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Propagation of Acoustic Waves in Porous Media and their Reflection and Transmission at a Pure Fluid/Porous Medium Permeable Interface

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Abstract

We find a sufficient condition of hyperbolicity for a differential system governing the motion of a one-dimensional porous medium, so ensuring the existence of a solution for the associated Cauchy problem. We study propagation of linear waves impacting at a pure-fluid/porous-medium interface and we deduce novel expressions for the reflection and transmission coefficients in terms of the spectral properties of the governing differential system. We show three dimensional plots drawing reflection and transmission coefficients as functions of Biot's parameters. In such a way we propose an indirect method for measuring Biot's parameters when the measurement of the reflection and transmission coefficients associated to the traveling waves is possible.

Keywords: Hyperbolicity, Propagation of Waves in Porous Media, Reflection and Transmission Coefficients, Indirect Measurements of Poroelastic Properties.

PACS: 47.56.+r, 46.40.Cd, *43.20.Gp, *43.20.Jr, *43.25.Cb, *43.40.Jc

1. Introduction

The problem of studying the propagation of acoustic waves in porous media and their reflection and transmission at a surface of discontinuity for material properties is an important challenge in engineering science. In fact, it is well known that this kind of studies may be useful for applications in petrol engineering, geophysics, seismology, submarine acoustic, etc. (see e.g. Coussy and Bourbie (1984), Snieder *et al.* (2007))

1.1. State of Art

The starting point of the theory of propagation of waves in porous media is universally identified in Biot (1956a, 1956b) who showed the existence of three types of waves propagating in a fluid-filled porous medium: two compressive waves and one shear wave. In particular, one of the compressive waves (called fast wave, or P_1 wave) is similar to a wave propagating in a solid continuum, while the second compressive wave (called slow wave, or P_2 wave) exists only in a porous medium. A necessary condition for the existence of these two types of wave is the continuous spatial distribution of both solid and fluid phase (see also Coussy and Bourbie (1984)). This means that the pores of the matrix are interconnected pores (open pore hypothesis). The existence of the P_2 wave remained unobserved for a long time, due to the difficulty of setting up an

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experimental framework sensitive enough to measure such a slow wave. The first author who experimentally proved the existence of a slow compressional wave propagating in a porous medium was Plona (1980).

Coussy and Bourbie (1984) found the conditions assuring that the effects described by Biot's theory may be negligible when dealing with waves propagating in infinite porous media. These effects become surely relevant when considering material discontinuities able to generate phenomena of reflection and transmission. The fact of considering a discontinuity surface inside a porous medium leads to some difficulties which are primarily concerned with the correct choice of the boundary conditions to be imposed at such a surface. We consider here only *solid-material* discontinuities, i.e. surfaces which are simultaneously *contact discontinuities* for the solid constituent (no solid transfer of mass through the surface) and *shock waves* for the fluid constituent (fluid transfer of mass allowed across the surface).

Many efforts were made to establish the boundary conditions at a solid-material surface discontinuity compatible with Biot's equations for the non-dissipative, linearized case. Deresiewicz and Skalak (1963) first derived a set of boundary conditions sufficient to ensure uniqueness of the solution of the boundary value problems to Biot's equations and these conditions were then used by several authors (see e.g. Aknine *et al.* (1997), Denneman *et al.* (2002), Sharma (2004)) to study the behavior of linear waves impacting at a solid-material discontinuity surface inside a porous medium.

The open pore boundary conditions established by Deresiewicz and Skalak (1963) for Biot's linear equations at a solid-material surface discontinuity are:

- continuity of normal and tangential velocity of the solid constituent
- fluid mass conservation across the surface
- continuity of normal and tangential stresses
- continuity of pore pressure.

The first condition is a purely kinematical condition related to the regularity assumptions made on the solid placement map and to the fact that the considered surface discontinuity is solid-material (see e.g. dell'Isola *et al.* (2009)). The second condition is due to the assumption that no creation or dissolution of fluid mass occurs when the fluid crosses the interface. The third and fourth conditions are the so-called natural boundary conditions which can be derived by means of a suitable Hamilton principle (see Rasolofosaon and Coussy (1985) and dell'Isola *et al.* (2009)).

Nevertheless, when deriving the natural boundary conditions to be imposed at a solid-material surface discontinuity inside a porous medium by means of a variational principle, an additional boundary condition appears which is not included in the aforementioned conditions. This condition, when considering the non-dissipative case, is:

- continuity of the tangential component of the relative velocity of the fluid constituent with respect to the solid one.

Since this condition is derived in dell'Isola *et al.* (2009) from a variational principle which also gives rise to Biot's bulk equation, it is clear that it is intrinsically compatible with Biot's equations themselves. Indeed, another slight difference between the boundary conditions derived by Deresiewicz and Skalak (1963) and those derived by means of a variational principle exists. It consists in the fact that continuity of the chemical potential must be imposed instead of continuity of the pore pressure. This seems implicitly stated in the late works of Biot (1972 and 1973) where the chemical potential is introduced as a consequence of the application of a variational principle to poroelasticity.

Rasolofosaon and Coussy (1985) and Lion *et al.* (2004) seem to be aware of Biot's late results and state that the aforementioned different conditions coincide when the considered system suffers small perturbations in the neighborhood of equilibrium states: in our opinion such a statement needs to be further investigated in the light of the principles of continuum thermodynamics. Nevertheless, when non-linear problems are addressed, the right boundary condition to be imposed is undoubtedly the continuity of chemical potential. The boundary conditions proposed in the first part of this paper are valid in the general framework of

non-linear poromechanics. They also take into account inertia terms which, however, are negligible when linearizing.

Deresiewicz and Skalak (1963) also derived closed pores and partially closed pores boundary conditions sufficient to ensure uniqueness of Biot's equations. Sharma (2008) proposed a modification of such conditions introducing a parameter representing the effective connections between the surface pores of two porous media at the interface. He derived a set of boundary conditions for linearized Biot's equations on the basis of physically based principles satisfying the conservation of energy. The fact of considering the case of two porous media with partially connected pores introduces reasonings on the meso-structure of the interface which are not taken into account in the model presented in this paper. Indeed, the starting point of our work is a set of differential equations and *natural* boundary conditions derived by means of a macroscopic variational principle. We refrain by the attempt of establishing "ad hoc" conditions without the guide of a suitably adapted variational principle. On the other hand, the case of closed pores interface is even more difficult to frame in the light of a variational approach. In this case, in fact, one should consider kinematical mappings which are discontinuous at the considered interface. This possibility is not contemplated by standard Hamilton principles which always deal with functions which are continuous across the interface. Some results on the case of discontinuous kinematical mappings are presented in dell'Isola *et al.* (2009), but an exhaustive treatment on this subject is still lacking in the literature. Gurevich and Schoenberg (1999) showed, by means of a limit procedure on a transition layer across the discontinuity, that only the open-pore interface conditions are fully consistent with the validity of Biot's equations. This result is not astonishing since a fundamental hypothesis of Biot's classical theory is the continuity of both solid and fluid placement maps.

The model used in the present paper, differently as done e.g. in dell'Isola and Hutter (1999), does not consider the porosity as an independent kinematical parameter in the description of considered porous medium. However, the problem of finding the correct boundary conditions to be used, when these more general models are considered, seems not to be satisfactorily solved.

1.2. Summary of the Paper

In the first part of this paper, we start by showing the set of solid-Lagrangian bulk equations and natural boundary conditions at a solid-material surface discontinuity between two dissimilar porous media as derived by dell'Isola *et al.* (2009). This differential system is valid in the general non-linear case. Dissipation is here neglected, while inertial effects are taken into account in the presented model. We also obtain the simplified case of a porous medium in contact with a pure fluid as a particular case of the aforementioned problem. These sets of solid-Lagrangian equations and boundary conditions are then rewritten in the Eulerian configuration of the porous system thus producing the bulk and boundary evolution equations for the considered porous system.

Subsequently, a general expression of conservation of total energy is derived both in its solid-Lagrangian and in its Eulerian form. The bulk form of conservation of total energy is seen to be a consequence of bulk evolution equations. The corresponding jump conditions associated to the solid-Lagrangian and Eulerian energy conservation laws are also derived.

As it is possible to explicitly derive a balance law for the conservation of total energy one can get important consequences when the governing bulk equations of the porous system are written in quasilinear, conservative form. In this case, in fact, the existence of a supplementary conservation law implies that, if the total energy is a convex function of the other conserved quantities, then the considered system of differential equations is *symmetric t-hyperbolic* (see Friedrichs and Lax (1971), Jeffrey (1976) and Godunov (2003)). The importance of the concept of *symmetric t-hyperbolic systems* is that the Cauchy's initial value problem for a differential system with this property is always well-posed. This means that a unique solution exists at least locally in time once the initial conditions are assigned and it depends continuously on initial data. Indeed, we can establish that the version of Biot's equations we have used to describe longitudinal wave propagation can be written in such a form, if the total energy is a convex function of conservative variables.

In the second part of the paper, we address the simplified one-dimensional problem of reflection and transmission of linear waves at the interface between a pure fluid and a porous medium. We start by writing

the one-dimensional governing bulk equations in quasilinear form and we then consider their corresponding matrix formulation: $\mathbf{V}_t + \mathbf{A} \cdot \mathbf{V}_x = 0$. We show that the matrix of coefficients \mathbf{A} can be decomposed as the product of an invertible, symmetric matrix \mathbf{P}^{-1} and of the Hessian matrix \mathbf{B} of the total energy of the porous system. This particular structure of the matrix \mathbf{A} implies important consequences on the existence of a solution for the considered differential system. In fact, we are able to easily prove a sufficient condition of hyperbolicity of the considered quasilinear differential system just by imposing definite positiveness of the Hessian matrix \mathbf{B} .

The Eulerian bulk equations and boundary conditions are then linearized around a suitable equilibrium configuration.

We look for a periodic wave solution in the region of hyperbolicity of the considered linearized differential system. The derived solution is found to be frequency-independent. A new compact and general form of the energy fluxes associated to the incident wave, to the reflected wave and to the P_1 and P_2 transmitted waves is evaluated in terms of the eigenvalues and eigenvectors of the linearized matrix of coefficients \mathbf{A} , of the linearized Hessian matrix of the total energy \mathbf{B} and of the wave amplitudes. As a consequence, the reflection and transmission coefficients are derived in terms of this simple form of the energy fluxes.

This compact form for computing reflection and transmission coefficients in porous media is very useful. Indeed, the simple expressions derived here for the reflection and transmission coefficients are explicitly related to some properties of the matrix form of the considered quasilinear differential system. In particular, a useful property is that the eigenvectors of the matrix of coefficients \mathbf{A} are orthogonal in the inner product associated to the Hessian Matrix of the total energy \mathbf{B} . To our knowledge, the results presented here are novel: indeed, most of the closed forms for reflection and transmission coefficients already available (see e.g. Denneman *et al.* (2002), Gurevich *et al.* (2004)) are based on reasonings which refrain from any consideration on the energy flow associated to wave transmission and reflection. In some cases (see e.g. Rubino *et al.* (2006) and Aknine *et al.* (1997)), expressions for the reflected and transmitted energy are given, but they are never put in relation with the spectral properties of the differential system governing the wave propagation. In our opinion, the new closed form for reflection and transmission coefficients presented in Eq.(49) is very general in nature. Indeed, no hypothesis on the constitutive behavior of the porous medium has been made at this level. In order to show a possible application of the obtained results, we decide to limit ourselves to the case of an isotropic porous medium and we arrive to determine reflection and transmission coefficients in terms of the Biot's parameters of the porous medium itself. Reflection and transmission coefficients versus the Biot's parameters are drawn in three-dimensional plots for different values of the undrained Lamé parameters of the solid matrix, some of which are those of a Limestone of Bourgogne (see Lion *et al.* (2005)). We show that the obtained three-dimensional graphics can be useful, from a practical point of view, to determine the Biot's coefficient and the Biot's modulus by means of indirect measurements. An interesting *threshold phenomenon*, which determines a sudden change in the amount of energy transported by P_1 and P_2 waves for certain values of the Biot's parameters, is also observed. More precisely, we show that some "threshold values" of b and M exist corresponding to which the greater amount of energy which is initially transported by the P_1 wave is transferred to the P_2 wave.

However, we are aware that, in order to supply an efficient tool for experimental purposes, more theoretical and numerical investigations are still necessary. For the time being, we remark that, while the determination of the undrained Lamé parameters is relatively easy to obtain by means of laboratory tests, the determination of Biot's parameters is more complicated and sometimes different methods lead to a different measured value of these coefficients (see e.g. Lion *et al.* (2004)). Hence, it is clear how a method allowing the determination of Biot's parameters in a simple way is desirable. The plots presented in the last part of this paper clearly show that, once the undrained Lamé parameters have been fixed by choosing the material, the simple measurement of the reflection and transmission coefficients associated to the compressional waves, can lead to the determination of both the Biot's parameters for the considered porous material. Of course, more general studies including shear waves and two-dimensional compressive waves can be performed starting from the general differential system presented in the first part. Indeed, the methods shown here can be seen as a basis for further, more refined analyses which can lead to new non-invasive methods for determining Biot's parameters *in situ* when it is possible to measure reflection and transmission coefficients.

2. Solid-Lagrangian and Eulerian Differential Systems

We refer to dell'Isola *et al.* (2009) for the extended deduction, using a variational principle, of the solid-Lagrangian equations governing the motion of a porous medium and of the jump conditions to be imposed at a fluid-permeable interface between two dissimilar porous media. We briefly recall here the kinematical framework needed for describing the motion of such a system. Let B_s and B_f be two open subsets of \mathbb{R}^3 (usually referred to as the Lagrangian configurations of the two constituents), $(0, T)$ be a time interval and let

$$\chi_s : B_s \times (0, T) \rightarrow \mathbb{R}^3, \quad \chi_f : B_f \times (0, T) \rightarrow \mathbb{R}^3$$

be the maps which represent the placement of the solid and fluid constituents respectively. Moreover, let $\phi_s : B_s \times (0, T) \rightarrow B_f$ be the map which locates, for any instant $t \in (0, T)$, the fluid material particle \mathbf{X}_f which is in contact with the solid material particle \mathbf{X}_s . The three introduced maps are related by $\phi_s = \chi_f^{-1} \circ \chi_s$. We also assume that χ_s and ϕ_s are *piecewise C^1 diffeomorphisms*, i.e. they are continuous in B_s , while their space gradients $\nabla \chi_s$ and $\nabla \phi_s$ are continuous a.e. in B_s except on a surface S_s which is assumed to be fixed in the solid-Lagrangian configuration. This means that the fields $\nabla \chi_s$ and $\nabla \phi_s$ suffer a jump at the surface S_s . Moreover, we assume that $\chi_s(B_s, t) = \chi_f(B_f, t)$ and we denote $B_e(t)$ this time-varying sub-domain of \mathbb{R}^3 usually referred to as Eulerian configuration of the porous system.

Notation 1. Given three fields $e(\mathbf{x}, t)$, $s(\mathbf{X}_s, t)$ and $f(\mathbf{X}_f, t)$ defined on B_e , B_s and B_f respectively, we denote the corresponding transported fields as $e^{\textcircled{S}} := e \circ \chi_s$, $e^{\textcircled{F}} := e \circ \chi_f$, $s^{\textcircled{F}} := s \circ \phi_s^{-1}$, $s^{\textcircled{S}} := s \circ \chi_s^{-1}$, $f^{\textcircled{S}} := f \circ \chi_f^{-1}$ and $f^{\textcircled{F}} := f \circ \phi_s$.

2.1. Governing Bulk Equations

In the conservative case, neglecting body forces and assuming that no creation or dissolution of mass occurs during the motion of the system, the solid-Lagrangian governing bulk equations for a porous medium read ¹

$$\text{div} \left(\mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \right) = \eta_s \gamma_s + m_f \gamma_f^{\textcircled{S}}, \quad (1)$$

$$\nabla \left(\frac{\partial \Psi}{\partial m_f} \right) + \mathbf{F}_s^T \cdot \gamma_f^{\textcircled{S}} = 0 \quad (2)$$

$$\frac{\partial m_f}{\partial t} + \text{div} \mathbf{D} = 0, \quad \frac{\partial \eta_s}{\partial t} = 0, \quad (3)$$

where η_s and m_f are the solid-Lagrangian densities of the solid and of the fluid respectively (both defined on B_s), γ_s and γ_f are the accelerations of the solid and of the fluid constituent (defined on B_s and B_f respectively), \mathbf{F}_s is the gradient of the solid placement map, $\boldsymbol{\varepsilon} := (\mathbf{F}_s^T \cdot \mathbf{F}_s - \mathbf{I})/2$ is the Green-Lagrange deformation tensor, Ψ is the solid-Lagrangian volume deformation energy of the porous medium and $\mathbf{D} := m_f \mathbf{F}_s^{-1} (\mathbf{v}_f^{\textcircled{S}} - \mathbf{v}_s)$ is the solid-Lagrangian mass-flux vector. Moreover, we remark that the operators ∇ and div operate on fields which are naturally defined on B_s : this means that, in the preceding formulas, these operators are defined in the variables $\mathbf{X}_s \in B_s$. Indeed, once the domain of the field is established, its partial differentiation is defined with respect to the independent variables. This is one of the reasons why we introduce Notation 1 which allow for the immediate identification of the domain of definition of a transported field: in this way we can avoid to use burdening notations for the differentiation operators as usually done in the literature. It is known that $\gamma_s = \partial \mathbf{v}_s / \partial t$ and $\gamma_f^{\textcircled{S}} = \partial \mathbf{v}_f^{\textcircled{S}} / \partial t + \nabla \mathbf{v}_f^{\textcircled{S}} \cdot \mathbf{F}_s^{-1} \cdot (\mathbf{v}_f^{\textcircled{S}} - \mathbf{v}_s)$, where \mathbf{v}_s and \mathbf{v}_f are the velocities of the two constituents (defined on B_s and B_f respectively). Equations (1), (2) and (3) represent the balance of total momentum, the balance of fluid momentum and the fluid and

¹Here and in the sequel we indicate by a central dot the single contraction between two tensor fields. Naturally, this notation will be extended for indicating the ordinary matrix product.

solid balances of masses respectively. These equations, which in their Eshelbian formulation were considered in Quiligotti *et al.* (2003), generalize those deduced and studied in Placidi *et al.* (2008) where, however, a purely Eulerian approach for only compressional waves is considered.

Recalling transport formulas from Lagrangian to Eulerian gradient and divergence (see e.g. dell'Isola *et al.* (2009)) and multiplying on the left equation (2) by \mathbf{F}_s^{-T} , the governing bulk equations (1), (2) and (3) can be rewritten in their Eulerian form as

$$\operatorname{div} \sigma = \rho_s \left(\frac{\partial \mathbf{v}_s^\ominus}{\partial t} + \nabla \mathbf{v}_s^\ominus \cdot \mathbf{v}_s^\ominus \right) + \rho_f \left(\frac{\partial \mathbf{v}_f^\ominus}{\partial t} + \nabla \mathbf{v}_f^\ominus \cdot \mathbf{v}_f^\ominus \right) \quad (4)$$

$$\frac{\partial \mathbf{v}_f^\ominus}{\partial t} + \nabla \mathbf{v}_f^\ominus \cdot \mathbf{v}_f^\ominus + \nabla g = 0, \quad (5)$$

$$\frac{\partial \rho_s}{\partial t} + \operatorname{div} (\rho_s \mathbf{v}_s^\ominus) = 0, \quad \frac{\partial \rho_f}{\partial t} + \operatorname{div} (\rho_f \mathbf{v}_f^\ominus) = 0, \quad (6)$$

where, setting $J_s := \det \mathbf{F}_s$, $\sigma = (J_s^{-1} \mathbf{F}_s \cdot \partial \Psi / \partial \boldsymbol{\varepsilon} \cdot \mathbf{F}_s^T)^\ominus$ is the usual Eulerian Cauchy stress tensor. Moreover, $\rho_s = (J_s^{-1} \eta_s)^\ominus$ and $\rho_f = (J_s^{-1} m_f)^\ominus$ are the Eulerian apparent densities of the solid and of the fluid constituent respectively, $g := (\partial \Psi / \partial m_f)^\ominus$ is the chemical (or Gibbs) potential of the porous medium. Clearly, here the ∇ and div operators operate on Eulerian fields and then the partial differentiation is intended with respect to the variables $\mathbf{x} \in B_e$.

2.2. Jump Conditions Between Two Dissimilar Porous Media

As for the jump conditions holding at a solid material, fluid-permeable interface between two dissimilar porous media, always assuming the conservative case, they read

$$\left[\left[\mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} - (\mathbf{v}_f^\ominus - \mathbf{v}_s) \otimes \mathbf{D} \right] \right] \cdot \mathbf{N}_s = 0, \quad (7)$$

$$[\mathbf{v}_s] = 0, \quad \boldsymbol{\tau}_e \cdot \left[\left[\mathbf{v}_f^\ominus - \mathbf{v}_s \right] \right] = 0, \quad (8)$$

$$\left[\left[\frac{1}{2} (\mathbf{v}_f^\ominus - \mathbf{v}_s)^2 + \frac{\partial \Psi}{\partial m_f} \right] \right] = 0, \quad (9)$$

$$[\mathbf{D}] \cdot \mathbf{N}_s = 0, \quad [\eta_s] \mathfrak{D}_s = 0. \quad (10)$$

where \mathbf{N}_s is the unit normal to the solid-material surface discontinuity S_s , $\boldsymbol{\tau}_e$ is any tangent vector to the Eulerian discontinuity surface corresponding to S_s and \mathfrak{D}_s is the celerity of the solid-Lagrangian discontinuity surface. Notice that the solid balance of mass on S_s is automatically satisfied since the surface S_s is assumed to be fixed and then $\mathfrak{D}_s = 0$.

Recalling the transport formulas for the unit normal vectors, the Eulerian form of the jump conditions simply reads

$$\left[\left[\sigma - (\mathbf{v}_f^\ominus - \mathbf{v}_s^\ominus) \otimes \rho_f (\mathbf{v}_f^\ominus - \mathbf{v}_s^\ominus) \right] \right] \cdot \mathbf{N}_e = 0, \quad (11)$$

$$[\mathbf{v}_s^\ominus] = 0, \quad \boldsymbol{\tau}_e \cdot \left[\left[\mathbf{v}_f^\ominus - \mathbf{v}_s^\ominus \right] \right] = 0, \quad (12)$$

$$\left[\left[\frac{1}{2} (\mathbf{v}_f^\ominus - \mathbf{v}_s^\ominus)^2 + g \right] \right] = 0, \quad (13)$$

$$\left[\left[\rho_f (\mathbf{v}_f^\ominus \cdot \mathbf{N}_e - \mathfrak{D}) \right] \right] = 0, \quad \left[\left[\rho_s (\mathbf{v}_s^\ominus \cdot \mathbf{N}_e - \mathfrak{D}) \right] \right] = 0, \quad (14)$$

where \mathbf{N}_e is the unit normal vector to the Eulerian discontinuity surface and \mathfrak{D} is the celerity of the Eulerian moving discontinuity surface. Notice that, since by definition $\mathfrak{D} := \mathbf{v}_s^\ominus \cdot \mathbf{N}_e$, also in this Eulerian form the solid balance of mass at the interface is automatically satisfied.

2.3. Boundary Conditions Between a Porous Medium and a Pure Fluid

As for the boundary conditions holding at a solid material interface between a porous medium and a pure fluid, they can be derived as a particular case of the jump conditions between two dissimilar porous media (see dell'Isola *et al.* 2009). Continuously extending the velocity \mathbf{v}_s^\ominus through the discontinuity surface, their Eulerian form can be simplified into

$$\sigma \cdot \mathbf{N}_e = -p_f \mathbf{N}_e + \left[\left[\mathbf{v}_f^\ominus \right] \right] \rho_f (\mathbf{v}_f^\ominus \cdot \mathbf{N}_e - \mathfrak{D}) \quad (15)$$

$$\left[\left[\mathbf{v}_s^\ominus \right] \right] = 0, \quad \boldsymbol{\tau}_e \cdot \left[\left[\mathbf{v}_f^\ominus \right] \right] = 0, \quad g = g_f - \frac{1}{2} \left[\left[\left(\mathbf{v}_f^\ominus - \mathbf{v}_s^\ominus \right)^2 \right] \right], \quad (16)$$

$$\left[\left[\rho_f (\mathbf{v}_f^\ominus \cdot \mathbf{N}_e - \mathfrak{D}) \right] \right] = 0, \quad (17)$$

where p_f and g_f are the pressure and the chemical potential of the pure fluid and \mathfrak{D} is the celerity of the Eulerian discontinuity surface.

2.4. Solid-Lagrangian and Eulerian Conservation of Total Energy

In this subsection we show that both the solid Lagrangian and Eulerian governing bulk equations admit conservation of energy. This means that the equation of conservation of energy is dependent from all the other governing equations both in the solid-Lagrangian and Eulerian case.

Let $\Psi(\boldsymbol{\varepsilon}, m_f)$ and $\Lambda(\eta_s, m_f, \mathbf{v}_s, \mathbf{v}_f^\ominus) := 1/2 \left(\eta_s (\mathbf{v}_s)^2 + m_f (\mathbf{v}_f^\ominus)^2 \right)$ be the solid-Lagrangian potential and kinetic energies of the porous system respectively. The solid-Lagrangian formulation of the conservation of total energy reads (see Appendix for calculations).

$$\frac{\partial}{\partial t} (\Psi + \Lambda) + \operatorname{div} \left(m_f \frac{\partial \Psi}{\partial m_f} \mathbf{F}_s^{-1} \cdot (\mathbf{v}_f^\ominus - \mathbf{v}_s) + \frac{1}{2} m_f (\mathbf{v}_f^\ominus)^2 \mathbf{F}_s^{-1} \cdot (\mathbf{v}_f^\ominus - \mathbf{v}_s) - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \cdot \mathbf{F}_s^T \cdot \mathbf{v}_s \right) = 0. \quad (18)$$

This solid-Lagrangian balance law admits the following corresponding jump condition (see dell'Isola *et al.* for details) at any solid-material surface discontinuity:

$$\left[\left[m_f \frac{\partial \Psi}{\partial m_f} \mathbf{F}_s^{-1} \cdot (\mathbf{v}_f^\ominus - \mathbf{v}_s) + \frac{1}{2} m_f (\mathbf{v}_f^\ominus)^2 \mathbf{F}_s^{-1} \cdot (\mathbf{v}_f^\ominus - \mathbf{v}_s) - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \cdot \mathbf{F}_s^T \cdot \mathbf{v}_s \right] \right] \cdot \mathbf{N}_s = 0. \quad (19)$$

Let now $W := J_s^{-1} \Psi$ be the Eulerian potential energy density. The Eulerian form of the conservation of total energy (18) reads (see Appendix for calculations)

$$\begin{aligned} & \frac{\partial}{\partial t} \left(W + \frac{1}{2} \rho_s (\mathbf{v}_s^\ominus)^2 + \frac{1}{2} \rho_f (\mathbf{v}_f^\ominus)^2 \right) \\ & + \operatorname{div} \left[\rho_f \left(\frac{\partial W}{\partial \rho_f} + \frac{1}{2} (\mathbf{v}_f^\ominus)^2 \right) \mathbf{v}_f^\ominus + \left(\frac{1}{2} \rho_s (\mathbf{v}_s^\ominus)^2 - \mathbf{F}_s \cdot \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \cdot \mathbf{F}_s^T \right) \cdot \mathbf{v}_s^\ominus \right] = 0. \end{aligned} \quad (20)$$

This Eulerian conservation law admits the following jump condition at the Eulerian discontinuity surface:

$$\begin{aligned} & \left[\left[\rho_f \left(\frac{\partial W}{\partial \rho_f} + \frac{1}{2} (\mathbf{v}_f^\ominus)^2 \right) \mathbf{v}_f^\ominus + \left(\frac{1}{2} \rho_s (\mathbf{v}_s^\ominus)^2 - \mathbf{F}_s \cdot \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \cdot \mathbf{F}_s^T \right) \cdot \mathbf{v}_s^\ominus \right] \right] \cdot \mathbf{N}_e \\ & - \left[\left[W + \frac{1}{2} \rho_s (\mathbf{v}_s^\ominus)^2 + \frac{1}{2} \rho_f (\mathbf{v}_f^\ominus)^2 \right] \right] \mathfrak{D} = 0. \end{aligned} \quad (21)$$

3. One Dimensional Case

We address the problem of a porous medium in contact with a pure fluid by limiting ourselves to a one-dimensional case. More precisely, we assume that in the porous medium the energy of shear waves can be neglected: this is coherent with a similar assumption made in Gurevich *et al.* (2004).

Moreover, since in the sequel we only deal with scalar fields, we neglect all the circled superscripts for fields in different configurations. In fact the value of scalar functions does not depend on the configuration in which they are evaluated.

3.1. Governing Equations of the Porous Medium and of the Pure Fluid

We consider the case in which all the fields defined in the porous medium region have only one non-vanishing component along a given Eulerian direction $x = \chi_s(X)$. This means that we are considering the case in which the motion of the solid and of the fluid constituents is assumed to be in one given direction X . In this one-dimensional case the deformation gradient and the Green-Lagrange deformation tensor take the simple form

$$\mathbf{F}_s = \begin{pmatrix} F & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\varepsilon} = \frac{1}{2} \begin{pmatrix} F^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This simplified expression of \mathbf{F}_s implies that $F = J_s = \eta_s/\rho_s$.

Since the Green-Lagrange deformation tensor $\boldsymbol{\varepsilon}$ only has one non vanishing component $\epsilon := 1/2(F^2 - 1) = 1/2((\eta_s/\rho_s)^2 - 1)$, we can introduce a deformation energy density $\tilde{\Psi}(\epsilon, m_f) := \Psi(\boldsymbol{\varepsilon}, m_f)$ which depends only on the scalar quantity ϵ . It is clear that $(\partial\Psi/\partial\boldsymbol{\varepsilon})_{11} = \partial\tilde{\Psi}/\partial\epsilon$. According to these definitions, it is easy to check that the first component of the stress tensor is given by $\sigma_{11} = \eta_s/\rho_s \partial\tilde{\Psi}/\partial\epsilon$. Let us now introduce the variable $\tau_s := 1/\rho_s = 1/\eta_s\sqrt{1+2\epsilon}$ and the real function $\hat{\Psi}(\tau_s, m_f) := \tilde{\Psi}(\epsilon, m_f)$. It is easy to recover that

$$\sigma_{11} = \frac{1}{\eta_s} \frac{\partial\hat{\Psi}}{\partial\tau_s}.$$

The governing equations of the porous medium are given by equations (4), (5) and (6) which in the considered one-dimensional case, substituting equation (5) in equation (4), dividing by ρ_s and recalling that $\rho_f = (\rho_s/\eta_s) m_f$, give

$$\begin{aligned} \frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial x} + \frac{1}{\eta_s} \frac{\partial}{\partial x} \left(\hat{\Psi} - \tau_s \frac{\partial\hat{\Psi}}{\partial\tau_s} - m_f \frac{\partial\hat{\Psi}}{\partial m_f} \right) &= 0, & \frac{\partial v_f}{\partial t} + v_f \frac{\partial v_f}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial\hat{\Psi}}{\partial m_f} \right) &= 0, \\ \frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x} (\rho_f v_f) &= 0, & \frac{\partial \rho_s}{\partial t} + \frac{\partial}{\partial x} (\rho_s v_s) &= 0, \end{aligned}$$

where we simply denoted by v_s, v_f the only non vanishing components of \mathbf{v}_s^\oplus and \mathbf{v}_f^\oplus along the x direction

Recalling that $J_s^{-1} = \rho_s/\eta_s$, $\tau_s = 1/\rho_s$ and $m_f = (\eta_s/\rho_s) \rho_f$, we can introduce the potential

$$W(\rho_s, \rho_f) := J_s^{-1} \hat{\Psi}(\tau_s, m_f) = \frac{\rho_s}{\eta_s} \hat{\Psi} \left(\frac{1}{\rho_s}, \eta_s \frac{\rho_f}{\rho_s} \right).$$

It is easy to check that

$$\frac{\partial\hat{\Psi}}{\partial m_f} = \frac{\partial W}{\partial \rho_f}, \quad \frac{\partial\hat{\Psi}}{\partial \tau_s} = \eta_s \left(W - \rho_s \frac{\partial W}{\partial \rho_s} - \rho_f \frac{\partial W}{\partial \rho_f} \right),$$

where it can be recognized that $-(1/\eta_s)(\partial\hat{\Psi}/\partial\tau_s)$ is the Legendre transform of W . Hence, we can rewrite the governing equations in terms of the potential W as

$$\begin{aligned} \frac{\partial v_s}{\partial t} + \frac{\partial}{\partial x} \left(\frac{v_s^2}{2} + \frac{\partial W}{\partial \rho_s} \right) &= 0, & \frac{\partial v_f}{\partial t} + \frac{\partial}{\partial x} \left(\frac{v_f^2}{2} + \frac{\partial W}{\partial \rho_f} \right) &= 0, \\ \frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x} (\rho_f v_f) &= 0, & \frac{\partial \rho_s}{\partial t} + \frac{\partial}{\partial x} (\rho_s v_s) &= 0. \end{aligned} \quad (22)$$

Equations (22) were obtained earlier for a different physical system in Gavriluk *et al.* (1997, 1998) where a sufficient criterion of hyperbolicity was formulated : if $W(\rho_s, \rho_f)$ is convex then the equations are hyperbolic for small relative velocity $w = v_f - v_s$. The proof was based on the fact the equations can be put in symmetric form by using the Godunov-Friedrichs-Lax method of symmetrization of conservation laws. Let us recall this method (Friedrichs and Lax (1971) and Godunov (2003)). Let a system of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0$$

admit an additional conservation law

$$\frac{\partial e(\mathbf{u})}{\partial t} + \frac{\partial g(\mathbf{u})}{\partial x} = 0,$$

where $e(\mathbf{u})$ is a convex function of \mathbf{u} . Then, if we introduce the variables $\mathbf{v} = \partial e(\mathbf{u}) / \partial \mathbf{u}$, the Legendre transform of $e(\mathbf{u})$ takes the simple form

$$e^*(\mathbf{v}) := \max_{\mathbf{u}} (\mathbf{u} \cdot \mathbf{v} - e(\mathbf{u})) = \mathbf{u} \cdot \mathbf{v} - e(\mathbf{u}).$$

Introducing now the function

$$g^*(\mathbf{v}) := \mathbf{f}(\mathbf{u}) \cdot \mathbf{v} - g(\mathbf{u}),$$

and noticing that $\mathbf{u} = \partial e^* / \partial \mathbf{v}$ and $\mathbf{f}(\mathbf{u}) = \partial g^* / \partial \mathbf{v}$, it is straightforward that the originary system of conservation laws can be rewritten in symmetric form as

$$e^{*''} \cdot \frac{\partial \mathbf{v}}{\partial t} + g^{*''} \cdot \frac{\partial \mathbf{v}}{\partial x} = 0, \quad (23)$$

where $e^{*''}$ and $g^{*''}$ denote the Hessian matrices of e^* and g^* , respectively.

Recalling the hypothesis of convexity of $e(\mathbf{u})$ -and consequently of $e^*(\mathbf{v})$ - it is straightforward that $e^{*''}$ is definite positive and then we can state that the system is *symmetric t-hyperbolic* in the sense of Friedrichs. To put equations (22) in such a form, we need to find an additional convex conservation law.

Notice that the governing equations admit conservation of energy in the form

$$\frac{\partial}{\partial t} \left(\frac{\rho_f v_f^2}{2} + \frac{\rho_s v_s^2}{2} + W \right) + \frac{\partial}{\partial x} \left(\rho_f v_f \left(\frac{v_f^2}{2} + \frac{\partial W}{\partial \rho_f} \right) + \rho_s v_s \left(\frac{v_s^2}{2} + \frac{\partial W}{\partial \rho_s} \right) \right) = 0. \quad (24)$$

The convexity region of the total energy is studied in subsection 3.3.

As for the pure fluid, the governing equations are

$$\frac{\partial v_f}{\partial t} + \frac{\partial}{\partial x} \left(\frac{v_f^2}{2} + \frac{\partial U}{\partial \rho_f} \right) = 0, \quad \frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x} (\rho_f v_f) = 0 \quad (25)$$

where $U(\rho_f)$ is the Eulerian potential of the pure fluid. It is an immediate consequence of the previous results that also the governing equations of a pure fluid admit conservation of total energy in the form

$$\frac{\partial}{\partial t} \left(\frac{\rho_f v_f^2}{2} + U \right) + \frac{\partial}{\partial x} \left(\rho_f v_f \left(\frac{v_f^2}{2} + \frac{\partial U}{\partial \rho_f} \right) \right) = 0. \quad (26)$$

So, this system can also put in the symmetric form if the Hessian matrix of the total energy as a function of conserved variables is positive definite.

Even if, mathematically, symmetric form (23) of equations is preferable to that expressed in terms of the conservative variables ((22) and (25)), it is not explicit because of the Legendre transform of the energy needed for the symmetrization. This is why, in the next section, we propose a different symmetric form written in terms of physical variables.

The one dimensional problem considered in the present instance is rather similar to the one studied by dell'Isola *et al.* (1997). However, in cited paper the fluid drainage through the solid matrix was not taken into account.

3.2. Matrix Formulation of the Differential Problem

As it will be justified in the sequel, we indicate by a superposed $-$ sign all quantities defined in the pure fluid region and by a superposed $+$ sign all quantities defined in the porous medium region. Since the fields ρ_s and v_s are only defined in the porous medium region, we can omit the $+$ superscript for “solid” variables.

For practical reasons, we use another approach by rewriting the governing equations (22) and (25) in the special non-symmetric form:

$$\mathbf{V}_t^+ + \mathbf{A}^+ \cdot \mathbf{V}_x^+ = 0, \quad \mathbf{V}_t^- + \mathbf{A}^- \cdot \mathbf{V}_x^- = 0, \quad (27)$$

where we denote by t and x subscripts the derivatives with respect to time and space respectively. Moreover, we set $\mathbf{V}^+ := (\rho_f^+, \rho_s, v_f^+, v_s)^T$, $\mathbf{V}^- := (\rho_f^-, v_f^-)^T$ and

$$\mathbf{A}^+ := \begin{pmatrix} v_f^+ & 0 & \rho_f^+ & 0 \\ 0 & v_s & 0 & \rho_s \\ W_{ff} & W_{fs} & v_f^+ & 0 \\ W_{fs} & W_{ss} & 0 & v_s \end{pmatrix}, \quad \mathbf{A}^- := \begin{pmatrix} v_f^- & \rho_f^- \\ U_{ff} & v_f^- \end{pmatrix},$$

with

$$W_{ij} := \frac{\partial^2 W}{\partial \rho_i^+ \partial \rho_j^+} \quad \text{and} \quad U_{ff} := \frac{\partial^2 U}{\partial (\rho_f^-)^2},$$

where clearly $i = f, s$ and $j = f, s$.

We notice that the matrices \mathbf{A}^+ and \mathbf{A}^- can be written as the product of two symmetric matrices :

$$\mathbf{A}^+ = (\mathbf{P}^+)^{-1} \cdot \mathbf{B}^+, \quad \mathbf{A}^- = (\mathbf{P}^-)^{-1} \cdot \mathbf{B}^-, \quad (28)$$

where

$$\begin{aligned} (\mathbf{P}^+)^{-1} = \mathbf{P}^+ = (\mathbf{P}^+)^T &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & (\mathbf{P}^-)^{-1} = \mathbf{P}^- = (\mathbf{P}^-)^T &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{B}^+ = (\mathbf{B}^+)^T &= \begin{pmatrix} W_{ff} & W_{fs} & v_f^+ & 0 \\ W_{fs} & W_{ss} & 0 & v_s \\ v_f^+ & 0 & \rho_f^+ & 0 \\ 0 & v_s & 0 & \rho_s \end{pmatrix}, & \mathbf{B}^- = (\mathbf{B}^-)^T &= \begin{pmatrix} U_{ff} & v_f^- \\ v_f^- & \rho_f^- \end{pmatrix}. \end{aligned}$$

It is immediate to check that the matrices \mathbf{B}^+ and \mathbf{B}^- are the *Hessian matrices* (matrices of second derivatives) of the *Eulerian total energy* of the porous medium and of the pure fluid respectively.

Moreover, due to decomposition (28) of the matrices of coefficients, it is clear that the differential systems (27) are equivalent to symmetric systems, even if their hyperbolicity is not straightforward.

3.3. A Sufficient Criterion of Hyperbolicity

Let us start by considering a quasilinear differential system of n independent equations written in the matrix form $\mathbf{V}_t + \mathbf{A} \cdot \mathbf{V}_x = 0$. Let us assume that the $n \times n$ matrix of coefficients can be written as $\mathbf{A} = \mathbf{P}^{-1} \cdot \mathbf{B}$ where \mathbf{P}^{-1} is an invertible and symmetric $n \times n$ matrix and \mathbf{B} is a symmetric and positive definite $n \times n$ matrix. We conclude that the eigenvalues of \mathbf{A} are real, i.e. the system is *hyperbolic*. Indeed, since $\lambda = 0$ is not an eigenvalue of $\mathbf{A} = \mathbf{P}^{-1} \cdot \mathbf{B}$, the characteristic equation is equivalent to $\det(\mathbf{P} - \mathbf{B}/\lambda) = 0$. This implies that all the eigenvalues of \mathbf{A} are real (see e.g. Taylor (2005)).

Remark 1. If $\mathbf{P}^{-1} = \mathbf{P}$ then distinct eigenvectors of $\mathbf{A} = \mathbf{P}^{-1} \cdot \mathbf{B}$ are orthogonal in the inner product associated to \mathbf{B} .

Proof. Let \mathbf{h}_1 and \mathbf{h}_2 be two distinct eigenvectors of \mathbf{A} and λ_1 and λ_2 the corresponding eigenvalues. Since \mathbf{B} and \mathbf{P}^{-1} are symmetric matrices we have that $(\mathbf{A} \cdot \mathbf{h}_1) \cdot (\mathbf{B} \cdot \mathbf{h}_2) = \lambda_1 (\mathbf{h}_1 \cdot \mathbf{B} \cdot \mathbf{h}_2) = \lambda_1 (\mathbf{B} \cdot \mathbf{h}_1) \cdot \mathbf{h}_2$. On the other hand, always accounting for the symmetry of \mathbf{P}^{-1} and recalling that $\mathbf{P}^{-1} = \mathbf{P}$, it is easy to get $(\mathbf{A} \cdot \mathbf{h}_1) \cdot (\mathbf{B} \cdot \mathbf{h}_2) = (\mathbf{P} \cdot \mathbf{B} \cdot \mathbf{h}_1) \cdot (\mathbf{B} \cdot \mathbf{h}_2) = (\mathbf{B} \cdot \mathbf{h}_1) \cdot (\mathbf{A} \cdot \mathbf{h}_2) = \lambda_2 (\mathbf{B} \cdot \mathbf{h}_1) \cdot \mathbf{h}_2$. These two equalities imply $(\lambda_1 - \lambda_2)(\mathbf{B} \cdot \mathbf{h}_1) \cdot \mathbf{h}_2 = 0$, which proves the Remark. \square

We stress the fact that requiring definite positiveness of \mathbf{B} is just a *sufficient* criterion of hyperbolicity for the considered differential system. In fact, this request does not exclude cases in which \mathbf{A} has real eigenvalues without \mathbf{B} being definite positive. This is immediately evident for the differential system (27)₂ governing the motion of the pure fluid: it is straightforward that the two eigenvalues of \mathbf{A}^- are $v_f^- \pm \sqrt{\rho_f^- U_{ff}}$, while the condition of definite positiveness of \mathbf{B}^- is $U_{ff} > (v_f^-)^2 / \rho_f^-$. It is clear that in the region $0 < U_{ff} < (v_f^-)^2 / \rho_f^-$ the system is still hyperbolic without \mathbf{B}^- being definite positive.

Studying the hyperbolicity of the system (27)₁ governing the motion of the porous medium is not so immediate, since the structure of the eigenvalues of \mathbf{A}^+ is complicated. This is why we limit ourselves to a sufficient criterion of hyperbolicity by demanding definite positiveness of the matrix \mathbf{B}^+ (i.e. convexity of the total energy of the porous system). In this way we are sure that the eigenvalues of \mathbf{A} are real and this property will be seen to be crucial for studying propagation of linear waves in the porous system. The criterion of positiveness of \mathbf{B}^+ is the following :

$$W_{ff} > 0, \quad W_{ff}W_{ss} - W_{fs}^2 > 0, \quad v_f^2 < \frac{\rho_f}{W_{ss}} (W_{ff}W_{ss} - W_{fs}^2),$$

$$v_f^2 v_s^2 - W_{ss} \rho_s v_f^2 - W_{ff} \rho_f v_s^2 + \rho_f \rho_s (W_{ff}W_{ss} - W_{fs}^2) > 0,$$

where we omitted all the $+$ superscripts. It is straightforward that if \mathbf{B}^+ is definite positive then the potential W is convex. Moreover, in the case $v_s = 0$ the third and the fourth inequalities coincide. It can be directly checked that these inequalities imply

$$y < \kappa, \quad z < \frac{\kappa - y}{1 - y}, \quad 0 < \kappa < 1, \quad (29)$$

where

$$y := \frac{v_f^2}{\rho_f W_{ff}}, \quad z := \frac{v_s^2}{\rho_s W_{ss}}, \quad \kappa := \frac{W_{ff}W_{ss} - W_{fs}^2}{W_{ff}W_{ss}}.$$

It is then easy to show that the matrix \mathbf{B}^+ is definite positive (and then the potential W is surely convex) in the curvilinear triangle shown in Fig.1

We want to underline that, although equations (22) are Galilean invariant, the considered matrix formulation (27)₁ is not. In fact, if we change the reference system, the matrix \mathbf{A}^+ changes as

$$\mathbf{A}' = \mathbf{A}^+ + u\mathbf{I}, \quad (30)$$

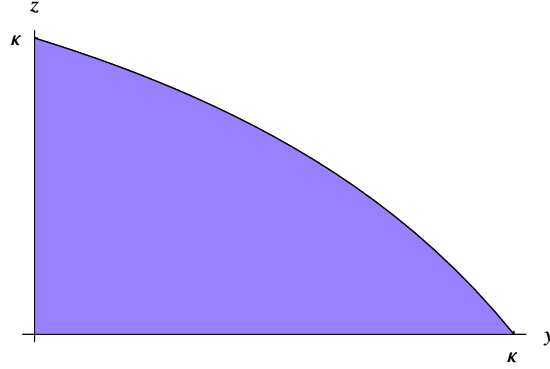


Figure 1: Region of definite positiveness of \mathbf{B}^+ (sufficient condition of convexity of W).

where u is any constant velocity. It is clear at this point that a sufficient criterion of hyperbolicity for the system (30) is

$$y' < \kappa, \quad z' < \frac{\kappa - y'}{1 - y'}, \quad 0 < \kappa < 1, \quad (31)$$

where

$$y' := \frac{(v_f + u)^2}{\rho_f W_{ff}}, \quad z' := \frac{(v_s + u)^2}{\rho_s W_{ss}}.$$

Clearly, if the velocities v_f and v_s are close to each other, then there always exists u such that the inequalities (31) are valid. In the case where the relative velocity $w := v_f - v_s$ is large our sufficient conditions (31) are violated and one can prove that the eigenvalues of \mathbf{A}' can be complex and the system of equations (30) is not hyperbolic. We conclude that the system is surely hyperbolic if the matrix \mathbf{B}^+ is definite positive and if the relative velocity w is small.

3.4. Linearized Problem

We start by linearizing the differential systems (27) around the equilibrium configuration

$$\begin{aligned} v_f^- &= 0, \quad \rho_f^- = (\rho_f^0)^- = \text{const}, \\ v_f^+ &= 0, \quad v_s = 0, \quad \rho_f^+ = (\rho_f^0)^+ = \text{const}, \quad \rho_s = \rho_s^0 = \text{const}. \end{aligned}$$

The linearized systems are

$$\tilde{\mathbf{V}}_t^- + \mathbf{A}_0^- \cdot \tilde{\mathbf{V}}_x^- = 0, \quad \tilde{\mathbf{V}}_t^+ + \mathbf{A}_0^+ \cdot \tilde{\mathbf{V}}_x^+ = 0, \quad (32)$$

where $\tilde{\mathbf{V}}^- := (\tilde{\rho}_f^-, \tilde{v}_f^-)$ and $\tilde{\mathbf{V}}^+ := (\tilde{\rho}_f^+, \tilde{\rho}_s^+, \tilde{v}_f^+, \tilde{v}_s^+)$ are the perturbed vectors of unknowns around the chosen equilibrium configuration. Moreover, $\mathbf{A}_0^- = \mathbf{P}^- \cdot \mathbf{B}_0^-$ and $\mathbf{A}_0^+ = \mathbf{P}^+ \cdot \mathbf{B}_0^+$, with

$$\mathbf{B}_0^- = \begin{pmatrix} U_{ff}^0 & 0 \\ 0 & (\rho_f^0)^- \end{pmatrix}, \quad \mathbf{B}_0^+ = \begin{pmatrix} W_{ff}^0 & W_{fs}^0 & 0 & 0 \\ W_{fs}^0 & W_{ss}^0 & 0 & 0 \\ 0 & 0 & (\rho_f^0)^+ & 0 \\ 0 & 0 & 0 & \rho_s^0 \end{pmatrix}$$

where we set $U_{ff}^0 := U_{ff}|_{(\rho_f^0)^-}$ and $W_{ij}^0 := W_{ij}|_{((\rho_f^0)^+, \rho_s^0)}$.

It is easy to check that in this linearized case, the definite positiveness of the matrices \mathbf{B}_0^- and \mathbf{B}_0^+ coincides with the convexity of the potentials U and W respectively. This means that if the potentials U and W are

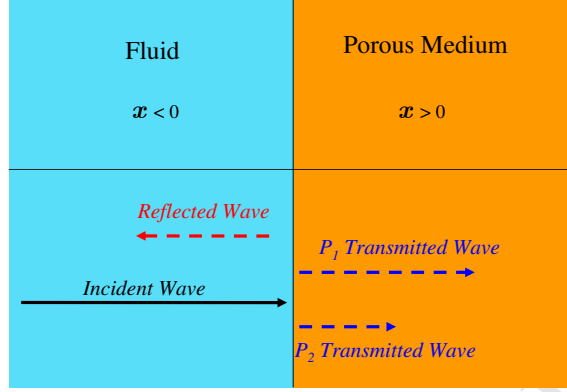


Figure 2: One dimensional wave problem

convex in the neighborhood of the chosen equilibrium solution, then the considered linearized differential systems are surely hyperbolic.

The characteristic polynomials of \mathbf{A}_0^- and \mathbf{A}_0^+ are :

$$\begin{aligned} &(\lambda^-)^2 - (\rho_f^0)^- U_{ff}^0, \\ &(\lambda^+)^4 - W_{ss}^0 \rho_s^0 (\lambda^+)^2 + (\rho_f^0)^+ \rho_s^0 \left(W_{ff}^0 W_{ss}^0 - (W_{fs}^0)^2 \right). \end{aligned} \quad (33)$$

In the region of definite positiveness of the matrix \mathbf{B}_0^- (convexity of U), it is straightforward that the characteristic polynomial of \mathbf{A}_0^- has a positive and a negative root which will be seen to be associated to an incident and to a reflected longitudinal wave respectively. Analogously, in the region of definite positiveness of the matrix \mathbf{B}_0^+ (convexity of W), it can be checked that the characteristic polynomial of \mathbf{A}_0^+ always has two positive and two negative roots corresponding to longitudinal waves.

The present treatment generalizes the one which can be found in Quiligotti *et al.* (2002).

4. Linear Waves at the Porous Medium-Pure Fluid Interface

Consider an initially fixed boundary $x = 0$ ($\mathfrak{D}_0 = 0$) between a pure fluid ($x < 0$) and a porous medium ($x > 0$) (see Fig. 2). The geometry of the problem justifies the chosen notations of a + and - superscripts for fields defined in the porous medium and in the pure fluid region respectively.

4.1. Boundary Conditions

The boundary conditions (15) and (16) in the considered one-dimensional, linearized case reduce to :

$$\sigma_{11}^+ = -p_f^-, \quad g^+ = g_f^-, \quad v_s^+ = \mathfrak{D},$$

or equivalently

$$(W - \rho_s W_s - \rho_f W_f)^+ = -p_f^-, \quad (34)$$

$$(W_f)^+ = g_f^-, \quad v_s^+ = \mathfrak{D}, \quad (35)$$

where we set $W_i := \partial W / \partial \rho_i$, $i = s, f$. We also recall that $g_f^- := \partial U / \partial \rho_f^- =: U_f$ is the pure fluid chemical potential, $p_f^- := -U + \rho_f^- g_f^-$ is the pure fluid pressure and \mathfrak{D} is the celerity of the Eulerian interface. It is easy to recover that the linearized boundary conditions (34), (35) around the equilibrium solution read

$$(-\rho_s^0 (W_{ss}^0 \tilde{\rho}_s + W_{fs}^0 \tilde{\rho}_f) - \rho_f^0 (W_{ff}^0 \tilde{\rho}_f + W_{fs}^0 \tilde{\rho}_s))^+ = -(\rho_f^0)^- U_{ff}^0 \tilde{\rho}_f^-, \quad (36)$$

$$(W_{ff}^0 \tilde{\rho}_f + W_{fs}^0 \tilde{\rho}_s)^+ = U_{ff}^0 \tilde{\rho}_f^-, \quad \tilde{v}_s^+ - \mathfrak{D} = 0. \quad (37)$$

Analogously the fluid balance of mass (14)₁ linearized around the considered equilibrium solution gives

$$(\rho_f^0 (\tilde{v}_f - \mathfrak{D}))^+ = (\rho_f^0 (\tilde{v}_f - \mathfrak{D}))^-; \quad (38)$$

notice that the solid balance of mass (14)₂ is automatically satisfied since $v_s = \mathfrak{D}$.

4.2. Propagation of Linear Waves

Consider now an incident longitudinal wave of frequency ω in a pure fluid situated in the region $x < 0$. When the incident wave (of amplitude α_i) reaches the interface between the fluid and the porous medium, a reflected wave (of amplitude α_r) appears in the fluid (see Fig. 2). To model this phenomenon, we assume that the perturbed solution in the pure fluid region takes the periodic form:

$$\tilde{\mathbf{V}}^- := (\tilde{\rho}_f^-, \tilde{v}_f^-)^T = \alpha_i^- \mathbf{h}_1^- e^{i(k^- x - \omega t)} + \alpha_r^- \mathbf{h}_2^- e^{i(-k^- x - \omega t)}. \quad (39)$$

where $\mathbf{h}_1^- = ((\rho_f^0)^-, \sqrt{(\rho_f^0)^- U_{ff}^0})^T$, $\mathbf{h}_2^- = (-(\rho_f^0)^-, \sqrt{(\rho_f^0)^- U_{ff}^0})^T$ are the two eigenvectors of the linearized matrix \mathbf{A}_0^- associated to the eigenvalues λ_1^- and λ_2^- respectively. We remark that the second component of the introduced eigenvectors has the dimension of a velocity and is commonly interpreted as a characteristic velocity of propagation in the pure fluid.

In expression (39) we have that the amplitude α_i^- of the incident wave is given, while the amplitude α_r^- of the reflected wave is unknown. Moreover, substituting expression (39) in the differential system (32)₁ we obtain the associated dispersion relations which imply that the wave number is given by $k^- = \omega / \sqrt{(\rho_f^0)^- U_{ff}^0}$.

In the domain $x > 0$ where the porous medium is situated two transmitted waves with positive velocities appear:

$$\tilde{\mathbf{V}}^+ := (\tilde{\rho}_f^+, \tilde{\rho}_s^+, \tilde{v}_f^+, \tilde{v}_s^+)^T = \beta_1^+ \mathbf{h}_1^+ e^{i(k_1^+ x - \omega t)} + \beta_2^+ \mathbf{h}_2^+ e^{i(k_2^+ x - \omega t)}, \quad (40)$$

where \mathbf{h}_j^+ , $j = 1, 2$ are the eigenvectors of the matrix \mathbf{A}_0^+ corresponding to two positive eigenvalues of \mathbf{A}_0^+ : $\lambda_1^+ > 0$ and $\lambda_2^+ > 0$ (we recall that \mathbf{A}_0^+ surely has two positive and two negative eigenvalues since we are assuming that \mathbf{B}_0^+ is definite positive). Moreover, $k_j^+ = \omega / \lambda_j^+$, $j = 1, 2$ are the wave numbers associated to the two waves transmitted in the porous medium. In the sequel we indicate by $\mathbf{h}_1^+ = (h_{11}, h_{12}, h_{13}, h_{14})$ and $\mathbf{h}_2^+ = (h_{21}, h_{22}, h_{23}, h_{24})$ the components of the two eigenvectors \mathbf{h}_j^+ . Notice that the first two components have the dimension of a density, while the last two components have the dimension of a velocity.

Finally, we assume that the free boundary moves with a celerity expressed in the form

$$\mathfrak{D} = \gamma h_{14} e^{-i\omega t}. \quad (41)$$

In this expression of the celerity, we multiply the right hand side by h_{14} just to deal with a dimensionless constant γ . As it will be clearer in the following, the fact of considering a periodic solution for the celerity of the interface and not for its displacement (as usually done) allows us to obtain a wave propagation problem which does not depend on frequency. We also underline that, since our solutions must be real (they represent physical quantities such as densities and velocities), the periodic expressions (39), (40) and (41) used to recover our solutions, actually mean that the real and the imaginary part of those expressions are separately solutions of our problem.

We conclude by remarking that we have four unknowns α_r^- , β_1^+ , β_2^+ and γ and four boundary conditions (36), (37)_{1,2} and (38). It is easy to check that replacing expressions (39), (40), 41 with $x = 0$ in these boundary conditions implies a set of four non-homogeneous algebraic equations with constant term proportional to the given amplitude α_i . This means that the four constants α_r^- , β_1^+ , β_2^+ and γ are themselves found to be proportional to α_i .

4.3. Energy Flux Associated to the Incident, Reflected and Transmitted Waves

We now consider the conservation of total energy (see also Eqs. (24) and (26)) for the pure fluid ($x < 0$) and for the porous medium ($x > 0$)

$$\frac{\partial E^-}{\partial t} + \frac{\partial H^-}{\partial x} = 0, \quad \frac{\partial E^+}{\partial t} + \frac{\partial H^+}{\partial x} = 0 \quad (42)$$

where clearly we set

$$\begin{aligned} E^- &:= U + \frac{\rho_f^- (v_f^-)^2}{2} & H^- &:= \rho_f^- v_f^- \left(\frac{(v_f^-)^2}{2} + \frac{\partial U}{\partial \rho_f^-} \right), \\ E^+ &:= W + \frac{\rho_f^+ (v_f^+)^2}{2} + \frac{\rho_s v_s^2}{2}, & H^+ &:= \rho_f^+ v_f^+ \left(\frac{(v_f^+)^2}{2} + \frac{\partial W}{\partial \rho_f^+} \right) + \rho_s v_s \left(\frac{v_s^2}{2} + \frac{\partial W}{\partial \rho_s} \right). \end{aligned}$$

The quantities H^+ , H^- are the so-called *energy fluxes* associated to the waves traveling in the porous medium and in the pure fluid respectively. It can be recognized with some calculation that the quadratic form of equations (42) around the considered equilibrium solution read

$$\frac{\partial \tilde{E}^-}{\partial t} + \frac{\partial \tilde{H}^-}{\partial x} = 0, \quad \frac{\partial \tilde{E}^+}{\partial t} + \frac{\partial \tilde{H}^+}{\partial x} = 0 \quad (43)$$

with

$$\tilde{E}^- := \frac{1}{2} \tilde{\mathbf{V}}^- \cdot \mathbf{B}_0^- \cdot \tilde{\mathbf{V}}^-, \quad \tilde{H}^- := \frac{1}{2} (\mathbf{B}_0^- \cdot \tilde{\mathbf{V}}^-) \cdot (\mathbf{A}_0^- \cdot \tilde{\mathbf{V}}^-) \quad (44)$$

$$\tilde{E}^+ := \frac{1}{2} \tilde{\mathbf{V}}^+ \cdot \mathbf{B}_0^+ \cdot \tilde{\mathbf{V}}^+, \quad \tilde{H}^+ := \frac{1}{2} (\mathbf{B}_0^+ \cdot \tilde{\mathbf{V}}^+) \cdot (\mathbf{A}_0^+ \cdot \tilde{\mathbf{V}}^+). \quad (45)$$

It is well known that, due to the conservation law of total energy (42) and to the fact that we are using periodic functions to build our solutions, the time integral over a period T of the energy flux H , takes the same value whatever the point x . This means the real number $J^+ := \int_0^T H^+(x, t) dt$ (or $J^- := \int_0^T H^-(x, t) dt$) is the same at any point x of the porous medium (or of the pure fluid) region. In particular, $J^+ = \int_0^T H^+(0, t) dt$ and $J^- = \int_0^T H^-(0, t) dt$.

Recalling expressions (44) and (45) for H^- and H^+ , expressions (39) and (40) for $\tilde{\mathbf{V}}^-$ and $\tilde{\mathbf{V}}^+$ and also Remark 1, it can be recovered after some calculation that

$$J^- = \frac{a}{2} (\mathbf{B}_0^- \cdot (\alpha_i^- \mathbf{h}_1^- + \alpha_r^- \mathbf{h}_2^-)) \cdot (\mathbf{A}_0^- \cdot (\alpha_i^- \mathbf{h}_1^- + \alpha_r^- \mathbf{h}_2^-)), \quad (46)$$

$$J^+ = \frac{a}{2} (\mathbf{B}_0^+ \cdot (\beta_1^+ \mathbf{h}_1^+ + \beta_2^+ \mathbf{h}_2^+)) \cdot (\mathbf{A}_0^+ \cdot (\beta_1^+ \mathbf{h}_1^+ + \beta_2^+ \mathbf{h}_2^+)), \quad (47)$$

where a is a constant coming from the integration over the period T of the considered periodic functions. Starting from this expression for J^- (J^+), recalling that \mathbf{h}_1^- , \mathbf{h}_2^- (\mathbf{h}_1^+ , \mathbf{h}_2^+) are eigenvectors of \mathbf{A}_0^- (\mathbf{A}_0^+) and that these eigenvectors are orthogonal in the inner product associated to \mathbf{B}_0^- (\mathbf{B}_0^+), it can be recovered that

$$J^- = \frac{a}{2} (J_i + J_r), \quad J^+ = \frac{a}{2} (J_{t_1} + J_{t_2}) \quad (48)$$

where we set

$$J_i := \lambda_1^-(\alpha_i^-)^2 (\mathbf{B}_0^- \cdot \mathbf{h}_1^-) \cdot \mathbf{h}_1^-, \quad J_r := \lambda_2^-(\alpha_r^-)^2 (\mathbf{B}_0^- \cdot \mathbf{h}_2^-) \cdot \mathbf{h}_2^-,$$

$$J_{t_1} := \lambda_1^+(\beta_1^+)^2 (\mathbf{B}_0^+ \cdot \mathbf{h}_1^+) \cdot \mathbf{h}_1^+, \quad J_{t_2} := \lambda_2^+(\beta_2^+)^2 (\mathbf{B}_0^+ \cdot \mathbf{h}_2^+) \cdot \mathbf{h}_2^+.$$

It is now possible to introduce a closed form for the reflection and transmission coefficients as:

$$r := \frac{J_r}{J_i}, \quad t_1 := \frac{J_{t_1}}{J_i}, \quad t_2 := \frac{J_{t_2}}{J_i}. \quad (49)$$

These expressions for reflection and transmission coefficients at a pure fluid/porous medium interface are deduced here for the first time. In fact, even if some efforts have been made to deduce such coefficients (see e.g. Denneman *et al.* (2002) and Gurevich *et al.* (2004)), a compact expression depending only on the spectral properties of the differential systems was not available up to now. It is easy to understand that these compact expressions can be very useful in order to deal with a wide class of problems of reflection and transmission of linear waves at a pure fluid/porous medium interface. We have already noticed that the reflected and transmitted amplitudes $(\alpha_r^-, \beta_1^+, \beta_2^+)$ are all proportional to the incident amplitude α_i^- : this means that the introduced reflection and transmission coefficients do not depend on the amplitude of the considered waves. Moreover, since conservation of energy must be verified at the interface $x = 0$, we also have $r + t_1 + t_2 = 1$.

4.4. Explicit Calculation of the Amplitudes α_r^- , β_1^+ , β_2^+ and γ

In this subsection we make an explicit calculation of the unknown amplitudes which may be useful for optimizing numeric calculations.

It can be recovered from equation (33) and assuming the convexity of W that the two positive eigenvalues of the matrix \mathbf{A}_0^+ are

$$\lambda_1^+ = \frac{\sqrt{2}}{2} \sqrt{c_1 + c_2}, \quad \lambda_2^+ = \frac{\sqrt{2}}{2} \sqrt{c_1 - c_2}, \quad (50)$$

with

$$c_1 := (\rho_f^0)^+ W_{ff}^0 + \rho_s^0 W_{ss}^0 > 0, \quad c_2 = \sqrt{c_1^2 - 4(\rho_f^0)^+ \rho_s^0 (W_{ss}^0 W_{ff}^0 - (W_{fs}^0)^2)} > 0.$$

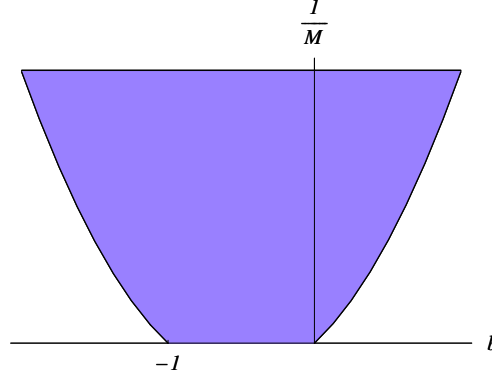
The corresponding eigenvectors are

$$\mathbf{h}_1^+ = \left(\frac{\sqrt{2}}{2} \frac{(\rho_f^0)^+ W_{ff}^0 - \rho_s^0 W_{ss}^0 + c_2}{W_{fs}^0}, \sqrt{2} \rho_s^0, \frac{((\rho_f^0)^+ W_{ff}^0 - \rho_s^0 W_{ss}^0 + c_2) \sqrt{c_1 + c_2}}{2W_{fs}^0 (\rho_f^0)^+}, \sqrt{c_1 + c_2} \right)^T,$$

$$\mathbf{h}_2^+ = \left(\frac{\sqrt{2}}{2} \frac{(\rho_f^0)^+ W_{ff}^0 - \rho_s^0 W_{ss}^0 - c_2}{W_{fs}^0}, \sqrt{2} \rho_s^0, \frac{((\rho_f^0)^+ W_{ff}^0 - \rho_s^0 W_{ss}^0 - c_2) \sqrt{c_1 - c_2}}{2W_{fs}^0 (\rho_f^0)^+}, \sqrt{c_1 - c_2} \right)^T.$$

We note that here, in analogy with the pure fluid case, the quantities $\sqrt{c_1 + c_2}$ and $\sqrt{c_1 - c_2}$ can be interpreted as two characteristic velocities of propagation in the porous medium. Substituting expressions (39), (40) and (41) in the boundary conditions (36), (37)_{1,2} and (38) (valid at $x = 0$) implies the following system of algebraic equations:

$$\begin{aligned} C_{11}\beta_1^+ + C_{12}\beta_2^+ + C_{13}\alpha_r^- &= C_{13}\alpha_i^- \\ C_{21}\beta_1^+ + C_{22}\beta_2^+ + C_{23}\alpha_r^- &= C_{23}\alpha_i^- \\ C_{31}\beta_1^+ + C_{32}\beta_2^+ + C_{33}\alpha_r^- + C_{34}\gamma &= -C_{33}\alpha_i^-, \\ C_{41}\beta_1^+ + C_{42}\beta_2^+ - C_{41}\gamma &= 0 \end{aligned}$$


 Figure 3: Region of local convexity of W .

where

$$\begin{aligned} C_{11} &= -h_{11} (W_{fs}^0 \rho_s^0 + W_{ff}^0 (\rho_f^0)^+) - h_{12} (W_{ss}^0 \rho_s^0 + W_{fs}^0 (\rho_f^0)^+), \\ C_{12} &= -h_{21} (W_{fs}^0 \rho_s^0 + W_{ff}^0 (\rho_f^0)^+) - h_{22} (W_{ss}^0 \rho_s^0 + W_{fs}^0 (\rho_f^0)^+), \quad C_{13} = -(c_f^0)^2 (\rho_f^0)^-, \\ C_{21} &= h_{11} W_{ff}^0 + h_{12} W_{fs}^0, \quad C_{22} = h_{21} W_{ff}^0 + h_{22} W_{fs}^0, \quad C_{23} = (c_f^0)^2, \\ C_{31} &= h_{13} (\rho_f^0)^+, \quad C_{32} = h_{23} (\rho_f^0)^+, \quad C_{33} = -c_f^0 (\rho_f^0)^-, \quad C_{34} = h_{14} ((\rho_f^0)^- - (\rho_f^0)^+), \\ C_{41} &= h_{14}, \quad C_{42} = h_{24}. \end{aligned}$$

It follows that the four unknowns can be found in terms of the amplitude α_i of the incident wave as

$$\beta_1^+ = \frac{B_1}{A} \alpha_i^-, \quad \beta_2^+ = \frac{B_2}{A} \alpha_i^-, \quad \alpha_r^- = \left(1 + \frac{B_3}{A}\right) \alpha_i^-, \quad \gamma = \frac{B_1 + B_4}{A} \alpha_i^-,$$

with

$$\begin{aligned} A &= C_{11} [C_{22} C_{33} C_{41} - C_{23} (C_{32} C_{41} + C_{34} C_{42})] + C_{12} C_{41} [-C_{21} C_{33} + C_{23} (C_{31} + C_{34})] \\ &\quad + C_{13} [-C_{22} (C_{31} + C_{34}) C_{41} + C_{21} (C_{32} C_{41} + C_{34} C_{42})] \\ B_1 &= 2 (C_{13} C_{22} - C_{12} C_{23}) C_{33} C_{41}, \quad B_2 = 2 (C_{11} C_{23} - C_{13} C_{21}) C_{33} C_{41}, \\ B_3 &= 2 (C_{12} C_{21} - C_{11} C_{22}) C_{33} C_{41}, \quad B_4 = \frac{C_{42}}{C_{41}} B_2. \end{aligned}$$

5. The Isotropic Case

Despite the general validity of the results obtained up to now, we now choose a particular constitutive behavior both for the pure fluid and the porous medium in order to show some numeric simulations arising from the theoretical scheme proposed here. We choose particular expressions for the potential energies of the two considered mechanical systems. As for the pure fluid, we assume the classical expression for the Eulerian volume potential energy density of a compressible fluid

$$U = \frac{(c_f^0)^2}{(\rho_f^0)^-} (\rho_f^-)^2,$$

where $c_f^0 > 0$ is the sound velocity in the pure fluid. We recall that this expression for the potential U is valid only in a neighborhood of the considered equilibrium configuration $(\rho_f^- = (\rho_f^0)^-)$. It is straightforward that the potential U is convex and this means that the linearized differential system $(32)_1$ is surely hyperbolic.

On the other hand, we introduce the following expression for the solid-Lagrangian volume deformation energy density valid for the isotropic case (see e.g. Coussy (2004) and Biot (1957))

$$\Psi(\boldsymbol{\varepsilon}, m_f) = \frac{\lambda}{2} (\text{tr } \boldsymbol{\varepsilon})^2 + \mu \text{tr } (\boldsymbol{\varepsilon}^2) + \frac{1}{2} M \left(\frac{m_f}{m_f^0} \right)^2 - b M \frac{m_f}{m_f^0} \text{tr } \boldsymbol{\varepsilon}. \quad (51)$$

Here λ and μ are the undrained Lamé coefficients, M is the Biot modulus and b is called the Biot coefficient. We recall that this expression of the energy density is only a local expression, i.e. it is valid only in a neighborhood of the configuration $\boldsymbol{\varepsilon} = 0$, $m_f = m_f^0$. It is easy to recognize (see section (3.1)) that in the considered one-dimensional isotropic case we have that the introduced potentials for the porous medium are given by

$$\begin{aligned} \tilde{\Psi}(\epsilon, m_f) &= \frac{1}{2} (\lambda + 2\mu) \epsilon^2 + \frac{1}{2} M \left(\frac{m_f}{m_f^0} \right)^2 - b M \frac{m_f}{m_f^0} \epsilon, \\ \hat{\Psi}(\tau_s, m_f) &= \frac{1}{4} \left(\frac{\lambda}{2} + \mu \right) (\eta_s^2 \tau_s^2 - 1)^2 + \frac{1}{2} M \left(\frac{m_f}{m_f^0} \right)^2 - \frac{1}{2} b M \frac{m_f}{m_f^0} (\eta_s^2 \tau_s^2 - 1), \\ W(\rho_s, \rho_f) &= \frac{1}{4} \left(\frac{\lambda}{2} + \mu \right) \frac{\rho_s}{\eta_s} \left(\left(\frac{\eta_s}{\rho_s} \right)^2 - 1 \right)^2 + \frac{1}{2} \frac{M}{(m_f^0)^2} \frac{\eta_s}{\rho_s} \rho_f^2 - \frac{1}{2} \frac{b M}{(m_f^0)} \left(\left(\frac{\eta_s}{\rho_s} \right)^2 - 1 \right) \rho_f, \end{aligned}$$

where here and in the sequel we omit all the + superscripts. All these expressions are valid locally. Hence, recalling that $1/\rho_s = 1/\eta_s \sqrt{1 + 2\epsilon}$ and since we are considering a perturbation in the neighborhood of $\epsilon = 0$, we assume from now on that $\rho_s^0 = \eta_s$. Recalling that $m_f^0 = \eta_s/\rho_s \rho_f$, it is easy to recover that

$$W_{ff}^0 = \frac{M}{(\rho_f^0)^2}, \quad W_{fs}^0 = \frac{M(b-1)}{\rho_s^0 \rho_f^0}, \quad W_{ss}^0 = \frac{(\lambda + 2\mu) + M(1-3b)}{(\rho_s^0)^2}.$$

Hence, it is easy to recover that the potential W is convex in the neighborhood of the chosen equilibrium solution (the matrix of its second derivatives is definite positive) only if the following inequalities are satisfied

$$M > 0, \quad \frac{1}{M} > \frac{b(1+b)}{\lambda + 2\mu}. \quad (52)$$

These inequalities establish the existence of a region in the plane $(b, 1/M)$ in which the potential W is locally convex (see Fig. 3) and then the differential system $(32)_2$ is surely hyperbolic.

In the sequel, it is useful to introduce the dimensionless variable $m \in [0, 1]$ defined by means of the relationship

$$M = m \frac{(\lambda + 2\mu)}{b(1+b)}, \quad (53)$$

in such a way the plots of the reflection and transmission coefficients as functions of the Biot coefficient b and of the scaled Biot modulus m seem to be most useful. Indeed, the presented plots show the values of transmission and reflection coefficients in the whole region of convexity of W . These plots have been obtained by fixing (i) a unique value for the pure fluid apparent density, (ii) the constitutive parameters of the pure fluid (water), (iii) a unique value for the solid and fluid apparent densities in the porous medium and (iv) three different values for the elastic coefficients of the solid matrix. On the other hand, we let the Biot parameters vary in the range compatible with the convexity of W (see equations (52)). In particular, in order to take a view to applications the values chosen in (iii) and one of those chosen in (iv) are relative to a water-filled limestone of Bourgogne (see Lion *et al.* 2005). The other two values chosen in (iv) are of an order of magnitude greater and smaller respectively.

In Figures 4 to 9 the variation of the reflection and transmission coefficients are shown as functions of the Biot's parameters of the porous medium. In particular, we let the Biot coefficient b vary in $[0, 1]$ and the scaled Biot modulus m vary in suitably adapted ranges (obviously all included in the $[0, 1]$ interval). These choices are suggested by a threshold phenomenon which is observed for every value of b , when the Biot modulus M varies. For values of M smaller than the threshold, P_1 is the dominant transmitted wave, while for values of M greater than the threshold, P_2 transports the greater amount of energy. We remark that the new parametrization accomplished by introducing the dimensionless parameter m , not only is suitable for representing the reflection and transmission coefficients on the whole W convexity domain, but also for easily rendering aforementioned threshold phenomenon. It has also to be remarked that there is no thermodynamical restriction imposing that b must vary in $[0, 1]$. We limit our attention to this case because, presently, measured values of b for soils (see e.g. Lion *et al.* (2004)) fall in this range.

Figure 4 shows how the energy of reflected wave is influenced by the “overall (averaged) mechanical properties” of the porous medium as determined by both b and M parameters: for small values of M there is a large amount of reflected energy, which decreases when the porous medium allows for the most similar wave propagation compared with that occurring in pure fluid. Indeed, it is easy to understand that the reflected energy decreases when the porous medium has averaged mechanical properties close to the mechanical properties of the pure fluid. Consequently, the amount of transmitted energy increases when the solid-fluid mixture constituting the porous medium can “mimic” sufficiently well the pure fluid. Hence the way according to which the total amount of energy transported by the incident wave is divided between reflected and transmitted energy depends on how “similar” are the overall mechanical properties of the porous medium with respect to those of the pure fluid which occupies the region $x < 0$.

Figure 7 shows how, with increasing values of M , one gets another maximum for reflected energy: this is sensible since increasing values of M are related to stiffer saturating fluid in the porous medium.

Figures 5 and 6 must be considered simultaneously. Indeed, they show that, once the total amount of energy which may be transmitted inside the porous medium is determined by its overall mechanical properties, the transmitted energy is divided between P_1 and P_2 waves depending on how relevant are solid-fluid coupling phenomena in the porous medium. More precisely, for a fixed value of b , once the value of M increases, initially the transmitted wave having the higher energetic content is P_1 (or fast) wave. This wave is mainly related to velocity and deformation of solid matrix: the values of M are not great enough, for given b coefficient, to trigger an energy transfer to a slow P_2 wave, which involves an energy transport mainly related to fluid relative flow and deformation. When M reaches the “coupling threshold value” the greater part of transmitted energy is instead carried by P_2 wave. This result is not astonishing if we consider the expression (51) for the porous medium deformation energy: indeed, if M approaches to zero the deformation energy of the porous medium is uniquely due to the solid constituent.

Figures 8 and 9 show that the phenomenon observed in Fig. 7, i.e. the increase for large M of reflected energy, is related to a corresponding decrease of the energy transported by P_2 waves, while the energy transported by P_1 wave remains almost constant. In other words, with increasing M there is a final cutoff of fluid-type waves in the porous medium.

To conclude, we note here again how the presented three-dimensional plots can be useful to determine the value of the Biot's coefficients b and M when the measurement of reflection and transmission coefficients is possible. Indeed, some flat regions are present in these plots and hence, sometimes, the indirect measure of the Biot's parameters may not be unique. Nevertheless, the fact of measuring three different waves (one reflected and two transmitted) drastically reduces this unfortunate possibility. It is sufficient to make the intersection of all the couples (b, M) corresponding to the measured values of the three waves to reasonably limit the number of admissible values of the Biot's parameters which are compatible to the three performed measurements. For these reasons, we believe that the presented method for calculating Biot's parameters from the measured values of the reflection and transmission coefficients may prove its effectiveness in a wide range of practical applications.

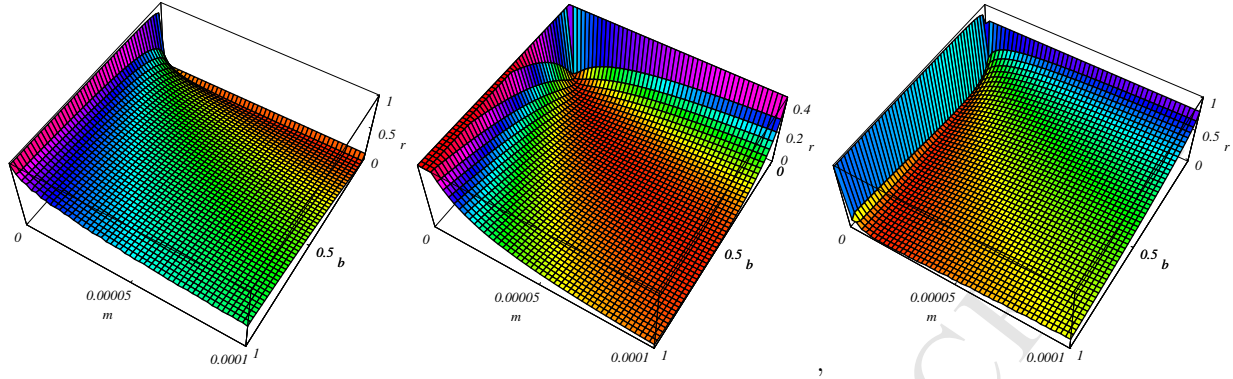


Figure 4: Reflection coefficient vs. Biot's Parameters for $m \in [0, 10^{-4}]$. From left to right: $\lambda + 2\mu = 3GPa$, $\lambda + 2\mu = 30GPa$, $\lambda + 2\mu = 300GPa$.

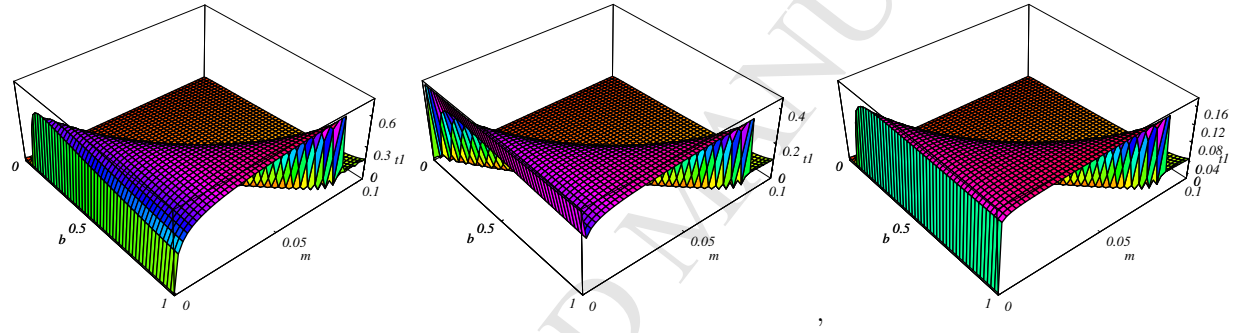


Figure 5: P_1 wave transmission coefficient vs. Biot's Parameters for $m \in [0, 0.1]$. From left to right: $\lambda + 2\mu = 3GPa$, $\lambda + 2\mu = 30GPa$, $\lambda + 2\mu = 300GPa$.

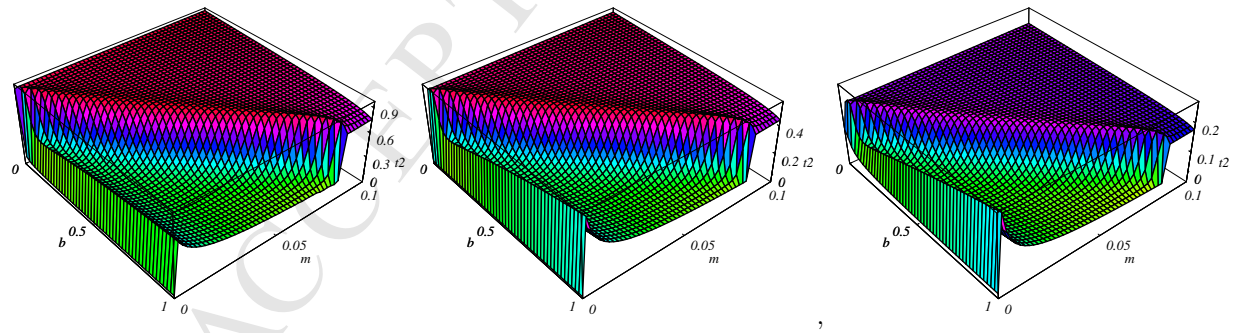


Figure 6: P_2 wave transmission coefficient vs. Biot's Parameters for $m \in [0, 0.1]$. From left to right: $\lambda + 2\mu = 3GPa$, $\lambda + 2\mu = 30GPa$, $\lambda + 2\mu = 300GPa$.

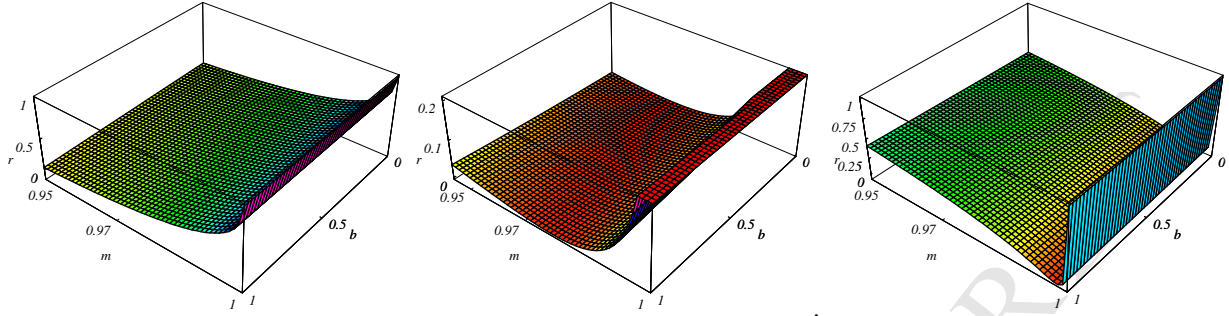


Figure 7: Reflection coefficient vs. Biot's Parameters for $m \in [0.95, 1]$. From left to right: $\lambda + 2\mu = 3GPa$, $\lambda + 2\mu = 30GPa$, $\lambda + 2\mu = 300GPa$.

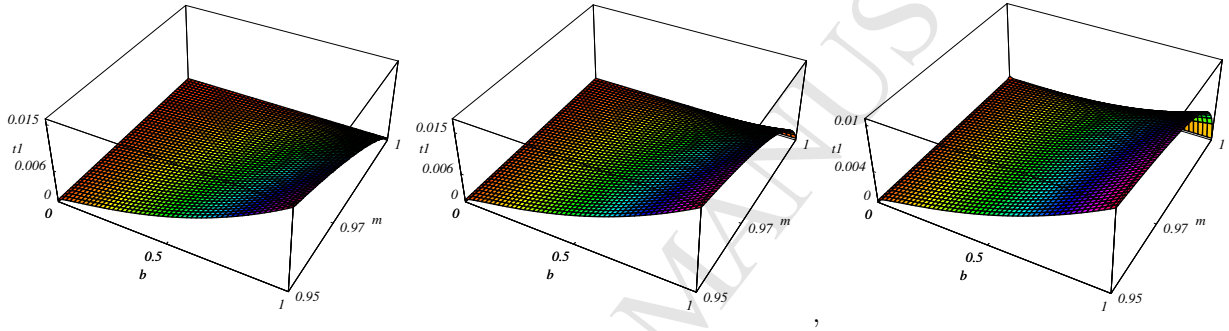


Figure 8: P_1 wave transmission coefficient vs. Biot's Parameters for $m \in [0.95, 1]$. From left to right: $\lambda + 2\mu = 3GPa$, $\lambda + 2\mu = 30GPa$, $\lambda + 2\mu = 300GPa$.

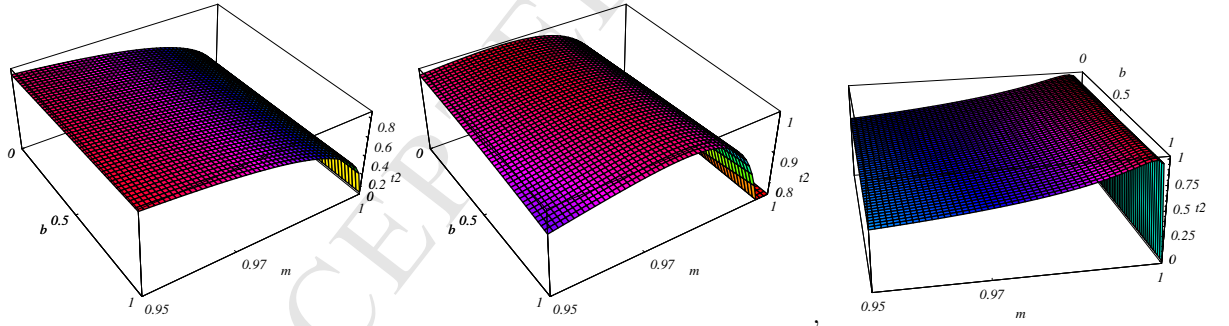


Figure 9: P_2 wave transmission coefficient vs. Biot's Parameters for $m \in [0.95, 1]$. From left to right: $\lambda + 2\mu = 3GPa$, $\lambda + 2\mu = 30GPa$, $\lambda + 2\mu = 300GPa$.

6. Conclusions

The main result of the first part of this paper could be considered, at a first sight, of purely mathematical interest. Indeed, we apply well-established results in the classical theory of partial differential equations to establish some sufficient conditions assuring the existence of propagating waves in Biot fluid-saturated porous media. The novel result we present, however, is not of purely mathematical interest. In fact, the reasonings leading to the statement about hyperbolicity of Biot's equations, allow for a very simple and compact representation of the boundary conditions which must be associated to them. These boundary conditions were deduced by means of a variational approach in dell'Isola *et al.* (2009) where the late results of Biot (1972) were suitably extended. We establish symmetric hyperbolicity of Biot's evolution equations

by remarking that they imply a form of balance of energy having a conservative structure with convex energy function. Moreover, boundary conditions associated to Biot's equation also imply jump form of energy conservation which is compatible with the deduced bulk energy conservation law. The insight gained by such a mathematical treatment allowed for a very effective representation of transmission and reflection coefficients of waves propagating in a pure fluid and impacting at a pure-fluid/porous-medium interface. Indeed, in section 4.3, the mathematical results previously established are used to determine a very simple form of such coefficients which are expressed in terms of the spectral properties of the governing differential system.

Although in the literature the problem of reflection and transmission of waves in porous media has been already attacked, up to now its solution was not attained in the extent which is required to guide experiments toward the indirect measurement of the constitutive properties of porous media. Many authors showed plots drawing the reflection and transmission coefficients as functions of the frequency (see e.g. Denneman *et al.* (2002)) or of the incidence angle of the impacting wave (see e.g. Rubino *et al.* (2006)). This choice implies that all the constitutive parameters of the considered media are assumed to be known by means of direct measurements performed on suitably prepared specimens. Indeed, as shown by Lion *et al.* (2004), the experimental devices needed to perform such direct measurements of constitutive poroelastic properties are quite sophisticated. These authors also show that, while the measurement of the undrained Lamé coefficients is relatively easy to obtain, measuring the Biot's parameters appears more difficult. For these reasons we propose, as an application of our more general results, an indirect method for the determination of Biot's parameters b and M by means of the measurement of energetic transmission and reflection coefficients of acoustic waves.

Future investigations should include the consideration of the effect on transmission and reflection phenomena of a stationary drainage flow in the porous medium. In this way one should be able to detect some of the phenomena described in dell'Isola and Hutter (1998).

7. Acknowledgements

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Appendix A. Solid-Lagrangian and Eulerian Conservation of Total Energy

We first derive the solid-Lagrangian conservation law for the total energy $\Psi(\varepsilon, m_f) + \Lambda(\eta_s, m_f, \mathbf{v}_s, \mathbf{v}_f^{\otimes})$. Recalling that η_s is constant, it is straightforward that

$$\frac{\partial}{\partial t}(\Psi + \Lambda) = \frac{\partial \Psi}{\partial \varepsilon} \mid \frac{\partial \varepsilon}{\partial t} + \frac{\partial \Psi}{\partial m_f} \frac{\partial m_f}{\partial t} + \eta_s \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial t} + m_f \mathbf{v}_f^{\otimes} \cdot \frac{\partial \mathbf{v}_f^{\otimes}}{\partial t} + \frac{1}{2}(\mathbf{v}_f^{\otimes})^2 \frac{\partial m_f}{\partial t}, \quad (\text{A.1})$$

where we clearly indicate by the symbol \mid the scalar product between two second order tensors. Moreover, since $\gamma_f^{\otimes} = \partial \mathbf{v}_f^{\otimes} / \partial t - \nabla \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s^{-1} \cdot (\mathbf{v}_s - \mathbf{v}_f^{\otimes})$, it can be proven that equations (1) and (2) imply

$$\frac{\partial \mathbf{v}_f^{\otimes}}{\partial t} = -\mathbf{F}_s^{-T} \cdot \nabla \left(\frac{\partial \Psi}{\partial m_f} \right) + \nabla \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s^{-1} \cdot (\mathbf{v}_s - \mathbf{v}_f^{\otimes}), \quad (\text{A.2})$$

$$\frac{\partial \mathbf{v}_s}{\partial t} = \frac{1}{\eta_s} \left(\operatorname{div} \left(\mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} \right) + m_f \mathbf{F}_s^{-T} \cdot \nabla \left(\frac{\partial \Psi}{\partial m_f} \right) \right). \quad (\text{A.3})$$

Using these expressions in equation (A.1), together with the fluid balance of mass (3) and the fact that $\partial \varepsilon / \partial t = \mathbf{F}_s^T \cdot \nabla \mathbf{v}_s$, equation (18) for the solid-Lagrangian energy conservation can be finally recovered.

Let us now introduce the real function

$$W(\varepsilon, \rho_f^{\otimes}) := J_s^{-1} \Psi(\varepsilon, J_s \rho_f^{\otimes}).$$

Recalling that $\partial J_s / \partial \varepsilon = J_s (\mathbf{F}_s^T \cdot \mathbf{F}_s)^{-1}$, is easy to show that

$$\frac{\partial \Psi}{\partial m_f} = \frac{\partial W}{\partial \rho_f^{\otimes}} \quad \text{and} \quad \frac{\partial \Psi}{\partial \varepsilon} = J_s \left(\frac{\partial W}{\partial \varepsilon} + (\mathbf{F}_s^T \cdot \mathbf{F}_s)^{-1} \left(W - \rho_f^{\otimes} \frac{\partial W}{\partial \rho_f^{\otimes}} \right) \right).$$

Hence, expression (18) for the solid-Lagrangian conservation of total energy is rewritten in terms of W as

$$\frac{\partial}{\partial t}(\Psi + \Lambda) + \operatorname{div} \left[J_s \rho_f^{\otimes} \left(\frac{\partial W}{\partial \rho_f^{\otimes}} \mathbf{F}_s^{-1} \cdot \mathbf{v}_f^{\otimes} + \frac{1}{2}(\mathbf{v}_f^{\otimes})^2 \mathbf{F}_s^{-1} \cdot (\mathbf{v}_f^{\otimes} - \mathbf{v}_s) \right) - J_s \left(W \mathbf{F}_s^{-1} + \frac{\partial W}{\partial \varepsilon} \cdot \mathbf{F}_s^T \right) \cdot \mathbf{v}_s \right] = 0. \quad (\text{A.4})$$

Let \mathbf{f} and f be respectively a vector and a scalar field defined on the reference configuration of the solid. It is known that, if one has a balance law on the solid-Lagrangian configuration in the form $\partial f / \partial t + \operatorname{div} \mathbf{f} = 0$, then its corresponding balance law on the Eulerian configuration is given by (see dell'Isola *et al.* 2009 for details)

$$\frac{\partial}{\partial t}(J_s^{-1} f)^{\otimes} + \operatorname{div}(J_s^{-1} \mathbf{f} \cdot \mathbf{F}_s^{-T} + J_s^{-1} f \mathbf{v}_s)^{\otimes}.$$

Transporting the solid-Lagrangian balance law (A.4) according to this transport formula and simplifying, the Eulerian balance law for the total energy is finally recovered in the form of equation (20).

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