Asymptotics of a vanishing period : General existence theorem and basic properties of frescos

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Abstract

In this paper we introduce the word "fresco" to denote a $[\lambda]-$primitive monogenic geometric (a,b)-module. The study of this "basic object" (generalized Brieskorn module with one generator) which corresponds to the minimal filtered (regular) differential equation satisfied by a relative de Rham cohomology class, began in [B.09] where the first structure theorems are proved. Then in [B.10] we introduced the notion of theme which corresponds in the $[\lambda]-$primitive case to frescos having a unique Jordan-Hölder sequence. Themes correspond to asymptotic expansion of a given vanishing period, so to the image of a fresco in the module of asymptotic expansions. For a fixed relative de Rham cohomology class (for instance given by a smooth differential form $d-$closed and $df-$closed) each choice of a vanishing cycle in the spectral eigenspace of the monodromy for the eigenvalue $exp(2i\pi.\lambda)$ produces a $[\lambda]-$primitive theme, which is a quotient of the fresco associated to the given relative de Rham class itself.
The first part of this paper shows that, for any $[\lambda]-$primitive fresco there exists an unique Jordan-Hölder sequence (called the principal J-H. sequence) with corresponding quotients giving the opposite of the roots of the Bernstein polynomial in a non decreasing order. Then we introduce and study the semi-simple part of a given fresco and we characterize the semi-simplicity of a fresco by the fact for any given order of the roots of its Bernstein polynomial we may find a J-H. sequence making them appear with this order. Then, using the parameter associated to a rank 2 $[\lambda]-$primitive theme, we introduce inductiveley a numerical invariant, that we call the $\alpha-$invariant, which depends polynomially on the isomorphism class of a fresco

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(in a sens which has to be defined) and which allows to give an inductive way to produce a sub-quotient rank 2 theme of a given $[\lambda]-$primitive fresco assuming non semi-simplicity.
In the last section we prove a general existence result which naturally associate a fresco to any relative de Rham cohomology class of a proper holomorphic function of a complex manifold onto a disc. This is, of course, the motivation for the study of frescos.

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Key words. Fresco, theme, (a,b)-module, asymptotic expansion, vanishing period, Gauss-Manin connection, filtered differential equation.

Contents

Introduction

Let $f : X \to D$ be an holomorphic function on a connected complex manifold. Assume that $\{df = 0\} \subset \{f = 0\} := X_0$. We consider X as a degenerating family of complex manifolds parametrized by $D^* := D \setminus \{0\}$ with a singular member X_0 at the origin of D. Let ω be a smooth $(p+1)-$differential form on X satisfying $d\omega = 0 = df \wedge \omega$. Then in many interesting cases (see for instance [B.II] , [B.III] for the case of a function with 1-dimensional singular set and the section 4 for the general proper case) the relative family of de Rham cohomology classes induced on the fibers $(X_s)_{s\in D^*}$ of f by ω/df is solution of a minimal filtered differential equation

defined from the Gauss-Manin connection of f. This object, called a **fresco** is a monogenic regular (a,b)-module satisfying an extra condition, called "geometric", which encodes simultaneously the regularity at 0 of the Gauss-Manin connection, the monodromy theorem and B. Malgrange's positivity theorem.
We study the structure of such an object in order to determine the possible quotient themes of a given fresco. Such a theme corresponds to a possible asymptotic expansion of vanishing periods constructed from ω by choosing a vanishing cycle $\gamma \in H_p(X_{s_0}, \mathbb{C})$ and definig

$$F_\gamma(s) := \int_{\gamma_s} \omega/df$$

where γ_s is the (multivalued) horizontal family of cycles defined from γ in the fibers of f (see [M.74]).

Let me describe the content of this article.
After some easy preliminaries, we prove in section 1 that a $[\lambda]-$primitive fresco E admits an *unique* Jordan-Hölder sequence, called the principal J-H. sequence, in which the opposite of the roots of the Bernstein polynomial of E appears in a non decreasing order. This uniqueness result is important because it implies, for instance, that the isomorphism classes of each quotient of terms of the principal J-H. sequence only depends on the isomorphism class of E.

In section 2 we define and study semi-simple regular (a,b)-modules and the corresponding semi-simple filtration. In the case of a $[\lambda]-$primitive fresco we prove that semi-simplicity is characterized by the fact that we may find a J-H. sequence in which the opposite of the roots of the Bernstein polynomial appears in strictely decreasing order (but also in any given order).

In section 3 we answer to the following question : How to recognize from the principal J-H. sequence of a $[\lambda]-$primitive fresco if it is semi-simple. Of course the answer is obvious if some rank 2 sub-quotient theme appears from this sequence. But when it is not the case, such a rank 2 sub-quotient theme may appear after commuting some terms in the J-H. sequence. The simplest example is when

$$E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b)(1+\alpha.b^{p_1+p_2})^{-1}.(a-\lambda_2.b).(a-\lambda_3.b)$$

with $\lambda_{i+1} = \lambda_i + p_i - 1$ for $i = 1, 2$ with $p_i \in \mathbb{N}^*, i = 1, 2$ and $\alpha \in \mathbb{C}^*$. Using the identity in $\tilde{\mathcal{A}}$

$$(a-\lambda_2.b).(a-\lambda_3.b) = (a-(\lambda_3+1).b).(a-(\lambda_2-1).b)$$

it is easy to see that $(a-(\lambda_2-1).b).[1]$ generates a (normal) rank 2 theme in E, because we assume $\alpha \neq 0$.

We solve this question introducing for a $[\lambda]-$primitive rank $k \geq 2$ fresco E such F_{k-1} and E/F_1 are semi-simple, where $(F_j)_{j\in[1,k]}$ is the principal J-H. sequence of E, the $\alpha-$invariant $\alpha(E)$. This complex number only depends on the isomorphism class of E and is zero if and only E is semi-simple. We also prove that when $\alpha(E)$ is not zero it gives the parameter of any normal rank 2 sub-theme of E.

In section 4 we give, using tools introduced in [B.II], a general existence theorem for the fresco associated to a relative de Rham cohomology class in the geometric situation described in the begining of this introduction, assuming the function f proper.

1 Preliminaries.

1.1 Definitions and characterization as $\tilde{\mathcal{A}}-$modules.

We are interested in "standard " formal asymptotic expansions of the following type

$$\sum_{q=1}^{p}\sum_{j=0}^{N} \mathbb{C}[[s]].s^{\lambda_q-1}.(Log\, s)^j$$

where $\lambda_1,\ldots,\lambda_q$ are positive rational numbers, and in fact in vector valued such expansions. The two basic operations on such expansions are

- the multiplication by s that we shall denote a,
- and the primitive in s without constant that we shall denote b.

This leads to consider on the set of such expansions a left module structure on the $\mathbb{C}-$algebra

$$\tilde{\mathcal{A}} := \{ \sum_{\nu=0}^{\infty} P_\nu(a).b^\nu, \quad \text{where} \quad P_\nu \in \mathbb{C}[x] \ \}$$

defined by the following conditions

- The commutation relation $a.b - b.a = b^2$ which is the translation of the Leibnitz rule ;
- The continuity for the $b-$adic filtration of $\tilde{\mathcal{A}}$ of the left and right multiplications by a.

Define now for λ in $\mathbb{Q}\cap]0,1]$ and $N\in\mathbb{N}$ the left $\tilde{\mathcal{A}}-$module

$\Xi_\lambda^{(N)} := \oplus_{j=0}^N \mathbb{C}[[s]].s^{\lambda-1}.(Log\, s)^j = \oplus_{j=0}^N \mathbb{C}[[a]].s^{\lambda-1}.(Log\, s)^j = \oplus_{j=0}^N \mathbb{C}[[b]].s^{\lambda-1}.(Log\, s)^j.$

Of course we let a and b act on $\Xi_\lambda^{(N)}$ as explained above.
Define also, when Λ is a finite subset in $\mathbb{Q}\cap]0,1]$, the $\tilde{\mathcal{A}}-$module

$$\Xi_\Lambda^{(N)} := \oplus_{\lambda\in\Lambda} \Xi_\lambda^{(N)}.$$

More generally, if V is a finite dimensional complex vector space we shall put a structure of left $\tilde{\mathcal{A}}-$module on $\Xi_\Lambda^{(N)}\otimes_{\mathbb{C}} V$ with the following rules :

$$a.(\varphi\otimes v) = (a.\varphi)\otimes v \quad \text{and} \quad b.(\varphi\otimes v) = (b.\varphi)\otimes v$$

for any $\varphi\in\Xi_\Lambda^{(N)}$ and any $v\in V$. It will be convenient to denote Ξ_λ the $\tilde{\mathcal{A}}-$module $\sum_{N\in\mathbb{N}} \Xi_\lambda^{(N)}$.

Definition 1.1.1 *A $\tilde{\mathcal{A}}-$module is call a* **fresco** *when it is isomorphic to a sub-module $\tilde{\mathcal{A}}.\varphi \subset \Xi_\Lambda^{(N)} \otimes V$ where φ is any element in $\Xi_\Lambda^{(N)} \otimes V$, for some choice of Λ, N and V as above.*
A fresco is a **theme** *when we may choose $V := \mathbb{C}$ in the preceeding choice.*

Now the characterization of frescos among all left $\tilde{\mathcal{A}}-$modules is not so obvious. The following theorem is proved in [B.09].

Theorem 1.1.2 *A left $\tilde{\mathcal{A}}-$module E is a fresco if and only if it is a geometric (a,b)-module which is generated (as a $\tilde{\mathcal{A}}-$module) by one element. Moreover the annihilator in $\tilde{\mathcal{A}}$ of any generator of E is a left ideal of the form $\tilde{\mathcal{A}}.P$ where P may be written as follows*

$$P = (a - \lambda_1.b).S_1^{-1}.(a - \lambda_2.b).S_2^{-1} \dots (a - \lambda_k.b).S_k^{-1}, \quad k := \dim_{\mathbb{C}}(E/b.E)$$

where λ_j are rational numbers such that $\lambda_j + j > k$ for $j \in [1, k]$ and where $S_1, \dots, S_k$ are invertible elements in the sub-algebra $\mathbb{C}[[b]]$ of $\tilde{\mathcal{A}}$.
Conversely, for such a $P \in \tilde{\mathcal{A}}$ the left $\tilde{\mathcal{A}}-$module $E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P$ is a fresco and it is a free rank k module on $\mathbb{C}[[b]]$.

Let me recall briefly for the convenience of the reader the definitions of the notions involved in the previous statement.

- A **(a,b)-module** E is a free finite rank $\mathbb{C}[[b]]$ module endowed with an $\mathbb{C}-$linear endomorphism a such that $a.S = S.a + b^2.S'$ for $S \in \mathbb{C}[[b]]$; or, in an equivalent way, a $\tilde{\mathcal{A}}-$module which is free and finite type over the subalgebra $\mathbb{C}[[b]] \subset \tilde{\mathcal{A}}$.
  It has a **simple pole** when a satisfies $a.E \subset b.E$. In this case the **Bernstein polynomial** B_E of E is defined as the minimal polynomial of $-b^{-1}.a$ acting on $E/b.E$.

- A (a,b)-module E is **regular** when it may be embedded in a simple pole (a,b)-module. In this case there is a minimal such embedding which is the inclusion of E in its saturation $E^\sharp$ by $b^{-1}.a$. The **Bernstein polynomial** of a regular (a,b)-module is, by definition, the Bernstein polynomial of its saturation $E^\sharp$.

- A regular (a,b)-module E is called **geometric** when all roots of its Bernstein polynomial are rational and strictly negative.

Note that the formal completion in b of the Brieskorn module of a function with an isolated singularity is a geometric (a,b)-module. The last section of this article shows that this structure appears in a rather systematic way in the study of the Gauss-Manin connection of a proper holomorphic function on a complex manifold.

Definition 1.1.3 *A submodule F in E is* **normal** *when $F \cap b.E = b.F$.*

For any sub-module F of a (a,b)-module E there exists a minimal normal sub-module $\tilde{F}$ of E containing F. We shall call it the **normalization** of F. It is easy to see that $\tilde{F}$ is the pull-back by the quotient map $E \to E/F$ of the $b-$torsion of E/F.

When F is normal the quotient E/F is again a (a,b)-module. Note that a sub-module of a regular (resp. geometric) (a,b)-module is regular (resp. geometric) and when F is normal E/F is also regular (resp. geometric).

Lemma 1.1.4 *Let E be a fresco and F be a normal sub-module in E. Then F is a fresco and also the quotient E/F.*

PROOF. The only point to prove, as we already know that F and E/F are geometric (a,b)-modules thanks to [B.09], is the fact that F and E/F are generated as $\tilde{\mathcal{A}}-$module by one element. This is obvious for E/F, but not for F. We shall use the theorem 1.1.2 for E/F to prove that F is generated by one element. Let e be a generator of E and let P as in the theorem 1.1.2 which generates the annihilator ideal of the image of e in E/F. Then $P.e$ is in F. We shall prove that $P.e$ generates F as a $\tilde{\mathcal{A}}-$module. Let y be an element in F and write $y = u.e$ where u is in $\tilde{\mathcal{A}}$. As P is, up to an invertible element in $\mathbb{C}[[b]]$, a monic polynomial in a with coefficients in $\mathbb{C}[[b]]$, we may write $u = Q.P + R$ where Q and R are in $\tilde{\mathcal{A}}$ and R is a polynomial in a with coefficient in $\mathbb{C}[[b]]$ of degree $r < deg(P) = rank(E/F)$. Now, as y is in F, the image in E/F of $u.e$ is 0, and this implies that R annihilates the image of e in E/F. So R lies in $\tilde{\mathcal{A}}.P$ and so $R = 0$. Then we have $u = Q.P$ and $y = Q.P.e$ proving our claim. ■

In the case of a fresco the Bernstein polynomial is more easy to describe, thanks to the following proposition proved in [B.09].

Proposition 1.1.5 *Soit $E = \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P$ be a rank k fresco as described in the previous theorem. The Bernstein polynomial of E is the* characteristic polynomial *of $-b^{-1}.a$ acting on $E^\sharp/b.E^\sharp$. And the* **Bernstein element** *P_E of E, which is the element in $\tilde{\mathcal{A}}$ defined by the Bernstein polynomial B_E of E by the following formula*

$$P_E := (-b)^k.B_E(-b^{-1}.a)$$

is equal to $(a - \lambda_1.b)\dots(a - \lambda_k.b)$ for any such choice of presentation of E.

As an easy consequence, the k roots of the Bernstein polynomial of E are the opposite of the numbers $\lambda_1 + 1 - k, \dots, \lambda_k + k - k$. So the Bernstein polynomial is readable on the element P : the initial form of P in (a,b) is the Bernstein element P_E of E.

REMARK. If we have an exact sequence of (a,b)-modules

$$0 \to F \to E \to G \to 0$$

where E is a fresco, then F and G are frescos and the Bernstein elements satisfy the equality $P_E = P_F.P_G$ in the algebra $\tilde{\mathcal{A}}$ (see [B.09] proposition 3.4.4.). □

EXAMPLE. The $\tilde{\mathcal{A}}-$module $\Xi_\lambda^{(N)}$ is a simple pole (a,b) with rank $N+1$. Its Bernstein polynomial is equal to $(x+\lambda)^{N+1}$.
The theme $\tilde{\mathcal{A}}.\varphi \subset \Xi_\lambda^{(N)}$ where $\varphi = s^{\lambda-1}.(Log\, s)^N$ has rank $N+1$ and Bernstein element $(a-(\lambda+N).b)(a-(\lambda+N-1).b)\dots(a-\lambda.b)$. Its saturation is $\Xi_\lambda^{(N)}$.

The dual of a regular (a,b)-module is regular, but duality does not preserve the property of being geometric because duality change the sign of the roots of the Bernstein polynomial. As duality preserves regularity (see [B.95]), to find again a geometric (a,b)-module it is sufficient to make the tensor product by a rank 1 (a,b)-module E_δ for δ a large enough rational number[1]. The next lemma states that this "twist duality" preserves the notion of fresco.

Lemma 1.1.6 *Let E be a fresco and let δ be a rational number such that $E^* \otimes E_\delta$ is geometric. Then $E^* \otimes E_\delta$ is a fresco.*

The proof is obvious. ■

The following definition is useful when we want to consider only the part of asymptotic expansions corresponding to prescribe eigenvalues of the monodromy.

Definition 1.1.7 *Let Λ be a subset of $\mathbb{Q} \cap]0,1]$. We say that a regular (a,b)-module E is* $[\Lambda]-$**primitive** *when all roots of its Bernstein polynomial are in $-\Lambda + \mathbb{Z}$.*

The following easy proposition is proved in [B.09]

Proposition 1.1.8 *Let E be a regular (a,b)-module and Λ a subset of $\mathbb{Q} \cap]0,1]$. Then there exists a maximal submodule $E_{[\Lambda]}$ in E which is $[\Lambda]-$primitive. Moreover the quotient $E/E_{[\Lambda]}$ is a $[\Lambda^c]-$primitive (a,b)-module, where we denote $\Lambda^c := \mathbb{Q} \cap]0,1] \setminus \Lambda$.*

We shall mainly consider the case where Λ is a single element. Note that the $[\lambda]-$primitive part of an (a,b)-module $E \subset \Xi_\Lambda^{(N)} \otimes V$ corresponds to its intersection with $\Xi_\lambda^{(N)} \otimes V$.

1.2 The principal Jordan-Hölder sequence.

The classification of rank 1 regular (a,b)-module is very simple : each isomorphy class is given by a complex number and to $\lambda \in \mathbb{C}$ corresponds the isomorphy class of

[1] to tensor by E_δ is the same that to replace a by $a+\delta.b$; see [B.I] for the general definition of the tensor product.

$E_\lambda := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda.b)$. Then **Jordan-Hölder sequence** for a regular (a,b)-module E is a sequence

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k = E$$

of normal sub-modules such that for each $j \in [1,k]$ the quotient F_j/F_{j-1} has rank 1. Then to each J-H. sequence we may associate an ordered sequence of complex numbers $\lambda_1, \ldots, \lambda_k$ such that $F_j/F_{j-1} \simeq E_{\lambda_j}$.

EXAMPLE. A regular rank 1 (a,b)-module is a fresco if and only if it is isomorphic to E_λ for some $\lambda \in \mathbb{Q}^{+*}$. All rank 1 frescos are themes. The classification of rank 2 regular (a,b)-modules given in [B.93] gives the list of $[\lambda]-$primitive rank 2 frescos which is the following, where $\lambda_1 > 1$ is a rational number :

$$E = E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b).(a-(\lambda_1-1).b) \quad (1)$$

$$E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b).(1+\alpha.b^p)^{-1}.(a-(\lambda_1+p-1).b) \quad (2)$$

where $p \in \mathbb{N} \setminus \{0\}$ and $\alpha \in \mathbb{C}$.
The themes in this list are these in (1) and these in (2) with $\alpha \neq 0$. For a $[\lambda]-$primitive fresco in case (2) the number α will be called the **parameter** of the rank 2 fresco. By convention we shall define $\alpha := 1$ in the case (1). □

The following existence result is proved in [B.93]

Proposition 1.2.1 *For any regular (a,b)-module E of rank k there exists a J-H. sequence. The numbers $exp(2i\pi.\lambda_j)$ are independant of the J-H. sequence, up to permutation. Moreover the number $\mu(E) := \sum_{j=1}^k \lambda_j$ is also independent of the choice of the J-H. sequence of E.*

EXERCICE. Let E be a regular (a,b)-module and $E' \subset E$ be a sub-(a,b)-module with the same rank than E. Show that E' has finite $\mathbb{C}-$codimension in E given by

$$\dim_{\mathbb{C}} E/E' = \mu(E') - \mu(E).$$

hint : make an induction on the rank of E. □

For a $[\lambda]-$primitive fresco a more precise result is proved in [B.09]

Proposition 1.2.2 *For any J-H. sequence of a rank k $[\lambda]-$primitive fresco E the numbers $\lambda_j + j - k, j \in [1,k]$ are the opposite of the roots (with multiplicities) of the Bernstein polynomial of E. So, up to a permutation, they are independant of the choice of the J-H. sequence. Moreover, there always exists a J-H. sequence such that the associated sequence $\lambda_j + j$ is non decreasing.*

For a $[\lambda]-$primitive theme the situation is extremely rigid (see [B.09]) :

Proposition 1.2.3 *Let E a rank k $[\lambda]-$primitive theme. Then, for each j in $[0,k]$, there exists an unique normal rank j submodule F_j. The numbers associated to the unique J-H. sequence satisfy the condition that $\lambda_j + j, j \in [1,k]$ is an non decreasing sequence.*

Definition 1.2.4 *Let E be a $[\lambda]-$primitive fresco of rank k and let*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k = E$$

be a J-H. sequence of E. Then for each $j \in [1,k]$ we have $F_j/F_{j-1} \simeq E_{\lambda_j}$, where $\lambda_1, \ldots, \lambda_k$ are in $\lambda + \mathbb{N}$. We shall say that such a J-H. sequence is **principal** *when the sequence $[1,k] \ni j \mapsto \lambda_j + j$ is non decreasing.*

Proposition 1.2.5 *Let E be a $[\lambda]-$primitive fresco. Then its principal J-H. sequence is unique.*

We shall prove the uniqueness by induction on the rank k of E.
We begin by the case of rank 2.

Lemma 1.2.6 *Let E be a rank 2 $[\lambda]-$primitive fresco and let λ_1, λ_2 the numbers corresponding to a principal J-H. sequence of E (so $\lambda_1 + 1 \leq \lambda_2 + 2$). Then the normal rank 1 submodule of E isomorphic to E_{λ_1} is unique.*

Proof. The case $\lambda_1 + 1 = \lambda_2 + 2$ is obvious because then E is a $[\lambda]-$primitive theme (see [B.10] corollary 2.1.7). So we may assume that $\lambda_2 = \lambda_1 + p_1 - 1$ with $p_1 \geq 1$ and that E is the quotient $E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(a - \lambda_2.b)$ (see the classification of rank 2 frescos in 2.2), because the result is clear when E is a theme. We shall use the $\mathbb{C}[[b]]-$basis e_1, e_2 of E where a is defined by the relations

$$(a - \lambda_2.b).e_2 = e_1 \quad (a - \lambda_1.b).e_1 = 0.$$

This basis comes from the isomorphism $E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(a - \lambda_2.b)$ deduced from the classification of rank 2 frescos with $e_2 = [1]$ and $e_1 = (a - \lambda_2.b).e_2$.
Let look for $x := U.e_2 + V.e_1$ such that $(a - \lambda_1.b).x = 0$. Then we obtain

$$b^2.U'.e_2 + U.(a - \lambda_2.b).e_2 + (\lambda_2 - \lambda_1).b.U.e_2 + b^2.V'.e_1 = 0$$

which is equivalent to the two equations :

$$b^2.U' + (p_1 - 1).b.U = 0 \quad \text{and} \quad U + b^2.V' = 0.$$

The first equation gives $U = 0$ for $p_1 \geq 2$ and $U \in \mathbb{C}$ for $p_1 = 1$. As the second equation implies $U(0) = 0$, in all cases $U = 0$ and $V \in \mathbb{C}$. So the solutions are in $\mathbb{C}.e_1$. ■

Remark that in the previous lemma, if we assume $p_1 \geq 1$ and E is not a theme, it may exist infinitely many different normal (rank 1) submodules isomorphic to E_{λ_2+1}. But then, $\lambda_2 + 2 > \lambda_1 + 1$. This happens in the following example.

EXAMPLE. Let $E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P$ with

$$P := (a - \lambda_1.b).(a - \lambda_2.b)(1 + \alpha.b^{p_2})^{-1}.(a - \lambda_3.b)$$

where $\lambda_1 > 2$ is rational, p_1 and p_2 are in $\mathbb{N}^*$, $\lambda_2 = \lambda_1 + p_1 - 1, \lambda_3 = \lambda_2 + p_2 - 1$ and α is in $\mathbb{C}^*$.
Then using the identity (see the commuting lemma in [B.09])

$$(a - \lambda_1.b).(a - \lambda_2.b) = U^{-1}.(a - (\lambda_2 + 1).b).U^2.(a - (\lambda_1 - 1).b).U^{-1}$$

with $U := 1 + \rho.b^{p_1}, \rho \in \mathbb{C}^*$, it is easy to see that the rank 3 fresco E admits the rank 2 theme T with fundamental invariants $(\lambda_1 - 1, \lambda_3)$ and parameter $\rho.\alpha$. Remark that if we use the previous identity with $\rho = 0$ (so $U = 1$), then using the identity

$$\begin{aligned}&(a - (\lambda_1 - 1).b).(1 + \alpha.b^{p_2})^{-1}.(a - \lambda_3.b) = \\ &V^{-1}.(a - (\lambda_3 + 1).b).V^2.(1 + \alpha.b^{p_2})^{-1}.(a - (\lambda_1 - 2).b).V^{-1}\end{aligned}$$

where $V = 1 + \beta.b^{p_2}$ and $\beta := (1 + p_2/p_1).\alpha$ we see that E contains a normal rank 2 sub-theme which has fundamental invariants equal to $(\lambda_2 + 1, \lambda_3 + 1)$ and parameter β. □

PROOF OF PROPOSITION 1.2.5. As the result is obvious for $k = 1$, we may assume $k \geq 2$ and the result proved in rank $\leq k - 1$. Let $F_j, j \in [1, k]$ and $G_j, j \in [1, k]$ two J-H. principal sequences for E. As the sequences $\lambda_j + j$ and $\mu_j + j$ coïncide up to the order and are both non decreasing, they coïncide. Now let j_0 be the first integer in $[1, k]$ such that $F_{j_0} \neq G_{j_0}$. If $j_0 \geq 2$ applying the induction hypothesis to E/F_{j_0-1} gives $F_{j_0}/F_{j_0-1} = G_{j_0}/F_{j_0-1}$ and so $F_{j_0} = G_{j_0}$.
So we may assume that $j_0 = 1$. Let H be the normalization of $F_1 + G_1$. As F_1 and G_1 are normal rank 1 and distinct, then H is a rank 2 normal submodule. It is a $[\lambda]-$primitive fresco of rank 2 with two normal rank 1 sub-modules which are isomorphic as $\lambda_1 = \mu_1$. Moreover the principal J-H. sequence of H begins by a normal submodule isomorphic to E_{λ_1}. So the previous lemma implies $F_1 = G_1$. So for any $j \in [1, k]$ we have $F_j = G_j$. ■

Definition 1.2.7 *Let E be a $[\lambda]-$primitive fresco and consider its principal J-H. sequence $F_j, j \in [1.k]$. Put $F_j/F_{j-1} \simeq E_{\lambda_j}$ for $j \in [1, k]$ (with $F_0 = \{0\}$). We shall call* **fundamental invariants** *of E the ordered k-tuple $(\lambda_1, \ldots, \lambda_k)$.*

Of course, if we have any J-H. sequence for a $[\lambda]-$primitive fresco E with rank 1 quotients associated to the rational numbers $\mu_1, \ldots, \mu_k$, it is easy to recover the fundamental invariants of E because the numbers $\mu_j + j, j \in [1, k]$ are the same than the numbers $\lambda_j + j$ up to a permutation. But as the sequence $\lambda_j + j$ is non decreasing with j, it is enough to put the $\mu_j + j$ in the non decreasing order with j to conclude.

Let $\lambda_1, \ldots, \lambda_k$ be the fundamental invariants of a $[\lambda]-$primitive fresco E_0, and not $\mathcal{F}(\lambda_1, \ldots, \lambda_k)$ the set of isomorphism classes of frescos with these fundamental invariants. The uniqueness of the principal J-H. sequence of a $[\lambda]-$primitive fresco allows to define for each (i,j) $1 \leq i < j \leq k$ a map

$$q_{i,j} : \mathcal{F}(\lambda_1, \ldots, \lambda_k) \to \mathcal{F}(\lambda_i, \ldots, \lambda_j)$$

defined by $q_{i,j}([E]) = [F_j/F_{i-1}]$ where $(F_h)_{h\in[0,k]}$ is the principal J-H. sequence of E. This makes sens because any isomorphism $\varphi : E^1 \to E^2$ between two $[\lambda]-$primitive frescos induces isomorphisms between each term of the corresponding principal J-H. sequences.
For instance classification of rank 2 $[\lambda]-$primitive frescos gives (see example before proposition 1.2.1) for any rational number $\lambda_1 > 1$ and $p_1 \in \mathbb{N}$:

$$\text{for} \quad p_1 = 0 \quad \mathcal{F}(\lambda_1, \lambda_1 - 1) = \{pt\}$$
$$\text{for} \quad p_1 \geq 1 \quad \alpha : \mathcal{F}(\lambda_1, \lambda_1 + p_1 - 1) \xrightarrow{\simeq} \mathbb{C}$$

and $\{pt\}$ is given by the isomorphism class of $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b).(a-(\lambda_1-1).b)$ and the isomorphism class associated to $\alpha^{-1}(x)$ for $x \in \mathbb{C}$ in the case $p_1 \geq 1$ is given by $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b).(1+x.b^{p_1})^{-1}.(a-(\lambda_1+p_1-1).b)$.

2 Semi-simplicity.

2.1 Semi-simple regular (a,b)-modules.

Definition 2.1.1 *Let E be a regular (a,b)-module. We say that E is* **semi-simple** *if it is a sub-module of a finite direct sum of rank 1 regular (a,b)-modules.*

Note that if E is a sub-module of a regular (a,b)-module it is necessary regular. As a direct sum of regular (a,b)-modules if regular, our assumption that E is regular is superfluous.
It is clear from this definition that a sub-module of a semi-simple (a,b)-module is semi-simple and that a (finite) direct sum of semi-simple (a,b)-modules is again semi-simple.

Remark. A rather easy consequence of the classification of rank 2 (a,b)-modules given in [B.93] is that the rank 2 simple pole (a,b)-modules defined in the $\mathbb{C}[[b]]-$basis x, y by the relations :

$$(a-(\lambda+p).b).x = b^{p+1}.y \quad \text{and} \quad (a-\lambda).y = 0$$

for any $\lambda \in \mathbb{C}$ and any $p \in \mathbb{N}$ are not semi-simple. We leave the verification of this point to the reader.
Let us begin by a characterization of the semi-simple (a,b)-modules which have a simple pole. First we shall prove that a quotient of a semi-simple (a,b)-module is semi-simple. This will be deduced from the following lemma.

Lemma 2.1.2 *Let E be a (a,b)-module which is direct sum of regular rank 1 (a,b)-modules, and let $F \subset E$ be a rank 1 normal sub-module. Then F is a direct factor of E.*

Corollary 2.1.3 *If E is a semi-simple regular (a,b)-module and F a normal sub-module of E, the quotient E/F is a (regular) semi-simple (a,b)-module.*

PROOF OF THE LEMMA. Let $E = \oplus_{j=1}^{k} E_{\lambda_j}$ and assume that $F \simeq E_\mu$. Let e_j be a standard generator of E_{λ_j} and e a standard generator of E_μ. Write

$$e = \sum_{j=1}^{k} S_j(b).e_j$$

and compute $(a - \mu.b).e = 0$ using the fact that $e_j, j \in [1,k]$ is a $\mathbb{C}[[b]]-$basis of E and the relations $(a - \lambda_j.b).e_j = 0$ for each j. We obtain for each $j \in [1,k]$ the relation

$$b.S_j' - (\mu - \lambda_j).S_j = 0.$$

So, if $\mu - \lambda_j$ is not in $\mathbb{N}$, we have $S_j = 0$. When $\mu = \lambda_j + p_j$ with $p_j \in \mathbb{N}$ we obtain $S_j(b) = \rho_j.b^{p_j}$ for some $\rho_j \in \mathbb{C}$. As we assume that e is not in $b.E$, there exists at least one $j_0 \in [1,k]$ such that $p_{j_0} = 0$ and $\rho_{j_0} \neq 0$. Then it is clear that we have

$$E = F \oplus \left(\oplus_{j \neq j_0} E_{\lambda_j} \right)$$

concluding the proof. ■

PROOF OF THE COROLLARY. We argue by induction on the rank of F. In the rank 1 case, we have $F \subset E \subset \mathcal{E} := \oplus_{j=1}^{k} E_{\lambda_j}$. Let $\tilde{F}$ the normalization of F in $\oplus_{j=1}^{k} E_{\lambda_j}$. Then the lemma shows that there exists a $j_0 \in [1,k]$ such that

$$\mathcal{E} = \tilde{F} \oplus \oplus_{j \neq j_0} E_{\lambda_j}.$$

Then, as $\tilde{F} \cap E = F$, the quotient map $\mathcal{E} \to \mathcal{E}/\tilde{F} \simeq \oplus_{j \neq j_0} E_{\lambda_j}$ induces an injection of E/F in a direct sum of regular rank 1 (a,b)-modules. So E/F is semi-simple. Assume now that the result is proved for F with rank $\leq d-1$ and assume that F has rank d. Then using a rank 1 normal sub-module G in F, we obtain that F/G is a normal rank $d-1$ sub-module of E/G. Using the rank 1 case we know that E/G is semi-simple, and the induction hypothesis gives that

$$E/F = (E/G)\Big/(F/G)$$

is semi-simple. ■

Proposition 2.1.4 *Let E be a simple pole semi-simple (a,b)-module. Then E is a direct sum of regular rank 1 (a,b)-modules.*

PROOF. We shall prove the proposition by induction on the rank k of E. As the rank 1 case is obvious, assume the proposition proved for $k-1$ and that E is a simple pole rank $k \geq 2$ semi-simple (a,b)-module. From the existence of J-H. sequence, we may find an exact sequence

$$0 \to F \to E \to E_\lambda \to 0$$

where F has rank $k-1$. By definition F is semi-simple, but it also has a simple pole because $a.F \subset a.E \subset b.E$ implies that $a.F \subset F \cap b.E = b.F$ as F is normal in E. So by the induction hypothesis F is a direct sum of regular rank 1 (a,b)-modules.
Let $e \in E$ such that its image in E_λ is e_λ a standard generator of E_λ satisfying $a, e_\lambda = \lambda.b.e_\lambda$. Then we have $(a - \lambda.b).e \in F$. We shall first look at the case $k = 2$. So F is a rank 1 and we have $F \simeq E_\mu$ for some $\mu \in \mathbb{C}$. Let e_μ be a standard generator in F and put

$$(a - \lambda.b).e = S(b).e_\mu.$$

Our simple pole assumption on E implies $S(0) = 0$ and we may write $S(b) = b.\tilde{S}(b)$. We look for $T \in \mathbb{C}[[b]]$ such that $\varepsilon := e + T(b).e_\mu$ satisfies

$$(a - \lambda.b).\varepsilon = 0.$$

If such a T exists, then we would have $E = E_\mu \oplus E_\lambda$ where E_λ is the submodule generated by ε, because it is clear that we have $E = F \oplus \mathbb{C}[[b]].e$ as a $\mathbb{C}[[b]]-$module. To find T we have to solve in $\mathbb{C}[[b]]$ the differential equation

$$b.\tilde{S}(b) + (\mu - \lambda).b.T(b) + b^2.T'(b) = 0.$$

If $\lambda - \mu$ is not a non negative integer, such a T exists and is unique. But when $\lambda = \mu + p$ with $p \in \mathbb{N}$, the solution exists if and only if the coefficient of b^p in $\tilde{S}$ is zero. If it is not the case, define $\tilde{T}$ as the solution of the differential equation

$$\tilde{S}(b) - \alpha.b^p + b.\tilde{T}'(b) - p.\tilde{T}(b) = 0$$

where α is the coefficient of b^p in $\tilde{S}$ and where we choose $\tilde{T}$ by asking that it has no b^p term. Then $\varepsilon_1 := e + \tilde{T}(b).e_{\lambda-p}$ satisfies

$$(a - (\lambda - p).b).\varepsilon_1 = \alpha.b^{p+1}.e_{\lambda-p}.$$

Then, after changing $\lambda - p$ in λ and $e_{\lambda-p}$ in $\alpha.e_{\lambda-p}$, we recognize one of the rank 2 modules which appears in the previous remark and which is not semi-simple. So we have a contradiction. This concludes the rank 2 case.

Now consider the case $k \geq 3$ and using the induction hypothesis write

$$F \simeq \oplus_{j=1}^{k-1} E_{\mu_j}$$

and denote e_j a standard generator for E_{μ_j}. Write

$$(a-\lambda.b).e=\sum_{j=1}^{k-1} S_j(b).e_j.$$

The simple pole assumption again gives $S_j(0)=0$ for each j; now we look for $T_1,\dots,T_{k-1}$ in $\mathbb{C}[[b]]$ such that, defining $\varepsilon:=e+\sum_{j=1}^{k-1} T_j(b).e_j$, we have

$$(a-\lambda.b).\varepsilon=0.$$

For each j this leads to the differential equation

$$b.\tilde{S}_j(b)+(\mu_j-\lambda).b.T_j(b)+b^2.T_j'(b)=0$$

where we put $S_j(b)=b.\tilde{S}_j(b)$. We are back, for each given $j\in[1,k-1]$, to the previous problem in the rank 2 case. So if for each $j\in[1,k-1]$ we have a solution $T_j\in\mathbb{C}[[b]]$ it is easy to conclude that

$$E\subset F\oplus\mathbb{C}[[b]].\varepsilon\simeq F\oplus E_\lambda.$$

If for some j_0 there is no solution, the coefficient of b^{j_0} in $\tilde{S}_{j_0}$ does not vanish and then the image of E in $E\big/\oplus_{j\neq j_0}E_{\mu_j}$ is a rank 2 not semi-simple (a,b)-module. This contradicts the corollary 2.1.3 and concludes the proof. ■

REMARK. Let E be a semi-simple rank k (a,b)-module and let $E^\sharp$ its saturation by $b^{-1}.a$, then $E^\sharp$ is semi-simple because there exists $N\in\mathbb{N}$ with an inclusion $E^\sharp\subset b^{-N}.E$. Also $E^\flat$ its maximal simple pole sub-module is semi-simple. Then we have

$$E^\flat\simeq\oplus_{j=1}^k E_{\lambda_j+d_j}\subset E\subset E^\sharp\simeq\oplus_{j=1}^k E_{\lambda_j}$$

where $d_1,\dots,d_k$ are non negative integers.

Corollary 2.1.5 *Let E be a semi-simple (a,b)-module. Then its dual E^* is semi-simple.*

PROOF. For a regular (a,b)-module E we have $(E^\flat)^*=(E^*)^\sharp$. But the dual of E_λ is $E_{-\lambda}$ and the duality commutes with direct sums. So we conclude that $E^*\subset(E^\flat)^*$ and so E^* is a semi-simple. ■

REMARK. The tensor product of two semi-simple (a,b)-modules is semi-simple as a consequence of the fact that $E_\lambda\otimes E_\mu=E_{\lambda+\mu}$.

2.2 The semi-simple filtration.

Definition 2.2.1 *Let E be a regular (a,b)-module and x an element in E. We shall say that x is* **semi-simple** *if $\tilde{\mathcal{A}}.x$ is a semi-simple (a,b)-module.*

It is clear that any element in a semi-simple (a,b)-module is semi-simple. The next lemma shows that the converse is true.

Lemma 2.2.2 *Let E be a regular (a,b)-module such that any $x \in E$ is semi-simple. Then E is semi-simple.*

PROOF. Let $e_1, \ldots, e_k$ be a $\mathbb{C}[[b]]-$basis of E. Then each $\tilde{\mathcal{A}}.e_j$ is semi-simple, and the map $\oplus_{j=1}^{k} \tilde{\mathcal{A}}.e_j \to E$ is surjective. So E is semi-simple thanks to the corollary 2.1.3 and the comment following the definition 2.1.1. ■

Lemma 2.2.3 *Let E be a regular (a,b)-module. The subset $S_1(E)$ of semi-simple elements in E is a normal submodule in E.*

PROOF. As a direct sum and quotient of semi-simple (a,b)-modules are semi-simple, it is clear that for x and y semi-simple the sum $\tilde{\mathcal{A}}.x + \tilde{\mathcal{A}}.y$ is semi-simple. So $x + y$ is semi-simple. This implies that $S_1(E)$ is a submodule of E. If $b.x$ is in $S_1(E)$, then $\tilde{\mathcal{A}}.b.x$ is semi-simple. Then $\tilde{\mathcal{A}}.x \simeq \tilde{\mathcal{A}}.b.x \otimes E_{-1}$ is also semi-simple, and so $S_1(E)$ is normal in E. ■

Definition 2.2.4 *Let E be a regular (a,b)-module. Then the submodule $S_1(E)$ of semi-simple elements in E will be called the* **semi-simple part** *of E.*
Defining $S_j(E)$ as the pull-back on E of the semi-simple part of $E/S_{j-1}(E)$ for $j \geq 1$ with the initial condition $S_0(E) = \{0\}$, we define a sequence of normal submodules in E such that $S_j(E)/S_{j-1}(E) = S_1(E/S_{j-1}(E)$. We shall call it the ***semi-simple filtration*** *of E. The smallest integer d such we have $S_d(E) = E$ will be called the* **semi-simple depth** *of E and we shall denote it $d(E)$.*

REMARKS.

i) As $S_1(E)$ is the maximal semi-simple sub-module of E it contains any rank 1 sub-module of E. So $S_1(E) = \{0\}$ happens if and only if $E = \{0\}$.

ii) Then the ss-filtration of E is strictly increasing for $0 \leq j \leq d(E)$.

iii) It is easy to see that for any submodule F in E we have $S_j(F) = S_j(E) \cap F$. So $S_j(E)$ is the subset of $x \in E$ such that $d(x) := d(\tilde{\mathcal{A}}.x) \leq j$ and $d(E)$ is the supremum of $d(x)$ for $x \in E$. □

Lemma 2.2.5 *Let* $0 \to F \to E \to G \to 0$ *be an exact sequence of regular (a,b)-modules such that* F *and* G *are semi-simple and such that for any roots* λ, μ *of the Bernstein polynomial of* F *and* G *respectively we have* $\lambda - \mu \notin \mathbb{Z}$. *Then* E *is semi-simple.*

PROOF. By an easy induction we may assume that F is rank 1 and also that G is rank 1. Then the conclusion follows from the classification of the rank 2 regular (a,b)-module. ■

APPLICATION. A regular (a,b)-module E is semi-simple if and only if for each $\lambda \in \mathbb{C}$ its primitive part $E[\lambda]$ is semi-simple.

The following easy facts are left as exercices for the reader.

1. Let $0 \to F \to E \to G \to 0$ be a short exact sequence of regular (a,b)-modules. Then we have the inequalities
$$\sup\{d(F), d(G)\} \leq d(E) \leq d(F) + d(G).$$

2. Let $E \subset \Xi_\lambda^{(N)} \otimes V$ be a sub-module where V is a finite dimensional complex space. Then for each $j \in [1, N+1]$ we have
$$S_j(E) = E \cap \left(\Xi_\lambda^{(j-1)} \otimes V\right).$$

3. In the same situation as above, the minimal N for such an embedding of a geometric $[\lambda]-$primitive E is equal to $d(E) - 1$.

4. In the same situation as above, the equality $S_1(E) = E \cap \Xi_\lambda^{(0)} \otimes V$ implies $rank(S_1(E)) \leq \dim_{\mathbb{C}} V$. It is easy to see that any $[\lambda]-$primitive semi-simple geometric (a,b)-module F may be embedded in $\Xi_\lambda^{(0)} \otimes V$ for some V of dimension $rank(F)$. But it is also easy to prove that when $F = S_1(E)$ with E as above, such an embedding may be extended to an embedding of E into $\Xi_\lambda^{(d(E)-1)} \otimes V$. □

2.3 Semi-simple frescos.

We begin by a simple remark : A $[\lambda]-$primitive theme is semi-simple if and only if it has rank ≤ 1. This is an easy consequence of the fact that the saturation of a rank 2 $[\lambda]-$primitive theme is one of the rank 2 (a,b)-modules considered in the remark following the definition 2.1.1 which are not semi-simple

Lemma 2.3.1 *A geometric (a,b)-module* E *is* **semi-simple** *if and only if any quotient of* E *which is a* $[\lambda]-$*primitive theme for some* $[\lambda] \in \mathbb{Q}/\mathbb{Z}$ *is of rank* ≤ 1.

PROOF. The condition is clearly necessary as a quotient of a semi-simple (a,b)-module is semi-simple (see corollary 2.1.3), thanks to the remark above. Using the application of the lemma 2.2.5, it is enough to consider the case of a $[\lambda]-$primitive fresco to prove that the condition is sufficient. Let $\varphi : E \to \Xi_\lambda^{(N)} \otimes V$ be an embedding of E which exists thanks to the embedding theorem 4.2.1. of [B.09], we obtain that each component of this map in a basis $v_1, \dots v_p$ of V has rank at most 1 as its image is a $[\lambda]-$primitive theme. Then each of these images is isomorphic to some $E_{\lambda+q}$ for some integer q. So we have in fact an embedding of E in a direct sum of $E_{\lambda+q_i}$ and E is semi-simple. ■

REMARK. The preceeding proof shows that a fresco is a non zero $[\lambda]-$primitive theme for some $[\lambda] \in \mathbb{Q}/\mathbb{Z}$ if and only if $S_1(E)$ has rank 1. □

Lemma 2.3.2 *Let E be a semi-simple fresco with rank k and let $\lambda_1, \dots, \lambda_k$ be the numbers associated to a J-H. sequence of E. Let $\mu_1, \dots, \mu_k$ be a twisted permutation[2] of $\lambda_1, \dots, \lambda_k$. Then there exists a J-H. sequence for E with quotients corresponding to $\mu_1, \dots, \mu_k$.*

PROOF. As the symetric group $\mathfrak{S}_k$ is generated by the transpositions $t_{j,j+1}$ for $j \in [1, k-1]$, it is enough to show that, if E has a J-H. sequence with quotients given by the numbers $\lambda_1, \dots, \lambda_k$, then there exists a J-H. sequence for E with quotients $\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1} + 1, \lambda_j - 1, \lambda_{j+2}, \dots, \lambda_k$ for $j \in [1, k-1]$. But $G := F_{j+1}/F_{j-1}$ is a rank 2 sub-quotient of E with an exact sequence

$$0 \to E_{\lambda_j} \to G \to E_{\lambda_{j+1}} \to 0.$$

As G is a rank 2 semi-simple fresco, it admits also an exact sequence

$$0 \to G_1 \to G \to G/G_1 \to 0$$

with $G_1 \simeq E_{\lambda_{j+1}+1}$ and $G/G_1 \simeq E_{\lambda_j - 1}$. Let $q : F_{j+1} \to G$ be the quotient map. Now the J-H. sequence for E given by

$$F_1, \dots, F_{j-1}, q^{-1}(G_1), F_{j+1}, \dots, F_k = E$$

satisfies our requirement. ■

Proposition 2.3.3 *Let E be a $[\lambda]-$primitive fresco. A necessary and sufficient condition in order that E is semi-simple is that it admits a J-H. sequence with quotient corresponding to $\mu_1, \dots, \mu_k$ such that the sequence $\mu_j + j$ is strictly decreasing.*

[2]This means that the sequence $\mu_j + j, j \in [1, k]$ is a permutation (in the usual sens) of $\lambda_j + j, j \in [1, k]$.

Remarks.

1. As a fresco is semi-simple if and only if for each $[\lambda]$ its $[\lambda]-$primitive part is semi-simple, this proposition gives also a criterium to semi-simplicity for any fresco.

2. This criterium is a very efficient tool to produce easily examples of semi-simple frescos.

Proof. Remark first that if we have, for a $[\lambda]-$primitive fresco E, a J-H. sequence $F_j, j \in [1,k]$ such that $\lambda_j + j = \lambda_{j+1} + j + 1$ for some $j \in [1, k-1]$, then F_{j+1}/F_{j-1} is a sub-quotient of E which is a $[\lambda]-$primitive theme of rank 2. So E is not semi-simple. As a consequence, when a $[\lambda]-$primitive fresco E is semi-simple the principal J-H. sequence corresponds to a strictly increasing sequence $\lambda_j + j$. Now, thanks to the previous lemma we may find a J-H. sequence for E corresponding to the strictly decreasing order for the sequence $\lambda_j + j$.

No let us prove the converse. We shall use the following lemma.

Lemma 2.3.4 *Let F be a rank k semi-simple $[\lambda]-$primitive fresco and let $\lambda_j + j$ the strictly increasing sequence corresponding to its principal J-H. sequence. Let $\mu \in [\lambda]$ such that $0 < \mu + k < \lambda_1 + 1$. Then any fresco E in an exact sequence*

$$0 \to F \to E \to E_\mu \to 0$$

is semi-simple (and $[\lambda]-$primitive).

Proof. The case $k = 1$ is obvious, so assume that $k \geq 2$ and that we have a rank 2 quotient $\varphi : E \to T$ where T is a $[\lambda]-$primitive theme. Then $Ker\,\varphi \cap F$ is a normal submodule of F of rank $k-2$ or $k-3$ (for $k \geq 3$). If $Ker\,\varphi \cap F$ is of rank $k-3$, the rank of $F/(Ker\,\varphi \cap F)$ is 2 and it injects in T via φ. So $F/(Ker\,\varphi \cap F)$ is a rank 2 $[\lambda]-$primitive theme. As it is semi-simple, because F is semi-simple, we get a contradiction.
So the rank of $F/(Ker\,\varphi \cap F)$ is 1 and we have an exact sequence

$$0 \to F/(Ker\,\varphi \cap F) \to T \to E/F \to 0.$$

Put $F/(Ker\,\varphi \cap F) \simeq F_1(T) \simeq E_\nu$. Because T is a $[\lambda]-$primitive theme, we have the inequality $\nu + 1 \leq \mu + 2$. Looking at a J-H. sequence of E ending by

$$\cdots \subset Ker\,\varphi \cap F \subset \varphi^{-1}(F_1(T)) \subset E$$

we see that $\nu + k - 1$ is in the set $\{\lambda_j + j, j \in [1,k]\}$ and as $\lambda_1 + 1$ is the infimum of this set we obtain $\lambda_1 + 1 \leq \mu + k$ contradicting our assumption. ■

END OF PROOF OF THE PROPOSITION 2.3.3. Now we shall prove by induction on the rank of a $[\lambda]-$primitive fresco E that if it admits a J-H. sequence corresponding to a strictly decreasing sequence $\mu_j + j$, it is semi-simple. As the result is obvious in rank 1, we may assume $k \geq 1$ and the result proved for k. So let E be a fresco of rank $k+1$ and let $F_j, j \in [1, k+1]$ a J-H. sequence for E corresponding to the strictly decreasing sequence $\mu_j + j, j \in [1, k+1]$. Put $F_j/F_{j-1} \simeq E_{\mu_j}$ for all $j \in [1, k+1]$, define $F := F_k$ and $\mu := \mu_{k+1}$; then the induction hypothesis gives that F is semi-simple and we apply the previous lemma to conclude. ■

The following interesting corollary is an obvious consequence of the previous proposition.

Corollary 2.3.5 *Let E be a fresco and let $\lambda_1, \ldots, \lambda_k$ be the numbers associated to any J-H. sequence of E. Let $\mu_1, \ldots, \mu_d$ be the numbers associated to any J-H. sequence of $S_1(E)$. Then, for $j \in [1, k]$, there exists a rank 1 normal submodule of E isomorphic to E_{λ_j+j-1} if and only if there exists $i \in [1, d]$ such that we have $\lambda_j + j - 1 = \mu_i + i - 1$.*

Of course, this gives the list of all isomorphy classes of rank 1 normal submodules contained in E. So, using shifted duality, we get also the list of all isomorphy classes of rank 1 quotients of E. It is interesting to note that this is also the list of the possible initial exponents for maximal logarithmic terms which appear in the asymtotic expansion of a given relative de Rham cohomology class after integration on any vanishing cycle in the spectral subspace of the monodromy associated to the eigenvalue $exp(2i\pi.\lambda)$.

3 A criterion for semi-simplicity.

3.1 Polynomial dependance.

All $\mathbb{C}-$algebras have a unit.

When we consider a sequence of algebraically independent variables $\rho := (\rho_i)_{i\in\mathbb{N}}$ we shall denote by $\mathbb{C}[\rho]$ the $\mathbb{C}-$algebra generated by these variables. Then $\mathbb{C}[\rho][[b]]$ will be the commutative $\mathbb{C}-$algebra of formal power series

$$\sum_{\nu=0}^{\infty} P_\nu(\rho).b^\nu$$

where $P_\nu(\rho)$ is an element in $\mathbb{C}[\rho]$ so a polynomial in $\rho_0, \ldots, \rho_{N(\nu)}$ where $N(\nu)$ is an integer depending on ν. So each coefficient in the formal power serie in b depends only on a finite number of the variables ρ_i.

Definition 3.1.1 *Let E be a (a,b)-module and let $e(\rho)$ be a family of elements in E depending on a family of variables $(\rho_i)_{i\in\mathbb{N}}$. We say that $e(\rho)$* ***depends polynomially on*** *ρ if there exists a fixed $\mathbb{C}[[b]]-$basis $e_1,\dots,e_k$ of E such that*

$$e(\rho)=\sum_{j=1}^{k} S_j(\rho).e_j$$

where S_j is for each $j\in[1,k]$ an element in the algebra $\mathbb{C}[\rho][[b]]$.

Remarks.

1. It is important to remark that when $e(\rho)$ depends polynomially of ρ, then $a.e(\rho)$ also. Then for any $u\in\tilde{\mathcal{A}}$, again $u.e(\rho)$ depends polynomially on ρ.

2. It is easy to see that we obtain an equivalent condition on the family $e(\rho)$ by asking that the coefficient of $e(\rho)$ are in $\mathbb{C}[\rho][[b]]$ in a $\mathbb{C}[[b]]-$basis of E whose elements depend polynomially of ρ.

3. The invertible elements in the algebra $\mathbb{C}[\rho][[b]]$ are exactly those elements with a constant term in b invertible in the algebra $\mathbb{C}[\rho]$. As we assume that the variables $(\rho)_{i\in\mathbb{N}}$ are algebraically independent, the invertible elements are those with a constant term in b in $\mathbb{C}^*$. □

Proposition 3.1.2 *Let E be a rank k $[\lambda]-$primitive fresco and let $\lambda_1,\dots,\lambda_k$ its fundamental invariants. Let $e(\rho)$ be a family of generators of E depending polynomially on a family of algebraically independant variables $(\rho_i)_{i\in\mathbb{N}}$. Then there exists $S_1,\dots,S_k$ in $\mathbb{C}[\rho][[b]]$ such that $S_j(\rho)[0]\equiv 1$ for each $j\in[1,k]$ and such that the annihilator of $e(\rho)$ in E is generated by the element of $\tilde{\mathcal{A}}$*

$$P(\rho):=(a-\lambda_1.b).S_1(\rho)^{-1}\dots S_{k-1}(\rho)^{-1}.(a-\lambda_k.b).S_k(\rho)^{-1}.$$

Proof. The key result to prove this proposition is the rank 1 case. In this case we may consider a standard generator e_1 of E which is a $\mathbb{C}[[b]]-$basis of E and satisfies

$$(a-\lambda_1.b).e_1=0.$$

Then, by definition, we may write

$$e(\rho)=S_1(\rho).e_1$$

where S_1 is in $\mathbb{C}[\rho][[b]]$ is invertible in this algebra, so has a constant term in $\mathbb{C}^*$. Up to normalizing e_1, we may assume that $S_1(\rho)[0]\equiv 1$ and then define

$$P(\rho):=(a-\lambda_1.b).S_1(\rho)^{-1}.$$

It clearly generates the annihilator of $e(\rho)$ for each ρ.
Assume now that the result is already proved for the rank $k-1 \geq 1$. Then consider the family $[e(\rho)]$ in the quotient E/F_{k-1} where F_{k-1} is the rank $k-1$ sub-module of E in its principal J-H. sequence. Remark first that $[e(\rho)]$ is a family of generators of E/F_{k-1} which depends polynomially on ρ. This is a trivial consequence of the fact that we may choose a $\mathbb{C}[[b]]-$basis $e_1, \ldots, e_k$ in E such that $e_1, \ldots, e_{k-1}$ is a $\mathbb{C}[[b]]$ basis of F_{k-1} and e_k maps to a standard generator of $E/F_{k-1} \simeq E_{\lambda_k}$. Then the rank 1 case gives $S_k \in \mathbb{C}[\rho][[b]]$ with $S_k(\rho)[0] = 1$ and such that $(a - \lambda_k.b).S_k(\rho)^{-1}.e(\rho)$ is in F_{k-1} for each ρ. But then it is a family of generators of F_{k-1} which depends polynomially on ρ and the inductive assumption allows to conclude. ■

Fix now the fundamental invariants $\lambda_1, \ldots, \lambda_k$ for a $[\lambda]-$primitive fresco.

Definition 3.1.3 *Consider now a complex valued function f defined on a subset $\mathcal{F}_0$ of the isomorphism classes $\mathcal{F}(\lambda_1, \ldots, \lambda_k)$ of $[\lambda]-$primitive frescos with fundamental invariants $\lambda_1, \ldots, \lambda_k$. We shall say that f* **depends polynomially on the isomorphism class** *$[E] \in \mathcal{F}_0$* **of the fresco** *E if the following condition is satisfied :*
Let s be the collection of algebraically independant variables corresponding to the non constant coefficients of k éléments $S_1, \ldots, S_k$ in $\mathbb{C}[[b]]$ satisfying $S_j(0) = 1 \quad \forall j \in [1, k]$ and consider for each value of s the rank k $[\lambda]-$primitive fresco $E(s) := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(s)$ where

$$P(s) := (a - \lambda_1.b)S_1(s)^{-1} \ldots (a - \lambda_k.b).S_k(s)^{-1}$$

where $S_1(s), \ldots, S_k(s)$ correspond to the given values for s.
Then there exists a polynomial $F \in \mathbb{C}[s]$ such that for each value of s such that $[E(s)]$ is in $\mathcal{F}_0$, the value of $F(s)$ is equal to $f([E(s)])$.

Example. Let $(s_i)_{i \in \mathbb{N}^*}$ a family of algebraically independant variables and let $S(s) := 1 + \sum_{i=1}^{\infty} s_i.b^i \in \mathbb{C}[s][[b]]$. Define $E(s) := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).S(s)[b]^{-1}.(a - \lambda_2.b)$ where $\lambda_1 > 1$ is rational and $\lambda_2 := \lambda_1 + p_1 - 1$ with $p_1 \in \mathbb{N}^*$. Define $\alpha(s) := s_{p_1}$. Then the number $\alpha(s)$ depends only of the isomorphism class of the fresco $E(s)$ and defines a function on $\mathcal{F}(\lambda_1, \lambda_2)$ which depends polynomially on $[E] \in \mathcal{F}(\lambda_1, \lambda_2)$. □

3.2 The $\alpha-$invariant.

The first proposition will be the induction step in the construction of the $\alpha-$invariant.

Proposition 3.2.1 *Fix $k \geq 2$. Denote $\mathcal{F}_0(\lambda_1, \ldots, \lambda_k)$ the subset of $\mathcal{F}(\lambda_1, \ldots, \lambda_k)$ of isomorphism class of rank k $[\lambda]-$primitive fresco E with invariants $\lambda_1, \ldots, \lambda_k$ such that F_{k-1} and E/F_1 are semi-simple where $(F_j)_{j \in [1,k]}$ is the principal J-H. sequence of E.*
We assume that, for $k \geq 3$ we have :

- *for rank* $\leq k-1$ *frescos the* α *invariant is defined on the corresponding subset* $\mathcal{F}_0$*;*
- *for* $[E] \in \mathcal{F}_0$ *the number* $\alpha(E)$ *is zero if and only if* E *is semi-simple*
- *and that* $[E] \mapsto \alpha(E)$ *depends polynomially of* $[E] \in \mathcal{F}_0$*.*

Fix E *with* $[E]$ *in* $\mathcal{F}_0(\lambda_1, \ldots, \lambda_k)$*. Let* e *be a generator of* E *such that* $(a-\lambda_{k-1}.b).(a-\lambda_k.b).e$ *lies in* F_{k-2}*. Then the sub-module* G_{k-1} *of* E *generated by* $(a-(\lambda_{k-1}-1).b).e$ *is a normal rank* $k-1$ *sub-module of* E *which is in* $\mathcal{F}_0(\lambda_1, \ldots, \lambda_{k-2}, \lambda_k+1)$ *and* $\alpha(G_{k-1})$ *is independant of the choice of such a generator* e*. Moreover, it defines a polynomial function on* $\mathcal{F}_0(\lambda_1, \ldots, \lambda_k)$*.*

Proof. We shall prove first that if e is a generator of e such that $(a-\lambda_{k-1}.b).(a-\lambda_k.b).e$ lies in F_{k-2} then $G_{k-1} := \tilde{\mathcal{A}}.(a-(\lambda_{k-1}-1).b.e$ is a normal rank $k-1$ sub-module of E such that $[G_{k-1}]$ belongs to $\mathcal{F}_0(\lambda_1, \ldots, \lambda_{k-2}, \lambda_k+1)$. Using the identity in $\tilde{\mathcal{A}}$:

$$(a-\lambda_{k-1}.b).(a-\lambda_k.b) = (a-(\lambda_k+1).b).(a-(\lambda_{k-1}-1).b)$$

we see that G_{k-1} contains F_{k-2}, that $G_{k-1}/F_{k-2} \simeq E_{\lambda_k+1}$ and G_{k-1} admits

$$F_1 \subset \cdots \subset F_{k-2} \subset G_{k-1}$$

as principal J-H. sequence. Then the fundamental invariants for G_{k-1} are equal to $\lambda_1, \ldots, \lambda_{k-2}, \lambda_k+1$, and the corank 1 term F_{k-2} is semi-simple. As G_{k-1}/F_1 is a sub-module of E/F_1 it is semi-simple and we have proved that

$$[G_{k-1}] \in \mathcal{F}_0(\lambda_1, \ldots, \lambda_{k-2}, \lambda_k+1).$$

Claim. If ε is another generator of E such that $(a-\lambda_{k-1}.b).(a-\lambda_k.b).\varepsilon$ is in F_{k-2} we have

$$\varepsilon = \rho.e + \sigma.b^{p_{k-1}-1}.(a-\lambda_k.b).e \quad \text{modulo } F_{k-2}$$

where (ρ, σ) is in $\mathbb{C}^* \times \mathbb{C}$.
Write $\varepsilon = U.e_k + V.e_{k-1}$ modulo F_{k-2} where $e_k := e$ and $e_{k-1} := (a-\lambda_k.b).e$ and where U, V are in $\mathbb{C}[[b]]$. Now

$$(a-\lambda_k.b).\varepsilon = b^2.U'.e_k + U.e_{k-1} + (\lambda_{k-1}-\lambda_k).b.V.e_{k-1} + b^2.V'.e_{k-1} \quad \text{modulo } F_{k-2}$$

and, as $(a-\lambda_{k-1}.b).b^2.U'e_k \in F_{k-1}$ implies $U'=0$, we have $U = \rho \in \mathbb{C}^*$, and we obtain

$$(a-\lambda_k.b).\varepsilon = \left[\rho + b^2.V' - (p_{k-1}-1).b.V\right].e_{k-1} \quad \text{modulo } F_{k-2}.$$

If $T := \rho + b^2.V' - (p_{k-1}-1).b.V$ we have

$$(a-\lambda_{k-1}.b).(a-\lambda_k.b).\varepsilon = b^2.T'.e_{k-1} \quad \text{modulo } F_{k-2}$$

and so T is a constant and equal to ρ. Then $V = \sigma.b^{p_{k-1}-1}$ for some complex number σ and our claim is proved.
For $\tau \in \mathbb{C}$ define

$$\varepsilon(\tau) := e_k + \tau.b^{p_{k-1}-1}.e_{k-1}$$

and let

$$G^\tau_{k-1} := \tilde{\mathcal{A}}.(a-(\lambda_{k-1}-1).b).(e_k+\tau.b^{p_{k-1}-1}.e_{k-1}) = \tilde{\mathcal{A}}.(a-(\lambda_{k-1}-1).b).\varepsilon(\tau).$$

Remark also that for any choice of ε such that $(a-\lambda_{k-1}.b).(a-\lambda_k.b).\varepsilon$ is in F_{k-2} the sub-module genrated by ε is equal to G^τ_{k-1} for some $\tau \in \mathbb{C}$.
The generator $(a-(\lambda_k+1).b).(a-(\lambda_{k-1}-1).b).\varepsilon(\tau)$ of F_{k-2} depends polynomially on τ. Then the proposition 3.1.2 gives $Q(\tau)$ depending polynomially on τ which annihilates for each τ this generator. Then $Q(\tau).(a-(\lambda_k+1).b)$ annihilates the generator $(a-(\lambda_{k-1}-1).b).\varepsilon(\tau)$ of G^τ_{k-1}. Remark now that for any $\tau \in \mathbb{C}$ the fresco G^τ_{k-1} has its principal J-H. sequence given by $F_1 \subset \cdots \subset F_{k-2} \subset G^\tau_{k-1}$. So G^τ_{k-1}/F_1 and F_{k-2} are semi-simple, and by our induction hypothesis, the number $\alpha(G^\tau_{k-1})$ is well defined and $\tau \mapsto \alpha(G^\tau_{k-1})$ is a polynomial in τ.

The key point will be now to show that either $\alpha(G^\tau_{k-1})$ is identically zero or it never vanishes. This will prove that this number is independant of τ concluding the proof.
But if for some $\tau \in \mathbb{C}$ we have $\alpha(G^\tau) = 0$, then G^τ is semi-simple. Now the sub-module $F_{k-1} + G^\tau_{k-1}$ has rank k and is semi-simple. So its normalization is also semi-simple. But this normalization is E. So any sub-module of E is semi-simple and we have $\alpha(G^{\tau'}_{k-1}) = 0$ for any τ'.
This also shows that when $\alpha(G^\tau_{k-1}) \neq 0$ for some τ then it is not zero for any choice of τ. Now a polynomial never vanishing is a constant, and this implies that we may define $\alpha(E) := \alpha(G^\tau_{k-1})$ for any value of τ.
To conclude the proof we have to show that, with this definition, the number $\alpha(E)$ depends polynomially on E. For that purpose it is enough, thanks to the previous computation, to produce a polynomial function $F \in \mathbb{C}[s]$ such that for each s with $E(s) \in \mathcal{F}_0(\lambda_1, \ldots, \lambda_k)$ we have $F(s) = \alpha([E(s)]$.
The first remark is that we may assume that $S_k = 1$. Then, thanks to the previous computation and the induction on the rank, it is enough to show that we may also reduce to the case where $S_{k-1} = 1$ because in this case we have

$$\alpha(E[s]) = \alpha([G_{k-1}(s)])$$

where

$$G_{k-1}(s) = \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b).S_1(s)^{-1}\ldots S_{k-2}(s)^{-1}.(a-(\lambda_k+1).b).$$

Denote $s^i_{k-1}, i \in \mathbb{N}^*$ the variables associated to the non constant coefficients of S_{k-1}. Looking to a generator of the form

$$\tilde{e} := e_k + X.e_{k-1}$$

with $X \in \mathbb{C}[s_{k-1}][[b]]$ such that $(a-\lambda_{k-1}.b).(a-\lambda_k.b).\tilde{e}$ is in F_{k-2} this leads to the equation

$$b^2.X' - (p_{k-1}-1).b.X = 1 - S_{k-1}(s)$$

which has an unique solution $X \in \mathbb{C}[s_{k-1}][[b]]$ without term in $b^{p_{k-1}-1}$ as we assume $E(s)/F_1(s)$ semi-simple, hypothesis which implies that S_{k-1} has no $b^{p_{k-1}}$ term. With this generator $\tilde{e}$ we have $(a-\lambda_{k-1}.b).(a-\lambda_k.b).\tilde{e}$ which is a generator of F_{k-2} which depends polynomially on s^i_{k-1}. Now we may consider F_{k-2} as a fix fresco (so we fix the coefficients in $S_1,\dots,S_{k-2}$ and apply the proposition 3.1.2 to obtain a $Q(s) := (a-\lambda_1.b).S_1(s)^{-1}\dots(a-\lambda_{k-2}.b).S_{k-2}(s)^{-1}$ which annihilates $(a-\lambda_{k-1}.b).(a-\lambda_k.b).\tilde{e}$ in $E(s)$ and depends polynomially on s. We may conclude by the induction assumption because we know that we have

$$\alpha([E(s)]) = \alpha([G_{k-1}(s)])$$

where $G_{k-1}(s) = \tilde{\mathcal{A}}.(a-\lambda_{k-1}-1).b).\tilde{e}(s)$ and because we know that the generator $(a-\lambda_{k-1}-1).b).\tilde{e}(s)$ is annihilated by $Q(s).(a-(\lambda_k+1).b)$ which depends polynomially on s. ■

Theorem 3.2.2 *Let $\lambda_1,\dots,\lambda_k$ be the fundamental invariants of a rank k $[\lambda]-$primitive fresco such that $p_j \geq 1$ for each $j \in [1,k-1]$, and let $\mathcal{F}_0(\lambda_1,\dots,\lambda_k)$ the subset of $\mathcal{F}(\lambda_1,\dots,\lambda_k)$ of isomorphism classes of E satisfying the condition that F_{k-1} and E/F_1 are semi-simple, where $(F_j)_{j\in[1,k]}$ is the principal J-H. sequence of E. Then there exists a unique polynomial function*

$$\alpha : \mathcal{F}_0(\lambda_1,\dots,\lambda_k) \to \mathbb{C}$$

with the following properties :

i) $\alpha([E]) = 0$ if and only if E is semi-simple.

ii) When E has a generator e such that $(a-\lambda_{k-1}.b).(a-\lambda_k.b).e$ is in F_{k-2}, we have $\alpha(E) = \alpha(G_{k-1})$ where $G_{k-1} := \tilde{\mathcal{A}}.(a-\lambda_{k-1}-1).b).e$.

iii) When $E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b).(1+x.b^{p_1})^{-1}.(a-\lambda_2.b)$ then $\alpha([E]) = x$.

PROOF. We define α for the rank 2 case as the coefficient of b^{p_1} in S_1 in any standard presentation $E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-\lambda_1.b).S_1^{-1}.(a-\lambda_2.b)$. This is given by the easy computation which gives the rank 2 classification for $[\lambda]-$primitive frescos, and fulfills condition iii). Then thanks to the previous proposition we may define $\alpha(E)$ for any rank $k \geq 3$ $[\lambda]-$primitive fresco E such that F_{k-1} and E/F_1 are semi-simple via condition ii). The condition i) and the fact that α is a polynomial function are proved in the induction step given by the proposition. ■

Let me give in rank 3 a polynomial $F \in \mathbb{C}[s]$ such that we have $F(s) = \alpha([E(s)])$ for those s for which $E(s)$ is in $\mathcal{F}_0(\lambda_1, \lambda_2, \lambda_3)$.
Assume $p_1 \geq 1$ and $p_2 \geq 1$. Then the necessary and sufficient condition to be in $\mathcal{F}_0(\lambda_1, \lambda_2, \lambda_3)$ is $s^1_{p_1} = s^2_{p_2} = 0$.

Lemma 3.2.3 *Let* $E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).S_1^{-1}.(a - \lambda_2.b).S_2^{-1}.(a - \lambda_3.b)$ *such that* $s^1_{p_1} = s^2_{p_2} = 0$*; the complex number* $\alpha(E)$ *is the coefficient of* $b^{p_1+p_2}$ *in* $p_2.V.S_1$ *for* $j = 1, 2$ *where* $V \in \mathbb{C}[[b]]$ *is a solution of the differential equation*

$$b.V' = p_2.(V - S_2).$$

The proof is left to the reader as an exercice.

This gives the following formula for F :

$$F(s) = p_2. \sum_{j \neq p_2, j=0}^{p_1+p_2} s^1_{p_1+p_2-j}.\frac{s^2_j}{p_2 - j}.$$

Remark that for $S_2 = 1$ we find that $F(s)$ reduces to $s^1_{p_1+p_2}$. Using the commutation relation $(a - \lambda_2.b).(a - \lambda_3.b) = (a - (\lambda_3 + 1).b).(a - (\lambda_2 - 1).b)$ we find in this case that $E(s)$ has a normal rank 2 sub-module with fundamental invariants $\lambda_1, \lambda_3 + 1$ and parameter $s^1_{p_1+p_2}$ which is precisely $\alpha(E(s))$.

Consequence. If we have rank 4 $[\lambda]-$primitive fresco E with its principal J-H. sequence such that F_{j+1}/F_{j-1} is semi-simple for $j \in [1, 3]$, we may look inductively to $\alpha(F_3)$. If it is zero then we may look at $\alpha(E/F_1)$, and if it is zero we may look at $\alpha(E)$. If this is zero then E is semi-simple, and so, the inductive vanishing of all $\alpha-$invariants of sub-quotients of the principal J-H. sequence give a necessary and sufficient condition for semi-simplicity of E.
This easily extends to any rank k $[\lambda]-$primitive fresco. □

Now we shall show that in the previous situation when the $\alpha-$invariant of E is not zero it is the parameter of any normal rank 2 sub-theme of E. Although such a rank 2 normal sub-theme is not unique (in general), its isomorphism class is uniquely determined from the fundamental invariants of E and from the $\alpha-$invariant of E.

Proposition 3.2.4 *Let* E *be a rank* $k \geq 2$ $[\lambda]-$*primitive fresco such that* F_{k-1} *and* E/F_1 *are semi-simple and with* $\alpha(E) \neq 0$*. Note* $\lambda_1, \ldots, \lambda_k$ *the fundamental invariants of* E *and* $p(E) := \sum_{j=1}^{p-1} p_j$*. Then there exists at least one rank* 2 *normal sub-theme in* E *and each rank* 2 *normal sub-theme of* E *is isomorphic to*

$$\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(1 + \alpha(E).b^{p(E)})^{-1}.(a - (\lambda_k + k - 2).b). \qquad (@)$$

PROOF. The case $k = 2$ is clear, so we may assume that $k \geq 3$ and we shall prove the proposition by induction on k. So assume that the proposition is knwon for the rank $k-1 \geq 2$ and let E be a rank k $[\lambda]-$primitive fresco satisfying our assumptions. The fact that there exists a rank 2 normal sub-theme is consequence of the induction hypothesis as G^τ for any $\tau \in \mathbb{C}$ is normal rank $k-1$ in E and satisfies again our assumptions.
Recall that the fundamental invariants of G^τ are $\mu_1, \ldots, \mu_{k-1}$ with $\mu_j = \lambda_j$ for $j \in [1, k-2]$ and $\mu_{k-1} = \lambda_k + 1$. We have also $\alpha(G^\tau) = \alpha(E)$ for each τ, thanks to the proof of the theorem 3.2.2. As we have $p(G^\tau) = p(E)$ for each τ, and $\mu_{k-1} = \mu_1 + p(E) - 1$ the inductive hypothesis implies that any rank 2 normal sub-theme of any G^τ is isomorphic to (@). Then, to complete the proof, it is enough to show that any rank 2 normal sub-theme of E is contained in some G^τ.

Let T be a rank 2 normal sub-theme of E. We shall first prove that its fundamental invariants are equal to $\lambda_1, \lambda_k + k - 2$. As E/F_1 is semi-simple, the image of T by the quotient map $E \to E/F_1$ has rank 1 and this implies that $F_1 \cap T$ is a rank 1 normal sub-module. Then this implies that F_1 is the unique normal rank 1 submodule of T, proving that $\lambda_1(T) = \lambda_1$. We shall prove now that $\lambda_2(T) = \lambda_k + k - 2$.
First remark that the uniqueness of the principal J-H. sequence of E implies the uniqueness of the quotient map $E \to E/F_{k-1} \simeq E_{\lambda_k}$ because any surjective map $q : E \to E_{\lambda_k}$ will produce principal a J-H. sequence for E by adjoining a principal J-H. sequence for $Ker\, q$. So a quotient E/H admits a surjective map on E_{λ_k} if (and only if) $H \subset F_{k-1}$. So H has to be semi-simple. Then the quotient E/T has no surjective map on E_{λ_k}.
The exact sequence of frescos

$$0 \to T/F_1 \to E/F_1 \to E/T \to 0$$

gives the equality in $\tilde{\mathcal{A}}$:

$$P_{T/F_1}.P_{E/T} = P_{E/F_1} = (a - \lambda_2.b) \ldots (a - \lambda_k.b).$$

But as E/T is semi-simple and does not has a quotient isomorphic to E_{λ_k} this implies $T/F_1 \simeq E_{\lambda_k + k - 2}$, proving our claim.
So T is isomorphic to $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(1 + \beta.b^{p(E)})^{-1}.(a - (\lambda_k + k - 2).b)$ for some $\beta \in \mathbb{C}^*$. Let $x \in T$ be a generator of T which is annihilated by

$$(a - \lambda_1.b).(1 + \beta.b^{p(E)})^{-1}.(a - (\lambda_k + k - 2).b).$$

It satisfies

$$(a - (\lambda_k + k - 2).b).x \in F_1 \subset F_{k-2}. \qquad (*)$$

We shall determine all $x \in E$ which satisfies the condition $(*)$ modulo F_{k-2}. Fix a generator $e := e_k$ of E such that $e_{k-1} := (a - \lambda_k.b).e_k$ is in F_{k-1} and satisfies

$(a-\lambda_{k-1}.b).e_{k-1} \in F_{k-2}$. Then we look for $U, V \in \mathbb{C}[[b]]$ such that

$$x := U.e_k + V.e_{k-1} \quad \text{satisfies} \quad (a-(\lambda_k+k-2).b).x \in F_{k-2}.$$

This leads to the equations

$$\begin{aligned} &b.U' - (k-2).b.U = 0 \\ &U + b^2.V' - (p_{k-1}+k-3).V = 0 \end{aligned}$$

and so we get

$$U = \rho.b^{k-2} \quad \text{and} \quad V = \frac{\rho}{p_{k-1}}.b^{k-3} + \sigma.b^{p_{k-1}+k-3}$$

Note that for $\rho = 0$ we would have $x \in F_{k-1}$ and this is not possible for the generator of a rank 2 theme as F_{k-1} is semi-simple. Then assuming $\rho \neq 0$ we may write

$$x = \frac{\rho.b^{k-3}}{p_{k-1}}.\left[p_{k-1}.b.e_k + e_{k-1} + \frac{p_{k-1}.\sigma}{\rho}.b^{p_{k-1}}.e_{k-1}\right] \quad \text{modulo} \quad F_{k-2}.$$

Now recall that the generator of G_τ is given by

$$(a-(\lambda_{k-1}-1).b).\varepsilon(\tau)$$

where $\varepsilon(\tau) = e_k + \tau.b^{p_{k-1}-1}.e_{k-1}$. A simple computation gives

$$(a-(\lambda_{k-1}-1).b).\varepsilon(\tau) = p_{k-1}.b.e_k + e_{k-1} + p_{k-1}.\tau.b^{p_{k-1}}.e_{k-1} \quad \text{modulo } F_{k-2}.$$

As we know that each G_τ contains F_{k-2}, we conclude that any such x is in G_τ for $\tau = \sigma/\rho$, concluding the proof. ■

Using duality we may deduce from this result the following corollary.

Corollary 3.2.5 *In the situation of the previous proposition E as a rank 2 quotient theme and any such rank 2 quotient theme is isomorphic to*

$$\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a-(\lambda_1-k+2).b).(1+\beta(E).b^{p(E)})^{-1}.(a-\lambda_k.b)$$

where

$$\beta(E) := (-1)^k \frac{p_1.(p_1+p_2)\dots(p_1+\dots+p_{k-2})}{p_{k-1}.(p_{k-2}+p_{k-1})\dots(p_2+\dots p_{k-1})}\alpha(E)$$

The proof is left as an exercice for the reader who may use the following remark.

REMARK. Let E be a rank 2 $[\lambda]-$primitive theme with fundamental invariants λ_1, λ_2 and parameter $\alpha(E)$. For $\delta \in \mathbb{N}, \delta \gg 1$ $E^* \otimes E_\delta$ is a 2 $[1-\lambda]-$primitive theme with fundamental invariants $(\delta-\lambda_2, \delta-\lambda_1$ and parameter $-\alpha(E)$.
Then in the situation of the proposition 3.2.4 $E^* \otimes E_\delta, \delta \gg 1$ satisfies again our assumptions and we have $\alpha(E^* \otimes E_\delta) = \beta(E)$.

4 The existence theorem for frescos.

The aim of this section is to prove the the following existence theorem for the fresco associated to a relative de Rham cohomology class :

Theorem 4.0.6 *Let X be a connected complex manifold of dimension $n+1$ where n is a natural integer, and let $f : X \to D$ be an non constant proper holomorphic function on an open disc D in $\mathbb{C}$ with center 0. Let us assume that df is nowhere vanishing outside of $X_0 := f^{-1}(0)$.*
Let ω be a $\mathscr{C}^\infty - (p+1)-$differential form on X such that $d\omega = 0 = df \wedge \omega$. Denote by E the geometric $\tilde{\mathcal{A}}-$module $\mathbb{H}^{p+1}(X, (\hat{K}^\bullet, d^\bullet))$ and $[\omega]$ the image of ω in $E/B(E)$. Then $\tilde{\mathcal{A}}.[\omega] \subset E/B(E)$ is a fresco.

Note that this result is an obvious consequence of the finiteness theorem 4.3.4 that we shall prove below. It gives the fact that E is naturally an $\tilde{\mathcal{A}}-$module which is of finite type over the subalgebra $\mathbb{C}[[b]]$ of $\tilde{\mathcal{A}}$, and so its $b-$torsion $B(E)$ is a finite dimensional $\mathbb{C}-$vector space. Moreover, the finiteness theorem asserts that $E/B(E)$ is a geometric (a,b)-module.

4.1 Preliminaries.

Here we shall complete and precise the results of the section 2 of [B.II]. The situation we shall consider is the following : let X be a connected complex manifold of dimension $n+1$ and $f : X \to \mathbb{C}$ a non constant holomorphic function such that $\{x \in X/\ df = 0\} \subset f^{-1}(0)$. We introduce the following complexes of sheaves supported by $X_0 := f^{-1}(0)$

1. The formal completion "in f" $(\hat{\Omega}^\bullet, d^\bullet)$ of the usual holomorphic de Rham complex of X.

2. The sub-complexes $(\hat{K}^\bullet, d^\bullet)$ and $(\hat{I}^\bullet, d^\bullet)$ of $(\hat{\Omega}^\bullet, d^\bullet)$ where the subsheaves $\hat{K}^p$ and $\hat{I}^{p+1}$ are defined for each $p \in \mathbb{N}$ respectively as the kernel and the image of the map
$$\wedge df : \hat{\Omega}^p \to \hat{\Omega}^{p+1}$$
given par exterior multiplication by df. We have the exact sequence
$$0 \to (\hat{K}^\bullet, d^\bullet) \to (\hat{\Omega}^\bullet, d^\bullet) \to (\hat{I}^\bullet, d^\bullet)[+1] \to 0. \tag{1}$$
Note that $\hat{K}^0$ and $\hat{I}^0$ are zero by definition.

3. The natural inclusions $\hat{I}^p \subset \hat{K}^p$ for all $p \geq 0$ are compatible with the différential d. This leads to an exact sequence of complexes
$$0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0. \tag{2}$$

4. We have a natural inclusion $f^*(\hat{\Omega}^1_{\mathbb{C}}) \subset \hat{K}^1 \cap Ker\, d$, and this gives a subcomplex (with zero differential) of $(\hat{K}^\bullet, d^\bullet)$. As in [B.07], we shall consider also the complex $(\tilde{K}^\bullet, d^\bullet)$ quotient. So we have the exact sequence

$$0 \to f^*(\hat{\Omega}^1_{\mathbb{C}}) \to (\hat{K}^\bullet, d^\bullet) \to (\tilde{K}^\bullet, d^\bullet) \to 0. \tag{3}$$

We do not make the assumption here that $f = 0$ is a reduced equation of X_0, and we do not assume that $n \geq 2$, so the cohomology sheaf in degree 1 of the complex $(\hat{K}^\bullet, d^\bullet)$, which is equal to $\hat{K}^1 \cap Ker\, d$ does not coincide, in general with $f^*(\hat{\Omega}^1_{\mathbb{C}})$. So the complex $(\tilde{K}^\bullet, d^\bullet)$ may have a non zero cohomology sheaf in degree 1.

Recall now that we have on the cohomology sheaves of the following complexes $(\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet), ([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ and $f^*(\hat{\Omega}^1_{\mathbb{C}}), (\tilde{K}^\bullet, d^\bullet)$ natural operations a and b with the relation $a.b - b.a = b^2$. They are defined in a naïve way by

$$a := \times f \quad \text{and} \quad b := \wedge df \circ d^{-1}.$$

The definition of a makes sens obviously. Let me precise the definition of b first in the case of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ with $p \geq 2$: if $x \in \hat{K}^p \cap Ker\, d$ write $x = d\xi$ with $\xi \in \hat{\Omega}^{p-1}$ and let $b[x] := [df \wedge \xi]$. The reader will check easily that this makes sens. For $p = 1$ we shall choose $\xi \in \hat{\Omega}^0$ with the extra condition that $\xi = 0$ on the smooth part of X_0 (set theoretically). This is possible because the condition $df \wedge d\xi = 0$ allows such a choice : near a smooth point of X_0 we can choose coordinnates such $f = x_0^k$ and the condition on ξ means independance of $x_1, \cdots, x_n$. Then ξ has to be (set theoretically) locally constant on X_0 which is locally connected. So we may kill the value of such a ξ along X_0.
The case of the complex $(\hat{I}^\bullet, d^\bullet)$ will be reduced to the previous one using the next lemma.

Lemma 4.1.1 *For each $p \geq 0$ there is a natural injective map*

$$\tilde{b} : \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \to \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$$

which satisfies the relation $a.\tilde{b} = \tilde{b}.(b + a)$. For $p \neq 1$ this map is bijective.

Proof. Let $x \in \hat{K}^p \cap Ker\, d$ and write $x = d\xi$ where $x \in \hat{\Omega}^{p-1}$ (with $\xi = 0$ on X_0 if $p = 1$), and set $\tilde{b}([x]) := [df \wedge \xi] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. This is independant on the choice of ξ because, for $p \geq 2$, adding $d\eta$ to ξ does not modify the result as $[df \wedge d\eta] = 0$. For $p = 1$ remark that our choice of ξ is unique.
This is also independant of the the choice of x in $[x] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ because adding $\theta \in \hat{K}^{p-1}$ to ξ does not change $[df \wedge \xi]$.
Assume $\tilde{b}([x]) = 0$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$; this means that we may find $\alpha \in \hat{\Omega}^{p-2}$ such $df \wedge \xi = df \wedge d\alpha$. But then, $\xi - d\alpha$ lies in $\hat{K}^{p-1}$ and $x = d(\xi - d\alpha)$ shows that $[x] = 0$. So $\tilde{b}$ is injective.
Assume now $p \geq 2$. If $df \wedge \eta$ is in $\hat{I}^p \cap Ker\, d$, then $df \wedge d\eta = 0$ and $y := d\eta$ lies

in $\hat{K}^p \cap Ker\, d$ and defines a class $[y] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ whose image by $\tilde{b}$ is $[df \wedge \eta]$. This shows the surjectivity of $\tilde{b}$ for $p \geq 2$.
For $p = 1$ the map $\tilde{b}$ is not surjective (see the remark below).
To finish the proof let us to compute $\tilde{b}(a[x] + b[x])$. Writing again $x = d\xi$, we get

$$a[x] + b[x] = [f.d\xi + df \wedge \xi] = [d(f.\xi)]$$

and so

$$\tilde{b}(a[x] + b[x]) = [df \wedge f.\xi] = a.\tilde{b}([x])$$

which concludes the proof. ■

Denote by $i : (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet)$ the natural inclusion and define the action of b on $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ by $b := \tilde{b} \circ \mathcal{H}^p(i)$. As i is $a-$linear, we deduce the relation $a.b - b.a = b^2$ on $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ from the relation of the previous lemma.

The action of a on the complex $([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ is obvious and the action of b is zero.

The action of a and b on $f^*(\hat{\Omega}^1_{\mathbb{C}}) \simeq E_1 \otimes \mathbb{C}_{X_0}$ are the obvious one, where E_1 is the rank 1 (a,b)-module with generator e_1 satisfying $a.e_1 = b.e_1$ (or, equivalentely, $E_1 := \mathbb{C}[[z]]$ with $a := \times z$, $b := \int_0^z$ and $e_1 := 1$).
Remark that the natural inclusion $f^*(\hat{\Omega}^1_{\mathbb{C}}) \hookrightarrow (\hat{K}^\bullet, d^\bullet)$ is compatible with the actions of a and b. The actions of a and b on $\mathcal{H}^1(\hat{K}^\bullet, d^\bullet)$ are simply induced by the corresponding actions on $\mathcal{H}^1(\hat{K}^\bullet, d^\bullet)$.

Remark. The exact sequence of complexes (1) induces for any $p \geq 2$ a bijection

$$\partial^p : \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \to \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$$

and a short exact sequence

$$0 \to \mathbb{C}_{X_0} \to \mathcal{H}^1(\hat{I}^\bullet, d^\bullet) \xrightarrow{\partial^1} \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \to 0 \qquad (@)$$

because of the de Rham lemma. Let us check that for $p \geq 2$ we have $\partial^p = (\tilde{b})^{-1}$ and that for $p = 1$ we have $\partial^1 \circ \tilde{b} = Id$. If $x = d\xi \in \hat{K}^p \cap Ker\, d$ then $\tilde{b}([x]) = [df \wedge \xi]$ and $\partial^p[df \wedge \xi] = [d\xi]$. So $\partial^p \circ \tilde{b} = Id \quad \forall p \geq 0$. For $p \geq 2$ and $df \wedge \alpha \in \hat{I}^p \cap Ker\, d$ we have $\partial^p[df \wedge \alpha] = [d\alpha]$ and $\tilde{b}[d\alpha] = [df \wedge \alpha]$, so $\tilde{b} \circ \partial^p = Id$. For $p = 1$ we have $\tilde{b}[d\alpha] = [df \wedge (\alpha - \alpha_0)]$ where $\alpha_0 \in \mathbb{C}$ is such that $\alpha_{|X_0} = \alpha_0$. This shows that in degree 1 $\tilde{b}$ gives a canonical splitting of the exact sequence $(@)$.

4.2 $\tilde{\mathcal{A}}-$structures.

Let us consider now the $\mathbb{C}-$algebra

$$\tilde{\mathcal{A}} := \{\sum_{\nu \geq 0} \quad P_\nu(a).b^\nu\}$$

where $P_\nu \in \mathbb{C}[z]$, and the commutation relation $a.b - b.a = b^2$, assuming that left and right multiplications by a are continuous for the $b-$adic topology of $\tilde{\mathcal{A}}$. Define the following complexes of sheaves of left $\tilde{\mathcal{A}}-$modules on X :

$$(\Omega'^\bullet[[b]], D^\bullet) \quad \text{and} \quad (\Omega''^\bullet[[b]], D^\bullet) \quad \text{where} \tag{4}$$

$$\Omega'^\bullet[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \quad \text{with} \quad \omega_0 \in \hat{K}^p$$

$$\Omega''^\bullet[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \quad \text{with} \quad \omega_0 \in \hat{I}^p$$

$$D(\sum_{j=0}^{+\infty} b^j.\omega_j) = \sum_{j=0}^{+\infty} b^j.(d\omega_j - df \wedge \omega_{j+1})$$

$$a.\sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=0}^{+\infty} b^j.(f.\omega_j + (j-1).\omega_{j-1}) \quad \text{with the convention} \quad \omega_{-1} = 0$$

$$b.\sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=1}^{+\infty} b^j.\omega_{j-1}$$

It is easy to check that D is $\tilde{\mathcal{A}}-$linear and that $D^2 = 0$. We have a natural inclusion of complexes of left $\tilde{\mathcal{A}}-$modules

$$\tilde{i} : (\Omega''^\bullet[[b]], D^\bullet) \to (\Omega'^\bullet[[b]], D^\bullet).$$

Remark that we have natural morphisms of complexes

$$u : (\hat{I}^\bullet, d^\bullet) \to (\Omega''^\bullet[[b]], D^\bullet)$$
$$v : (\hat{K}^\bullet, d^\bullet) \to (\Omega'^\bullet[[b]], D^\bullet)$$

and that these morphisms are compatible with i. More precisely, this means that we have the commutative diagram of complexes

$$\begin{array}{ccc} (\hat{I}^\bullet, d^\bullet) & \xrightarrow{u} & (\Omega''^\bullet[[b]], D^\bullet) \\ \downarrow i & & \downarrow \tilde{i} \\ (\hat{K}^\bullet, d^\bullet) & \xrightarrow{v} & (\Omega'^\bullet[[b]], D^\bullet) \end{array}$$

The following theorem is a variant of theorem 2.2.1. of [B.II].

Theorem 4.2.1 *Let X be a connected complex manifold of dimension $n+1$ and $f : X \to \mathbb{C}$ a non constant holomorphic function such that*

$$\{x \in X/\ df = 0\} \subset f^{-1}(0).$$

Then the morphisms of complexes u and v introduced above are quasi-isomorphisms. Moreover, the isomorphims that they induce on the cohomology sheaves of these complexes are compatible with the actions of a and b.

This theorem builds a natural structure of left $\tilde{\mathcal{A}}-$modules on each of the complex $(\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet), ([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ and $f^*(\hat{\Omega}^1_{\mathbb{C}}), (\tilde{K}^\bullet, d^\bullet)$ in the derived category of bounded complexes of sheaves of $\mathbb{C}-$vector spaces on X.
Moreover the short exact sequences

$$0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0$$
$$0 \to f^*(\hat{\Omega}^1_{\mathbb{C}}) \to (\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet) \to (\tilde{K}^\bullet, d^\bullet) \to 0$$

are equivalent to short exact sequences of complexes of left $\tilde{\mathcal{A}}-$modules in the derived category.

PROOF. We have to prove that for any $p \geq 0$ the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ are bijective and compatible with the actions of a and b. The case of $\mathcal{H}^p(v)$ is handled (at least for $n \geq 2$ and f reduced) in prop. 2.3.1. of [B.II]. To seek for completeness and for the convenience of the reader we shall treat here the case of $\mathcal{H}^p(u)$.
First we shall prove the injectivity of $\mathcal{H}^p(u)$. Let $\alpha = df \wedge \beta \in \hat{I}^p \cap Ker\, d$ and assume that we can find $U = \sum_{j=0}^{+\infty} b^j.u_j \in \Omega''^{p-1}[[b]]$ with $\alpha = DU$. Then we have the following relations

$$u_0 = df \wedge \zeta, \quad \alpha = du_0 - df \wedge u_1 \quad \text{and} \quad du_j = df \wedge u_{j+1}\ \forall j \geq 1.$$

For $j \geq 1$ we have $[du_j] = b[du_{j+1}]$ in $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$; using corollary 2.2. of [B.II] which gives the $b-$separation of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$, this implies $[du_j] = 0, \forall j \geq 1$ in $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$. For instance we can find $\beta_1 \in \hat{K}^{p-1}$ such that $du_1 = d\beta_1$. Now, by de Rham, we can write $u_1 = \beta_1 + d\xi_1$ for $p \geq 2$, where $\xi_1 \in \hat{\Omega}^{p-2}$. Then we conclude that $\alpha = -df \wedge d(\xi_1 + \zeta)$ and $[\alpha] = 0$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$.
For $p = 1$ we have $u_1 = 0$ and $[\alpha] = [-df \wedge d\xi_1] = 0$ in $\mathcal{H}^1(\hat{I}^\bullet, d^\bullet)$.
We shall show now that the image of $\mathcal{H}^p(u)$ is dense in $\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ for its $b-$adic topology. Let $\Omega := \sum_{j=0}^{+\infty} b^j.\omega_j \in \Omega''^p[[b]]$ such that $D\Omega = 0$. The following relations holds $d\omega_j = df \wedge \omega_{j+1}$ $\forall j \geq 0$ and $\omega_0 \in \hat{I}^p$. The corollary 2.2. of [B.II] again allows to find $\beta_j \in \hat{K}^{p-1}$ for any $j \geq 0$ such that $d\omega_j = d\beta_j$. Fix $N \in \mathbb{N}^*$. We have

$$D(\sum_{j=0}^{N} b^j.\omega_j) = b^N.d\omega_N = D(b^N.\beta_N)$$

and $\Omega_N := \sum_{j=0}^{N} b^j.\omega_j - b^N.\beta_N$ is $D-$closed and in $\Omega''^p[[b]]$. And we have $\Omega - \Omega_N \in b^N.\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$, so the sequence $(\Omega_N)_{N\geq 1}$ converges to Ω in $\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ for its $b-$adic topology. Let us show that each Ω_N is in the image of $\mathcal{H}^p(u)$.
Write $\Omega_N := \sum_{j=0}^{N} b^j.w_j$. The condition $D\Omega_N = 0$ implies $dw_N = 0$ and $dw_{N-1} = df \wedge w_N = 0$. If we write $w_N = dv_N$ we obtain $d(w_{N-1} + df \wedge v_N) = 0$ and $\Omega_N - D(b^N.v_N)$ is of degree $N-1$ in b. For $N = 1$ we are left with $w_0 + b.w_1 - (-df \wedge v_1 + b.dv_1) = w_0 + df \wedge v_1$ which is in $\hat{I}^p \cap Ker\, d$ because

$dw_0 = df \wedge dv_1$.
To conclude it is enough to know the following two facts

i) The fact that $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ is complete for its $b-$adic topology.

ii) The fact that $Im(\mathcal{H}^p(u)) \cap b^N.\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet) \subset Im(\mathcal{H}^p(u) \circ b^N) \quad \forall N \geq 1$.

Let us first conclude the proof of the surjectivity of $\mathcal{H}^p(u)$ assuming i) and ii).
For any $[\Omega] \in \mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ we know that there exists a sequence $(\alpha_N)_{N\geq 1}$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ with $\Omega - \mathcal{H}^p(u)(\alpha_N) \in b^N.\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$. Now the property ii) implies that we may choose the sequence $(\alpha_N)_{N\geq 1}$ such that $[\alpha_{N+1}] - [\alpha_N]$ lies in $b^N.\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. So the property i) implies that the Cauchy sequence $([\alpha_N])_{N\geq 1}$ converges to $[\alpha] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. Then the continuity of $\mathcal{H}^p(u)$ for the $b-$adic topologies coming from its $b-$linearity, implies $\mathcal{H}^p(u)([\alpha]) = [\Omega]$.
The compatibility with a and b of the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ is an easy exercice.

Let us now prove properties i) and ii).
The property i) is a direct consequence of the completion of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ for its $b-$adic topology given by the corollary 2.2. of [B.II] and the $b-$linear isomorphism $\tilde{b}$ between $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ and $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ constructed in the lemma 2.1.1. above.
To prove ii) let $\alpha \in \hat{I}^p \cap Ker\, d$ and $N \geq 1$ such that

$$\alpha = b^N.\Omega + DU$$

where $\Omega \in \Omega''^p[[b]]$ satisfies $D\Omega = 0$ and where $U \in \Omega''^{p-1}[[b]]$. With obvious notations we have

$$\begin{aligned}
&\alpha = du_0 - df \wedge u_1 \\
&\cdots \\
&0 = du_j - df \wedge u_{j+1} \quad \forall j \in [1, N-1] \\
&\cdots \\
&0 = \omega_0 + du_N - df \wedge u_{N+1}
\end{aligned}$$

which implies $D(u_0 + b.u_1 + \cdots + b^N.u_N) = \alpha + b^N.du_N$ and the fact that du_N lies in $\hat{I}^p \cap Ker\, d$. So we conclude that $[\alpha] + b^N.[du_N]$ is in the kernel of $\mathcal{H}^p(u)$ which is 0. Then $[\alpha] \in b^N.\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. ■

Remark. The map

$$\beta : (\Omega'[[b]]^\bullet, D^\bullet) \to (\Omega''[[b]]^\bullet, D^\bullet)$$

defined by $\beta(\Omega) = b.\Omega$ commutes to the differentials and with the action of b. It induces the isomorphism $\tilde{b}$ of the lemma 4.1.1 on the cohomology sheaves. So it is a quasi-isomorphism of complexes of $\mathbb{C}[[b]]-$modules.

To prove this fact, it is enough to verify that the diagram

$$\begin{array}{ccc} (\hat{K}^\bullet, d^\bullet) & \xrightarrow{v} & (\Omega'[[b]]^\bullet, D^\bullet) \\ \downarrow{\scriptstyle \tilde{b}} & & \downarrow{\scriptstyle \beta} \\ (\hat{I}^\bullet, d^\bullet) & \xrightarrow{u} & (\Omega''[[b]]^\bullet, D^\bullet) \end{array}$$

induces commutative diagams on the cohomology sheaves.
But this is clear because if $\alpha = d\xi$ lies in $\hat{K}^p \cap Ker\, d$ we have $D(b.\xi) = b.d\xi - df \wedge \xi$ so $\mathcal{H}^p(\beta) \circ \mathcal{H}^p(v)([\alpha]) = \mathcal{H}^p(u) \circ \mathcal{H}^p(\tilde{b})([\alpha])$ in $\mathcal{H}^p(\Omega''[[b]]^\bullet, D^\bullet)$. ■

4.3 The finiteness theorem.

Let us recall some basic definitions on the left modules over the algebra $\tilde{\mathcal{A}}$.
Now let E be any left $\tilde{\mathcal{A}}-$module, and define $B(E)$ as the $b-$torsion of E. that is to say

$$B(E) := \{x \in E \ / \ \exists N \quad b^N.x = 0\}.$$

Define $A(E)$ as the $a-$torsion of E and

$$\hat{A}(E) := \{x \in E \ / \ \mathbb{C}[[b]].x \subset A(E)\}.$$

Remark that $B(E)$ and $\hat{A}(E)$ are sub-$\tilde{\mathcal{A}}-$modules of E but that $A(E)$ is not stable by b.

Definition 4.3.1 *A left $\tilde{\mathcal{A}}-$module E is called* **small** *when the following conditions hold*

1. *E is a finite type $\mathbb{C}[[b]]-$module ;*
2. *$B(E) \subset \hat{A}(E)$;*
3. *$\exists N \ / \ a^N.\hat{A}(E) = 0$;*

Recall that for E small we have always the equality $B(E) = \hat{A}(E)$ (see [B.I] lemme 2.1.2) and that this complex vector space is finite dimensional. The quotient $E/B(E)$ is an (a,b)-module called **the associate (a,b)-module** to E.
Conversely, any left $\tilde{\mathcal{A}}-$module E such that $B(E)$ is a finite dimensional $\mathbb{C}-$vector space and such that $E/B(E)$ is an (a,b)-module is small.
The following easy criterium to be small will be used later :

Lemma 4.3.2 *A left $\tilde{\mathcal{A}}-$module E is small if and only if the following conditions hold :*

1. *$\exists N \ / \ a^N.\hat{A}(E) = 0$;*
2. *$B(E) \subset \hat{A}(E)$;*

3. $\cap_{m\geq 0} b^m.E \subset \hat{A}(E)$ *;*

4. $Ker\, b$ *and* $Coker\, b$ *are finite dimensional complex vector spaces.*

As the condition 3 in the previous lemma has been omitted in [B.II] (but this does not affect this article because this lemma was used only in a case were this condition 3 was satisfied, thanks to proposition 2.2.1. of *loc. cit.*), we shall give the (easy) proof.

PROOF. First the conditions 1 to 4 are obviously necessary. Conversely, assume that E satisfies these four conditions. Then condition 2 implies that the action of b on $\hat{A}(E)/B(E)$ is injective. But the condition 1 implies that $b^{2N} = 0$ on $\hat{A}(E)$ (see [B.I]). So we conclude that $\hat{A}(E) = B(E) \subset Ker\, b^{2N}$ which is a finite dimensional complex vector space using condition 4 and an easy induction. Now $E/B(E)$ is a $\mathbb{C}[[b]]-$module which is separated for its $b-$adic topology. The finitness of $Coker\, b$ now shows that it is a free finite type $\mathbb{C}[[b]]-$module concluding the proof. ■

Definition 4.3.3 *We shall say that a left* $\tilde{\mathcal{A}}-$*module* E *is* **geometric** *when* E *is small and when it associated (a,b)-module* $E/B(E)$ *is geometric.*

The main result of this section is the following theorem, which shows that the Gauss-Manin connection of a proper holomorphic function produces geometric $\tilde{\mathcal{A}}-$modules associated to vanishing cycles and nearby cycles.

Theorem 4.3.4 *Let* X *be a connected complex manifold of dimension* $n+1$ *where* n *is a natural integer, and let* $f : X \to D$ *be an non constant proper holomorphic function on an open disc* D *in* $\mathbb{C}$ *with center* 0*. Let us assume that* df *is nowhere vanishing outside of* $X_0 := f^{-1}(0)$*.*
Then the $\tilde{\mathcal{A}}-$*modules*

$$\mathbb{H}^j(X, (\hat{K}^\bullet, d^\bullet)) \quad \text{and} \quad \mathbb{H}^j(X, (\hat{I}^\bullet, d^\bullet))$$

are geometric for any $j \geq 0$*.*

In the proof we shall use the $\mathscr{C}^\infty$ version of the complex $(\hat{K}^\bullet, d^\bullet)$. We define K^p_∞ as the kernel of $\wedge df : \mathscr{C}^{\infty,p} \to \mathscr{C}^{\infty,p+1}$ where $\mathscr{C}^{\infty,j}$ denote the sheaf of $\mathscr{C}^\infty-$ forms on X of degree p, let $\hat{K}^p_\infty$ be the $f-$completion and $(\hat{K}^\bullet_\infty, d^\bullet)$ the corresponding de Rham complex.
The next lemma is proved in [B.II] (lemma 6.1.1.)

Lemma 4.3.5 *The natural inclusion*

$$(\hat{K}^\bullet, d^\bullet) \hookrightarrow (\hat{K}^\bullet_\infty, d^\bullet)$$

induce a quasi-isomorphism.

REMARK. As the sheaves $\hat{K}^\bullet_\infty$ are fine, we have a natural isomorphism

$$\mathbb{H}^p(X,(\hat{K}^\bullet,d^\bullet)) \simeq H^p\big(\Gamma(X,\hat{K}^\bullet_\infty),d^\bullet\big).$$

Let us denote by X_1 the generic fiber of f. Then X_1 is a smooth compact complex manifold of dimension n and the restriction of f to $f^{-1}(D^*)$ is a locally trivial $\mathscr{C}^\infty$ bundle with typical fiber X_1 on $D^* = D\setminus\{0\}$, if the disc D is small enough around 0. Fix now $\gamma \in H_p(X_1,\mathbb{C})$ and let $(\gamma_s)_{s\in D^*}$ the corresponding multivalued horizontal family of $p-$cycles $\gamma_s \in H_p(X_s,\mathbb{C})$. Then, for $\omega \in \Gamma(X,\hat{K}^p_\infty \cap Ker\, d)$, define the multivalued holomorphic function

$$F_\omega(s) := \int_{\gamma_s} \frac{\omega}{df}.$$

Let now

$$\Xi := \oplus_{\alpha\in\mathbb{Q}\cap]-1,0],j\in[0,n]} \quad \mathbb{C}[[s]].s^\alpha.\frac{(Logs)^j}{j!}.$$

This is an $\tilde{\mathcal{A}}-$modules with a acting as multiplication by s and b as the primitive in s without constant. Now if $\hat{F}_\omega$ is the asymptotic expansion at 0 of F_ω, it is an element in Ξ, and we obtain in this way an $\tilde{\mathcal{A}}-$linear map

$$Int : \mathbb{H}^p(X,(\hat{K}^\bullet,d^\bullet)) \to H^p(X_1,\mathbb{C})\otimes_{\mathbb{C}}\Xi.$$

To simplify notations, let $E := \mathbb{H}^p(X,(\hat{K}^\bullet,d^\bullet))$. Now using Grothendieck theorem [G.65], there exists $N\in\mathbb{N}$ such that $Int(\omega)\equiv 0$, implies $a^N.[\omega]=0$ in E. As the converse is clear we conclude that $\hat{A}(E) = Ker(Int)$. It is also clear that $B(E)\subset Ker(Int)$ because Ξ has no $b-$torsion. So we conclude that E satisfies properties 1 and 2 of the lemma 4.3.2. The property 3 is also true because of the regularity of the Gauss-Manin connection of f.

END OF THE PROOF OF THEOREM 4.3.4. To show that $E := \mathbb{H}^p(X,(\hat{K}^\bullet,d^\bullet))$ is small, it is enough to prove that E satisfies the condition 4 of the lemma 4.3.2. Consider now the long exact sequence of hypercohomology of the exact sequence of complexes

$$0\to(\hat{I}^\bullet,d^\bullet)\to(\hat{K}^\bullet,d^\bullet)\to([\hat{K}/\hat{I}]^\bullet,d^\bullet)\to 0.$$

It contains the exact sequence

$$\mathbb{H}^{p-1}(X,([\hat{K}/\hat{I}]^\bullet,d^\bullet))\to\mathbb{H}^p(X,(\hat{I}^\bullet,d^\bullet))\overset{\mathbb{H}^p(i)}{\to}\mathbb{H}^p(X,(\hat{K}^\bullet,d^\bullet))\to\mathbb{H}^p(X,([\hat{K}/\hat{I}]^\bullet,d^\bullet))$$

and we know that b is induced on the complex of $\tilde{\mathcal{A}}-$modules quasi-isomorphic to $(\hat{K}^\bullet,d^\bullet)$ by the composition $i\circ\tilde{b}$ where $\tilde{b}$ is a quasi-isomorphism of complexes of $\mathbb{C}[[b]]-$modules. This implies that the kernel and the cokernel of $\mathbb{H}^p(i)$ are isomorphic (as $\mathbb{C}-$vector spaces) to $Ker\, b$ and $Coker\, b$ respectively. Now to prove that E satisfies condition 4 of the lemma 4.3.2 it is enough to prove finite dimensionality for the vector spaces $\mathbb{H}^j(X,([\hat{K}/\hat{I}]^\bullet,d^\bullet))$ for all $j\geq 0$.

But the sheaves $[\hat{K}/\hat{I}]^j \simeq [Ker\, df/Im\, df]^j$ are coherent on X and supported in X_0. The spectral sequence

$$E_2^{p,q} := H^q\big(H^p(X, [\hat{K}/\hat{I}]^\bullet), d^\bullet\big)$$

which converges to $\mathbb{H}^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$, is a bounded complex of finite dimensional vector spaces by Cartan-Serre. This gives the desired finite dimensionality.
To conclude the proof, we want to show that $E/B(E)$ is geometric. But this is an easy consequence of the regularity of the Gauss-Manin connexion of f and of the Monodromy theorem, which are already incoded in the definition of Ξ : the injectivity on $E/B(E)$ of the $\tilde{\mathcal{A}}-linear$ map Int implies that $E/B(E)$ is geometric. Remark now that the piece of exact sequence above gives also the fact that $\mathbb{H}^p(X, (\hat{I}^\bullet, d^\bullet))$ is geometric, because it is an exact sequence of $\tilde{\mathcal{A}}-$modules. ■

Remark. It is easy to see that the properness assumption on f is only used for two purposes :
–To have a (global) $\mathscr{C}^\infty$ Milnor fibration on a small punctured disc around 0, with a finite dimensional cohomology for the Milnor fiber.
– To have compactness of the singular set $\{df = 0\}$, which contains the supports of the coherent sheaves $(Ker\, df/Im\, df)^i$.
This allows to give with the same proof an analoguous finiteness result in many other situations.

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