A Reduced Basis Method for the Simulation of American Options

Bernard Haasdonk, Julien Salomon, Barbara Wohlmuth

A Reduced Basis Method for the Simulation of American Options

Bernard Haasdonk, Julien Salomon and Barbara Wohlmuth

Abstract We present a reduced basis method for the simulation of American option pricing. To tackle this model numerically, we formulate the problem in terms of a time dependent variational inequality. Characteristic ingredients are a POD-greedy and an angle-greedy procedure for the construction of the primal and dual reduced spaces. Numerical examples are provided, illustrating the approximation quality and convergence of our approach.

1 Introduction

We consider the problem of American option pricing and refer to (Achdou and Pironneau, 2005) and the references therein for an introduction into computational methods for option pricing. While European options can be modelled by a parabolic partial differential equation, American options result in additional inequality constraints. We refer to (Hager et al, 2010) for a possible numerical treatment by primal-dual finite elements and to (Glowinski, 2008; Geiger and Kanzow, 2002) for an abstract framework on the theory of constrained variational problems. We are interested in providing a fast numerical algorithm to solve accurately the variational inequality system of an American put option for a large variety of different parameter values such as interest rate, dividend, strike prize and volatility. Reduced basis (RB) methods are an appropriate means for standard parametrized parabolic partial differential equations, cf. (Haasdonk and Ohlberger, 2008; Rozza, 2005; Veroy

Bernard Haasdonk
IANS, Universität Stuttgart, Germany, e-mail: haasdonk@mathematik.uni-stuttgart.de

Julien Salomon
CEREMADE, Université Paris-Dauphine, e-mail: salomon@ceremade.dauphine.fr

Barbara Wohlmuth
M2 - Zentrum Mathematik, Technische Universität München, e-mail: wohlmuth@ma.tum.de

et al, 2003; Buffa et al, 2011) and the references therein. These are based on low-dimensional approximation spaces, that are constructed by greedy procedures. Convergence behavior of these procedures are known in some cases (Buffa et al, 2011; Haasdonk, 2011). The computational advantage of RB-methods over standard discretization methods is obtained by its possible offline/online decomposition: First, a typically expensive offline-phase involving the computation of the reduced spaces is performed. This phase only needs to be precomputed once. Then, the online phase allows an extremely fast computation of the RB solutions for many new parameters as only low dimensional systems need to be solved. Recently, we adopted the RB methodology to constrained stationary elliptic problems (Haasdonk et al, 2011), which we extend here to the instationary case. We refer to the recent contribution (Cont et al, 2011) for a tailored RB approach in option pricing. In contrast to our setting no inequality constraints are taken into account. The main challenge is the construction of a suitable low dimensional approximation of the dual cone required for the approximation of the constraints. In this contribution, we introduce a new greedy strategy based on an angle criteria and show numerical results.

2 American Option Model

An American option is a contract which permits its owner to receive a certain payoff $\psi(S,\tau) \geq 0$ at any time τ between 0 and $T > 0$. The variable T indicates the maturity. Introducing the backward time variable $t := T - \tau$, we can use, e.g., (Achdou and Pironneau, 2005) the following non linear model

$$\partial_t P - \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 P - (r-q)s\partial_s P + rP \geq 0, \quad P - \psi \geq 0,$$
$$\left(\partial_t P - \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 P - (r-q)s\partial_s P + rP\right) \cdot (P - \psi) = 0,$$

where $P = P(s,t)$ is the price of an American put, with $s \in \mathbb{R}_+$ the asset's value, σ is the volatility, r is the interest rate, q is the dividend payment and $\psi = \psi(s,t)$ is the payoff function. The boundary and initial conditions are as follows: $P(s,0) = \psi(s)$, $P(0,t) = K$, $\lim_{s\to+\infty} P(s,t) = 0$, where $K > 0$ is a fixed strike price that satisfies $K = \psi(0,0)$. In what follows, we focus on the case $\psi(s,t) = (K-s)_+$ with $(\cdot)_+ = \max(0,\cdot)$, but our method applies as well to other types of payoff functions. For the implementation, we restrict the values of s to a bounded interval $\Omega := (0, s_f)$, where s_f is large enough to make the assumption $P(s_f,t) = 0$ realistic. Let us also set $\widetilde{P} = P - P_0$, with initial data $P_0(s,t) = K(1 - s/s_f)$, so that $\widetilde{P}$ satisfies homogeneous Dirichlet conditions. Our aim is now to reformulate the last system in a weak form, where our reduced basis method applies. In this view, we introduce the following functional spaces:

$$V := \left\{ v \in L^2(\Omega) | s\partial_s v \in L^2(\Omega), v_{|\partial\Omega} = 0 \right\}, \quad W := V'.$$

The scalar product $\langle\cdot,\cdot\rangle_V$ associated with V is defined by $\langle u,v\rangle_V := \langle s\partial_s u, s\partial_s v\rangle_{L^2(\Omega)} + \langle u,v\rangle_{L^2(\Omega)}$, where $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ is the usual scalar product on $L^2(\Omega)$. The operators are specified as follows:

$$a(u,v;\mu) = \frac{1}{2}\sigma^2\langle\partial_s u, \partial_s(s^2 v)\rangle_{L^2(\Omega)} + \langle -(r-q)s\partial_s u + ru, v\rangle_{L^2(\Omega)},$$

$$f(v;\mu) = \langle F, v\rangle_{L^2(\Omega)},\ g(\eta;\mu) = \langle\widetilde{\psi},\eta\rangle_W,$$

with $F := -\left(\partial_t P_0 - \frac{1}{2}\sigma^2 s^2\partial^2_{ss}P_0 - (r-q)s\partial_s P_0 + rP_0\right)$, i.e. $F = K\left(\frac{s}{s_f}q - r\right)$ and $\widetilde{\psi} := \psi - P_0$. For $\eta \in W = V'$, we also define $b(\eta,v) = \eta(v)$. We can now recast our problem in the following weak form, parametrized by $\mu = (K,r,q,\sigma) \in \mathscr{P} \subset \mathbb{R}^4$. We now introduce u as a weak representant of the solution $\widetilde{P}$, as this is the standard notation in reduced basis literature:

$$\langle\partial_t u, v\rangle_{L^2(\Omega)} + a(u,v;\mu) - b(\lambda,v) = f(v;\mu), \qquad v \in V \tag{1}$$

$$b(\eta-\lambda,u) \geq g(\eta-\lambda;\mu), \qquad \eta \in M, \tag{2}$$

where $M \subset W$ is a closed convex cone. Various methods can be considered to solve numerically Equations (1–2). In what follows, we use a θ-scheme for the time discretization. Given $\mu \in \mathscr{P}$, $L \in \mathbb{N}$ and $\Delta t := T/L$, this method corresponds to the following iteration.

Given $0 < n \leq L-1$ and $u^n \in V$, find $u^{n+1} \in V$ and $\lambda^{n+1} \in M$ that satisfy $\forall v \in V, \forall \eta \in M$,

$$\left\langle\frac{u^{n+1}-u^n}{\Delta t}, v\right\rangle_{L^2(\Omega)} + a(\theta u^{n+1} + (1-\theta)u^n, v;\mu) - b(\lambda^{n+1},v) = f(v;\mu), \tag{3}$$

$$b(\eta-\lambda^{n+1}, u^{n+1}) \geq g(\eta-\lambda^{n+1};\mu). \tag{4}$$

This recursive definition is initialized with $u^0 := \widetilde{\psi}$. Note that in this scheme, the definition of λ^n is not recursive.

3 Reduced Basis Method

Standard finite element approaches do not exploit the structure of the solution and for a given parameter value, a high dimensional system has to be solved. In what follows, we introduce a specific Galerkin approximation of the solution, based on the reduced basis method and present algorithms to compute the corresponding bases. The principle of the reduced basis method consists in computing parametric solutions in low dimensional subspaces of V and W that are generated with particular solutions of our problem. Let us explain in more detail the corresponding formulation. For $N \in \mathbb{N}$, consider a finite subset $\mathscr{P}_N := \{\mu_1,\ldots,\mu_N\} \subset \mathscr{P}$ with $\mu_i \neq \mu_j$, $\forall i \neq j$. The reduced spaces V_N and W_N are defined by $V_N := \operatorname{span}\{\psi_1,\ldots,\psi_{N_V}\}$ and

$W_N := \text{span}\{\xi_1, \dots, \xi_{N_W}\}$ where ψ_i and ξ_i are defined from the large set of snapshot solutions $u^n(\mu_i)$ and $\lambda^n(\mu_i)$, $i = 1, \dots, N$, $n = 0, \dots, L$. Here $u^n(\mu_i)$ and $\lambda^n(\mu_i)$ denote the solution of Equations (3–4) at the time $t_n := n\Delta t$ for the parameter value $\mu = mu_i$. The functions ψ_j and ξ_j are suitably selected elements spanning V_N and W_N with $N_V, N_W \leq N(L+1)$ preferably small. Both families $\Psi_N = (\psi_j)_{j=1,\dots,N_V}$ and $\Xi_N = (\xi_j)_{j=1,\dots,N_W}$ are supposed to be composed of linearly independent functions, hence are so called reduced bases. Numerical algorithms to build these two sets will be presented in Section 4. We define the reduced cone $M_N \subset M$ as

$$M_N = \left\{ \sum_{j=1}^{N_W} \alpha_j \xi_j,\ \alpha_j \geq 0 \right\}.$$

In this setting, the reduced problem reads:

Given $\mu \in \mathscr{P}$, $0 \leq n \leq L-1$, $u_N^n \in V_N$, find $u_N^{n+1} \in V_N$ and $\lambda_N^{n+1} \in M_N$ that satisfy $\forall v_N \in V_N, \forall \eta_N \in M_N$,

$$\left\langle \frac{u_N^{n+1} - u_N^n}{\Delta t}, v_N \right\rangle_{L^2(\Omega)} + a(\theta u_N^{n+1} + (1-\theta)u_N^n, v_N; \mu) - b(\lambda_N^{n+1}, v_N) = f(v_N; \mu), \quad (5)$$

$$b(\eta_N - \lambda_N^{n+1}, u_N^{n+1}) \geq g(\eta_N - \lambda_N^{n+1}; \mu), \quad (6)$$

where the initial value u_N^0 is chosen as the orthogonal projection of u_0 on V_N, i.e.

$$\langle u_N^0 - u_0, v_N \rangle_V = 0,\ \forall v_N \in V_N.$$

4 Reduced Basis Construction

In this section, we present two methods to extract a basis $\Psi_N \subset V$ and $\Xi_N \subset M$ from the snapshots. Both are greedy procedures based on a finite training set $\mathscr{P}_{train} \subset \mathscr{P}$ small enough such that it can be scanned quickly. Given an arbitrary integer N_W, the dual reduced basis $\Xi_N = (\xi_j)_{j=1,\dots,N_W}$ is built iteratively according to the following algorithm. The goal of the approach is to obtain a reduced cone $M_N \subset M$ capturing as much "volume" as possible.

Algorithm 1 *(Angle-greedy algorithm) Given N_W, $\mathscr{P}_{train} \subset \mathscr{P}$, choose arbitrarily $0 \leq n_1 \leq L$ and $\mu_1 \in \mathscr{P}_{train}$ and do*

1. *set* $\Xi_N^1 = \left\{ \frac{\lambda^{n_1}(\mu_1)}{\|\lambda^{n_1}(\mu_1)\|_W} \right\}$, $W_N^1 := \text{span}(\Xi_N^1)$,
2. *for* $k = 1, \dots, N_W - 1$, *do*
   a. *find* $(n_{k+1}, \mu_{k+1}) := \text{argmax}_{n=0,\dots,L,\ \mu \in \mathscr{P}_{train}} \left(\measuredangle \left(\lambda^n(\mu), W_N^k \right) \right)$,
   b. *set* $\xi_{k+1} := \frac{\lambda^{n_{k+1}}(\mu_{k+1})}{\|\lambda^{n_{k+1}}(\mu_{k+1})\|_W}$,
   c. *define* $\Xi_N^{k+1} = \Xi_N^k \cup \{\xi_{k+1}\}$, $W_N^{k+1} := \text{span}(\Xi_N^{k+1})$,

3. *define* $\Xi_N := \Xi_N^{N_W}$, $W_N := \mathrm{span}(\Xi_N)$.

Here we have used the notation $\measuredangle(v,S)$ to denote the angle between a vector v and a linear space $S \subset W$, which is simply obtained via the orthogonal projection Π_S from W on S by

$$\measuredangle(v,S) = \arccos \frac{||\Pi_S v||_W}{||v||_W}, \quad v \in W.$$

We apply the POD-greedy algorithm (Haasdonk and Ohlberger, 2008) to design the primal reduced basis Ψ_N. This procedure is standard in RB-methods for evolution problems. In RB-methods, frequently *weak* greedy procedures are used, which make beneficial use of rapidly computable error estimators and allow to handle large sets $\mathscr{P}_{train}$ (Buffa et al, 2011). However, as our analysis does not yet provide a-posteriori error estimators, we use the true projection errors as error indicators. This corresponds to the so called *strong* greedy procedure (Buffa et al, 2011; Haasdonk, 2011).

Algorithm 2 *(POD-greedy algorithm) Given* $\widetilde{N}_V > 0$, $\mathscr{P}_{train} \subset \mathscr{P}$, *choose arbitrarily* $\mu_1 \in \mathscr{P}_{train}$,

1. *set* $\widetilde{\Psi}_N^1 = \left\{ \frac{u^0(\mu_1)}{\|u^0(\mu_1)\|_V} \right\}$, $\widetilde{V}_N^1 := \mathrm{span}(\widetilde{\Psi}_N^1)$,
2. *for* $k = 1, \ldots, \widetilde{N}_V - 1$, *do*
   a. *define* $\mu_{k+1} := \mathrm{argmax}_{\mu \in \mathscr{P}_{train}} \left(\sum_{n=0}^{L} \|u^n(\mu) - \Pi_{\widetilde{V}_N^k}(u^n(\mu))\|_V^2 \right)$,
   b. *define* $\widetilde{\psi}_{k+1} := POD_1 \left(u^n(\mu_{k+1}) - \Pi_{\widetilde{V}_N^k}(u^n(\mu_{k+1})) \right)_{n=0,\ldots,L}$,
   c. *define* $\widetilde{\Psi}_N^{k+1} := \widetilde{\Psi}_N^k \cup \{\widetilde{\psi}_{k+1}\}$,
3. *define* $\widetilde{\Psi}_N := \widetilde{\Psi}_N^{\widetilde{N}_V}$, $\widetilde{V}_N := \mathrm{span}\widetilde{\Psi}_N$.

Here, we have denoted by $\Pi_{\widetilde{V}_N^k}$ the orthogonal projection on $\widetilde{V}_N^k$ with respect to $\langle \cdot, \cdot \rangle_V$, and by POD_1 the routine that extracts from a family of vectors the first Proper Orthogonal Decomposition (POD) mode that can be obtained via the best approximation property

$$POD_1 (v^n)_{n=0,\ldots,L} := \arg \min_{||z||_V = 1} \sum_{n=0}^{L} ||v^n - \langle v^n, z \rangle_V z||_V^2.$$

In this definition V is spanned by v^n, $n = 0, \ldots, L$. A convergence analysis of the POD-greedy procedure is provided in (Haasdonk, 2011). Note that Algorithm 2 always returns an orthonormal basis. This is even the case if a parameter value $\mu \in \mathscr{P}_{train}$ is selected more than once. We point out that our System (5–6) has a saddle point structure. Thus taking $\mathrm{span}\Psi_N$ as reduced basis for the primal variable might result in an ill posed problem. To guarantee the inf-sup stability of our approach, we follow an idea introduced in (Rozza, 2005) for the Stokes problem, see also (Haasdonk et al, 2011) for variational inequalities. It consists in the enrichment $\Psi_N := \widetilde{\Psi}_N^{\widetilde{N}_V} \cup (B\xi_i)_{i=1,\ldots,N_W}$, where $B\xi_i$ is the solution of $b(\xi_i, v) = \langle B\xi_i, v \rangle_V$,

for $v \in V$. We conclude with the final reduced space $V_N := \text{span}\Psi_N$ of dimension $N_V :=$dim V_N. By construction we have $\widetilde{N}_V \leq N_V \leq \widetilde{N}_V + N_W$.

5 Numerical Results

In this section, we present some numerical results obtained on the American Option model. We start with a description of the numerical values and methods we use. In order to compute snapshots, we use a standard finite element method for the space discretization and the θ-scheme presented in Section 2 for the time-discretization. The time domain $[0,T] = [0,1]$ is discretized with a uniform mesh of step size $\Delta t := T/L$, $L = 20$. The θ-scheme is used with $\theta = 1/2$, i.e. we apply a Crank-Nicolson method. The space domain $\Omega = (0, s_f) = (0, 300)$ is discretized with a uniform mesh of step size $\Delta s := s_f/S$, $S = 101$. For the function space, we use standard conforming nodal first order finite elements. For the sake of simplicity, we keep the notation V for the discrete high dimensional space and define it by $V := \{v \in H_0^1(\Omega) | v_{|[s_m, s_{m+1}]} \in P_1, m = 0, \ldots, S-1\}$ of dimension $H_V = H := S - 2 = 99$ with $s_m := m\Delta s$. We associate the basis function $\phi_i \in V$ with its Lagrange node $s_i \in \Omega$, i.e., $\phi_i(s_j) = \delta_{ij}, i, j = 1, \ldots, H$. The discretization of the Lagrange multipliers is performed using a dual finite element basis χ_j of $W := V'$ having the same support as ϕ_j, so that $b(\phi_i, \chi_j) = \delta_{ij}$, $i, j = 1, \ldots, H_W = H$. The cone M is defined by: $M = \left\{ \sum_{i=1}^{H_W} \eta_i \chi_i,\ \eta_i \geq 0 \right\}$. To build the basis, we consider a subset $\mathscr{P}_{train}$ of $\mathscr{P}$ that is composed of $N = 16$ values chosen randomly in the set

$$\begin{aligned} \mathscr{P} &= [(1-\tfrac{\varepsilon}{2})K_0, (1+\tfrac{\varepsilon}{2})K_0] \times [(1-\tfrac{\varepsilon}{2})r_0, (1+\tfrac{\varepsilon}{2})r_0] \\ &\times [(1-\tfrac{\varepsilon}{2})q_0, (1+\tfrac{\varepsilon}{2})q_0] \times [(1-\tfrac{\varepsilon}{2})\sigma_0, (1+\tfrac{\varepsilon}{2})\sigma_0]. \end{aligned}$$

with the numerical values $\varepsilon = 0.1$, $K_0 = 100$, $r_0 = 0.05$, $q_0 = 0.0015$, $\sigma_0 = 0.5$. To define the basis Ψ_N and the convex set Ξ_N, we use Algorithm 2 combined with the enlargement by the supremizers and Algorithm 1. The eight first vectors of Ψ_N, Ξ_N and the supremizers are represented in Figure 1. We simulate two trajectories corresponding to the values $(\widetilde{N}_V, N_W) = (8,8)$ and $(\widetilde{N}_V, N_W) = (16,16)$ respectively. The corresponding bases Ψ_N are of size $N_V = 16$ and $N_V = 32$ respectively. We chose randomly a parameter vector μ corresponding to the values $K = 106.882366$, $r = 0.048470$, $d = 0.007679$, $\sigma = 0.418561$ in $\mathscr{P}$. Some steps of the simulation are represented in Figure 2, the top and lower row refer to the smaller and larger reduced spaces, respectively. We clearly see the improvement in the approximation by increasing the reduced dimensions. In order to evaluate the efficiency of the greedy algorithms proposed in Section 4, we plot the evolution of the quantities

$$\varepsilon_N^u := \max_{\mu \in \mathscr{P}_{train}} \sqrt{\sum_{n=0}^{L} \|u^n(\mu) - \Pi_{V_N^k}(u^n(\mu))\|_V^2},\ \varepsilon_N^\lambda := \max_{\substack{n=0,\ldots,L,\\ \mu \in \mathscr{P}_{train}}} \left(\measuredangle \left(\lambda^n(\mu), W_N^k \right) \right)$$

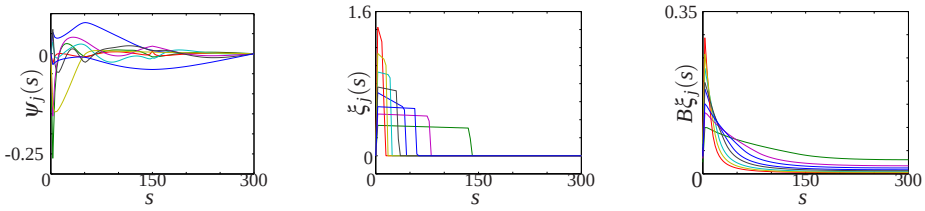

Fig. 1 Eight first vectors of the reduced basis Ψ_N, Ξ_N and the corresponding supremizers.

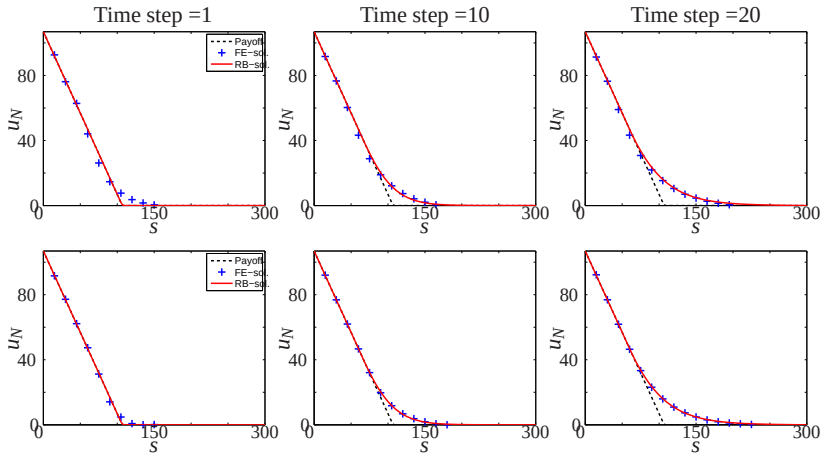

Fig. 2 Finite element approximation (solid red line) and Reduced basis approximation (blue $+$) at time steps $t/\Delta t = 1$, $t/\Delta t = 10$ and $t/\Delta t = T/\Delta t = 20$. The payoff function ψ is represented with the black dashed line. The reduced bases that are used have been generated by $(\widetilde{N}_V, N_W) = (8,8)$ (plots on the top) or $(\widetilde{N}_V, N_W) = (16,16)$ (plots on the bottom).

during their iterations. The results are plotted in the first two diagrams in Figure 3. We observe an excellent exponential convergence of the approximation measures.As final experiment, we address the generalization ability of the RB-model to parameters outside the training set. We consider $\mathscr{P}_{test} \subset \mathscr{P}$, a random set of $N_{test} = 10$ parameter vectors and estimate, for a given $\mu \in \mathscr{P}$, the efficiency of our method through these quantities:

$$err_N(\mu) = \sqrt{\Delta t \sum_{n=0}^{L} \|u^n(\mu) - u_N^n(\mu)\|_V^2}, \quad Err_N^{L^\infty} = \max_{\mu \in \mathscr{P}_{test}} (err_N(\mu)).$$

Note that $err_N(\mu)$ actually depends on Ψ_N ; for the sake of simplicity, we have omitted this reliance in the notation. As a test, we evaluate the influence of the parameters $\widetilde{N}_V, N_W$ determining the sizes of the bases Ψ_N and Ξ_N on $Err_N^{L^\infty}$. The results are plotted in the right diagram of Figure 3. In our example we numerically obtain $N_V = \widetilde{N}_V + N_W$ in all cases, indicating, that the primal snapshots and supremizers are linearly independent. We observe a reasonable good error decay when simultaneously increasing $\widetilde{N}_V$ and N_W, indicating that the reduced method is working well.

We also note that in our case, the size of the dual basis has a limited impact on the results.

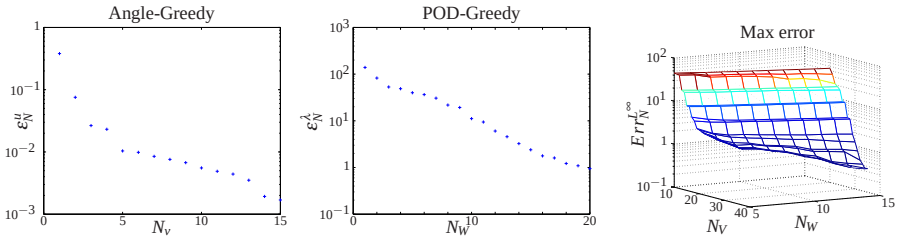

Fig. 3 Values of ε_N^u and ε_N^λ during the iterations of the greedy Algorithms 1 (left) and 2 (middle). Right: Values of $Err_N^{L^\infty}$ with respect to N_V and N_W.

References

Achdou Y, Pironneau O (2005) Computational methods for option pricing. Frontiers in applied mathematics, Society for Industrial and Applied Mathematics

Buffa A, Maday Y, Patera AT, Prud'homme C, Turinici G (2011) A priori convergence of the greedy algorithm for the parametrized reduced basis. To be published ESAIM-Math Model Numer Anal

Cont R, Lantos N, Pironneau O (2011) A reduced basis for option pricing. SIAM Journal on Financial Mathematics 2:287–316

Geiger C, Kanzow C (2002) Theorie und Numerik restringierter Optimierungsaufgaben. Springer-Lehrbuch Masterclass, Springer

Glowinski R (2008) Numerical Methods for Nonlinear Variational Problems. Scientific Computation, Springer

Haasdonk B (2011) Convergence rates of the POD-greedy method. Submitted SimTech Preprint 2011-23, University of Stuttgart, submitted.

Haasdonk B, Ohlberger M (2008) Reduced basis method for finite volume approximations of parametrized linear evolution equations. M2AN, Math Model Numer Anal 42(2):277–302

Haasdonk B, Salomon J, Wohlmuth B (2011) A reduced basis method for parametrized variational inequalities. submitted SimTech Preprint 2011-16, University of Stuttgart

Hager C, Hüeber S, Wohlmuth B (2010) Numerical techniques for the valuation of basket options and its greeks. J Comput Fin 13(4):1–31

Rozza G (2005) Shape design by optimal flow control and reduced basis techniques: applications to bypass configurations in haemodynamics. PhD thesis, EPFL, Lausanne

Veroy K, Prud'homme C, Rovas DV, Patera AT (2003) A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations. In: In Proceedings of 16th AIAA computational fluid dynamics conference, paper 2003-3847