(p)}_{B\varepsilon}$ coincides in each set $B^i_{R_\varepsilon}$ with the solution of $\mathcal{P}_N(B'^i_{r_\varepsilon};B^i_{R_\varepsilon};(\boldsymbol{e}_p,0))$ and is equal to zero in $\Omega\setminus B'_{R_\varepsilon}$. As the last mentioned upper bounds hold true under (18) if T is a ball, the technical difficulty is overcome. This method is appropriate to the case $N=2$, because then the effective problem does not depend on the choice of T, unlike the case $N=3$.

7.1 Case $N=3$, $0<\gamma^{(3)}<+\infty$

We fix three arbitrary fields $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\zeta}$ such that

$$\begin{aligned}&\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta}\in C^\infty(\overline{\Omega\times(0,t_1)},\mathbb{R}^3),\\&\boldsymbol{\varphi}=\boldsymbol{\psi}=\boldsymbol{\zeta}=\frac{\partial\boldsymbol{\varphi}}{\partial t}=\frac{\partial\boldsymbol{\psi}}{\partial t}=\frac{\partial\boldsymbol{\zeta}}{\partial t}=0\text{ on }(\partial\Omega\times]0,t_1])\cup(\Omega\times\{t_1\}),\end{aligned}\tag{100}$$

and set

$$\boldsymbol{\chi}_\varepsilon(x,t):=\sum_{i\in I_\varepsilon}\left(\fint_{T^i_{r_\varepsilon}}\boldsymbol{\chi}(y,t)dy\right)1_{Y^i_\varepsilon}(x),\qquad\boldsymbol{\chi}\in\{\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta}\}.\tag{101}$$

For each $p\in\{1,2,3\}$, we denote by $\boldsymbol{\theta}^{(p)}_\infty$ (resp., $\boldsymbol{\eta}^{(p)}_\infty$) the solution of the problem $\mathcal{P}_{K_0}(T;\mathbb{R}^3;(\boldsymbol{e}_p,0))$ (resp., $\mathcal{P}_{K_0}(T;\mathbb{R}^3;(0,\boldsymbol{e}_p))$) (see (10)). We fix a sequence (R_ε) of positive reals satisfying (62), set

$$g_\varepsilon(r):=\begin{cases}1&\text{if }0\le r\le R_\varepsilon,\\-\frac{t}{R_\varepsilon}+2&\text{if }R_\varepsilon\le r\le 2R_\varepsilon,\\0&\text{if }r\ge 2R_\varepsilon,\end{cases}\tag{102}$$

and define (see (33))

$$\begin{aligned}\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}(x)&:=g_\varepsilon\left(|y_\varepsilon(x)|\right)\boldsymbol{\theta}^{(p)}_\infty\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right),\\\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}(x)&:=g_\varepsilon\left(|y_\varepsilon(x)|\right)\boldsymbol{\eta}^{(p)}_\infty\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right).\end{aligned}\tag{103}$$

The next equations, satisfied for each $p,q\in\{1,2,3\}$, $i\in I_\varepsilon$, will be used to prove (112) and to pass from (130) to (132):

$$\begin{aligned}r_\varepsilon(\mathbf{Cap}_3(T;\mathbb{R}^3))_{pq}&=\int_{\mathbb{R}^3\setminus T^i_{r_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}\left(\boldsymbol{\theta}^{(p)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)\right):\boldsymbol{e}\left(\boldsymbol{\theta}^{(q)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)\right)dx\\&=\boldsymbol{e}_q.\int_{\partial T^i_{r_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}\left(\boldsymbol{\theta}^{(p)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)\right)\boldsymbol{\nu}d\mathcal{H}^2\\&=\boldsymbol{e}_q.\int_{\partial T^i_{r_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2=-\boldsymbol{e}_q.\int_{\partial B^i_{R_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2,\end{aligned}\tag{104}$$

$$
\begin{aligned}
r_\varepsilon(\mathbf{Cap}_3(T;\mathbb{R}^3))_{p(q+3)} &= \int_{\mathbb{R}^3\setminus T^i_{r_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}\left(\boldsymbol{\theta}^{(p)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)\right):\boldsymbol{e}\left(\boldsymbol{\eta}^{(q)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)\right)dx\\
&= \boldsymbol{e}_p.\int_{\partial T^i_{r_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(q)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2 = -\boldsymbol{e}_p.\int_{\partial B^i_{R_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(q)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2\\
&= \frac{2}{\mathrm{diam}T}\int_{\partial T^i_{r_\varepsilon}}\left(\boldsymbol{e}_q\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right).(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu})d\mathcal{H}^2,
\end{aligned}
\tag{105}
$$

$$
\begin{aligned}
&r_\varepsilon(\mathbf{Cap}_3(T;\mathbb{R}^3))_{(p+3)(q+3)}\\
&\quad= \int_{\mathbb{R}^3\setminus T^i_{r_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}\left(\boldsymbol{\eta}^{(p)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)\right):\boldsymbol{e}\left(\boldsymbol{\eta}^{(q)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)\right)dx\\
&\quad= \frac{2}{\mathrm{diam}T}\boldsymbol{e}_p.\int_{\partial T^i_{r_\varepsilon}}\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\wedge(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(q)}_{\infty,\varepsilon})\boldsymbol{\nu})d\mathcal{H}^2.
\end{aligned}
\tag{106}
$$

Proof of (104), (105), (106). By (33), (102), (103), the field $\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}$ (resp., $\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}$) coincides in each set $H^i_\varepsilon = B^i_{R_\varepsilon}\setminus T^i_{r_\varepsilon}$ with the solution $\boldsymbol{\theta}^{(p)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)$ of $\mathcal{P}_{K_0}(T^i_{r_\varepsilon};\mathbb{R}^3;(\boldsymbol{e}_p,0))$ (resp., $\boldsymbol{\eta}^{(p)}_\infty\left(\frac{x-\varepsilon i}{r_\varepsilon}\right)$ of $\mathcal{P}_{K_0}(T^i_{r_\varepsilon};\mathbb{R}^3;(0,\boldsymbol{e}_p))$), therefore

$$
\mathbf{div}\left(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\right)=\mathbf{div}\left(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\right)=0\quad\text{in}\quad B^i_{R_\varepsilon}\setminus T^i_{r_\varepsilon}.
\tag{107}
$$

Denoting by $\boldsymbol{\nu}$ the outward unit normal to $\partial(B^i_{R_\varepsilon}\setminus T^i_{r_\varepsilon})$, we deduce from the divergence formula that (by Lemma 11 below, the next integrals are well defined)

$$
\begin{aligned}
\int_{\partial T^i_{r_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2 &= -\int_{\partial B^i_{R_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2,\\
\int_{\partial T^i_{r_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2 &= -\int_{\partial B^i_{R_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}d\mathcal{H}^2.
\end{aligned}
\tag{108}
$$

By (16) we have

$$
\mathbf{Cap}_3(T^i_{r_\varepsilon};\mathbb{R}^3)=\mathbf{Cap}_3(r_\varepsilon T;\mathbb{R}^3)=r_\varepsilon\mathbf{Cap}_3(T;\mathbb{R}^3).
\tag{109}
$$

The equations (104), (105), (106) are deduced by integration by parts, taking (107), (108), (109), and the definition of $\mathbf{Cap}_3(T^i_{r_\varepsilon};\mathbb{R}^3)$ into account (see (6)). □

In the next Lemma, we establish some suitable uniform upper bounds for the norm of the gradients of $\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}$ and $\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}$ in the matrix $\Omega\setminus T_{r_\varepsilon}$.

Lemma 11. *Under (18), there holds*

$$
\begin{aligned}
&\left|\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}(x)\right|+\left|\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}(x)\right|\le C\frac{r_\varepsilon}{|y_\varepsilon(x)|}\quad\forall x\in\overline{\Omega\setminus T_{r_\varepsilon}},\\
&\left|\boldsymbol{\nabla}\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}(x)\right|+\left|\boldsymbol{\nabla}\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}(x)\right|\le C\frac{r_\varepsilon}{|y_\varepsilon(x)|^2}\quad\forall x\in\overline{\Omega\setminus T_{r_\varepsilon}}.
\end{aligned}
\tag{110}
$$

In particular, we have

$$
\begin{aligned}
&|\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}|_{L^\infty(\partial T_{r_\varepsilon})}\le\frac{C}{r_\varepsilon},\qquad |\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}|_{L^\infty(\partial B_{R_\varepsilon})}\le C\frac{r_\varepsilon}{R^2_\varepsilon},\\
&|\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}|_{L^\infty(\partial T_{r_\varepsilon})}\le\frac{C}{r_\varepsilon},\qquad |\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}|_{L^\infty(\partial B_{R_\varepsilon})}\le C\frac{r_\varepsilon}{R^2_\varepsilon},
\end{aligned}
\tag{111}
$$

and

$$\int_\Omega |\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}|^2 + |\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}|^2 dx \le C R_\varepsilon r_\varepsilon \gamma^{(3)}_\varepsilon, \quad \int_\Omega |\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})|^2 + |\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})|^2 dx \le C\gamma^{(3)}_\varepsilon. \tag{112}$$

Proof. Let us fix $\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in (\mathbb{R}^3)^2$ and let $\boldsymbol{w}$ be the solution of $\mathcal{P}_{K_0}(T;\mathbb{R}^3;\boldsymbol{\xi})$ (see (10)). The field $\boldsymbol{w}$, which represents a displacement in the homogeneous isotropic elastic space $\mathbb{R}^3 \setminus T$ corresponding to zero body forces, is biharmonic in $\mathbb{R}^3 \setminus T$ (see [24, p. 133]), therefore it satisfies the following mean value formula (see [24, p. 21, (4)])

$$\boldsymbol{w}(y) = \frac{3}{8\pi}\left(5\int_{y+B} \boldsymbol{w} dx - \int_{y+\partial B} \boldsymbol{w} d\mathcal{H}^2\right), \tag{113}$$

provided $\boldsymbol{y} + B \subset \mathbb{R}^3 \setminus T$. Let us fix $\varepsilon > 0$. Since $|\boldsymbol{w}|_{K_0(\mathbb{R}^3;\mathbb{R}^3)} < +\infty$ (see (8)), there exists $r_0 > 0$ such that $T \subset r_0 B$ and

$$\int_{\mathbb{R}^3 \setminus r_0 B} |\boldsymbol{w}|^6 + |\boldsymbol{\nabla}\boldsymbol{w}|^2 dx \le \varepsilon. \tag{114}$$

If $|y| > r_0 + 1$, then $y + B \subset \mathbb{R}^3 \setminus r_0 B$ and by (114) and Hölder inequality we have

$$\begin{aligned} &\left|\int_{y+B} \boldsymbol{w} dx\right| \le C\left(\int_{y+B} |\boldsymbol{w}|^6 dx\right)^{\frac{1}{6}} \le C\varepsilon^{\frac{1}{6}}, \\ &\int_{y+B} |\boldsymbol{w}|^2 dx \le C\left(\int_{y+B} |\boldsymbol{w}|^6 dx\right)^{\frac{1}{3}} \le C\varepsilon^{\frac{1}{3}}, \end{aligned} \tag{115}$$

yielding, by the continuity of the trace application from $H^1(y+B)$ to $L^1(y+\partial B)$,

$$\left|\int_{y+\partial B} \boldsymbol{w} d\mathcal{H}^2\right|^2 \le C\int_{y+B} |\boldsymbol{w}|^2 + |\boldsymbol{\nabla}\boldsymbol{w}|^2 dx \le C(\varepsilon^{\frac{1}{3}} + \varepsilon). \tag{116}$$

Joining (113), (115), (116), we infer that $|\boldsymbol{w}(y)| < C\varepsilon^{\frac{1}{6}}$ provided $|y| > r_0$, hence $\lim_{|y|\to+\infty} |\boldsymbol{w}(y)| = 0$. Being biharmonic in a neighborhood of infinity and vanishing at infinity, the field $\boldsymbol{w}$ satisfies the following estimates in a neighborhood of infinity (see [24, p. 23]): $|\boldsymbol{w}(y)| \le \frac{C}{|y|}$, $|\boldsymbol{\nabla}\boldsymbol{w}(y)| \le \frac{C}{|y|^2}$. On the other hand, as ∂T is of class C^3 (see (36)), by the classical boundary regularity results for solutions to elliptic systems of partial differential equations (see [1]) there holds $\boldsymbol{w} \in C^1(\overline{\mathbb{R}^3 \setminus T};\mathbb{R}^3)$, therefore the last mentioned estimates hold true in $\overline{\mathbb{R}^3 \setminus T}$ (by (37) there holds $0 \in T$). Choosing $(\boldsymbol{a},\boldsymbol{b}) \in \{(\boldsymbol{e}_p,0),(0,\boldsymbol{e}_p)\}$ (that is $\boldsymbol{w} \in \{\boldsymbol{\theta}^{(p)}_\infty, \boldsymbol{\eta}^{(p)}_\infty\}$), we obtain, by making suitable changes of variables, the following estimates in $\mathbb{R}^3 \setminus T_{r_\varepsilon}$ (see (33))

$$\begin{aligned} &\left|\boldsymbol{\theta}^{(p)}_\infty\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right)\right| + \left|\boldsymbol{\eta}^{(p)}_\infty\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right)\right| \le C\frac{r_\varepsilon}{|y_\varepsilon(x)|}, \\ &\left|\boldsymbol{\nabla}\left(\boldsymbol{\theta}^{(p)}_\infty\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right)\right)\right| + \left|\boldsymbol{\nabla}\left(\boldsymbol{\eta}^{(p)}_\infty\left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right)\right)\right| \le C\frac{r_\varepsilon}{|y_\varepsilon(x)|^2}. \end{aligned}$$

Taking (102) and (103) into account, we deduce (110), (111), and (112). □

With a view to applying Theorem 2, we adopt the following notations

$$\begin{aligned}
&H := (L^2(\Omega;\mathbb{R}^3))^3,\ V := H_0^1(\Omega;\mathbb{R}^3)\times(L^2(\Omega;\mathbb{R}^3))^2,\\
&((\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega}),(\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta}))_H := \int_\Omega \rho_0\boldsymbol{u}.\boldsymbol{\varphi}dx\\
&\qquad + \frac{1}{|T|}\int_{\Omega\times T}\rho\left(\boldsymbol{v}+\frac{2\boldsymbol{\omega}}{\mathrm{diam}T}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)\right).\left(\boldsymbol{\psi}+\frac{2\boldsymbol{\zeta}}{\mathrm{diam}T}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)\right)dxdy,\\
&(((\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega}),(\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta})))_V := ((\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega}),(\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta}))_H + \int_\Omega \boldsymbol{\nabla u}.\boldsymbol{\nabla\varphi}dx,
\end{aligned} \tag{117}$$

and consider the continuous bilinear symmetric form a on V and the fields $\xi,\tilde{\xi}\in L^2(0,t_1;V)$, $h\in L^2(0,t_1;H)$, $\xi_0\in V$, $\xi_1\in H$ defined by

$$\begin{aligned}
a(\xi(\tau),\tilde{\xi}(\tau)) &:= \int_\Omega \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}):\boldsymbol{e}(\boldsymbol{\varphi})(\tau)dx\\
&\qquad + \gamma^{(3)}\int_\Omega \begin{pmatrix}\boldsymbol{\psi}-\boldsymbol{\varphi}\\ \boldsymbol{\zeta}\end{pmatrix}.\mathbf{Cap}_3(T;\mathbb{R}^3)\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \boldsymbol{\omega}\end{pmatrix}(\tau)dx,\\
\xi = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega}),\ \tilde{\xi} &= (\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta}),\ \xi_0 = (\boldsymbol{b}_0,\boldsymbol{b}_0,0),\ \xi_1 = (\boldsymbol{c}_0,\boldsymbol{c}_0,0),\ h = (\boldsymbol{f},\boldsymbol{f},0).
\end{aligned} \tag{118}$$

Setting

$$\boldsymbol{\phi}_\varepsilon := \sum_{p=1}^{3}\varphi_p\left(\boldsymbol{e}_p-\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}\right)+\psi_{\varepsilon p}\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}+\zeta_{\varepsilon p}\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}, \tag{119}$$

we multiply the first line of (2) by $\boldsymbol{\phi}_\varepsilon$ and integrate by parts over $\Omega\times(0,t_1)$:

$$\begin{aligned}
&\int_{\Omega\times(0,t_1)}\rho_\varepsilon\boldsymbol{u}_\varepsilon.\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}dxdt+\int_\Omega\rho_\varepsilon\left(\boldsymbol{b}_0\frac{\partial\boldsymbol{\phi}_\varepsilon(0)}{\partial t}dx-\boldsymbol{c}_0\boldsymbol{\phi}_\varepsilon(0)\right)dx\\
&\qquad+\int_{\Omega\times(0,t_1)}\boldsymbol{a}_\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt-\int_{\Omega\times(0,t_1)}\rho_\varepsilon\boldsymbol{f}.\boldsymbol{\phi}_\varepsilon dxdt=0.
\end{aligned} \tag{120}$$

It is easy to check that $\boldsymbol{\phi}_\varepsilon=\boldsymbol{\psi}_\varepsilon+\frac{2}{\mathrm{diam}T}\boldsymbol{\zeta}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)$ in T_{r_ε}. Taking (32) and (57) into account, we infer

$$\begin{aligned}
&\int_{\Omega\times(0,t_1)}\rho_\varepsilon\boldsymbol{u}_\varepsilon.\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}dxdt=\int_{(\Omega\setminus T_{r_\varepsilon})\times(0,t_1)}\rho_0\boldsymbol{u}_\varepsilon.\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}dxdt\\
&\quad+\int_0^{t_1}\int\rho\left(\frac{y_\varepsilon}{r_\varepsilon}\right)\boldsymbol{u}_\varepsilon.\left(\frac{\partial^2\boldsymbol{\psi}_\varepsilon}{\partial t^2}+\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\zeta}_\varepsilon}{\partial t^2}\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right)dm_\varepsilon dt.
\end{aligned} \tag{121}$$

The sequence $\left(\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}1_{\Omega\setminus T_{r_\varepsilon}}\right)$ converges strongly to $\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}$ in $L^2(\Omega\times(0,t_1);\mathbb{R}^3)$, hence by the strong convergence of $(\boldsymbol{u}_\varepsilon)$ to $\boldsymbol{u}$ in $L^2(\Omega\times(0,t_1);\mathbb{R}^3)$ (see (73)), we have

$$\lim_{\varepsilon\to0}\int_{(\Omega\setminus T_{r_\varepsilon})\times(0,t_1)}\rho_0\boldsymbol{u}_\varepsilon.\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}dxdt=\int_{\Omega\times(0,t_1)}\rho_0\boldsymbol{u}.\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}dxdt. \tag{122}$$

By (72) and (101) there holds

$$\left|\int_0^{t_1}\int \rho\left(\frac{y_\varepsilon}{r_\varepsilon}\right)\boldsymbol{u}_\varepsilon.\left[\left(\frac{\partial^2\boldsymbol{\psi}_\varepsilon}{\partial t^2}+\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\zeta}_\varepsilon}{\partial t^2}\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right)\right.\right.$$
$$\left.\left.-\left(\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}+\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\zeta}}{\partial t^2}\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)\right)\right]dm_\varepsilon dt\right| \tag{123}$$
$$\leq Cr_\varepsilon\sqrt{\int_0^{t_1}\int|\boldsymbol{u}_\varepsilon|^2dm_\varepsilon dt}\leq Cr_\varepsilon.$$

By testing the two-scale convergence of $(\boldsymbol{u}_\varepsilon)$ to $\boldsymbol{v}+\boldsymbol{\omega}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)$ with respect to (m_ε) (see (58), (75)) with the test field $\rho(y)(\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}+\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\zeta}}{\partial t^2}\wedge(\boldsymbol{y}-\boldsymbol{y}_T))$, taking (35) and (123) into account, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_0^{t_1}\int\rho\left(\frac{y_\varepsilon}{r_\varepsilon}\right)\boldsymbol{u}_\varepsilon.\left(\frac{\partial^2\boldsymbol{\psi}_\varepsilon}{\partial t^2}+\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\zeta}_\varepsilon}{\partial t^2}\wedge\left(\frac{\boldsymbol{y}_\varepsilon}{r_\varepsilon}-\boldsymbol{y}_T\right)\right)dm_\varepsilon dt\\
&=\frac{1}{|T|}\int_{\Omega\times(0,t_1)\times T}\rho\left(\boldsymbol{v}+\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)\right).\\
&\qquad\left(\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}+\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\zeta}}{\partial t^2}\wedge(\boldsymbol{y}-\boldsymbol{y}_T)\right)dxdtdy\\
&=\int_{\Omega\times(0,t_1)}\left(\overline{\rho}\boldsymbol{v}.\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}+(\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\wedge\boldsymbol{v}).\frac{2}{\mathrm{diam}T}\frac{\partial^2\boldsymbol{\zeta}}{\partial t^2}\right.\\
&\quad\left.+\left(\frac{2}{\mathrm{diam}T}\boldsymbol{\omega}\wedge\overline{\rho}(\boldsymbol{y}_G-\boldsymbol{y}_T)\right).\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}+\left(\frac{2}{\mathrm{diam}T}\right)^2\boldsymbol{J}^\rho\boldsymbol{\omega}.\frac{\partial^2\boldsymbol{\zeta}}{\partial t^2}\right)dxdt.
\end{aligned}\tag{124}$$

Joining (117), (118), (121), (122), and (124), we get

$$\lim_{\varepsilon\to0}\int_{\Omega\times(0,t_1)}\rho_\varepsilon\boldsymbol{u}_\varepsilon.\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}dxdt=\int_0^{t_1}(\xi(t),\tilde{\xi}"(t))_H dt. \tag{125}$$

By repeating the same argument, replacing $\boldsymbol{u}_\varepsilon$ by the continuous fields $\boldsymbol{b}_0,\boldsymbol{c}_0,\boldsymbol{f}$ and the sequence $\left(\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}\right)$ by $\left(\frac{\partial\boldsymbol{\phi}_\varepsilon}{\partial t}(0)\right)$, $(\boldsymbol{\phi}_\varepsilon(0))$, $(\boldsymbol{\phi}_\varepsilon)$, we find

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_\Omega\rho_\varepsilon\left(\boldsymbol{b}_0.\frac{\partial\boldsymbol{\phi}_\varepsilon}{\partial t}(0)-\boldsymbol{c}_0.\boldsymbol{\phi}_\varepsilon(0)\right)dx-\int_{\Omega\times(0,t_1)}\rho_\varepsilon\boldsymbol{f}.\boldsymbol{\phi}_\varepsilon dxdt\\
&\qquad=(\xi_0,\tilde{\xi}'(0))_H-(\xi_1,\tilde{\xi}(0))_H-\int_0^{t_1}(h(t),\tilde{\xi}(t))_H dt.
\end{aligned}\tag{126}$$

Noticing that $\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)=0$ on T_{r_ε}, we set (see (63))

$$\begin{aligned}
&\int_{\Omega\times(0,t_1)}\boldsymbol{a}_\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt=I_{1\varepsilon}+I_{2\varepsilon}+I_{3\varepsilon};\quad I_{1\varepsilon}:=\int_{\Omega\setminus(B_{2R_\varepsilon})\times(0,t_1)}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\varphi})dxdt,\\
&I_{2\varepsilon}:=\int_{(B_{2R_\varepsilon}\setminus B_{R_\varepsilon})\times(0,t_1)}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt,\quad I_{3\varepsilon}:=\int_{H_\varepsilon\times(0,t_1)}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt.
\end{aligned}\tag{127}$$

By (73) the sequence $(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon))$ converges weakly to $\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u})$ in $L^2(\Omega\times(0,t_1);\mathbb{S}^3)$. As $(\boldsymbol{e}(\boldsymbol{\varphi})\mathbf{1}_{\Omega\setminus(B_{2R_\varepsilon})})$ converges strongly to $\boldsymbol{e}(\boldsymbol{\varphi})$ in $L^2(\Omega\times(0,t_1);\mathbb{S}^N)$, we infer

$$\lim_{\varepsilon\to 0} I_{1\varepsilon} = \int_{\Omega\times(0,t_1)} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi})dxdt. \tag{128}$$

By (72), (110), (119), and (127), there holds

$$\begin{aligned} I_{2\varepsilon} &\le C\sqrt{\int_{(B_{2R_\varepsilon}\setminus B_{R_\varepsilon})\times(0,t_1)} |\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)|^2 dxdt} \\ &\le C\sqrt{\sum_{p=1}^{3}\int_{B_{2R_\varepsilon}\setminus B_{R_\varepsilon}} 1+|\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}|^2+|\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}|^2+|\boldsymbol{\nabla}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})(x)|^2+|\boldsymbol{\nabla}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})(x)|^2\, dx} \\ &\le C\sqrt{\left(1+\frac{r_\varepsilon^2}{R_\varepsilon^4}\right)\mathcal{L}^3(B_{2R_\varepsilon})} \le C\sqrt{\frac{r_\varepsilon}{R_\varepsilon}\gamma_\varepsilon^{(3)}+\frac{R_\varepsilon^3}{\varepsilon^3}} = o(1). \end{aligned} \tag{129}$$

Denoting by $\boldsymbol{\nu}$ the outward unit normal to $\partial(B_{R_\varepsilon}\setminus T_{r_\varepsilon})$, we set (see (38), (64))

$$\begin{aligned} \tilde{I}_{3\varepsilon} := \sum_{p=1}^{3}\int_{\partial T_{r_\varepsilon}\times(0,t_1)} &\left(\tilde{\boldsymbol{v}}_\varepsilon + \frac{2\tilde{\boldsymbol{\omega}}_\varepsilon}{\mathrm{diam}T}\wedge\left(\frac{\boldsymbol{y}_\varepsilon}{r_\varepsilon}-\boldsymbol{y}_T\right)\right) \\ &\cdot\left((\psi_{\varepsilon p}-\varphi_{\varepsilon p})\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}+\zeta_{\varepsilon p}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}\right)d\mathcal{H}^{N-1}dt \\ +\sum_{p=1}^{3}\int_{\partial B_{R_\varepsilon}\times(0,t_1)} &\tilde{\boldsymbol{u}}_\varepsilon\cdot\left((\psi_{\varepsilon p}-\varphi_{\varepsilon p})\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}+\zeta_{\varepsilon p}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}\right)d\mathcal{H}^{N-1}dt. \end{aligned} \tag{130}$$

We establish below that

$$|I_{3\varepsilon}-\tilde{I}_{3\varepsilon}| = o(1). \tag{131}$$

For a fixed $t\in(0,t_1)$, the fields $\boldsymbol{\varphi}_\varepsilon, \boldsymbol{\psi}_\varepsilon, \boldsymbol{\zeta}_\varepsilon, \tilde{\boldsymbol{v}}_\varepsilon, \tilde{\boldsymbol{\omega}}_\varepsilon$, and $\tilde{\boldsymbol{u}}_\varepsilon$ are constant in each connected components of H_ε. By (16), (104)-(106), and (130), we have

$$\begin{aligned} \tilde{I}_{3\varepsilon} &= \int_0^{t_1}\sum_{i\in I_\varepsilon}\begin{pmatrix}\boldsymbol{\psi}_\varepsilon-\boldsymbol{\varphi}_\varepsilon\\ \boldsymbol{\zeta}_\varepsilon\end{pmatrix}.\mathbf{Cap}_3(r_\varepsilon T;\mathbb{R}^3)\begin{pmatrix}\tilde{\boldsymbol{v}}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon\\ \tilde{\boldsymbol{\omega}}_\varepsilon\end{pmatrix}(\varepsilon i,t)dt \\ &= \frac{r_\varepsilon}{\varepsilon^3}\sum_{i\in I_\varepsilon}\int_{Y_\varepsilon^i\times(0,t_1)}\begin{pmatrix}\boldsymbol{\psi}_\varepsilon-\boldsymbol{\varphi}_\varepsilon\\ \boldsymbol{\zeta}_\varepsilon\end{pmatrix}.\mathbf{Cap}_3(T;\mathbb{R}^3)\begin{pmatrix}\tilde{\boldsymbol{v}}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon\\ \tilde{\boldsymbol{\omega}}_\varepsilon\end{pmatrix}dxdt \\ &= \gamma_\varepsilon^{(3)}\int_{\Omega\times(0,t_1)}\begin{pmatrix}\boldsymbol{\psi}_\varepsilon-\boldsymbol{\varphi}_\varepsilon\\ \boldsymbol{\zeta}_\varepsilon\end{pmatrix}.\mathbf{Cap}_3(T;\mathbb{R}^3)\begin{pmatrix}\tilde{\boldsymbol{v}}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon\\ \tilde{\boldsymbol{\omega}}_\varepsilon\end{pmatrix}dxdt. \end{aligned} \tag{132}$$

By passing to the limit as $\varepsilon\to 0$ in (132), thanks to the uniform convergence of $(\boldsymbol{\varphi}_\varepsilon,\boldsymbol{\psi}_\varepsilon,\boldsymbol{\zeta}_\varepsilon)$ to $(\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta})$ in $\Omega\times(0,t_1)$ (see (101)) and to the convergences stated in (73) and (75), we obtain

$$\lim_{\varepsilon\to 0}\tilde{I}_{3\varepsilon} = \gamma^{(3)}\int_{\Omega\times(0,t_1)}\begin{pmatrix}\boldsymbol{\psi}-\boldsymbol{\varphi}\\ \boldsymbol{\zeta}\end{pmatrix}.\mathbf{Cap}_3(T;\mathbb{R}^3)\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \boldsymbol{\omega}\end{pmatrix}dxdt. \tag{133}$$

Joining (118), (127), (128), (129), (131), and (133), we get

$$\lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)}\boldsymbol{a}_\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt = \int_0^{t_1} a(\xi,\tilde{\xi})dt. \tag{134}$$

We deduce from (120), (125), (126), and (134) that

$$\int_0^{t_1} (\xi, \tilde{\xi}")_H + a(\xi, \tilde{\xi}) - (h, \tilde{\xi})_H dt + (\xi_0, \tilde{\xi}'(0))_H - (\xi_1, \tilde{\xi}(0))_H = 0. \tag{135}$$

By the arbitrary choice of $\tilde{\xi} = (\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\zeta})$ and by a density argument, the variational formulation (135) is equivalent to (98). It is easy to check that under (117) and (118) the assumptions of Theorem 2 are satisfied. We infer that $\xi = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\omega})$ satisfies (95) and is the unique solution of the problem (94), equivalent to (40). The proof of Theorem 1 is achieved in the case $N = 3$. □

Proof of (131). We set

$$\begin{aligned} \overline{I}_{3\varepsilon} &:= \int_{H_\varepsilon \times (0,t_1)} \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\overline{\boldsymbol{\phi}}_\varepsilon) dx dt, \\ \overline{\boldsymbol{\phi}}_\varepsilon &:= \sum_{p=1}^{3} \varphi_{\varepsilon p} \left(\boldsymbol{e}_p - \boldsymbol{\theta}^{(p)}_{\infty,\varepsilon} \right) + \psi_{\varepsilon p} \boldsymbol{\theta}^{(p)}_{\infty,\varepsilon} + \zeta_{\varepsilon p} \boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}. \end{aligned} \tag{136}$$

Thanks to (72), (112), and to the estimate $|\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon| \leq C R_\varepsilon$ in H_ε (easily deduced from (101)), we get

$$\begin{aligned} |I_{3\varepsilon} - \overline{I}_{3\varepsilon}| &\leq C \sqrt{\int_{H_\varepsilon \times (0,t_1)} \left| \boldsymbol{e} \left(\sum_{p=1}^{3} (\varphi_{\varepsilon p} - \varphi_p)(\boldsymbol{e}_p - \boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}) \right) \right|^2 dx dt} \\ &\leq C \sqrt{\int_{H_\varepsilon \times (0,t_1)} \left(R_\varepsilon^2 \sum_{p=1}^{3} \left| \boldsymbol{e} \left(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon} \right) \right|^2 + 1 + \sum_{p=1}^{3} |\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}|^2 \right) dx dt} \\ &\leq C \sqrt{R_\varepsilon^2 \gamma_\varepsilon^{(3)} + \mathcal{L}^3(H_\varepsilon)} = o(1). \end{aligned} \tag{137}$$

As for fixed t the fields $\boldsymbol{\varphi}_\varepsilon$, $\boldsymbol{\psi}_\varepsilon$, $\boldsymbol{\zeta}_\varepsilon$ are constant in each connected components of H_ε, by (136) there holds

$$\overline{I}_{3\varepsilon} = \sum_{p=1}^{3} \int_{H_\varepsilon \times (0,t_1)} (\psi_{\varepsilon p} - \varphi_{\varepsilon p}) \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}) + \zeta_{\varepsilon p} \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}) dx dt.$$

By integration by parts, taking (107) into account, we infer

$$\begin{aligned} \overline{I}_{3\varepsilon} = &\sum_{p=1}^{3} \int_{\partial T_{r_\varepsilon} \times (0,t_1)} \boldsymbol{u}_\varepsilon . \left((\psi_{\varepsilon p} - \varphi_{\varepsilon p}) \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}) \boldsymbol{\nu} + \zeta_{\varepsilon p} \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}) \boldsymbol{\nu} \right) d\mathcal{H}^{N-1}(x) dt \\ &+ \sum_{p=1}^{3} \int_{\partial B_{R_\varepsilon} \times (0,t_1)} \boldsymbol{u}_\varepsilon . \left((\psi_{\varepsilon p} - \varphi_{\varepsilon p}) \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}) \boldsymbol{\nu} + \zeta_{\varepsilon p} \boldsymbol{a}_0 \boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon}) \boldsymbol{\nu} \right) d\mathcal{H}^{N-1}(x) dt, \end{aligned}$$

where $\boldsymbol{\nu}$ denotes the outward unit normal to ∂H_ε. It follows then from (130) that

$$\begin{aligned}\left|\overline{I}_{3\varepsilon}-\tilde{I}_{3\varepsilon}\right| &\le C\sum_{p=1}^{3}\sqrt{\int_{\partial T_{r_\varepsilon}\times(0,t_1)}|\tilde{\boldsymbol{v}}_\varepsilon+\frac{2}{\mathrm{diam}T}\tilde{\boldsymbol{\omega}}_\varepsilon\wedge\left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon}-\boldsymbol{y}_T\right)-\boldsymbol{u}_\varepsilon|^2d\mathcal{H}^2dt}\\ &\quad\times\sqrt{\mathcal{H}^2(\partial T_{r_\varepsilon})}\left(\left|\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}\right|_{L^\infty(r_\varepsilon T)}+\left|\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}\right|_{L^\infty(r_\varepsilon T)}\right)\\ &\quad+C\sum_{p=1}^{3}\sqrt{\int_{\partial B_{R_\varepsilon}\times(0,t_1)}|\tilde{\boldsymbol{u}}_\varepsilon-\boldsymbol{u}_\varepsilon|^2d\mathcal{H}^2dt}\sqrt{\mathcal{H}^2(\partial B_{R_\varepsilon})}\\ &\quad\times\left(\left|\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}\right|_{L^\infty(R_\varepsilon B)}+\left|\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\eta}^{(p)}_{\infty,\varepsilon})\boldsymbol{\nu}\right|_{L^\infty(R_\varepsilon B)}\right).\end{aligned}$$

Thanks to (66), (72), and (111) we infer (by (34) we have $\lim_{\varepsilon\to0}c_\varepsilon=+\infty$)

$$\begin{aligned}\left|\overline{I}_{3\varepsilon}-\tilde{I}_{3\varepsilon}\right| &\le C\sqrt{\int_{T_{r_\varepsilon}\times(0,t_1)}r_\varepsilon\left|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)\right|^2dxdt}\sqrt{\frac{r_\varepsilon^2}{\varepsilon^3}}\frac{1}{r_\varepsilon}\\ &\quad+C\sqrt{R_\varepsilon\int_{B_{R_\varepsilon}\times(0,t_1)}|\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon|^2dxdt}\sqrt{\frac{R_\varepsilon^2}{\varepsilon^3}}\frac{r_\varepsilon}{R_\varepsilon^2}\qquad(138)\\ &\le C\sqrt{\frac{r_\varepsilon}{\varepsilon^3}}\left(\sqrt{\frac{1}{c_\varepsilon}}+\sqrt{\frac{r_\varepsilon}{R_\varepsilon}}\right)=C\sqrt{\gamma^{(3)}_\varepsilon}\left(\sqrt{\frac{1}{c_\varepsilon}}+\sqrt{\frac{r_\varepsilon}{R_\varepsilon}}\right)=o(1).\end{aligned}$$

The estimate (131) follows from (137) and (138). □

7.2 Case $N=2$, $0<\gamma^{(2)}<+\infty$

We fix $\boldsymbol{\varphi},\boldsymbol{\psi}$ such that

$$\begin{aligned}&\boldsymbol{\varphi},\boldsymbol{\psi}\in C^\infty(\overline{\Omega\times(0,t_1)},\mathbb{R}^2),\\ &\boldsymbol{\varphi}=\boldsymbol{\psi}=\frac{\partial\boldsymbol{\varphi}}{\partial t}=\frac{\partial\boldsymbol{\psi}}{\partial t}=0\quad\text{ on }\quad(\partial\Omega\times]0,t_1])\cup(\Omega\times\{t_1\}),\end{aligned}$$

and set

$$\boldsymbol{\phi}_\varepsilon:=\sum_{\alpha=1}^{2}\varphi_\alpha\left(\boldsymbol{e}_\alpha-\boldsymbol{\theta}^{(\alpha)}_\varepsilon\right)+\psi_{\varepsilon\alpha}\boldsymbol{\theta}^{(\alpha)}_\varepsilon,\qquad(139)$$

where $\boldsymbol{\psi}_\varepsilon$ is defined by (101) and $\boldsymbol{\theta}^{(\alpha)}_\varepsilon$ ($\alpha\in\{1,2\}$) denotes the solution of

$$\inf_{\boldsymbol{\theta}\in H^1_0(\Omega,\mathbb{R}^2)}\left\{\int_\Omega\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}):\boldsymbol{e}(\boldsymbol{\theta})dx\;\middle|\;\begin{matrix}\boldsymbol{\theta}=\boldsymbol{e}_\alpha\ \text{ in }\ T_{r_\varepsilon},\\ \boldsymbol{\theta}=0\text{ in }\ \ \Omega\setminus B_{R_\varepsilon}\end{matrix}\right\}.\qquad(140)$$

Hence the field $\boldsymbol{\theta}^{(\alpha)}_\varepsilon$ coincides in each set $B^i_{R_\varepsilon}$ with the solution of $\mathcal{P}_2(T^i_{r_\varepsilon};B^i_{R_\varepsilon};(\boldsymbol{e}_\alpha,0))$ (see (7)) and is equal to zero in $\Omega\setminus B_{R_\varepsilon}$. We have $\mathbf{Cap}_2(T^i_{r_\varepsilon};B^i_{R_\varepsilon})=\mathbf{Cap}_2(r_\varepsilon T;R_\varepsilon B)$ and the next formula (similar to (107)-(104)) hold true ($\alpha,\beta\in\{1,2\}$, $i\in I_\varepsilon$)

$$\begin{aligned}&\mathbf{div}\left(\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(\alpha)}_\varepsilon)\right)=0\quad\text{ in }\ H_\varepsilon\ (=B_{R_\varepsilon}\setminus T_{r_\varepsilon}),\\ &\int_{\partial T^i_{r_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(\alpha)}_\varepsilon)\boldsymbol{\nu}d\mathcal{H}^1=-\int_{\partial B^i_{R_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(\alpha)}_\varepsilon)\boldsymbol{\nu}d\mathcal{H}^1,\\ &(\mathbf{Cap}_2(r_\varepsilon T;R_\varepsilon B))_{\alpha\beta}=\boldsymbol{e}_\beta.\int_{\partial T^i_{r_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(\alpha)}_\varepsilon)\boldsymbol{\nu}d\mathcal{H}^1=-\boldsymbol{e}_\beta.\int_{\partial B^i_{R_\varepsilon}}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}^{(\alpha)}_\varepsilon)\boldsymbol{\nu}d\mathcal{H}^1,\end{aligned}$$

where $\boldsymbol{\nu}$ denotes the outward unit normal to $\partial(B_{R_\varepsilon} \setminus T_{r_\varepsilon})$. There holds

$$\int_{H_\varepsilon} \left|\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})\right|^2 dx \le C\gamma_\varepsilon^{(2)}, \qquad \int_\Omega |\boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 dx \le CR_\varepsilon^2\gamma_\varepsilon^{(2)}. \tag{141}$$

Proof of (141). By (5), (21) and (39), we have

$$\begin{aligned}\int_{H_\varepsilon} \left|\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})\right|^2 dx &\le C\sum_{i\in I_\varepsilon}\int_{B^i_{R_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}) : \boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})dx \\ &\le \frac{C}{\varepsilon^2}\ (\mathbf{Cap}_2(r_\varepsilon T; R_\varepsilon B))_{\alpha\alpha} \le \frac{C}{\varepsilon^2|\log r_\varepsilon|} = C\gamma_\varepsilon^{(2)}.\end{aligned} \tag{142}$$

For each $i \in I_\varepsilon$ and $\alpha \in \{1,2\}$, the restriction of $\boldsymbol{\theta}_\varepsilon^{(\alpha)}$ to $B^i_{R_\varepsilon}$ belongs to $H^1_0(B^i_{R_\varepsilon}, \mathbb{R}^2)$. By applying Poincaré inequality in $H^1_0(B^i_{R_\varepsilon}; \mathbb{R}^2)$ and Korn inequality in $H^1_0(B^i_{R_\varepsilon}; \mathbb{R}^2)$, taking (62) and (142) into account, we obtain

$$\begin{aligned}\int_\Omega |\boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 dx &\le \sum_{i\in I_\varepsilon}\int_{B^i_{R_\varepsilon}} |\boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 dx \le CR_\varepsilon^2 \sum_{i\in I_\varepsilon}\int_{B^i_{R_\varepsilon}} |\ \boldsymbol{\nabla}\boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 dx \\ &\le CR_\varepsilon^2\sum_{i\in I_\varepsilon}\int_{B^i_{R_\varepsilon}} |\ \boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})|^2 dx = CR_\varepsilon^2\int_{H_\varepsilon} |\ \boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})|^2 dx \le CR_\varepsilon^2\gamma_\varepsilon^{(2)}.\square\end{aligned}$$

Repeating the argument of the case $N = 3$, we multiply the first line of (2) by $\boldsymbol{\phi}_\varepsilon$ and integrate by parts, to obtain (120). Then we set

$$\begin{aligned}&\int_{\Omega\times(0,t_1)} \boldsymbol{a}_\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt = I_{1\varepsilon} + I_{2\varepsilon}; \quad I_{1\varepsilon} := \int_{\Omega\setminus(H_\varepsilon\cup T_{r_\varepsilon})\times(0,t_1)} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\varphi})dxdt, \\ &I_{2\varepsilon} := \int_{H_\varepsilon\times(0,t_1)} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}\left(\boldsymbol{\varphi} + \sum_{\alpha=1}^2(\psi_{\varepsilon\alpha} - \varphi_\alpha)\boldsymbol{\theta}_\varepsilon^{(\alpha)}\right)dxdt.\end{aligned} \tag{143}$$

In the spirit of Theorem 2, we set

$$\begin{aligned}&H := (L^2(\Omega;\mathbb{R}^2))^2,\ V := H^1_0(\Omega;\mathbb{R}^2)\times L^2(\Omega;\mathbb{R}^2), \\ &((\boldsymbol{u},\boldsymbol{v}),(\boldsymbol{\varphi},\boldsymbol{\psi}))_H := \int_\Omega \rho_0\boldsymbol{u}.\boldsymbol{\varphi} + \overline{\rho}\boldsymbol{v}.\boldsymbol{\psi}dx, \\ &(((\boldsymbol{u},\boldsymbol{v}),(\boldsymbol{\varphi},\boldsymbol{\psi})))_V := ((\boldsymbol{u},\boldsymbol{v}),(\boldsymbol{\varphi},\boldsymbol{\psi}))_H + \int_\Omega \boldsymbol{\nabla}\boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{\varphi}dx, \\ &a((\boldsymbol{u},\boldsymbol{v}),(\boldsymbol{\varphi},\boldsymbol{\psi})) := \int_\Omega \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi})dxdt \\ &\qquad\qquad + \gamma^{(2)}4\pi\mu_0\frac{\lambda_0+2\mu_0}{\lambda_0+3\mu_0}\int_\Omega(\boldsymbol{\psi}-\boldsymbol{\varphi}).(\boldsymbol{v}-\boldsymbol{u})dx, \\ &\xi = (\boldsymbol{u},\boldsymbol{v}),\ \tilde{\xi} = (\boldsymbol{\varphi},\boldsymbol{\psi}),\ \xi_0 = (\boldsymbol{b}_0,\boldsymbol{b}_0),\ \xi_1 = (\boldsymbol{c}_0,\boldsymbol{c}_0),\ h = (\boldsymbol{f},\boldsymbol{f}).\end{aligned} \tag{144}$$

By the same reasoning used to get (125), (126), (128) in the case $N = 3$, we find

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)}\rho_\varepsilon \boldsymbol{u}_\varepsilon.\frac{\partial^2\boldsymbol{\phi}_\varepsilon}{\partial t^2}dxdt=\int_0^{t_1}(\xi,\tilde{\xi}")_H,\\
&\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\left(\boldsymbol{b}_0.\frac{\partial\boldsymbol{\phi}_\varepsilon}{\partial t}(0)-\boldsymbol{c}_0.\boldsymbol{\phi}_\varepsilon(0)\right)dx-\int_{\Omega\times(0,t_1)}\rho_\varepsilon\boldsymbol{f}.\boldsymbol{\phi}_\varepsilon dxdt\\
&\qquad\qquad=(\xi_0,\tilde{\xi}'(0))_H-(\xi_1,\tilde{\xi}(0))_H-\int_0^{t_1}(h,\tilde{\xi})_H dt,\\
&\lim_{\varepsilon\to 0} I_{1\varepsilon}=\int_{\Omega\times(0,t_1)}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}):\boldsymbol{e}(\boldsymbol{\varphi})dxdt.
\end{aligned}\tag{145}$$

We prove below that

$$\lim_{\varepsilon\to 0} I_{2\varepsilon}=\gamma^{(2)}4\pi\mu_0\frac{\lambda_0+2\mu_0}{\lambda_0+3\mu_0}\int_{\Omega\times(0,t_1)}(\boldsymbol{\psi}-\boldsymbol{\varphi}).(\boldsymbol{v}-\boldsymbol{u})dxdt.\tag{146}$$

By passing to the limit as $\varepsilon\to 0$ in (120), thanks to (143), (144), (145), and (146), we obtain the variational formulation (135) (with data given now by (144)), equivalent to (98). We deduce from Theorem 2 that $\xi=(\boldsymbol{u},\boldsymbol{v})$ satisfies (95) and is the unique solution of the problem (94), equivalent to (41). The proof of Theorem 1 is achieved. □

Proof of (146). Setting

$$\overline{I}_{2\varepsilon}:=\int_{H_\varepsilon\times(0,t_1)}\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}(\overline{\boldsymbol{\phi}}_\varepsilon)dxdt,\quad \overline{\boldsymbol{\phi}}_\varepsilon:=\sum_{\alpha=1}^{2}\varphi_{\varepsilon\alpha}\left(\boldsymbol{e}_\alpha-\boldsymbol{\theta}_\varepsilon^{(\alpha)}\right)+\psi_{\varepsilon\alpha}\boldsymbol{\theta}_\varepsilon^{(\alpha)},$$

and taking (72), (141) and the estimate $|\boldsymbol{\varphi}-\boldsymbol{\varphi}_\varepsilon|\le CR_\varepsilon$ in H_ε into account, we infer (as in (137))

$$\begin{aligned}
|I_{2\varepsilon}-\overline{I}_{2\varepsilon}|&\le C\sqrt{\int_{H_\varepsilon\times(0,t_1)}\left|\boldsymbol{e}\left(\sum_{\alpha=1}^{2}(\varphi_{\varepsilon\alpha}-\varphi_\alpha)(\boldsymbol{e}_\alpha-\boldsymbol{\theta}_\varepsilon^{(\alpha)})\right)\right|^2 dxdt}\\
&\le C\sqrt{\int_{H_\varepsilon\times(0,t_1)}\left(R_\varepsilon^2\sum_{\alpha=1}^{2}\left|\boldsymbol{e}\left(\boldsymbol{\theta}_\varepsilon^{(\alpha)}\right)\right|^2+1+\sum_{\alpha=1}^{2}|\boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2\right)dxdt}\\
&\le C\sqrt{\mathcal{L}^2(H_\varepsilon)+R_\varepsilon^2\gamma_\varepsilon^{(2)}}=o(1).
\end{aligned}\tag{147}$$

As $\boldsymbol{\psi}_\varepsilon(.,t)$ and $\boldsymbol{\varphi}_\varepsilon(.,t)$ are constant in each cell Y_ε^i, we have (see (101), (140))

$$\overline{I}_{2\varepsilon}:=\sum_{\alpha=1}^{2}\int_{B_{R_\varepsilon}\times(0,t_1)}(\psi_{\varepsilon\alpha}-\varphi_{\varepsilon\alpha})\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{e}\left(\boldsymbol{\theta}_\varepsilon^{(\alpha)}\right)dxdt.\tag{148}$$

Let B' denote the ball of center 0 and radius diam$T$ (by (37), $T\subset B'$) and, for each $\alpha\in\{1,2\}$, let $\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)}$ denote the solution of (see (63))

$$\inf_{\boldsymbol{\theta}\in H_0^1(\Omega;\mathbb{R}^2)}\left\{\int_\Omega\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}):\boldsymbol{e}(\boldsymbol{\theta})dx\;\middle|\;\begin{array}{l}\boldsymbol{\theta}=\boldsymbol{e}_\alpha\ \text{ in }\ B'_{r_\varepsilon},\\ \boldsymbol{\theta}=0\ \text{ in }\ \ \Omega\setminus B_{R_\varepsilon}\end{array}\right\}.\tag{149}$$

We prove below that

$$|\overline{I}_{2\varepsilon} - \overline{I}_{B2\varepsilon}| \leq C\sqrt{\varepsilon}\left(\gamma_\varepsilon^{(2)}\right)^{\frac{3}{4}} = o(1), \tag{150}$$

where

$$\overline{I}_{B2\varepsilon} = \sum_{\alpha=1}^{2} \int_{B_{R_\varepsilon}\times(0,t_1)} (\psi_{\varepsilon\alpha} - \varphi_{\varepsilon\alpha})\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}\left(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)}\right) dxdt. \tag{151}$$

The next convergence is established in [10, Formula (5.36)]

$$\lim_{\varepsilon\to 0} \overline{I}_{B2\varepsilon} = \gamma^{(2)} 4\pi\mu_0 \frac{\lambda_0 + 2\mu_0}{\lambda_0 + 3\mu_0} \int_{\Omega\times(0,t_1)} (\boldsymbol{\psi} - \boldsymbol{\varphi}).(\boldsymbol{v} - \boldsymbol{u}) dxdt. \tag{152}$$

Joining (147), (150), (152), we obtain (146). □

Proof of (150). By (72), (101), (148), and (151), there holds

$$\begin{aligned}
\left|\overline{I}_{2\varepsilon} - \overline{I}_{B2\varepsilon}\right| &\leq C\sqrt{\int_{\Omega\times(0,t_1)} |\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dxdt}\sqrt{\int_{\Omega\times(0,t_1)} \left|\boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)} - \boldsymbol{\theta}_\varepsilon^{(\alpha)})\right|^2 dxdt} \\
&\leq C\sqrt{\int_\Omega \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)} - \boldsymbol{\theta}_\varepsilon^{(\alpha)}) : \boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)} - \boldsymbol{\theta}_\varepsilon^{(\alpha)}) dx} \leq C\sqrt{J_{1\varepsilon} + J_{2\varepsilon}}, \\
J_{1\varepsilon} &:= \int_{B_{R_\varepsilon}\setminus B'_{r_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)} - \boldsymbol{\theta}_\varepsilon^{(\alpha)}) : \boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)} - \boldsymbol{\theta}_\varepsilon^{(\alpha)}) dx, \\
J_{2\varepsilon} &:= \int_{B'_{r_\varepsilon}\setminus T_{r_\varepsilon}} \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}) : \boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}) dx = \int_{B'_{r_\varepsilon}\setminus T_{r_\varepsilon}} \boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)} - \boldsymbol{e}_\alpha) : \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)}) dx.
\end{aligned} \tag{153}$$

By (140) and (149), we have $\boldsymbol{\theta}_\varepsilon^{(\alpha)} = \boldsymbol{\theta}_{B\varepsilon}^{(\alpha)} = 0$ on $\partial B_{R_\varepsilon}$, $\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)} = \boldsymbol{e}_\alpha$ on $\partial B'_{r_\varepsilon}$, $\boldsymbol{\theta}_\varepsilon^{(\alpha)} = \boldsymbol{e}_\alpha$ on $\partial T_{r_\varepsilon}$. By integration by parts, denoting by $\boldsymbol{\nu}$ the outward unit normal to $\partial B'_{r_\varepsilon}$, we infer

$$\begin{aligned}
J_{1\varepsilon} &= -\int_{\partial B'_{r_\varepsilon}} (\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}).\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)})\boldsymbol{\nu} d\mathcal{H}^1 + \int_{\partial B'_{r_\varepsilon}} (\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}).\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})\boldsymbol{\nu} d\mathcal{H}^1, \\
J_{2\varepsilon} &= -\int_{\partial B'_{r_\varepsilon}} (\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}).\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})\boldsymbol{\nu} d\mathcal{H}^1.
\end{aligned}$$

Taking the estimate $|\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(p)})\boldsymbol{\nu}|_{L^\infty(\partial(r_\varepsilon B'))} \leq \frac{C}{r_\varepsilon|\log r_\varepsilon|}$ (see [10, (5.12)]) into account, we deduce

$$\begin{aligned}
J_{1\varepsilon} + J_{2\varepsilon} &= -\int_{\partial B'_{r_\varepsilon}} (\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}).\boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{\theta}_{B\varepsilon}^{(\alpha)})\boldsymbol{\nu} d\mathcal{H}^1 \leq \frac{C}{r_\varepsilon|\log r_\varepsilon|}\int_{\partial B'_{r_\varepsilon}} |\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}| d\mathcal{H}^1 \\
&\leq \frac{C}{r_\varepsilon|\log r_\varepsilon|}\sqrt{\int_{\partial B'_{r_\varepsilon}} |\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 d\mathcal{H}^1}\sqrt{\mathcal{H}^1(\partial B'_{r_\varepsilon})} \\
&\leq \frac{C}{\sqrt{r_\varepsilon\varepsilon}|\log r_\varepsilon|}\sqrt{\int_{\partial B'_{r_\varepsilon}} |\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 d\mathcal{H}^1}.
\end{aligned} \tag{154}$$

By the continuity of the trace application from $H^1(B' \setminus T;\mathbb{R}^2)$ to $L^2(\partial B';\mathbb{R}^2)$ and by Poincaré inequality and Korn inequality in $\mathcal{H} := \{\boldsymbol{w} \in H^1(B' \setminus T;\mathbb{R}^2), \boldsymbol{w} = 0 \text{ on } \partial T\}$, there holds

$$\int_{\partial B'} |\boldsymbol{w}|^2 d\mathcal{H}^1 \leq C \int_{B' \setminus T} |\boldsymbol{\nabla w}|^2 dx \leq C \int_{B' \setminus T} |\boldsymbol{e}(\boldsymbol{w})|^2 dx \quad \forall\, \boldsymbol{w} \in \mathcal{H}.$$

By making suitable changes of variable (notice that $\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)} = 0$ on $\partial T_{r_\varepsilon}$), we deduce

$$\int_{\partial B'^i_{r_\varepsilon}} |\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 d\mathcal{H}^1 \leq C r_\varepsilon \int_{B'^i_{r_\varepsilon} \setminus T^i_{r_\varepsilon}} |\boldsymbol{e}(\boldsymbol{\theta}_\varepsilon^{(\alpha)})|^2 dx \quad \forall i \in I_\varepsilon. \tag{155}$$

Summing (155) over $i \in I_\varepsilon$, taking (141) into account, we obtain

$$\int_{\partial B'_{r_\varepsilon}} |\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 d\mathcal{H}^1 \leq C \frac{r_\varepsilon}{\varepsilon^2 |\log(r_\varepsilon)|}. \tag{156}$$

Joining (39), (153), (154), and (156), we get (150). □

7.3 Corrector result

The next proposition enhances the results stated in Theorem 1 by displaying an approximation of the solution $\boldsymbol{u}_\varepsilon$ of (2) in the strong topology of $L^2(0,t_1;H_0^1(\Omega;\mathbb{R}^N))$.

Proposition 2. *Under the hypotheses of Theorem 1, assume that $\boldsymbol{b}_0 = 0$, and that the solution $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega})$ of (40) if $N = 3$ (resp., of (41) if $N = 2$) is smooth, say $\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega} \in C^2([0,t_1];C^1(\overline{\Omega};\mathbb{R}^N))$. Let $\boldsymbol{u}_\varepsilon$ be the solution of (2) and let $\boldsymbol{\phi}_\varepsilon$ denote the field defined by substituting $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega})$ for $(\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta})$ in (119) if $N = 3$ (resp., in (139) if $N = 2$). Then*

$$\lim_{\varepsilon \to 0} \int_0^{t_1} |\boldsymbol{u}_\varepsilon - \boldsymbol{\phi}_\varepsilon|^2_{H_0^1(\Omega;\mathbb{R}^N)} dt = 0. \tag{157}$$

Remark 6. *(i) The approximation $\boldsymbol{\phi}_\varepsilon$ of $\boldsymbol{u}_\varepsilon$ introduced in Proposition 2 satisfies $\boldsymbol{\phi}_\varepsilon = \boldsymbol{v}_\varepsilon + \frac{2}{\mathrm{diam}T}\boldsymbol{\omega}_\varepsilon \wedge \left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon} - \boldsymbol{y}_T\right)$ in T_{r_ε} ($\boldsymbol{v}_\varepsilon$, $\boldsymbol{\omega}_\varepsilon$ being defined by (101)) hence the field $\boldsymbol{v}(x,t) + \frac{2}{r_\varepsilon \mathrm{diam}T}\boldsymbol{\omega}(x,t) \wedge \left(\frac{\boldsymbol{y}_\varepsilon(x)}{r_\varepsilon} - \boldsymbol{y}_T\right)$ (resp., $\frac{2}{r_\varepsilon \mathrm{diam}T}\boldsymbol{\omega}$) is a local approximation of the displacement in the inclusions (resp., of the rotation vector of the particles).*
(ii) A similar result can be established if $\gamma_N \in \{0, +\infty\}$ and for equilibrium equations.

Proof. We sketch the proof only in the case $N = 3$. The proof is similar in the case $N = 2$. By (34) and Korn inequality in $H_0^1(\Omega;\mathbb{R}^3)$ we have, for all $\tau \in (0,t_1)$,

$$|\boldsymbol{u}_\varepsilon - \boldsymbol{\phi}_\varepsilon(\tau)|^2_{H_0^1(\Omega;\mathbb{R}^3)} \leq C \int_\Omega \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon - \boldsymbol{\phi}_\varepsilon(\tau)) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon - \boldsymbol{\phi}_\varepsilon(\tau)) dx.$$

We deduce that $\int_0^{t_1} |\boldsymbol{u}_\varepsilon - \boldsymbol{\phi}_\varepsilon|^2_{H_0^1(\Omega;\mathbb{R}^3)} dt \leq C\,(J_{1\varepsilon} - 2J_{2\varepsilon} + J_{3\varepsilon})$, where

$$\begin{aligned}
J_{1\varepsilon} &:= \int_{\Omega \times (0,t_1)} \rho_\varepsilon \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right|^2 + \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon) dx dt, \\
J_{2\varepsilon} &:= \int_{\Omega \times (0,t_1)} \rho_\varepsilon \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} . \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t} + \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\phi}_\varepsilon) dx dt, \\
J_{3\varepsilon} &:= \int_{\Omega \times (0,t_1)} \rho_\varepsilon \left|\frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}\right|^2 + \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{\phi}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\phi}_\varepsilon) dx dt.
\end{aligned} \tag{158}$$

We prove below that (see (117), (118))

$$\lim_{\varepsilon\to 0} J_{1\varepsilon} = \lim_{\varepsilon\to 0} J_{2\varepsilon} = \lim_{\varepsilon\to 0} J_{3\varepsilon} = \int_0^{t_1} \left(\xi', \xi'\right)_H + a(\xi,\xi)dt, \tag{159}$$

yielding (157). □

Proof of (159). By (78) we have, since $\boldsymbol{b}_0 = 0$,

$$J_{1\varepsilon} = \int_{\Omega\times(0,t_1)} \rho_\varepsilon |\boldsymbol{c}_0|^2 dxdt + 2\int_0^{t_1} \left(\int_{\Omega\times(0,t)} \rho_\varepsilon \boldsymbol{f}.\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dxds \right) dt. \tag{160}$$

By (32) there holds

$$\int_{\Omega\times(0,t)} \rho_\varepsilon \boldsymbol{f}.\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dxds = \int_{\Omega\setminus T_{r_\varepsilon}\times(0,t)} \rho_0 \boldsymbol{f}.\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dxds + \int_{T_{r_\varepsilon}\times(0,t)} \rho \left(\frac{y_\varepsilon(x)}{r_\varepsilon}\right) \boldsymbol{f}.\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dm_\varepsilon ds.$$

By(73) and (75), the sequence $\left(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right)$ converges weak-star to $\frac{\partial \boldsymbol{u}}{\partial t}$ in $L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^3))$ and two-scale with respect to (m_ε) to $\frac{\partial \boldsymbol{v}}{\partial t} + \frac{2}{\mathrm{diam}T}\frac{\partial \boldsymbol{\omega}}{\partial t} \wedge (\boldsymbol{y}_G - \boldsymbol{y}_T)$. We easily deduce that for each $t \in (0,t_1)$, there holds

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\Omega\times(0,t)} \rho_\varepsilon \boldsymbol{f}.\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dxds \\ &\qquad = \int_{\Omega\times(0,t)} \rho_0 \boldsymbol{f}.\frac{\partial \boldsymbol{u}}{\partial t} + \overline{\rho}\boldsymbol{f}.\left(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{2}{\mathrm{diam}T}\frac{\partial \boldsymbol{\omega}}{\partial t} \wedge (\boldsymbol{y}_G - \boldsymbol{y}_T)\right) dxds. \end{aligned} \tag{161}$$

Since $|\boldsymbol{f}|$ is bounded in $\overline{\Omega\times(0,t_1)}$ (see (2)), by (72) and (79) we have, for all $t \in [0,t_1]$

$$\int_{\Omega\times(0,t)} \rho_\varepsilon \boldsymbol{f}.\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dxds \le C \left(\int_{\Omega\times(0,t)} \rho_\varepsilon \left|\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}\right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega\times(0,t)} \rho_\varepsilon |\boldsymbol{f}|^2 dx \right)^{\frac{1}{2}} \le C. \tag{162}$$

Joining (160), (161), (162), thanks to the dominated convergence theorem and to the weak-star convergence in the sense of measures of (ρ_ε) to $(\rho_0+\overline{\rho})$, taking the continuity of $\boldsymbol{c}_0$ into account (see (2)), we get, by (117), (118),

$$\begin{aligned} \lim_{\varepsilon\to 0} J_{1\varepsilon} &= \int_{\Omega\times(0,t_1)} (\rho_0 + \overline{\rho}) |\boldsymbol{c}_0|^2 dxdt \\ &\quad + 2\int_0^{t_1}\int_{\Omega\times(0,t)} \rho_0 \boldsymbol{f}.\frac{\partial \boldsymbol{u}}{\partial t} + \overline{\rho}\boldsymbol{f}.\left(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{2}{\mathrm{diam}T}\frac{\partial \boldsymbol{\omega}}{\partial t} \wedge (\boldsymbol{y}_G - \boldsymbol{y}_T)\right) dxdsdt \\ &= \int_0^{t_1} (\xi_1,\xi_1)_H + 2\left(\int_0^t (h,\xi')_H ds\right) dt. \end{aligned}$$

Since $\boldsymbol{b}_0 = 0$ there holds $\xi_0 = 0$ (see (118)) and, by (96), $(\xi_1,\xi_1)_H = 2e(0)$. We deduce from (97) that

$$\begin{aligned} \lim_{\varepsilon\to 0} J_{1\varepsilon} &= 2\int_0^{t_1} \left(e(0) + \left(\int_0^t \left(h,\xi'\right)_H ds\right)\right) dt \\ &= 2\int_0^{t_1} e(t)dt = \int_0^{t_1} \left(\xi',\xi'\right)_H + a(\xi,\xi)dt. \end{aligned}$$

To compute the limit of $(J_{2\varepsilon})$, we repeat the argument used to pass to the limit in (120), choosing $(\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta}) = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{\omega})$, that is $\xi = \tilde{\xi}$. The assumption of the second line of (100) is not required, because we don't need to integrate by parts with respect to t. We get (see (134)) $\lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)} \boldsymbol{a}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\phi}_\varepsilon)dxdt = \int_0^{t_1} a(\xi,\xi)dt$. Thanks to the second line of (73) and the fourth line of (75), by mimicking the proof of (122) and (124) we find $\lim_{\varepsilon\to 0}\int_{\Omega\times(0,t_1)} \rho_\varepsilon \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}.\frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}dxdt = \int_0^{t_1} \left(\xi',\xi'\right)_H dt$, and deduce that $\lim_{\varepsilon\to 0} J_{2\varepsilon} = \lim_{\varepsilon\to 0} J_{1\varepsilon}$. In order to determine $\lim_{\varepsilon\to 0} J_{3\varepsilon}$, we check that Proposition 1 stays the same if we substitute $\boldsymbol{\phi}_\varepsilon$ for $\boldsymbol{u}_\varepsilon$, and deduce that the limit of $(J_{2\varepsilon})$ is unchanged if we substitute $\boldsymbol{\phi}_\varepsilon$ for $\boldsymbol{u}_\varepsilon$ in the second line of (158), therefore $\lim_{\varepsilon\to 0} J_{3\varepsilon} = \lim_{\varepsilon\to 0} J_{2\varepsilon}$. □

7.4 Case $\gamma^{(N)} \in \{0,+\infty\}$

We sketch the proof of the results stated in Remark 3 in the cases $N = 2$, $\gamma^{(2)} = 0$ and $N = 3$, $\gamma^{(3)} = +\infty$. The other cases are similar.

Case $N = 2$ and $\gamma^{(2)} = 0$. We repeat the argument of the proof of the case $N = 2$ of Theorem 1, setting $(\boldsymbol{\psi},\boldsymbol{\zeta}) = (0,0)$. We get (145). By (72) and (141) there holds

$$\begin{aligned} |I_{2\varepsilon}| &= \left| \int_{H_\varepsilon\times(0,t_1)} \boldsymbol{a}_0\boldsymbol{e}\left(\sum_{\alpha=1}^{2}\varphi_\alpha(\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)})\right) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon)dxdt \right| \\ &\le C\sqrt{\int_{H_\varepsilon\times(0,t_1)} \left|\boldsymbol{e}\left(\sum_{\alpha=1}^{2}\varphi_\alpha(\boldsymbol{e}_\alpha - \boldsymbol{\theta}_\varepsilon^{(\alpha)})\right)\right|^2 dxdt} \\ &\le C\sqrt{\sum_{\alpha=1}^{2}\int_{H_\varepsilon\times(0,t_1)} \left|\boldsymbol{e}\left(\boldsymbol{\theta}_\varepsilon^{(\alpha)}\right)\right|^2 + 1 + |\boldsymbol{\theta}_\varepsilon^{(\alpha)}|^2 dxdt} \\ &\le C\sqrt{\gamma_\varepsilon^{(2)} + \mathcal{L}^2(H_\varepsilon)} = o(1). \end{aligned} \tag{163}$$

By passing to the limit in (120), thanks to (145) and (163), we obtain the variational formulation

$$\int_{\Omega\times(0,t_1)} \rho_0\boldsymbol{u}.\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2} + \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi})dxdt + \int_\Omega \rho_0\boldsymbol{b}_0.\frac{\partial\boldsymbol{\varphi}}{\partial t}(0) - \rho_0\boldsymbol{c}_0.\boldsymbol{\varphi}(0)dx = \int_{\Omega\times(0,t_1)} \rho_0\boldsymbol{\varphi}.\boldsymbol{f}dxdt,$$

equivalent to the the limit problem stated in Remark 3 (i). □

Case $N = 3$, $\gamma^{(3)} = +\infty$. By (76) we have $(\boldsymbol{v},\boldsymbol{\omega}) = (\boldsymbol{u},0)$, accordingly we set $(\boldsymbol{\psi},\boldsymbol{\zeta}) = (\boldsymbol{\varphi},0)$ and, following the argument of the proof of the case $0 < \gamma^{(3)} < +\infty$, we find (125), (126), (128). Thanks to (42) we can choose a sequence (R_ε) satisfying (62) and $R_\varepsilon\gamma_\varepsilon^{(3)} = o(1)$. We infer from (62), (72), and (112), that

$$|I_{2\varepsilon}| + |I_{3\varepsilon}| \leq C\sqrt{\int_{B_{2R_\varepsilon}\setminus T_{r_\varepsilon}\times(0,t_1)} \left| \boldsymbol{e}\left(\boldsymbol{\varphi} + \sum_{p=1}^{3}(\varphi_{\varepsilon p} - \varphi_p)\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}\right)\right|^2 dxdt}$$

$$\leq C\sqrt{\mathcal{L}^3(B_{2R_\varepsilon})} + C\sqrt{\sum_{p=1}^{3}\int_\Omega \left|\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}\right|^2 dx} + C|\varphi_\varepsilon - \varphi|_{L^\infty}\sqrt{\sum_{p=1}^{3}\int_\Omega \left|\boldsymbol{e}\left(\boldsymbol{\theta}^{(p)}_{\infty,\varepsilon}\right)\right|^2 dx}$$

$$\leq C\sqrt{\frac{R_\varepsilon^3}{\varepsilon^3}} + C\sqrt{R_\varepsilon r_\varepsilon \gamma_\varepsilon^{(3)}} + CR_\varepsilon\sqrt{\gamma_\varepsilon^{(3)}} = o(1).$$

Our sending $\varepsilon \to 0$ in (120) yields the variational formulation

$$\int_{\Omega\times(0,t_1)} (\rho_0 + \overline{\rho})\boldsymbol{u}.\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2} + \boldsymbol{a}_0\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi})dxdt$$
$$+ \int_\Omega (\rho_0 + \overline{\rho})\boldsymbol{b}_0.\frac{\partial\boldsymbol{\varphi}}{\partial t}(0) - (\rho_0 + \overline{\rho})\boldsymbol{c}_0.\boldsymbol{\varphi}(0)dx = \int_{\Omega\times(0,t_1)} (\rho_0 + \overline{\rho})\boldsymbol{f}.\boldsymbol{\varphi}dxdt.$$

which, joined with Proposition 1, is equivalent to (43). Furthermore, by (73) and (87) there holds $(\tilde{\boldsymbol{v}}_\varepsilon) \to \boldsymbol{u}$ strongly in $L^\infty(0,t_1;\, L^2(\Omega;\mathbb{R}^N))$ and, by (90), $(\tilde{\boldsymbol{\omega}}_\varepsilon) \to 0$ strongly in $L^\infty(0,t_1;\, L^2(\Omega;\mathbb{R}^N))$. □

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