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APPROXIMATELY RATIONALLY OR ELLIPTICALLY CONNECTED VARIETIES

CLAIRE VOISIN

To Slava Shokurov, on his 60th birthday

Abstract. We discuss a possible approach to the study of the vanishing of the Kobayashi pseudometric of a projective variety X, using chains of rational or elliptic curves contained in an arbitrarily small neighborhood of X in projective space for the Euclidean topology.

0. Introduction

In the paper [18], Lang made some conjectures concerning entire curves in complex projective varieties X. He conjectured for example that the Zariski closure of the locus in X swept-out by entire curves is equal to the locus swept-out by images of abelian varieties under non constant rational maps $\phi : A \dashrightarrow X$. When X is a very general quintic threefold in $\mathbb{P}^4$, this has been shown to be incompatible with Clemens conjecture [6] by the following arguments : (i) X contains countably many rational curves, and they are Zariski dense in X. (ii) On the other hand, X is not swept out by images of non constant generically finite rational maps $\phi : A \dashrightarrow X$ with $\dim A \geq 2$ (cf. [24]). (iii) Finally, if X was swept-out by elliptic curves, this would contradict Clemens conjecture on the discreteness of rational curves in X (see [8, Lecture 22] or [24]).

The goal of this note is to discuss and illustrate by examples several possible notions of *approximate rational connectedness* or *approximate elliptic connectedness* concerning complex projective manifolds. The general hope would be that approximately elliptically connected varieties are exactly varieties with trivial Kobayashi pseudodistance (cf. [16]). The main idea is that instead of looking at rational or elliptic curves (or abelian varieties) sitting in X, we should study rational or elliptic curves contained in arbitrarily small neighborhood of X in projective space (for the Euclidean topology).

We assume X is embedded in some projective space $\mathbb{P}^N$. We start with the following naïve definition:

Definition 0.1. *X is said to be approximately rationally connected in $\mathbb{P}^N$* **in the naïve sense** *if for any neighborhood (for the Euclidean topology) $U \subset \mathbb{P}^N$ of X, any two points of X are contained in a rational curve $C \subset U$.*

Remark 0.2. An equivalent definition is that any two points of X can be joined in an arbitrarily small neighborhood of X by a chain of rational curves, since such chains can be made irreducible by a small deformation in U, because U has positive tangent bundle.

The reason why this definition cannot be interesting, from the point of view of the study of the Kobayashi pseudodistance of X is the following fact.

Lemma 0.3. *i) Let Y be a connected projective variety. Then $X := Y \times \mathbb{P}^1$ is approximately rationally connected in the naïve sense in any projective embedding $X \subset \mathbb{P}^N$.*

ii) More generally, any connected variety X such that the union of rational curves contained in X is dense in X for the Euclidean topology is approximately rationally connected in the naïve sense in any projective embedding.

iii) Assume $X \subset \mathbb{P}^N$ has the property that for any neighborhood U of X, and for any point $x \in X$, there is a rational curve $C \subset U$ passing through x. Then X is approximately rationally connected in the naïve sense.

Proof. i) Indeed, let U be a neighborhood of X in $\mathbb{P}^N$. Then for any automorphism g of $\mathbb{P}^N$ sufficiently close to the identity, and any curve $C_y = y \times \mathbb{P}^1 \subset Y \times \mathbb{P}^1 = X$, the curve $g(C_y)$ is contained in U. It immediately follows that for any $x \in X$, the set of points z in X such that there is a rational curve $C \subset U$ passing through x and z contains an open neighborhood of the curve $C_{pr_1(x)} \setminus \{x\}$ in $X \setminus \{x\}$. Applying this argument to any $z \neq x$ in this neighborhood, we find that the set of points $x' \in X$ such that there is a chain of two rational curves C_1, $C_2 \subset U$ passing through x and x' contains an open neighborhood of x in X. As X is connected and compact, this easily implies that any two points of X can be connected by a chain of rational curves contained in U.

ii) Let U be a neighborhood of X in $\mathbb{P}^N$. For some $\epsilon > 0$, U contains $U_\epsilon(X) = \{y \in \mathbb{P}^N, d(y, X) < \epsilon\}$. For any point $x \in X$, there is by assumption a rational curve $C \subset X$ such that $d(x, C) < \epsilon$. Applying an automorphism g of $\mathbb{P}^N$ such that $d(g, Id) < \epsilon$, we can thus find a curve $g(C)$ contained in U and passing through x. We then conclude using iii).

iii) For any point $x \in X$, there is a rational curve $C_x \subset U$ passing through x. Applying to them automorphisms of $\mathbb{P}^N$ close to the identity and fixing a point $y \in C_x$, $y \neq x$, we conclude as in i) that there is a neighborhood V_x of x in X such that any point $y \in V_x$ is connected to x by a chain of two rational curves $C_x \cup g(C_x)$ contained in U. By compactness and connectedness of X, we conclude that any two points of X can be joined by a chain of rational curves contained in U. ■

Remark 0.4. The statement in i) of the above lemma shows that the Kobayashi pseudodistance $d_{X,K}$ of a subvariety $X \subset \mathbb{P}^N$ may be different from the limit over the open sets $U \subset \mathbb{P}^N$ of the restrictions $d_{U,K_{|X}}$. Indeed, in the above notation, if one chooses Y to be Kobayashi hyperbolic, then the Kobayashi pseudodistance of $X = Y \times \mathbb{P}^1$ is non zero, while the restrictions $d_{U,K_{|X}}$ are all 0.

The main defect of Definition 0.1 is the fact that it is not stable under surjective morphisms, that is, if $\phi : X \to Y$ is surjective and X is approximately rationally connected in the naïve sense, Y needs not satisfy the same property. This follows indeed from Lemmas 0.3, i), and 0.13. We could try to correct the definition by asking that not only X but also all varieties Y such that there is a surjective morphism from X to Y, are approximately rationally connected in the naïve sense (say in any projective embedding). However the following example shows that this is not strong enough:

Example 0.5. Consider the case where $X = (C \times S)/\iota$, where C is a curve of genus ≥ 2 with hyperelliptic involution i, S is a $K3$ surface which is the universal cover $S \to T$ of an Enriques surface, the involution ι acting on $C \times E$ acts as the

hyperelliptic involution on C and as the involution over T on S. This involution ι has no fixed points. By Lemma 0.3, ii) X is approximately rationally connected in any projective embedding, because rational curves are topologically dense in the fibers of $X \to \mathbb{P}^1$. Consider any surjective morphism $X \to Y$, where Y is normal. We claim that Y is approximately rationally connected in the naïve sense in any projective embedding. Indeed, if $\dim Y = 1$, one has $h^{1,0}(Y) = 0$, so $Y = \mathbb{P}^1$. If $\dim Y = 2$, as Y is dominated by $C \times S$, either it is dominated by the $K3$ surface S, or for each $c \in C$ the morphism from $c \times S$ to Y has for image a curve D, and then, Y is rationally dominated by a product $C \times \mathbb{P}^1$, since for any dominating rational map from a $K3$ surface to a smooth curve D, one has $D \cong \mathbb{P}^1$. In both cases, it is approximately rationally connected in the naïve sense in any projective embedding, using Lemma 0.3. The case where $\dim Y = 3$ is done similarly.

There are several ways to correct the naïve definition and we will propose two of them.

Definition 0.6. *X is* **strongly** *approximately rationally connected if for any embedding j of X in a product P of two projective spaces, $j(X)$ is approximately rationally connected inside P in the naïve sense (cf. Definition 0.1).*

Let us now give another variant of Definition 0.1, which might be easier to relate to the vanishing of the Kobayashi pseudometric. For any smooth $X \subset \mathbb{P}^N$, the projectivized tangent bundle $\mathbb{P}(T_X)$ is naturally contained in the projectivized tangent bundle $\mathbb{P}(T_{\mathbb{P}^N})$. Any curve $C \subset \mathbb{P}^N$ has a tangent lift $\widetilde{C}$ to $\mathbb{P}(T_{\mathbb{P}^N})$. Let $\widetilde{U}$ be a neighborhood of $\mathbb{P}(T_X)$ in $\mathbb{P}(T_{\mathbb{P}^N})$. We will say that a curve $C \subset \mathbb{P}^N$ is $\widetilde{U}$-close to X if $\widetilde{C} \subset \widetilde{U}$. Hence, not only C is close to X, but its tangent space at any point is close to T_X. The following definition takes into account the cohomology classes of the curves considered. Here we use the fact that if U is a tubular neighborhood of X in $\mathbb{P}^N$, U and X have the same homology. Let now $\widetilde{U}$ be a neighborhood of $\mathbb{P}(T_X)$ in $\mathbb{P}(T_{\mathbb{P}^N})$. Note that if for any U, $\widetilde{U}$, and for any point x of X, there passes a curve $C_x \subset U$ passing through x, which is $\widetilde{U}$-close to X, C_x can be chosen to vary locally continuously with x, hence to have a cohomology class $[C] \in H_2(U, \mathbb{Z}) = H_2(X, \mathbb{Z})$ locally independent of x. By considering chains, and by smoothing them, we conclude using arguments similar to the proof of Lemma 0.3, that if X is connected, we can assume that the class of the covering curves C_x is in fact independent of x.

Definition 0.7. *A connected variety $X \subset \mathbb{P}^N$ is* **cohomologically** *approximately rationally connected if for any U, $\widetilde{U}$ as above, through any point x of X there passes a rational curve C_x contained in U and $\widetilde{U}$-close to X, of class $[C]$ independent of x, and furthermore the convex cone generated by the $(n-1, n-1)$-components of classes $[C_i] \in H_2(U, \mathbb{Z}) = H_2(X, \mathbb{Z}) = H^{2n-2}(X, \mathbb{Z})$ of such covering curves $C_{i,x}$ contains a strongly positive class.*

Here we say that a class of type $(n-1, n-1)$ on an n-dimensional variety X is strongly positive if it has positive intersection with pseudoeffective $(1,1)$-classes (represented by weakly positive currents of type $(1,1)$). When the class belongs to the space $N_1(X)$ generated by curve classes, this is equivalent to being in the interior of the convex cone generated by classes of moving curves (cf. [1]).

Remark 0.8. Since the class of a curve $C \subset U$ is the class of the current of integration over C, this is the class in U of a closed current of type $(N-1, N-1)$,

$N = \dim U$. Using a differentiable retraction $\pi : U \to X$, we also have the current of integration over $\pi(C)$, whose class is the cohomology class $[C]$ above. This last class is not in general of type $(n-1, n-1)$ (see examples in section 1). However, when $\widetilde{U}$ is small, it is close to be of type $(n-1, n-1)$, as C is $\widetilde{U}$- close to X.

The cohomological condition in Definition 0.7 addresses the weakness of Definition 0.1: indeed, the rational curves $g(C_y)$ used in the proof of Lemma 0.3, i) are in the same class as the fibers of pr_1, and this class is not strongly positive.

Remark 0.9. If $H^2(X, \mathbb{Q}) = \mathbb{Q}$, Definitions 0.7 and 0.1 are quite close. Indeed, in this case, the cohomological condition in Definition 0.7 is empty, and we thus just ask that for any neighborhoods U of X, $\widetilde{U}$ of $\mathbb{P}(T_X)$, and any general point $x \in X$, there is a rational curve in U passing through x and $\widetilde{U}$-close to X. The last condition is satisfied by the examples of Lemma 0.3, i) (but they do not satisfy $H^2(X, \mathbb{Q}) = \mathbb{Q}$).

We believe that these notions should be related to the triviality of the Kobayashi pseudodistance (see [16]) of X, although it is quite hard to establish precise relations. This is due to the notorious difficulty to localize Ahlfors currents or Brody curves (see [11], [12], [21] for important progresses on this subject). The motivation for introducing these geometric definitions is the lack of progress on the understanding of complex varieties with vanishing Kobayashi pseudodistance (by contrast to the recent progresses made on the Green-Griffiths conjecture, eg for high degree hypersurfaces in projective space, see [10]).

However, the following easy lemma shows that approximate rational connectedness in either of the above senses is too restrictive topologically:

Lemma 0.10. *Abelian varieties are not approximately rationally connected (in the naïve sense). More precisely, if $A \subset \mathbb{P}^N$ is an abelian variety, and U is a tubular neighbourhood of A, U does not contain any rational curve.*

Proof. A and U have the same homotopy type. Hence, as $\pi_2(A) = 0$, we also have $\pi_2(U) = 0$. Thus a rational curve contained in U should be homologous to 0 in U, hence in $\mathbb{P}^N$, which is absurd. ■

This lemma (and the fact that abelian varieties have trivial Kobayashi pseudodistance) is the motivation for the following variant of Definitions 0.6, 0.7, 0.1:

Definition 0.11. *i) $X \subset \mathbb{P}^N$ is said to be approximately elliptically connected in the naïve sense if for any neighborhood U of X in $\mathbb{P}^N$, any two points $x, y \in X$ can be joined by a chain of elliptic curves in U.*

ii) $X \subset \mathbb{P}^N$ is said to be strongly approximately elliptically connected if Definition 0.6 holds with rational curves replaced by chains of elliptic curves.

iii) $X \subset \mathbb{P}^N$ is said to be cohomologically approximately elliptically connected if Definition 0.7 holds with rational curves replaced by elliptic curves.

Remark 0.12. If X is connected and approximately rationally (resp. elliptically) connected in the cohomological sense, then it is also approximately rationally (resp. elliptically) connected in the naïve sense, as follows from Lemma 0.3, iii) (or its obvious extension to the elliptic case).

Properties i), ii) and iii) are satisfied by (very general) Calabi-Yau varieties obtained as the double cover of projective space $\mathbb{P}^n$ ramified along a degree $2n+2$

hypersurface as they are covered (in infinitely many different ways) by families of elliptic curves (cf. [25]).

We will give however in section 1 (see Theorem 1.5) examples of varieties containing only finitely many rational curves, but which are approximately rationally connected in the cohomological sense. Similarly, abelian varieties are approximately cohomologically elliptically connected (see Theorem 1.1), while the general ones do not contain any elliptic curve. From this one can deduce that Fano varieties of lines of very general cubic threefolds satisfy this property, as they are covered in infinitely many different ways by a 2-dimensional family of surfaces birationally equivalent to abelian surfaces (cf. [23]).

Our hope is that approximately elliptically connected varieties in one of the strengthened senses described above are the same as the "special varieties" invented by Campana [3] (which are also conjectured to be the complex projective varieties with trivial Kobayashi pseudodistance). Notice that Demailly in [9] gives a description of the Kobayashi pseudometric of X involving algebraic curves in X, together with their intrinsic hyperbolic metric. This says that if X has a trivial Kobayashi pseudodistance, there are many algebraic curves in X for which the intrinsic hyperbolic metric is small compared to the metric obtained by restricting a given metric on X. In particular this compares the genus of these curves to their degrees, but this does not say anything on the genus alone.

To start with, we have the following easy lemma.

Lemma 0.13. *If a projective variety X is Kobayashi hyperbolic, it is not approximately elliptically connected (in the naïve sense) in any projective embedding.*

Proof. Indeed, if there is an elliptic curve E_n in any neighborhood $U_{\epsilon_n}(X)$ of X in $\mathbb{P}^N$, with $\lim_{n\to\infty}\epsilon_n = 0$, we can choose for each n a holomorphic map $f_n : \Delta \to E_n \to \mathbb{P}^N$ from the unit disk to E_n, such that $|f_n'(0)| = n$, where the modulus of the derivative is computed with respect to the ambient metric. By Brody's lemma [2], there is an entire curve in $\mathbb{P}^N$ obtained as the limit of a subsequence of the f_n's conveniently reparametrized. This entire curve is contained in $\cap_n U_{\epsilon_n}(X) = X$ and X is not Kobayashi hyperbolic.

■

We will also prove in section 2 the following property:

Proposition 0.14. *(see Proposition 2.3) If X is strongly approximatively rationally (resp. elliptically) connected and $\phi : X \to Y$ is a surjective morphism, then Y is approximately rationally (resp. elliptically) connected in the naïve sense. In particular, Y is not Kobayashi hyperbolic.*

We do not know whether this result holds for cohomological approximate elliptic or rational connectedness, but we know by Lemmas 0.3 and 0.13 that it does not hold for naïve approximate elliptic or rational connectedness.

Next, as the definitions are stable under étale covers (see Lemma 2.1), one crucial point needed in order to make the class of approximately elliptically connected varieties close to Campana's special manifolds would be the following:

Conjecture 0.15. *A variety of general type is not approximately elliptically connected in the naïve sense.*

As we do not even know that elliptic or rational curves are not topologically dense in a variety of general type (a weak version of Green-Griffiths-Lang conjecture),

this question seems to be out of reach at the moment. For example, we know that general hypersurfaces in $\mathbb{P}^n$ of degree $\geq 2n-2$ do not contain any rational curve for $n \geq 4$ (see [22], giving an optimal bound which is slightly better than [7]) and that the only rational curves contained in general hypersurfaces in $\mathbb{P}^n$ of degree $2n-3$, for $n \geq 6$, are lines (cf. [19]), but general hypersurfaces in $\mathbb{P}^n$ of degree $n+2 \leq d \leq 2n-4$ are not known to carry finitely many families of rational curves and not even known not to contain a dense set covered by rational curves.

In the other direction, we do not know if rational curves in Calabi-Yau hypersurfaces are topologically dense, except in dimension 2, that is for $K3$ surfaces, for a Baire second category subset of the moduli space by [5]. One question implicitly raised in the present paper is whether it is easier to study rational (or elliptic) curves contained in small neighbourhoods of such hypersurfaces.

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1. Some examples

Let us give two examples of classes of varieties which do not contain many rational (resp. elliptic curves) but are approximately rationally (resp. elliptically) connected in the cohomological sense.

Theorem 1.1. *Abelian varieties are approximately elliptically connected (for any projective embedding) in the cohomological sense.*

Proof. Let $A \subset \mathbb{P}^N$ and let $\widetilde{U} \subset \mathbb{P}(T_{\mathbb{P}^N})$ be an Euclidean neighborhood of $\mathbb{P}(T_A)$. For a small deformation A_ϵ of A in $\mathbb{P}^N$, $\mathbb{P}(T_{A_\epsilon}) \subset \mathbb{P}(T_{\mathbb{P}^N})$ is a small deformation of $\mathbb{P}(T_A)$, hence will be contained in $\widetilde{U}$ when the deformation is small enough. It follows that for any curve $C \subset A_\epsilon$, its tangent lift $\widetilde{C}$ is contained in $\widetilde{U}$.

We use now the well-known fact that abelian varieties isogenous to a product $E_1 \times \ldots \times E_n$, where each E_i is an elliptic curve, are dense (for the Euclidean topology) in the moduli space of n-dimensional polarized abelian varieties. On the other hand, inside $E_1 \times \ldots \times E_n$, the elliptic curves obtained as the images of E_i under the natural morphisms $\phi_i : E_i \to E_1 \times \ldots \times E_n$, $x \mapsto (e_1, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)$, for given points $e_j \in E_j$ can be chosen to pass through any point. Of course, the same is true for any abelian variety isogenous to E^n.

Let now $A_\epsilon \subset \mathbb{P}^N$ be a sufficiently small deformation of A which is isogenous to $E_1 \times \ldots \times E_n$. Then the elliptic curves $\phi_i(E_i)$ contained in A_ϵ sweep-out A_ϵ and their tangent lift is contained in $\widetilde{U}$. For any point $x \in A$, we can find an automorphism of $\mathbb{P}^N$ which is close to the identity and such that $x \in g(A_\epsilon)$. Thus, the curves $g(\phi(E))$ can be chosen to pass through any point of A and to have their tangent lift contained in $\widetilde{U}$.

Finally, when there is no nonzero morphism between E_i and E_j for $i \neq j$, the classes of the curves $\phi_i(E_i)$ generate the space of Hodge classes $Hdg^{2n-2}(A_\epsilon)$. In particular, a convex combination of these classes contains the class $h_{A_\epsilon}^{n-1}$, where $h_{A_\epsilon} = c_1(\mathcal{O}_{A_\epsilon}(1))$. Using the canonical isomorphism $H^{2n-2}(A_\epsilon, \mathbb{R}) \cong H^{2n-2}(A, \mathbb{R})$

we conclude that a convex combination of these classes, transported to A, contains the class h_A^{n-1}, where $h_A = c_1(\mathcal{O}_A(1))$. Taking the $(n-1, n-1)$-component, we conclude as well hat a convex combination of the $(n-1, n-1)$-components of these classes transported to A contains the class h_A^{n-1}, which finishes the proof.

■

Remark 1.2. The example of abelian varieties also illustrates why the notion of approximate elliptic (or rational) connectedness might be easier to study than the property of being swept out by entire curves, or of having arbitrarily small neighborhoods in $\mathbb{P}^N$ swept-out by entire curves. Indeed, there are of course a lot of entire curves in abelian varieties. However, the elliptic curves E exhibited above, contained in a small deformation of a given abelian variety A in projective space, are much more reasonable, since their induced metric is uniformly equivalent to their flat metric k_E (normalized so that the volume is equal to the degree). This is because the flat metric of A is equivalent to the induced metric on A, which easily implies that there is a flat metric h_{A_ϵ} on A_ϵ equivalent to the induced metric on A_ϵ with constants depending only on A. The restriction $h_{A_\epsilon|E}$ of this flat metric to E is a flat metric on E which is then uniformly equivalent to the induced metric $h_{|E}$. In other words we have $ch_{|E} \leq h_{A_\epsilon|E} \leq Ch_{|E}$, for some constants c, C depending only on A. Integrating over E the corresponding Kähler forms, we get $c \deg E \leq \int_E \omega_{A_\epsilon|E} \leq C \deg E$, which tells, since $\omega_{A_\epsilon|E}$ is the flat metric, that the normalized metric k_E, which is equal to $\frac{deg\, E}{\int_E \omega_{A_\epsilon|E}} h_{A_\epsilon|E}$ satisfies

$$\frac{c}{C} h_{|E} \leq k_E \leq \frac{C}{c} h_{|E}. \tag{1.1}$$

One interesting question is the following:

Question 1.3. *i) Does any elliptic curve close enough in the usual topology to an abelian subvariety A of $\mathbb{P}^N$ satisfy (1.1) for some constants depending only on A?*

ii) Does the above question have an affirmative answer for elliptic curves $\widetilde{U}$-close to A, for a small neighborhood $\widetilde{U}$ of $\mathbb{P}(T_A)$ in $\mathbb{P}(T_{\mathbb{P}^N})$?

An affirmative answer to these questions would have the following consequence:

Proposition 1.4. *Assume Question 1.3, i) has an affirmative answer for a given abelian variety $A \subset \mathbb{P}^N$. Then a subvariety $X \subset A$ which is of general type is not approximately elliptically connected in the naïve sense. If Question 1.3, ii) has an affirmative answer for $A \subset \mathbb{P}^N$, then a subvariety $X \subset A$ which is of general type is not approximately elliptically connected in the cohomological sense.*

Proof. Indeed, one knows by [15] that X satisfies the Green-Griffiths conjecture, so that the union of the entire curves contained in X is not Zariski dense in X. On the other hand, assume that for any $x \in X$, there is an elliptic curve $E_n \subset \mathbb{P}^N$ passing through x such that $E_n \subset V_{\frac{1}{n}}(X) = \{y \in \mathbb{P}^N,\, d(y, X) < \frac{1}{n}\}$. Then consider the flat uniformization $f_n : \mathbb{C} \to E_n$ such that $f_n(0) = x_n$ and $f_n^* k_{E_n}$ is the standard metric (so f_n is defined up to the action of $U(1)$). Then if Question 1.3, i) has a positive answer, as E_n is close to A, the derivatives $|f_n'|$ (computed with respect to the ambient metric on $\mathbb{P}^N$) are bounded above and below in modulus, so that we can extract a subsequence which converges uniformly on compact sets of $\mathbb{C}$ to a non constant entire curve passing through x and contained in X. As x is arbitrary, this contradicts Kawamata's result. Similarly, if question ii) has an affirmative answer,

elliptic curves $\widetilde{U}$-close to A satisfy the above property for $\widetilde{U}$ small. This is then also true for elliptic curves $\widetilde{U}$-close to X. Hence, by the above argument, one cannot have an elliptic curve $\widetilde{U}$-close to X passing through any point of X for $\widetilde{U}$ arbitrarily small. ■

The second example we will consider is the example of elliptic surfaces with finitely many rational curves. More precisely, we consider a very general hypersurface $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2l, 3)$ with $l \geq 2$.

Theorem 1.5. *i) S contains finitely many rational curves, namely the singular fibers of the elliptic fibration $f := pr_{1|S} : S \to \mathbb{P}^1$.*

ii) S is approximately rationally connected (relative to the Segre embedding) in the cohomological sense.

Proof. i) A smooth surface $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 3)$ is a $K3$ surface, which contains only countably many rational curves. If Σ is chosen to be very general,

$$\operatorname{Pic} \Sigma = (\operatorname{Pic} (\mathbb{P}^1 \times \mathbb{P}^2))_{|\Sigma}$$

hence no algebraic curve in Σ has degree 1 over $\mathbb{P}^1$. Take such a Σ and consider a very general morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1_0$ of degree l. Then the surface $S = \Sigma \times_{\mathbb{P}^1_0} \mathbb{P}^1$ is of bidegree $(2l, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ and for each rational curve $C \subset \Sigma$ not contained in a fiber of $\Sigma \to \mathbb{P}^1_0$, the curve $C \times_{\mathbb{P}^1_0} \mathbb{P}^1 \subset S$ is irreducible of positive geometric genus. Hence S does not contain any other rational curves than those contained in a fiber of $pr_1 : S \to \mathbb{P}^1$.

ii) First of all, we apply the criterion for density of the Noether-Lefschetz locus due to M. Green (cf. [27, Proposition 5.20]) to show that arbitrarily small deformations $S_\epsilon \subset \mathbb{P}^1 \times \mathbb{P}^2$ of a general surface S as above admit sections of $f_\epsilon : S_\epsilon \to \mathbb{P}^1$.

We recall the statement of the criterion in the form which we will use here : consider the universal family $\pi : \mathcal{S} \to B$, $\mathcal{S} \subset B \times \mathbb{P}^1 \times \mathbb{P}^2$ of such smooth surfaces. Let $0 \in B$ and let S_0 be the fiber $\pi^{-1}(0)$. Then we have:

Proposition 1.6. *Assume there is a $\lambda \in H^1(S_0, \Omega_{S_0})$ such that the map $\overline{\nabla}_\lambda : T_{B,0} \to H^2(S_0, \mathcal{O}_{S_0})$ is surjective. Then for any open set $U \subset B$ (for the Euclidean topology), the set of classes $\alpha \in H^2(S_0, \mathbb{Z})$ which become algebraic on some fiber $\mathcal{S}_t$ for some $t \in U$ contains the set of integral points in a non-empty open cone of $H^2(S_0, \mathbb{R})$.*

In this statement, the map $\overline{\nabla}_\lambda$ is the composition of the Kodaira-Spencer map $T_{B,0} \to H^1(S_0, T_{S_0})$ and the cup-product/contraction map $\lambda \lrcorner : H^1(S_0, T_{S_0}) \to H^2(S_0, \mathcal{O}_{S_0})$.

Using the Griffiths description of the IVHS of hypersurfaces (see [27, 6.2.1]), we find that this map identifies to the multiplication

$$\mu_{P_\lambda} : R_{2l,3}(S_0) \to R_{6l-2,6}(S_0)$$

by a certain polynomial $P_\lambda \in H^0(S_0, \mathcal{O}_{S_0}(4l - 2, 3))$, where $R(S_0)$ is the Jacobian ring of the defining equation of S_0. One checks explicitly that for generic S_0 and generic P_λ, the map μ_{P_λ} is surjective.

Using Proposition 1.6, we conclude that for generic S_0 and for any small simply connected neighborhood U of 0 in B, there is a non-empty open cone C in $H^2(S_0, \mathbb{R})$ such that any integral class $\alpha \in H^2(S_0, \mathbb{Z}) \cap C$ becomes (by parallel transport) algebraic on some fiber S_t, for some $t \in U$.

Let now $F \in H^2(S_0, \mathbb{Z})$ be the class of a fiber of f_0. Then its parallel transport to S_t is the class F_t of a fiber of f_t and the elliptic fibration $f_t : S_t \to \mathbb{P}^1$ admits a section if and only if there is an algebraic class $\alpha \in H^2(S_t, \mathbb{Z}) \cong H^2(S_0, \mathbb{Z})$ such that $< \alpha, F >= 1$. Note furthermore that the class $h := c_1(pr_2^* \mathcal{O}_{\mathbb{P}^2}(1))$, which is of degree 3 on the class F, remains algebraic on any deformation S_t of S_0 in $\mathbb{P}^1 \times \mathbb{P}^2$. Hence, f_t has a section if and only if there is an algebraic class $\alpha \in H^2(S_t, \mathbb{Z}) \cong H^2(S_0, \mathbb{Z})$ such that $< \alpha, F >= 1$ mod. 3. To conclude that the set of surfaces S_t having a section is dense, we then use the following lemma (which is used implicitly in [20, Remark 1]).

Lemma 1.7. *For any non-empty open cone $C \subset H^2(S_0, \mathbb{R})$, there are elements in $C \cap \{\alpha \in H^2(S_0, \mathbb{Z}), < \alpha, F >= 1$ mod. $3\}$.*

The second and final step of the proof is the following lemma, due to Chen and Lewis [5]:

Lemma 1.8. *Let $f : S \to \mathbb{P}^1$ be an elliptic fibration, and L a line bundle on S_t, of degree $d \neq 0$ on fibers. Assume the fibers of f are irreducible and reduced and that the monodromy group $\pi_1(\mathbb{P}^1_{reg}, t_0) \to Aut\, H^1(S_{t_0}, \mathbb{Z})$ is the full symplectic group $\mathrm{SL}(2, \mathbb{Z})$. Then for any section $\sigma : \mathbb{P}^1 \to S$ of f such that the class $d[\sigma] - c_1(L)$ is not of torsion, the curves $C_n := \sigma_n(\mathbb{P}^1)$ have the property that $\cup_n C_n$ is dense for the Euclidean topology in S.*

Here $\sigma_n := \mu_n \circ \sigma$, where $\mu_n : S \dashrightarrow S$ is the self-rational map which to $x \in S$ with $f(x) = u \in \mathbb{P}^1$ associates the point y of the fiber S_u such that $(dn+1)x - nL_{|S_u} = y$ in $\mathrm{Pic}\, S_u$.

■

The proof of Theorem 1.5 is now concluded as follows: let S_0 be generic. Let U be a neighborhood of S_0 in $\mathbb{P}^5$ and $\widetilde{U}$ be a neighborhood of $\mathbb{P}(T_{S_0})$ in $\mathbb{P}(T_{\mathbb{P}^5})$. As S_0 is generic, the fibration $f_0 : S_0 \to \mathbb{P}^1$ is a Lefschetz fibration with irreducible fibers and with monodromy equal to the full symplectic group. These properties remain true for a small deformation of S_0.

By Lemma 1.7, there is a surface S_t which is a small deformation of S_0 in $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ such that $f_t : S_t \to \mathbb{P}^1$ has a section C. In particular S_t is contained in U and $\mathbb{P}(T_{S_t})$ is contained in $\widetilde{U}$. Thus any curve contained in S_t is $\widetilde{U}$-close to S_0. By lemma 1.8, the union $\cup_n C_n$ is dense in S_t. Applying automorphisms of $\mathbb{P}^5$ close to the identity if needed, we conclude that for any point $x \in S_0$, there is an arbitrarily small deformation S_t of S_0 in $\mathbb{P}^5$ containing a section C_n passing through x. These curves are contained in U and $\widetilde{U}$-close to S_0.

To conclude, it only remains to check that a convex combination of the $(1,1)$-components of the classes of (a variant of) the curves C_n (transported to S_0) contains an ample class on S_0. For this we observe that the sum $2[C_n] + [C_{-2n-1}] \in H^2(S_t, \mathbb{Z}) = H^2(S_0, \mathbb{Z})$ is a combination of the class h and the class F. The coefficient in h is obviously positive. This class may not be ample, but we observe that the class F is in S_0 the class of a rational curve (namely a singular fiber). Instead of the curves C_n, we can thus consider curves C'_n obtained by smoothing in $\mathbb{P}^5$ the union of C_n and a covering of large degree of a singular fiber. The resulting curves can be chosen to stay $\widetilde{U}$-close to S_0 and the sum $2[C'_n] + [C'_{-2n-1}] \in H^2(S_t, \mathbb{Z}) = H^2(S_0, \mathbb{Z})$ is an ample class.

■

These two examples might give the feeling that the natural way to produce elliptic or rational curves in a (arbitrarily) small neighborhood of a subvariety $X \subset \mathbb{P}^N$ is by studying elliptic or rational curves lying in some small deformation X_ϵ of X in $\mathbb{P}^N$. This is however not true at all, as shows the following example, obtained by mimicking the trick of [26]: start with an abelian variety $A \cong \mathbb{C}^n/\Gamma$ admitting an endomorphism ϕ such that the corresponding endomorphism $\phi_{\mathbb{Q}}$ of $\Gamma_{\mathbb{Q}} = H_1(A, \mathbb{Q})$ has only eigenvalues of multiplicity 1. Then it is immediate to see that the pair (A, ϕ) is rigid. Furthermore, we can construct such ϕ's so that A does not contain any elliptic curve (one considers for example simple abelian varieties with complex multiplication).

Starting from such pair (A, ϕ) we consider the projective variety X obtained by blowing-up successively $A \times A \times \mathbb{P}^1$ along $A \times x_0 \times t_1$, $x_1 \times A \times t_2$, $\operatorname{diag} A \times t_3$, $\operatorname{graph} \phi \times t_4$, for generic choices of x_0, x_1 and distinct points $t_1, \dots, t_4 \in \mathbb{P}^1$. Choose a projective embedding of X in $\mathbb{P}^N$. As A does not contain elliptic or rational curves, the only rational or elliptic curves contained in X are contained in the union D of the exceptional divisors of the blow-ups or in proper transforms of the fibers of the map $p_{A\times A} \circ \tau : X \to A \times A$, where $\tau : X \to A \times A \times \mathbb{P}^1$ is the blow-up map. Furthermore, the deformations of X preserve the exceptional divisors hence are all of the same type as X, and as the pair (A, ϕ) is rigid, it follows that elliptic or rational curves contained in a small deformation X_ϵ of X in $\mathbb{P}^N$ are close (for the usual topology) to either a curve contained in D_ϵ, or to a fiber of the map $p_{A\times A} \circ \tau : X \to A \times A$. Hence, for a general point x of X, elliptic curves passing through x and contained in a small deformation X_ϵ of X have for homology class a multiple of the class of a fiber of $p_{A\times A} \circ \tau$. As this class is not strongly positive, we cannot use such curves to prove that X is approximately elliptically connected in the cohomological sense.

We have however the following result:

Lemma 1.9. *X is approximately elliptically connected (for any projective embedding $X \subset \mathbb{P}^N$) in the cohomological sense.*

Proof. Indeed, recall that $\tau : X \to A \times A \times \mathbb{P}^1$ is the blow-up map. Choose a neighborhood $\widetilde{U}$ of $\mathbb{P}(T_X)$ in $\mathbb{P}(T_{\mathbb{P}^N})$ and let $x \in X$. Let $\tau(x) = (y, t)$ with $y \in A \times A$ and $t \in \mathbb{P}^1$. We choose x so that $t \notin \{t_1, \dots, t_4\}$, so that in particular x does not belong to the exceptional divisor of τ, and there is a copy $A \times A \times t \subset X$ passing through x. Then by theorem 1.1, $A \times A \times t$ is approximately elliptically connected in $\mathbb{P}^N$ in the infinitesimal and cohomological sense. Thus there is an elliptic curve E_x passing through x, whose tangent lift is contained in $\widetilde{U}$. Furthermore the class of these elliptic curves can be chosen to be independent of $x \in X$ and a convex combination of them generate an ample class on $A \times A \times t \subset X$. On the other hand, assume now that the line $y \times \mathbb{P}^1$ does not meet the locus of $A \times A \times \mathbb{P}^1$ blown-up under τ. Then this line is a rational curve C_x contained in X and passing through x. In $\mathbb{P}^N$, we can smooth the curve $E_x \cup_x C_x$ and it is easy to see that the smoothed curve can be chosen to pass through x and to stay $\widetilde{U}$-close to X. This proves the result since a convex combination of the classes $[C_x] + [E_x]$ is strongly positive. ∎

2. Stability results and further questions

We start with the following results concerning stability under étale covers. Here P is any smooth complex projective variety.

Proposition 2.1. *Assume $X \subset P$ is connected and approximately rationally or elliptically connected in the naïve, resp. cohomological sense. Let $U \subset P$ be a neighborhood of X which has the same homotopy type as X (eg a tubular neighborhood). Then any étale connected proper cover $X' \to X$ is approximately rationally connected in the corresponding neighborhood $f : U' \to U$ of U, in the naïve, resp. cohomological sense.*

Proof. We give the proof for the rational case, the elliptic case working similarly, due to the fact that étale covers of elliptic curves are again elliptic curves. Let us first consider approximate connectedness in the naïve sense. Any small neighborhood $V_\epsilon(X')$ of X' in U' contains a neighborhood of the form $f^{-1}(U_{\epsilon'}(X))$, for some ϵ'. Let now x', $y' \in X'$ and let x, y be their images in X. There is a smooth rational curve C contained in $U_{\epsilon'}(X)$, (we assume here $\dim U \geq 3$ since the case where $\dim X = 1$ is completely understood by Lemma 0.13,) and containing x and y. The inverse image of C in $f^{-1}(U_{\epsilon'}(X))$ is a finite union of rational curves, and one of them, say $C_{x'}$, passes through x'. As $C_{x'}$ maps onto C, it contains one point y'' of X' over y. In conclusion, the set of points $y'' \in X'$ which are joined to x' by a rational curve in $V_\epsilon(X')$ contains an open subset $W_{x'} \subset X'$ which maps onto X. For any point $z \in W_{x'}$, the open set W_z must be equal to $W_{x'}$, since a point in W_z is joined to x' by a chain of two rational curves passing through z, and this chain can be smoothed. We may assume that the cover $f : X' \to X$ is Galois. Let $g \in \mathrm{Gal}\,(X'/X)$. Then $gW_{x'} = W_{gx'}$, and by the above we conclude that X' is the finite disjoint union of open sets of the form $W_{x'}$. As X' is connected, it follows that $X' = W_{x'}$.

For the approximate rational connected in the cohomological sense, we have to add the following argument: recall from Definition 0.7 that we need to have for any tubular neighborhood V' of X', any neighborhood $\widetilde{V}'$ of $\mathbb{P}(T_{X'})$ in $\mathbb{P}(T_{U'})$, and any point $x' \in X'$ a finite number of rational curves $C_{i,x'}$ passing through x', contained in V', $\widetilde{V}'$-close to X', such that the class of the curve $C_{i,x'}$ does not depend on x' and the $(n-1, n-1)$-part of the class $\sum_i n_i[C_{i,x'}] \in H_2(V', \mathbb{Z}) = H_2(X', \mathbb{Z})$ is strongly positive for some $n_i > 0$. Of course we may assume that V' and $\widetilde{V}'$ are inverse images under f of similar neighborhoods V, $\widetilde{V}$ for $X \subset U$. If we start from such data $C_{i,x}$ for $x \in X$ and for the neighborhoods V, $\widetilde{V}$, we observe now that the class in $H_2(X', \mathbb{Z})$ of the unique lift $C_{i,x'}$ of $C_{i,x}$ passing through x' does not depend on x', because it can be chosen to vary continuously with x' and X' is connected. In particular, we find that the sum $\sum_{g \in \mathrm{Gal}\,(X'/X)} g_*[C_{i,x'}]$ is equal to $\mathrm{card}\, G\, [C_{i,x'}]$ and it is the pull-back under f of the class $[C_{i,x}] \in H_2(X, \mathbb{Z})$. The fact that there exist such $C_{i,x'}$'s with $\sum_i n_i[C_{i,x'}]^{n-1,n-1} \in H_2(V', \mathbb{R}) = H_2(X', \mathbb{R})$ strongly positive is thus equivalent to the fact that there exist such $C_{i,x}$'s with $\sum_i n_i[C_{i,x}] \in H_2(V, \mathbb{R}) = H_2(X, \mathbb{R})$ strongly positive. ■

The following consequence of Proposition 2.1 illustrates the power of the cohomological condition in Definition 0.11 :

Corollary 2.2. *The varieties X in Example 0.5 are not approximately elliptically connected in the cohomological sense in any projective embedding.*

Proof. Recall that X is a quotient of a product $S \times C$ by a free involution ι, where $g(C) \geq 2$. If it was approximately elliptically connected in the cohomological sense in some projective embedding, by Proposition 2.1, the product $S \times C$ would be approximately rationally connected in the cohomological sense in some embedding. In particular, there would be elliptic curves E_i contained in a tubular neighborhood U of $S \times C$, with the property that some combination of the $(2,2)$-component of the classes $[E_i] \in H_2(U, \mathbb{Z}) = H_2(S \times C, \mathbb{Z}) = H^4(S \times C, \mathbb{Z})$ is strongly positive. But for any continuous map ϕ from an elliptic curve E to a genus ≥ 2 curve C, the induced map $\phi_* : H_2(E_i) \to H_2(C)$ is trivial. Thus the classes $pr_{2*}[E_i]$ vanish in $H_2(C, \mathbb{Z})$ and for any line bundle L of positive degree on C, $< pr_2^* c_1(L), [E_i] >= 0$. Thus $\sum_i n_i < pr_2^* c_1(L), [E_i]^{2,2} >= 0$ for any n_i's, which provides a contradiction. ■

Concerning the stability under morphism, we have the following easy result:

Proposition 2.3. *Let $\phi : X \to Y$ be a surjective morphism, where X and Y are smooth projective and $\dim Y > 0$. If X is strongly approximately rationally (resp. elliptically) connected, Y is approximately rationally (resp. elliptically) connected in the naïve sense in any projective embedding.*

Proof. Let $j_Y : Y \hookrightarrow \mathbb{P}^N$ be a projective embedding. Choose a projective embedding j_X of X in some projective space $\mathbb{P}^M$, and consider the embedding $j'_X = (j_X, j_Y \circ \phi) : X \hookrightarrow P = \mathbb{P}^M \times \mathbb{P}^N$. By assumption $j'_X(X)$ is approximately rationally (resp. elliptically) connected in the naïve sense in P. The morphism $pr_2 : P \to \mathbb{P}^N$ sends rational curves (resp. chain of elliptic curves) passing through any two given points of $j'_X(X)$ and contained in a sufficiently small neighborhood of $j'_X(X)$ to rational curves (resp. chain of elliptic curves) passing through any two given points of $j_Y(Y)$ and contained in a given neighborhood of Y in $\mathbb{P}^N$. ■

Corollary 2.4. *A fibration $X \to Y$ over a Kobayashi hyperbolic variety Y is not strongly approximately elliptically connected.*

Proof. Indeed, if it was strongly approximately elliptically connected, the variety Y would be approximately elliptically connected, hence in particular not Kobayashi hyperbolic by Lemma 0.13. This gives a contradiction. ■

As we mentioned in the introduction, Proposition 2.3 is not true for naïve approximate rational or elliptic connectedness. This implies a negative answer to the following question:

Question 2.5. *Let $Z \subset \mathbb{P}^N$ be the Segre embedding of $\mathbb{P}^k \times \mathbb{P}^l$ for some integers k, l. Fix a distance d on $\mathbb{P}^N$. Is it true that for any $\epsilon > 0$, there exists $\eta(\epsilon) > 0$, such that $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$, and that for any rational (resp. elliptic) curve C contained in $U_\epsilon(Z)$, there is a rational (resp. elliptic) curve $C' \subset Z$ such that $d(C, C') := \mathrm{Sup}_{c \in C, c' \in C'}\{d(c, C'), d(c', C)\} \leq \eta$?*

If we look at the proof of Lemma 0.3, i), a counterexample is obtained by constructing in a small neighborhood of the union of a large number of lines $l_i = x_i \times \mathbb{P}^1$, $i = 1, \ldots, M$ contained in Z, with $d(l_i, l_{i+1}) < \epsilon$, a chain $\cup_i l_{i,\epsilon}$ of rational curves in $\mathbb{P}^N$ obtained by deforming the l_i in such a way that $l_{i,\epsilon}$ meets $l_{i+1,\epsilon}$, and then by smoothing the resulting chain to a rational curve C. If the diameter of the set $\{x_i,\ i = 1, \ldots, M\}$ is large, and the points x_i are taken in a

Kobayashi hyperbolic subvariety $Y \subset \mathbb{P}^k$, the distance $d(C, C')$ between C and any elliptic or rational curve C' in Z is bounded below by a positive constant.

2.1. **Further questions and remarks.** The first obvious question is the following:

Question 2.6. *Let $X \subset \mathbb{P}^N$ be approximately elliptically connected (in the strong or cohomological sense). Is the Kobayashi pseudo-distance of X trivial?*

As our motivation was to understand the class of varieties with trivial Kobayashi pseudodistance, which includes conjecturally Calabi-Yau manifolds (cf. [16]), it is also natural to ask the following:

Question 2.7. *Let X be a Calabi-Yau manifold. Is X approximately rationally or elliptically connected (in any of the senses introduced in this paper)?*

Another important question is the following:

Question 2.8. *Is the property of cohomological approximate rational or elliptic connectedness independent of the choice of projective embedding?*

The following question is related to the work of Graber, Harris and Starr [13]:

Question 2.9. *Let $\phi : X \to Y$ be a surjective morphism. Assume that the fibers of ϕ are rationally connected (see [17]) and that Y is approximately rationally (resp. elliptically) connected in the strong or cohomological sense. Then is X approximately rationally (resp. elliptically) connected in the same sense?*

Let us give one result in this direction: let $P = \mathbb{P}^n$ and let $Q = \mathbb{P}(H^0(P, \mathcal{O}_P(d)))$, where $d \leq n$ and $n \geq 2$ so that the general hypersurface of degree d in P is Fano of dimension ≥ 1. There is a universal subvariety $Z \subset Q \times P$, defined by the tautological equation $F_d \in H^0(\mathcal{O}_{Q\times P}(1, d))$. Via the second projection, Z is a fibration in projective spaces over P.

Proposition 2.10. *If $Y \subset Q$ is rationally (resp. elliptically) connected in the cohomological sense, and $X := Y \times_Q Z \to Y$ is smooth of the expected dimension (hence the generic fiber of $X \to Y$ is a smooth hypersurface in P), X is approximately rationally (resp. elliptically) connected in the cohomological sense in Z, hence in $Q \times P$.*

Proof. Let $V \subset Z$ be a tubular neighborhood of X and $\widetilde{V} \subset \mathbb{P}(T_Z)$ be a neighborhood of $\mathbb{P}(T_X)$. There are neighborhoods $U \subset Q$ of Y and $\tilde{U} \subset \mathbb{P}(T_Q)$ of $\mathbb{P}(T_Y)$ such that V contains $\pi^{-1}(U)$ and $\widetilde{V}$ is contained in $\pi_*^{-1}(\tilde{U})$ where $\pi := pr_{1|Z} : Z \to Q$. As $Y \subset Q$ is approximately rationally, resp. elliptically, connected in the cohomological sense, there is a curve E which is rational (resp. elliptic), contained in U and passing through any point y of Y. Furthermore, E can be chosen to be $\widetilde{U}$-close to Y, of class independent of y, and finally a convex combination of the $(n-1, n-1)$-part of these classes contains a strongly positive class in Y, where $n = \dim Y$. Moving E if needed, we may assume that Z_E is smooth with irreducible fibers and $Z_E \to E$ is a smooth Fano complete intersection over the generic point of E. By the Tsen-Lang theorem, the family $Z_E \to E$ has a section. Results of [17] even show that such a section $\widetilde{E}$ can be chosen to have an arbitrarily positive class in Z_E. These sections produce elliptic curves $\widetilde{E} \subset V$ which are then $\widetilde{V}$-close to X, and pass through the general point of X. Finally, under our assumptions, (and because we may assume that $\dim X \geq 3$, otherwise the result is obvious) the

Lefschetz hyperplane section theorem says that $H^2(X,\mathbb{Z}) = H^2(Y,\mathbb{Z}) \oplus H^2(P,\mathbb{Z})$. It is then immediate to conclude that if the curves $\widetilde{E}$ have an ample class in Z_E and a convex combination of the $(n-1,n-1)$-components of the pushforward of their classes in $H_2(U,\mathbb{Z}) = H_2(Y,\mathbb{Z}) = H^{2n-2}(Y,\mathbb{Z})$ contains a strongly positive class, then a convex combination of the $(m-1,m-1)$-components of their classes in $H_2(V,\mathbb{Z}) = H_2(X,\mathbb{Z}) = H^{2m-2}(X,\mathbb{Z})$ contains a strongly positive class.

■

Remark 2.11. The analogous result, if one only assumes that the fibers of $X \to Y$ are approximately elliptically or even rationally connected in the strong sense, is not true. Indeed, consider the example 0.5 where $X = (C \times S)/\iota$, and $Y = \mathbb{P}^1$, where C is a curve of genus ≥ 2 with hyperelliptic involution i, S is a $K3$ surface which is the universal cover $S \to T$ of an Enriques surface. The morphism $\phi : X \to Y$ is induced by passing to the quotient from the projection $p_2 : C \times S \to C$ using the isomorphisms $X \cong (C \times S)/\iota$, $C/i \cong \mathbb{P}^1$ and equivariance of p_2. The fibers of ϕ are isomorphic to S or to T, hence are strongly approximately rationally connected. However X is not strongly approximately rationally or elliptically connected by Corollary 2.2.

To finish, let us conclude with the following questions:

Question 2.12. *(Campana) Assume X is approximately rationally connected (in the adequate sense). Is $\pi_1(X)$ finite?*

The following similar question is very much related to the results of [4].

Question 2.13. *(Campana) Assume X is strongly approximately elliptically connected (in the adequate sense). Is $\pi_1(X)$ virtually abelian?*

The two questions (for cohomological approximate connectedness) are related as follows.

Proposition 2.14. *Suppose Question 2.13 has a positive answer for cohomological approximate connectedness, then Question 2.12 also has a positive answer for cohomological approximate connectedness.*

Proof. Let $X \subset \mathbb{P}^N$ be approximately rationally connected in the cohomological sense. We know, assuming Question 2.13 has a positive answer, that $\pi_1(X)$ is virtually abelian. Passing to an étale cover of X, and using Lemma 2.1, we may assume that X is approximately rationally connected in the cohomological sense in an adequate variety U, and furthermore has torsion free abelian π_1. We want to prove that $\pi_1(X)$ is trivial. Equivalently, if $a_X : X \to Alb\, X$ is the Albanese map, letting $Y := a_X(X) \subset Alb\, X$, one wants to prove that Y is a point. Assume the contrary. Then choosing an ample line bundle on $Alb\, X$, its pull-back $a_X^* L$ to X is a semi-positive line bundle which is not numerically trivial. Consider now rational curves C in a tubular neighborhood U of X in projective space. Then their class $[C] \in H_2(U,\mathbb{Z}) \cong H_2(X,\mathbb{Z})$ factors through $\pi_2(X)$, hence vanishes in $H_2(Alb\, X)$ under the map a_{X*}. Hence we conclude that $< [C], \phi^* c_1(a_X^* L) >= 0$.

This contradicts the fact that X is approximately rationally connected in the cohomological sense, because the later implies in particular the existence of rational curves C_i in any small neighborhood of X in U, with the property that the class

$\sum_i n_i[C_i]^{n-1,n-1} \in H^{n-1,n-1}(X)$ is strongly positive, so that

$$\sum_i n_i < [C_i], \phi^* c_1(a_X^* L) > \neq 0.$$

■

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