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# Sobolev Extension Property for Tree-shaped Domains with Self-contacting Fractal Boundary

Thibaut Deheuvels\*

## Abstract

In this paper, we investigate the existence of extension operators from  $W^{1,p}(\Omega)$  to  $W^{1,p}(\mathbb{R}^2)$  ( $1 \leq p < \infty$ ) for a class of tree-shaped domains  $\Omega$  with a self-similar fractal boundary previously studied by Mandelbrot and Frame. Such a geometry can be seen as a bidimensional modelization of the bronchial tree. When the fractal boundary has no self-contact, Jones proved that there exist such extension operators for all  $p \in [1, \infty]$ . In the case when the fractal boundary self-intersects, this result does not hold. Here, we prove however that extension operators exist for  $p < p^*$  where  $p^*$  depends only on the dimension of the self-intersection of the boundary. The construction of these operators mainly relies on the self-similar properties of the domains.

## 1 Introduction

Extension results for Sobolev function spaces are of significant importance in analysis. In particular, it is useful to know if an open domain in  $\mathbb{R}^n$  has the  $W^{1,p}$ -extension property for some  $p \in [1, \infty]$ , that is, there exists a bounded linear operator

$$\Lambda : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n),$$

such that  $\Lambda u|_{\Omega} = u$  for all  $u \in W^{1,p}(\Omega)$ . Such domains are called  $W^{1,p}$ -extension domains. A domain that has the  $W^{1,p}$ -extension property for all  $p \in [1, \infty]$  is sometimes referred to as a Sobolev extension domain.

Calderón proved that every Lipschitz domain in  $\mathbb{R}^n$ , that is every domain whose boundary is locally the graph of a Lipschitz function, has the  $W^{1,p}$ -extension property for all  $p \in (1, \infty)$ . Stein extended his result to the cases  $p = 1$  and  $p = \infty$ .

In [14], Jones improved this result by introducing a class of domains in  $\mathbb{R}^n$  called  $(\varepsilon, \delta)$ -domains, every member of which is a Sobolev extension domain. Jones' proof uses as a main ingredient Whitney's extension theory. Such domains were also defined by Martio and Sarvas who referred to them as locally uniform domains (see [22]). This result is almost optimal in the plane, in the sense that every plane finitely connected Sobolev extension domain is an  $(\varepsilon, \delta)$ -domain.

In [20], Koskela proved that in the general case, if an open domain  $\Omega \subset \mathbb{R}^n$  has the  $W^{1,n}$ -extension property, then it has the  $W^{1,p}$ -extension property for all  $p \geq n$ . He also showed that if the embedding  $W^{1,p}(\Omega) \hookrightarrow C^{0,1-n/p}(\overline{\Omega})$  holds for some  $p > n$ , then  $\Omega$  has the  $W^{1,q}$ -extension property for all  $q > p$ . The case  $p < n$  is not as well understood.

In the present work, we consider a class of domains  $\Omega \subset \mathbb{R}^2$  with a self-similar fractal boundary that do not have the  $W^{1,p}$ -extension property for any  $p > 2$ , and we will study the case when

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$p < 2$ .

We focus on a class of tree-shaped domains  $\Omega$  in  $\mathbb{R}^2$  with a self-similar fractal boundary  $\Gamma^\infty$  which self-intersects, see for example figure 1. The set  $\Gamma^\infty$  is defined as the unique compact set such that

$$\Gamma^\infty = f_1(\Gamma^\infty) \cup f_2(\Gamma^\infty),$$

where  $f_1$  and  $f_2$  are two contracting similitudes with opposite rotation angles  $\pm\theta$  ( $0 \leq \theta < \frac{\pi}{2}$ ) and contraction ratio  $a \in [0, 1)$ . This type of fractal sets were first studied by Mandelbrot and Frame in [21].

We will see in paragraph 2.1.1 that there exists a critical ratio  $a_\theta^*$  dependent on the rotation angle of the similitude such that for every  $a < a_\theta^*$ , the set  $\Gamma^\infty$  is totally disconnected, and for  $a = a_\theta^*$ , it is connected. In the first case, the domain  $\Omega$  is an  $(\varepsilon, \delta)$ -domain and  $\Omega$  is a Sobolev extension domain. In this paper, emphasis will be put on the latter case, in which we will see that  $\Omega$  is not an  $(\varepsilon, \delta)$ -domain and  $\Omega$  is not a Sobolev extension domain. In this case, the assumptions required in [14] for the construction of Whitney extension operators are not satisfied.

Particular care will be given to the notion of trace on the set  $\Gamma^\infty$  for functions in  $W^{1,p}(\Omega)$ . We will use two different definitions of trace on  $\Gamma^\infty$ .

- The first one, referred to as the classical or strictly defined trace below, relies on the notion of the strict definition of a locally integrable function, see for instance [17] page 206. For a function  $u$  in  $W^{1,p}(\Omega)$  or  $W^{1,p}(\mathbb{R}^d)$ , this trace, noted  $u|_{\Gamma^\infty}$  below, is defined as its strictly defined counterpart on the subset of  $\Gamma^\infty$  where  $u$  is strictly defined.
- The second one was first introduced in [6]. Its construction is recalled in §3. This trace operator, noted  $\ell^\infty$  below, is obtained by exploiting the self-similarity as the limit of a sequence of operators  $\ell^n$  which map  $W^{1,p}(\Omega)$  to piecewise constant functions on a partition of  $\Gamma^\infty$  into  $2^n$  sets whose measure is  $2^{-n}$ .

A consequence of the main result of this paper is that these two definitions of trace on  $\Gamma^\infty$  in fact coincide (almost everywhere) on the set  $\Gamma^\infty$ ; this is proved in [5].

Jonsson and Wallin have proved extension and trace results for Besov and Sobolev spaces on  $d$ -sets (see [17]). See §2.2.3 for a definition of the Sobolev spaces on such sets in the special case of  $\Gamma^\infty$  which is a  $d$ -set where  $d$  is its Hausdorff dimension. In particular, see [17] page 183,  $W^{1,p}(\mathbb{R}^2)|_{\Gamma^\infty} = W^{1-\frac{2-d}{p},p}(\Gamma^\infty)$  for  $p \in (1, \infty)$ , where the trace is meant in the classical sense. The extension part of the theorem mainly relies on Whitney's extension theorem.

It has been proved in [1] that there exists a real number  $p^* > 1$  such that  $p^*$  only depends on the Hausdorff dimension of the self-intersection of  $\Gamma^\infty$  and

- if  $p < p^*$ , then  $\ell^\infty(W^{1,p}(\Omega)) = W^{1-\frac{2-d}{p},p}(\Gamma^\infty)$ ,
- if  $p > p^*$ , then the previous result does not hold.

The main extension result of this paper (Theorem 7) states that when  $p < p^*$ , the domains  $\Omega$  in fact have the  $W^{1,p}$ -extension property. To prove this result, we prove in Theorem 6 that there exists a continuous lifting operator  $\mathcal{E}$  in the sense of  $\ell^\infty$  from the trace space on  $\Gamma^\infty$  of functions in  $W^{1,p}(\Omega)$  to  $W^{1,p}(\mathbb{R}^2)$ : for all  $v \in \ell^\infty(W^{1,p}(\Omega))$ ,  $\ell^\infty((\mathcal{E}v)|_\Omega) = v$ . This last property, which will be crucial in the proof of Theorem 7, would not be guaranteed *a priori* by the lifting operator of Jonsson and Wallin in the classical sense.

An immediate consequence of this extension theorem is that, for  $p \in (1, p^*)$ , the Sobolev embeddings hold in  $\Omega$ . Theorem 7 is sharp in the sense that whenever  $p > p^*$ ,  $\Omega$  is not a  $W^{1,p}$ -extension domain, in Remark 5 below. The case  $p = p^*$  is partially discussed in Remark 5.

Note that the question of extensions or traces naturally arises in boundary value or transmission problems in domains with fractal boundaries. Boundary value problems posed in the domains  $\Omega$  displayed in Figure 1 were studied in [2], [3], [4] and [6], and numerical methods were proposed to compute the solutions in subdomains of  $\Omega$ . Such a geometry can be seen as a bidimensional idealization of the bronchial tree, for example. The problems studied in the latter papers aim at simulating the diffusion of medical sprays in human lungs.

The paper is organized as follows: the geometry of the studied domains is presented in Section 2. In Section 3, we briefly treat the less interesting sub-critical case when  $a < a_\theta^*$  and recall the construction of the trace operator introduced in [6]. The theory proposed in [16] is reviewed in Section 4, where we also recall the characterization of the trace space proved in [7] and the trace theorems proved in [1]. The main results of the paper are Theorems 6 and 7 which are stated in section 5 and proved in sections 6 and 7.

For the ease of the reader, the geometrical lemmas, which are crucial but technical, are proved in the Appendix at the end of the paper.

## 2 The Geometry

In this section, we define the geometry of fractal self-similar sets  $\Gamma^\infty$ , and ramified domains  $\Omega$  whose boundary contains  $\Gamma^\infty$ , see for example Figure 1.

### 2.1 The similitudes $f_1$ and $f_2$ and the self-similar set $\Gamma^\infty$

#### 2.1.1 Definitions and notations

Consider four real numbers  $a, \alpha, \beta, \theta$  such that  $0 < a < 1/\sqrt{2}$ ,  $\alpha > 0$ ,  $\beta > 0$  and  $0 < \theta < \pi/2$ . Let  $f_i$ ,  $i = 1, 2$  be the two similitudes in  $\mathbb{R}^2$  given by

$$\begin{aligned} f_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -\alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}, \\ f_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + a \begin{pmatrix} x_1 \cos \theta + x_2 \sin \theta \\ -x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}. \end{aligned} \tag{1}$$

The two similitudes have the same dilation ratio  $a$  and opposite angles  $\pm\theta$ . One can obtain  $f_2$  by composing  $f_1$  with the symmetry with respect to the axis  $\{x_1 = 0\}$ .

We denote by  $\Gamma^\infty$  the self-similar set associated to the similitudes  $f_1$  and  $f_2$ , i.e. the unique compact subset of  $\mathbb{R}^2$  such that

$$\Gamma^\infty = f_1(\Gamma^\infty) \cup f_2(\Gamma^\infty).$$

**Notations** We denote by

- $\mathcal{A}_n$  the set containing all the  $2^n$  mappings from  $\{1, \dots, n\}$  to  $\{1, 2\}$  also called strings of length  $n$  for  $n \geq 1$ ,
- $\mathcal{A}_0$  the set containing only one element called the empty string, that we agree to note  $\epsilon$ ,

- $\mathcal{A}$  the set defined by  $\mathcal{A} = \cup_{n \geq 0} \mathcal{A}_n$  containing the empty string and all the finite strings,
- $\mathcal{A}_\infty = \{1, 2\}^{\mathbb{N} \setminus \{0\}}$  the set of the sequences  $\sigma = (\sigma(i))_{i=1, \dots, \infty}$  with values  $\sigma(i) \in \{1, 2\}$ , i.e. the set of all infinite strings.

We will use the following notations:

- if  $n, m$  are nonnegative integers and  $\sigma \in \mathcal{A}_n, \sigma' \in \mathcal{A}_m$ , define:

$$\sigma\sigma' = (\sigma(1), \dots, \sigma(n), \sigma'(1), \dots, \sigma'(m)) \in \mathcal{A}_{n+m}, \quad (2)$$

if  $m = \infty$ , we define similarly  $\sigma\sigma' \in \mathcal{A}_\infty$ ,

- for  $n > 0, \sigma \in \mathcal{A}_n$ , and  $k \geq 0$ , we define

$$\sigma^k = \underbrace{\sigma\sigma \dots \sigma}_k \in \mathcal{A}_{nk}, \quad \sigma^\infty = \sigma\sigma \dots \sigma \dots \in \mathcal{A}_\infty, \quad (3)$$

- for  $\sigma, \tau \in \mathcal{A}$ , define:

$$\sigma|\tau = \{\sigma, \tau\} \subset \mathcal{A}, \quad (4)$$

- for  $\sigma \in \mathcal{A}$  and  $\mathcal{X} \subset \mathcal{A} \cup \mathcal{A}_\infty$ , define the set:

$$\sigma\mathcal{X} = \{\sigma\tau, \tau \in \mathcal{X}\}, \quad (5)$$

similarly, if  $\mathcal{X} \subset \mathcal{A}$ , define the set  $\mathcal{X}\sigma = \{\tau\sigma, \tau \in \mathcal{X}\}$ ,

- for  $\mathcal{X} \subset \mathcal{A}$  and  $k \in \mathbb{N}$ , introduce the sets:

$$\mathcal{X}^k = \{\sigma_1 \dots \sigma_k, \sigma_1, \dots, \sigma_k \in \mathcal{X}\}, \quad \mathcal{X}^\infty = \{\sigma_1\sigma_2 \dots \in \mathcal{A}_\infty, \forall i, \sigma_i \in \mathcal{X}\}, \quad \mathcal{X}^* = \bigcup_{k \in \mathbb{N}} \mathcal{X}^k.$$

**Example 1** The set  $(12|21)^\infty$  is the set of infinite strings  $\sigma \in \mathcal{A}_\infty$  such that  $\sigma(2k) \neq \sigma(2k-1)$  for all positive integers  $k$ .

For  $n \geq 0$ , the set  $(12|21)^n(1|2|\epsilon)$  is the set of strings  $\sigma \in \mathcal{A}_{2n} \cup \mathcal{A}_{2n+1}$  such that  $\sigma(2k) \neq \sigma(2k-1)$  for all integers  $k \in [1, n]$ .

The set  $(12|21)^*(1|2|\epsilon)$  is the set of strings  $\sigma \in \mathcal{A}$  such that  $\sigma \in (12|21)^n(1|2|\epsilon)$  for some  $n \geq 0$ .

We say that  $\sigma \in \mathcal{A}$  is a prefix of  $\tau \in \mathcal{A} \cup \mathcal{A}_\infty$  if  $\tau = \sigma\sigma'$  for some  $\sigma' \in \mathcal{A} \cup \mathcal{A}_\infty$ . For  $\sigma \in \mathcal{A}_n$  ( $n \in \mathbb{N}$ ) and  $k \leq n$ , we denote by  $\sigma|_k$  the only prefix of  $\sigma$  in  $\mathcal{A}_k$ :

$$\sigma|_k = (\sigma(1), \dots, \sigma(k)) \in \mathcal{A}_k. \quad (6)$$

Similarly, we say that  $\sigma' \in \mathcal{A} \cup \mathcal{A}_\infty$  is a suffix of  $\tau \in \mathcal{A} \cup \mathcal{A}_\infty$  if  $\tau = \sigma\sigma'$  for some  $\sigma \in \mathcal{A}$ .

For a positive integer  $n$  and  $\sigma \in \mathcal{A}_n$ , we define the similitude  $f_\sigma$  by

$$f_\sigma = f_{\sigma(1)} \circ \dots \circ f_{\sigma(n)}. \quad (7)$$

We also agree that  $f_\epsilon = \text{Id}$ . Similarly, if  $\sigma \in \mathcal{A}_\infty$ , define

$$f_\sigma = \lim_{n \rightarrow \infty} f_{\sigma(1)} \circ \dots \circ f_{\sigma(n)}. \quad (8)$$

For  $\sigma \in \mathcal{A}_\infty$ , the point  $f_\sigma(x)$  where  $x \in \mathbb{R}^2$  does not depend on  $x$ , and will be referred to as the limit point of the string  $\sigma$ . It should be noted that the set of all limit points of strings in  $\mathcal{A}$  is

exactly  $\Gamma^\infty$  (see for example [19]).

For  $\sigma \in \mathcal{A}$ , let the subset  $\Gamma^{\infty, \sigma}$  of  $\Gamma^\infty$  be defined by

$$\Gamma^{\infty, \sigma} = f_\sigma(\Gamma^\infty). \quad (9)$$

The definition of  $\Gamma^\infty$  implies that for all  $n > 0$ ,  $\Gamma^\infty = \bigcup_{\sigma \in \mathcal{A}_n} \Gamma^{\infty, \sigma}$ . We also define the set

$$\Xi^\infty = f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty). \quad (10)$$

**The critical contraction ratio  $a_\theta^*$**  The following theorem was stated by Mandelbrot *et al.* in [21] (a complete proof is given in [10]):

**Theorem 1** *For any  $\theta$ ,  $0 < \theta < \pi/2$ , there exists a unique positive number  $a_\theta^* < 1/\sqrt{2}$ , (which does not depend on  $(\alpha, \beta)$  see [7]) such that*

$$\begin{aligned} 0 < a < a_\theta^* &\Rightarrow \Xi^\infty = \emptyset &\Rightarrow \Gamma^\infty \text{ is totally disconnected,} \\ a = a_\theta^* &\Rightarrow \Xi^\infty \neq \emptyset &\Rightarrow \Gamma^\infty \text{ is connected, (from Th. 1.6.2 in [19]).} \end{aligned} \quad (11)$$

The critical parameter  $a_\theta^*$  is the unique positive root of the polynomial equation:

$$\sum_{i=0}^{\mathfrak{m}_\theta - 1} X^{i+2} \cos i\theta = \frac{1}{2}, \quad (12)$$

where

$$\mathfrak{m}_\theta \text{ is the smallest integer such that } \mathfrak{m}_\theta \theta \geq \pi/2. \quad (13)$$

**Remark 1** *From (12), it can be seen that  $\theta \mapsto a_\theta^*$  is a continuous and increasing function from  $(0, \pi/2)$  onto  $(1/2, 1/\sqrt{2})$  and that  $\lim_{\theta \rightarrow 0} a_\theta^* = 1/2$ .*

**Example 2** *We examine the cases  $\theta = \pi/4$  and  $\theta = \pi/5$  (see figure 1):*

- $\mathfrak{m}_{\pi/4} = 2$  and the critical parameter  $a_{\pi/4}^*$  is the unique positive solution of  $X^3 + \sqrt{2}X^2 - \sqrt{2}/2 = 0$ , i.e.  $a_{\pi/4}^* \simeq 0.593465$ ,
- $\mathfrak{m}_{\pi/5} = 3$ , and  $a_{\pi/5}^* \simeq 0.56658$ .

Hereafter, for a given  $\theta$ ,  $0 < \theta < \pi/2$ , we will write for brevity  $\mathfrak{m}$  instead of  $\mathfrak{m}_\theta$  and  $a^*$  instead of  $a_\theta^*$ , and we will only consider  $a$  such that  $0 < a \leq a^*$ .

### 2.1.2 Characterization of $\Xi^\infty$

We aim at characterizing  $\Xi^\infty$  defined in (10). We already know that  $\Xi^\infty \neq \emptyset$  if and only if  $a = a^*$ . Let us denote by  $\Lambda$  the vertical axis:  $\Lambda = \{x : x_1 = 0\}$  and by  $O$  the origin  $(0, 0)$ . Since  $f_1(\Gamma^\infty) = \Gamma^\infty \cap \{x_1 \leq 0\}$  and  $f_2(\Gamma^\infty) = \Gamma^\infty \cap \{x_1 \geq 0\}$ , we immediately see that  $\Xi^\infty = \Gamma^\infty \cap \Lambda$ . It can be observed (see [10] for the proof) that the sequences  $\sigma \in \mathcal{A}_\infty$  such that the limit point  $f_\sigma(O)$  of  $\sigma$  lies on  $\Lambda$  and that  $\sigma(1) = 1$  are characterized by the following property: for all  $n \leq 1$ , the truncated sequence  $\sigma_{\upharpoonright n}$  achieves the maximum of the abscissa of  $f_\eta(O)$  over all  $\eta \in \mathcal{A}_n$  such that  $\eta(1) = 1$ .

Let us make out two cases, according to the value of  $\mathfrak{m}$  defined in (13):

**The case when  $m\theta > \pi/2$**

**Proposition 1** *If  $m\theta > \pi/2$  and  $a = a^*$ , then  $\Xi^\infty$  contains the single point  $f_\sigma(O)$ , where*

$$\sigma = 12^{m+1}(12)^\infty \text{ or } \sigma = 21^{m+1}(21)^\infty, \quad (14)$$

*see paragraph 2.1.1 and Example 1 for the notations.*

**Proof.** For brevity, we skip the proof, which is available in [21, 10]. ■

**The case when  $m\theta = \pi/2$**

**Proposition 2** *If  $m\theta = \pi/2$  and  $a = a^*$ , then*

$$\Xi^\infty = \{f_\sigma(O), \sigma \in 12^{m+1}(12|21)^\infty\} = \{f_\sigma(O), \sigma \in 21^{m+1}(12|21)^\infty\} \quad (15)$$

**Proof.** For brevity, we skip the proof, which is available in [10]. ■

## 2.2 Ramified domains

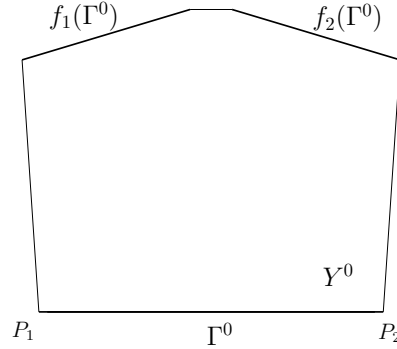
### 2.2.1 The construction

Call  $P_1 = (-1, 0)$  and  $P_2 = (1, 0)$  and  $\Gamma^0$  the line segment  $\Gamma^0 = [P_1P_2]$ . We impose that  $f_2(P_1)$ , and  $f_2(P_2)$  have positive coordinates, i.e. that

$$a \cos \theta < \alpha \text{ and } a \sin \theta < \beta. \quad (16)$$

We also impose that the open domain  $Y^0$  inside the closed polygonal line joining the points  $P_1, P_2, f_2(P_2), f_2(P_1), f_1(P_2), f_1(P_1), P_1$  in this order must be convex and hexagonal except if  $\theta = 0$ . With (16), this is true if and only if

$$(\alpha - 1) \sin \theta + \beta \cos \theta > 0. \quad (17)$$



The first cell  $Y^0$

Under assumptions (16) and (17), the domain  $Y^0$  is contained in the half-plane  $x_2 > 0$  and symmetric w.r.t. the vertical axis  $x_1 = 0$ .

We introduce  $K^0 = \overline{Y^0}$ . It is possible to glue together  $K^0, f_1(K^0)$  and  $f_2(K^0)$  and obtain a new polygonal domain, also symmetric with respect to the axis  $\{x_1 = 0\}$ . The assumptions (16) and (17) imply that  $Y^0 \cap f_1(Y^0) = \emptyset$  and  $Y^0 \cap f_2(Y^0) = \emptyset$ . We define the ramified open domain  $\Omega$  (see figure 1):

$$\Omega = \text{Interior} \left( \bigcup_{\sigma \in \mathcal{A}} f_\sigma(K^0) \right). \quad (18)$$

Note that  $\Omega$  is symmetric with respect to the axis  $x_1 = 0$ .

For a given  $\theta$ , with  $a^*$  defined as above, we shall make the following assumption on  $(\alpha, \beta)$ :

**Assumption 1** For  $0 < \theta < \pi/2$ , the parameters  $\alpha$  and  $\beta$  satisfy (17) and (16) for  $a = a^*$ , and are such that

- i) for all  $a, 0 < a \leq a^*$ , the sets  $Y^0, f_\sigma(Y^0), \sigma \in \mathcal{A}_n, n > 0$ , are disjoint
- ii) for all  $a, 0 < a < a^*$ ,  $f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) = \emptyset$
- iii) for  $a = a^*$ ,  $f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) \neq \emptyset$ .

**Remark 2** *Assumption 1 implies that if  $a = a^*$ , then  $f_1(\Omega) \cap f_2(\Omega) = \emptyset$ .*

The following theorem proved in [7] asserts that  $\forall \theta, 0 < \theta < \pi/2$ , there exists  $(\alpha, \beta)$  satisfying Assumption 1.

**Theorem 2** *If  $\theta \in (0, \pi/2)$ , then for all  $\alpha > a^* \cos \theta$ , there exists  $\bar{\beta} > 0$  such that  $\bar{\beta} > a^* \sin \theta$  and  $(\alpha - 1) \sin \theta + \bar{\beta} \cos \theta \geq 0$  and for all  $\beta \geq \bar{\beta}$ ,  $(\alpha, \beta)$  satisfies Assumption 1.*

On figure 1, we have shown two examples where Assumption 1 is satisfied. In the left part, we made the choice  $\theta = \frac{\pi}{5}$ , and in the right part, we chose  $\theta = \frac{\pi}{4}$  (see also example 2). Note the difference between the two cases: in the former case  $m_\theta \theta = \pi/2$  and the set  $\Xi^\infty$  defined in (10) is not countable whereas in the latter case,  $m_\theta \theta > \pi/2$  and the set  $\Xi^\infty$  is a singleton.

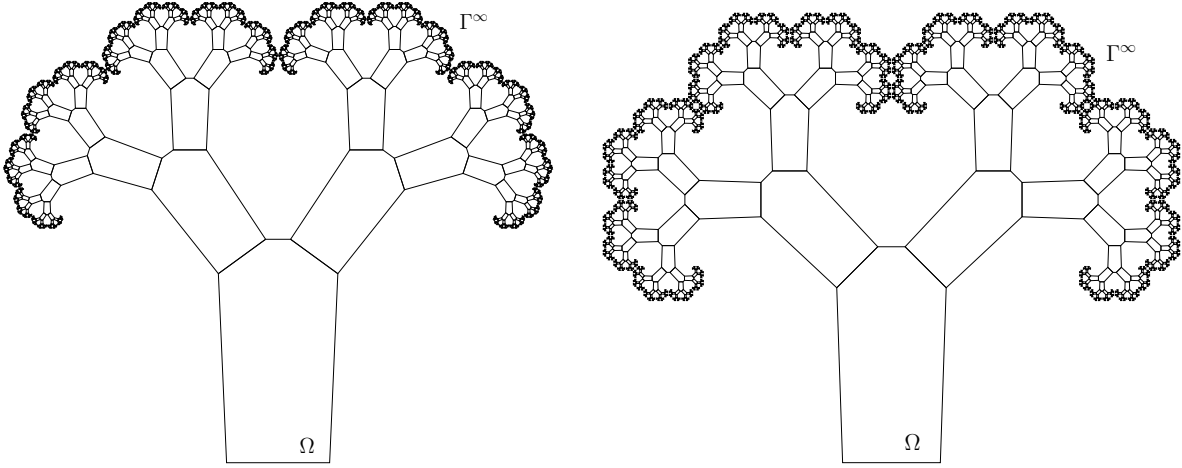


Figure 1: The ramified domain  $\Omega$  for  $\theta = \pi/5$  and  $\theta = \pi/4$  when  $a = a^*$ ,  $\alpha = 0.7$ ,  $\beta = 1.5$ .

### 2.2.2 The Moran condition

The Moran condition (or open set condition), see [23, 19], is that there should exist a nonempty bounded open subset  $\omega$  of  $\mathbb{R}^2$  such that  $f_1(\omega) \cap f_2(\omega) = \emptyset$  and  $f_1(\omega) \cup f_2(\omega) \subset \omega$ . For a given  $\theta \in (0, \pi/2)$ , let  $(\alpha, \beta)$  satisfy Assumption 1; for  $0 < a \leq a^*$ , the Moran condition is satisfied with  $\omega = \Omega$  because

- $f_1(\Omega) \cap f_2(\Omega) = \emptyset$ , which stems from point ii) in Assumption 1 if  $a < a^*$ , and from Remark 2 if  $a = a^*$ ;
- by construction of  $\Omega$ , we also have  $f_1(\Omega) \cup f_2(\Omega) \subset \Omega$ .

The Moran condition implies that the Hausdorff dimension of  $\Gamma^\infty$  is

$$\dim_H(\Gamma^\infty) = d \equiv -\log 2 / \log a,$$

see [23, 19]. Note that if  $a > 1/2$ , then  $d > 1$ . For instance, if  $\theta = \pi/4$  and  $a = a_{\pi/4}^*$ , then  $\dim_H(\Gamma^\infty) \simeq 1.3284371$ . It can be shown that if  $0 < \theta < \pi/2$ , we have  $0 < a \leq a^* < 1/\sqrt{2}$  and thus  $d < 2$ .

It can also be seen that if  $m_\theta = \pi/2$  and  $a = a^*$ , then the Hausdorff dimension of  $\Xi^\infty$  is  $d/2$ .



### 2.2.3 The self-similar measure $\mu$ and the spaces $W^{s,p}(\Gamma^\infty)$

To define traces on  $\Gamma^\infty$ , we recall the classical result on self-similar measures, see [11, 13] and [19] page 26:

**Theorem 3** *There exists a unique Borel regular probability measure  $\mu$  on  $\Gamma^\infty$  such that for any Borel set  $A \subset \Gamma^\infty$ ,*

$$\mu(A) = \frac{1}{2}\mu(f_1^{-1}(A)) + \frac{1}{2}\mu(f_2^{-1}(A)). \quad (19)$$

The measure  $\mu$  is called the *self-similar measure defined in the self-similar triplet*  $(\Gamma^\infty, f_1, f_2)$ . We define  $L^p(\Gamma^\infty)$ ,  $p \in [1, +\infty)$  as the space of the measurable functions  $v$  on  $\Gamma^\infty$  such that  $\int_{\Gamma^\infty} |v|^p d\mu < \infty$ , endowed with the norm  $\|v\|_{L^p(\Gamma^\infty)} = \left(\int_{\Gamma^\infty} |v|^p d\mu\right)^{1/p}$ . We also introduce  $L^\infty(\Gamma^\infty)$ , the space of essentially bounded functions with respect to the measure  $\mu$ . A Hilbertian basis of  $L^2(\Gamma^\infty)$  can be constructed with e.g. Haar wavelets.

We also define the space  $W^{s,p}(\Gamma^\infty)$  for  $s \in (0, 1)$  and  $p \in [1, \infty)$  as the space of functions  $v \in L^p(\Gamma^\infty)$  such that  $|v|_{W^{s,p}(\Gamma^\infty)} < \infty$ , where

$$|v|_{W^{s,p}(\Gamma^\infty)} = \left( \int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|v(x) - v(y)|^p}{|x - y|^{d+ps}} d\mu(x) d\mu(y) \right)^{\frac{1}{p}},$$

endowed with the norm  $\|v\|_{W^{s,p}(\Gamma^\infty)} = \|v\|_{L^p(\Gamma^\infty)} + |v|_{W^{s,p}(\Gamma^\infty)}$ .

### 2.2.4 Additional notations

We define the sets  $\Gamma^\sigma = f_\sigma(\Gamma^0)$  and  $\Gamma^N = \cup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma$ . The one-dimensional Lebesgue measure of  $\Gamma^\sigma$  for  $\sigma \in \mathcal{A}_N$  and of  $\Gamma^N$  are

$$|\Gamma^\sigma| = a^N |\Gamma^0| \text{ and } |\Gamma^N| = (2a)^N |\Gamma^0|.$$

We also introduce the sets  $\Omega^\sigma = f_\sigma(\Omega)$  for all  $\sigma \in \mathcal{A}$ .

We will sometimes use the notation  $\lesssim$  or  $\gtrsim$  to indicate that there may arise constants in the estimates, which are independent of the index  $n$  in  $\Gamma^n$ , or of the index  $\sigma$  in  $\Gamma^\sigma$  or  $\Gamma^{\infty,\sigma}$ . We may also write  $A \simeq B$  if  $A \lesssim B$  and  $B \lesssim A$ .

## 3 The space $W^{1,p}(\Omega)$

Hereafter, we take  $\theta$  in  $[0, \pi/2)$  and suppose that the parameters  $(\alpha, \beta)$  satisfy Assumption 1.

**Basic facts** For a real number  $p \geq 1$ , let  $W^{1,p}(\Omega)$  be the space of functions in  $L^p(\Omega)$  with first order partial derivatives in  $L^p(\Omega)$ . The space  $W^{1,p}(\Omega)$  is a Banach space with the norm

$\left(\|u\|_{L^p(\Omega)}^p + \|\frac{\partial u}{\partial x_1}\|_{L^p(\Omega)}^p + \|\frac{\partial u}{\partial x_2}\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}$ , see for example [8], p 60. Elementary calculus shows that  $\|u\|_{W^{1,p}(\Omega)} \equiv \left(\|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}$  is an equivalent norm, with  $\|\nabla u\|_{L^p(\Omega)}^p \equiv \int_\Omega |\nabla u|^p$  and  $|\nabla u| = \sqrt{|\frac{\partial u}{\partial x_1}|^2 + |\frac{\partial u}{\partial x_2}|^2}$ .

The spaces  $W^{1,p}(\Omega)$  as well as elliptic boundary value problems in  $\Omega$  have been studied in [6], with, in particular Poincaré inequalities and a Rellich compactness theorem. The same results in a similar but different geometry were proved by Berger [9] with other methods.

**Extension result in the case  $a < a^*$**  We first briefly discuss the less interesting case when  $a < a^*$  and the ramified domain  $\Omega$  is totally disconnected, see [12], Lemma 4.1 page 54. In this case, as seen in paragraph 2.2.1,  $\Omega$  is an  $(\varepsilon, \delta)$ -domain, and Theorem 1 in [14] yields a continuous extension operator from  $W^{1,p}(\Omega)$  to  $W^{1,p}(\mathbb{R}^2)$  for every  $p \in (1, \infty)$ .

**The case  $a = a^*$**  We will focus on that case in the rest of the present paper. As was seen in paragraph 2.2.1, the domain  $\Omega$  is not an  $(\varepsilon, \delta)$ -domain in this case, and the previous argument does not hold. However, it will be proved in Theorem 7 that the same result is true when  $1 < p < 2$  for angles  $\theta \in [0, \frac{\pi}{2})$  such that  $m\theta > \frac{\pi}{2}$ , and when  $1 < p < 2 - \frac{d}{2}$  for angles such that  $m\theta = \frac{\pi}{2}$ . It will also be seen (see Remark 5) that if  $p > 2$  in the first case, and if  $p > 2 - \frac{d}{2}$  in the second case, the extension result cannot hold.

We construct a sequence  $(\ell^n)_n$  of approximations of the trace operator: consider the sequence of linear operators  $\ell^n : W^{1,p}(\Omega) \rightarrow L^p(\Gamma^\infty)$ ,

$$\ell^n(v) = \sum_{\sigma \in \mathcal{A}_n} \left( \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right) \mathbf{1}_{f_\sigma(\Gamma^\infty)}, \quad (20)$$

where  $|\Gamma^\sigma|$  is the one-dimensional Lebesgue measure of  $\Gamma^\sigma$ . The following result was proved in [6].

**Proposition 3** *The sequence  $(\ell^n)_n$  converges in  $\mathcal{L}(W^{1,p}(\Omega), L^p(\Gamma^\infty))$  to an operator that we call  $\ell^\infty$ .*

In the following, we will sometimes note  $\ell^\infty(u)$  instead of  $\ell^\infty(u|_\Omega)$  for functions  $u \in W^{1,p}(\mathbb{R}^2)$ .

## 4 The spaces $JLip(t, p, q; 0; \Gamma^\infty)$ for $0 < t < 1$

In [16], A. Jonsson has introduced Haar wavelets of arbitrary order on self-similar fractal sets and has used these wavelets for constructing a family of Lipschitz spaces. These function spaces are named  $JLip(t, p, q; m; \mathcal{S})$ , where  $\mathcal{S}$  is the fractal set,  $t$  is a nonnegative real number,  $p, q$  are two real numbers not smaller than 1 and  $m$  is an integer ( $m$  is the order of the Haar wavelets used for constructing the space). Here  $J$  stands for jumps, since the considered functions may jump at some points of  $\mathcal{S}$ . If the fractal set  $\mathcal{S}$  is totally disconnected, then these spaces coincide with the Lipschitz spaces  $Lip(t, p, q; m; \mathcal{S})$  also introduced in [16]. The latter are a generalization of the more classical spaces  $Lip(t, p, q; \mathcal{S})$  introduced in [17] since  $Lip(t, p, q; [t]; \mathcal{S}) = Lip(t, p, q; \mathcal{S})$ . Note that  $Lip(t, p, q; [t]; \mathcal{S}) = B_t^{p,q}(\mathcal{S})$ , see [18]. We will focus on the case when  $\mathcal{S} = \Gamma^\infty$ ,  $m = 0$  and  $p = q$ , since this is sufficient for what follows.

### 4.1 Definition of the spaces $JLip(t, p, p; 0; \Gamma^\infty)$

The definition of  $JLip(t, p, p; 0; \Gamma^\infty)$  for  $p \in [1, \infty)$  presented below is adapted to the class of fractal sets  $\Gamma^\infty$  considered in the present paper. It was proved in [7] that this definition coincides with the original and more general one that was proposed in [16].

Consider a real number  $t$ ,  $0 < t < 1$ . Following [16], it is possible to characterize  $JLip(t, p, p; 0; \Gamma^\infty)$  by using expansions in the standard Haar wavelet basis on  $\Gamma^\infty$ . Consider the Haar mother wavelet  $g_0$  on  $\Gamma^\infty$ ,

$$g_0 = \mathbf{1}_{f_1(\Gamma^\infty)} - \mathbf{1}_{f_2(\Gamma^\infty)}, \quad (21)$$

and for  $n \in \mathbb{N}$ ,  $n > 0$ ,  $\sigma \in \mathcal{A}_n$ , let  $g_\sigma$  be given by

$$g_\sigma|_{\Gamma^{\infty,\sigma}} = 2^{n/2}g_0 \circ f_\sigma^{-1}, \text{ and } g_\sigma|_{\Gamma^\infty \setminus \Gamma^{\infty,\sigma}} = 0. \quad (22)$$

It is proved in [15] §5 that a function  $f \in L^p(\Gamma^\infty)$  can be expanded on the Haar basis as follows:

$$f = P_0f + \sum_{n \geq 0} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma g_\sigma, \quad (23)$$

where  $P_0f = \int_{\Gamma^\infty} f d\mu$ . For any function  $f \in L^p(\Gamma^\infty)$ , we define  $|f|_{JLip(t,p,p;0;\Gamma^\infty)}$  by:

$$|f|_{JLip(t,p,p;0;\Gamma^\infty)} = \left( \sum_{n \geq 0} 2^{n \frac{pt}{d}} 2^{n(\frac{p}{2}-1)} \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p \right)^{\frac{1}{p}}, \quad (24)$$

where the numbers  $\beta_\sigma$ ,  $\sigma \in \mathcal{A}$  are the coefficients of  $f$  in the Haar wavelet basis expansion given in (23).

**Definition 1** A function  $f \in L_\mu^p$  belongs to  $JLip(t,p,p;0;\Gamma^\infty)$  if and only if the norm

$$\|f\|_{JLip(t,p,p;0;\Gamma^\infty)} = |P_0f| + |f|_{JLip(t,p,p;0;\Gamma^\infty)} \quad (25)$$

is finite.

**Remark 3** An equivalent definition of  $JLip(t,p,p;0;\Gamma^\infty)$  can be given using projection of  $f$  on constants on  $\Gamma^{\infty,\sigma}$ , see [16, 7].

## 4.2 Characterization of the traces on $\Gamma^\infty$ of functions in $W^{1,p}(\Omega)$

The following theorem was proved in [7].

**Theorem 4** For a given  $\theta$ ,  $0 \leq \theta < \pi/2$ , if  $(\alpha, \beta)$  satisfies Assumption 1 and  $\Omega$  is constructed as in § 2.2.1 with  $1/2 \leq a \leq a^*$ , then for all  $p$ ,  $1 < p < \infty$ ,

$$\ell^\infty(W^{1,p}(\Omega)) = JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty). \quad (26)$$

## 4.3 $JLip$ versus Besov spaces on $\Gamma^\infty$

Define  $p_\theta^*$  by:

$$p_\theta^* = \begin{cases} 2 & \text{if } \theta \notin \{\frac{\pi}{2k}, k > 0\}, \\ 2 - d/2 & \text{if } \theta = \frac{\pi}{2k}, k > 0. \end{cases} \quad (27)$$

Note that  $p_\theta^*$  depends only on the dimension of the self-intersection of the fractal set  $\Gamma^\infty$ . The following theorem has been proved in [1].

**Theorem 5** • If  $a = a^*$  and  $1 < p < p_\theta^*$ , then

$$JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty) = W^{1 - \frac{2-d}{p}, p}(\Gamma^\infty), \quad (28)$$

• if  $a = a^*$  and  $p > p_\theta^*$ , then

$$JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty) \not\subset W^{1 - \frac{2-d}{p}, p}(\Gamma^\infty). \quad (29)$$

## 5 Extension results

In this section, we state and prove the two main extension results of this paper. The first one (Theorem 6) states that when  $p \in (1, p_\theta^*)$ , there exist liftings in  $W^{1,p}(\mathbb{R}^2)$  for functions in  $JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty)$ , that is the trace space of  $W^{1,p}(\Omega)$  on  $\Gamma^\infty$  (see (26)). The proof of this result relies on the construction of liftings for the Haar wavelets on  $\Gamma^\infty$ , and on Theorem 4.

**Remark 4** *Using Theorem 5 above and the trace theorem proved by Jonsson and Wallin in [17] stating that  $W^{1-\frac{2-d}{p}, p}(\Gamma^\infty) = W^{1,p}(\mathbb{R}^2)|_{\Gamma^\infty}$  for  $p \in (1, \infty)$ , it can be proved that*

$$JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty) = W^{1,p}(\mathbb{R}^2)|_{\Gamma^\infty}, \quad (30)$$

where we recall that the trace  $u|_{\Gamma^\infty}$  of a function  $u$  is meant in the classical sense (see [17]). It should be noted that Theorem 6 below differs from the previous result in that the trace is meant in the sense of  $\ell^\infty$ , which will be of particular importance, especially in the proof of Theorem 7 below.

The method proposed in the proof of Theorem 7 uses the lifting of Theorem 6 and the self-similar properties of the trace operator. Another choice could have been to work with the Whitney extension operator of (30), but we then could not have exploited the self-similar properties of the geometry as is done to prove Theorem 7.

In [5], Theorems 6 and 7 below are key ingredients in the proof that  $u|_{\Gamma^\infty} = \ell^\infty(u)$   $\mu$ -almost everywhere for all  $p > 1$  and  $u \in W^{1,p}(\Omega)$ .

The second result (Theorem 7) states that there exists a continuous extension operator from  $W^{1,p}(\Omega)$  to  $W^{1,p}(\mathbb{R}^2)$  when  $1 < p < p_\theta^*$ . The proof consists in the construction of the extension operator, with the help of the lifting introduced in Theorem 6.

**Theorem 6** 1. *If  $\theta \notin \frac{\pi}{2\mathbb{N}}$  and  $p \in (1, 2)$ , then there exists a continuous linear lifting operator  $\mathcal{E}$  from  $JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty)$  to  $W^{1,p}(\mathbb{R}^2)$  in the sense of  $\ell^\infty$ , i.e.*

$$\forall v \in JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty), \quad \ell^\infty((\mathcal{E}v)_\Omega) = v. \quad (31)$$

2. *If  $\theta \in \frac{\pi}{2\mathbb{N}}$  and  $p \in (1, 2 - \frac{d}{2})$ , then the conclusion remains true.*

We immediately deduce the following result for functions in  $W^{1,p}(\Omega)$  with  $1 < p < p_\theta^*$ .

**Corollary 1** *If  $p \in (1, p_\theta^*)$  and  $u \in W^{1,p}(\Omega)$ , then the function  $\bar{u} = \mathcal{E}\ell^\infty(u) \in W^{1,p}(\mathbb{R}^2)$  satisfies  $\ell^\infty(\bar{u}) = \ell^\infty(u)$ .*

We will deduce the main extension result of this paper:

**Theorem 7** 1. *If  $\theta \notin \frac{\pi}{2\mathbb{N}}$  and  $p \in (1, 2)$ , then there exists a continuous linear operator  $\mathcal{F}$  from  $W^{1,p}(\Omega)$  to  $W^{1,p}(\mathbb{R}^2)$  such that, for all  $u \in W^{1,p}(\Omega)$ ,  $\mathcal{F}u|_\Omega = u$ .*

2. *If  $\theta \in \frac{\pi}{2\mathbb{N}}$  and  $p \in (1, 2 - \frac{d}{2})$ , then the extension result remains true.*

*In other words, if  $p \in (1, p_\theta^*)$ , then  $\Omega$  has the  $W^{1,p}$ -extension property.*

**Remark 5** *The extension result of Theorem 7 is sharp in the following sense. As was seen in Remark 4, it is proved a posteriori in [5] that the trace operator  $\ell^\infty$  coincides with the trace operator introduced by Jonsson and Wallin in [17]  $\mu$ -almost everywhere. Therefore, if*

$\Omega$  is a  $W^{1,p}$ -extension domain for some  $p > p_\theta^*$ , then, by the trace theorem in [17] (p.182),  $\ell^\infty(W^{1,p}(\Omega)) = W^{1-\frac{2-d}{p},p}(\Gamma^\infty)$ , which contradicts (29).

In the case when  $\theta \notin \frac{\pi}{2\mathbb{N}}$ , we have  $p_\theta^* = 2$ , and we can conclude that  $\Omega$  does not have the  $W^{1,p_\theta^*}$ -extension property: if it did, then Koskela's theorem in [20] (Theorem B) would imply that  $\Omega$  has the  $W^{1,p}$ -extension property for all  $p > p_\theta^*$ . The case  $\theta \in \frac{\pi}{2\mathbb{N}}$  is open.

## 6 Proof of Theorem 6

### 6.1 Proof of point 1 in Theorem 6

Recall that in this case,  $\theta \notin \{\frac{\pi}{2k}, k > 0\}$ , and  $\mathbf{m}\theta > \frac{\pi}{2}$ , where  $\mathbf{m}$  was introduced in (13).

We start by lifting the Haar wavelets on  $\Gamma^\infty$  into functions in  $W^{1,p}(\mathbb{R}^2)$ ,  $1 < p < \infty$ . This will yield a natural lifting for functions in  $JLip(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty)$ , defined as the lifting of their expansion in the Haar wavelets basis.

#### 6.1.1 Lifting of the Haar wavelets

In this section, we define liftings  $\bar{g}_\sigma$  of the Haar wavelets  $g_\sigma$ ,  $\sigma \in \mathcal{A}$ , such that  $\bar{g}_\sigma \in W^{1,p}(\mathbb{R}^2)$  for all  $p \in (1, \infty)$ , and the pairwise intersections of the sets  $\text{supp } \nabla \bar{g}_\sigma$  ( $\sigma \in \mathcal{A}$ ) are contained in some cones centered at the points  $f_\eta(A)$  ( $\eta \in \mathcal{A}$ ), where  $A$  is the single point contained in  $f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty)$ .

More precisely, define  $C$  to be the vertical cone centered at  $A$  with angle  $\varphi_0 < \min(\mathbf{m}\theta - \pi/2, \pi/2 - (\mathbf{m} - 1)\theta)$  small enough so that  $C$  does not intersect  $f_1(\Omega)$  or  $f_2(\Omega)$ . We impose that the liftings  $\bar{g}_\sigma$ ,  $\sigma \in \mathcal{A}$  should satisfy the following condition: if  $\sigma \neq \tau$ , then

$$\text{supp } \nabla \bar{g}_\sigma \cap \text{supp } \nabla \bar{g}_\tau \subset \bigcup_{\eta \in \mathcal{A}} f_\eta(C). \quad (32)$$

Proposition 5 below will imply in particular that this condition is verified.

We proceed in three steps: we successively lift the constants, the mother Haar wavelet, and the other Haar wavelets.

**Lifting of the constants** We will introduce a lifting  $\chi$  of the constant function 1 on  $\Gamma^\infty$  that will allow to define liftings for the Haar wavelets by self-similarity.

The proof of Lemma 3 in [1] (see especially Figure 3 in [1]) can be easily modified to obtain the existence of a constant  $c > 0$  such that for all  $\sigma \in \mathcal{A}_n$  ( $n > 0$ ) with  $\sigma \notin \mathcal{B}$ ,

$$d(f_\sigma(\Omega), C) > ca^n, \quad (33)$$

where

$$\mathcal{B} = \{\sigma \in \mathcal{A}, \sigma \text{ is a prefix of } 12^{\mathbf{m}+1}(12)^\infty \text{ or } 21^{\mathbf{m}+1}(21)^\infty\}, \quad (34)$$

see paragraph 2.1.1 for the notations. Recall that the elements of  $\mathcal{B}$  are those strings  $\sigma \in \mathcal{A}$  such that  $d(f_\sigma(\Omega), \Lambda) = 0$  where  $\Lambda$  is the axis given by  $\{x_1 = 0\}$  (cf. Proposition 1).

Define  $\mathcal{H}$  to be the horizontal line tangent to the upper part of  $\Gamma^\infty$ . Write  $\delta = d(C \cap \mathcal{H}, \Omega)$ , note that  $\delta > 0$ .

We consider a smooth compactly supported function  $\chi$  on  $\mathbb{R}^2$  such that  $\chi = 1$  on a neighborhood

of the closure of the ramified domain  $\Omega$ , and such that  $\chi$  is symmetric with respect to the axis  $\Lambda$ . Write  $d_1 = d(\Omega, \text{supp } \nabla \chi)$  and  $d_2 = \sup\{d(x, \Omega), x \in \text{supp } \chi\}$ . We impose that

$$d_2 < \delta \tag{35}$$

$$d_2 < c \tag{36}$$

$$d_1 > ad_2, \tag{37}$$

see Figure 2.

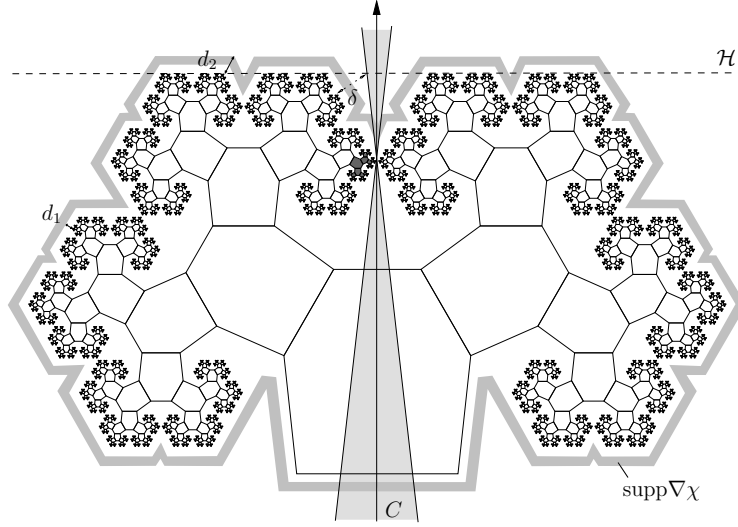


Figure 2: The function  $\chi$

The condition (37) will play an important role in the proof that (32) is fulfilled. Conditions (36) and (35) will help dealing with the pairwise intersections of the sets  $\text{supp } \nabla \bar{g}_\sigma$  within the cones.

**Lifting of the Haar mother wavelet** We introduce the polar coordinates  $(r, \varphi)$  centered at the point  $A$  such that the vertical half-line starting from  $A$  and pointing up is given by  $\{\varphi = 0\}$ . Define the function  $\psi$  as follows: if  $(r, \varphi) \in \mathbb{R}_+^* \times (-\pi, \pi]$ ,

$$\psi(r, \varphi) = \begin{cases} \frac{1}{2} \left(1 + \frac{\varphi}{\varphi_0}\right) & \text{if } |\varphi| \leq \varphi_0, \\ 1 & \text{if } \varphi_0 < \varphi < \pi - \varphi_0, \\ \frac{1}{2} \left(1 + \frac{\pi - \varphi}{\varphi_0}\right) & \text{if } \pi - |\varphi| \leq \varphi_0, \\ 0 & \text{if } -\pi + \varphi_0 < \varphi < -\varphi_0. \end{cases} \tag{38}$$

Note that the cone  $C$  is the support of  $\nabla \psi$ .

Define the lifting  $\bar{g}_0$  of the Haar mother wavelet by:

$$\bar{g}_0 = \psi \cdot (\chi \circ f_1^{-1}) - (1 - \psi) \cdot (\chi \circ f_2^{-1}). \tag{39}$$

Note that  $\bar{g}_0 \in W^{1,p}(\mathbb{R}^2)$  if and only if  $p < 2$ .

**Lifting of the Haar wavelets** We use the function  $\bar{g}_0$  and the self-similarity to define the liftings of the other Haar wavelets. We first define the natural lifting  $\tilde{g}_\sigma$  of  $g_\sigma$ , for  $\sigma \in \mathcal{A}_n$  by

$$\tilde{g}_\sigma = 2^{\frac{n}{2}} \bar{g}_0 \circ f_\sigma^{-1}.$$

Note that the functions  $\tilde{g}_\sigma$ ,  $\sigma \in \mathcal{A}$  do not satisfy condition (32) (take for example  $\sigma = 1$  and  $\tau = 2$ ). Hence, we will define cut-off functions whose gradients are supported in the cones  $f_\tau(C)$ .

Take  $\sigma \in \mathcal{A} \setminus \{\epsilon\}$ , we define for any prefix  $\tau \in \mathcal{A}_k$  of  $\sigma$  such that  $\tau \neq \sigma$ ,

$$\gamma_\tau^\sigma = \mathbb{1}_{\sigma(k+1)=1} \psi \circ f_\tau^{-1} + \mathbb{1}_{\sigma(k+1)=2} (1 - \psi) \circ f_\tau^{-1}. \quad (40)$$

Note that the function  $\gamma_\tau^\sigma$  is 1 on one connected component of  $\mathbb{R}^2 \setminus f_\tau(C)$ , and 0 on the other. This definition is based on the observation that for all prefix  $\tau \in \mathcal{A}_k$  of  $\sigma$  such that  $\tau \neq \sigma$ ,  $\Omega^\sigma \subset f_\tau(\{x_1 < 0\})$  if  $\sigma(k+1) = 1$ , and  $\Omega^\sigma \subset f_\tau(\{x_1 > 0\})$  if  $\sigma(k+1) = 2$ .

For  $\sigma \in \mathcal{A}$ , we introduce the set  $\mathcal{M}(\sigma)$  of all those prefixes  $\tau \in \mathcal{A}$  of  $\sigma$  such that  $f_\tau(A) \in \Gamma^{\infty, \sigma}$ . It is easily checked that

$$\mathcal{M}(\sigma) = \{\tau \in \mathcal{A}, \sigma = \tau\sigma', \sigma' \in \mathcal{B}\}, \quad (41)$$

where  $\mathcal{B}$  is defined in (34). If  $\tau \in \mathcal{M}(\sigma) \setminus \{\sigma\}$ , then the function  $\tilde{g}_\sigma$  needs to be multiplied by the cut-off function  $\gamma_\tau^\sigma$ .

**Remark 6** 1. If  $\sigma \in \mathcal{B}$ , then  $\epsilon \in \mathcal{M}(\sigma)$ .

2. For all  $n > 0$  and  $\sigma \in \mathcal{A}_n$ , one has  $\sigma, \sigma_{\uparrow n-1} \in \mathcal{M}(\sigma)$ , since the empty string and the string  $(\sigma(n))$  belong to  $\mathcal{B}$ .

We can now define the cut off lifting  $\bar{g}_\sigma$  of the Haar wavelet  $g_\sigma$ ,  $\sigma \in \mathcal{A}_n$ :

$$\bar{g}_\sigma = \left( \prod_{\tau \in \mathcal{M}(\sigma), \tau \neq \sigma} \gamma_\tau^\sigma \right) \tilde{g}_\sigma. \quad (42)$$

**Example** We present an example where  $\theta = \frac{\pi}{3}$  (hence  $\mathfrak{m} = 2$ ), and  $\sigma = 12^3 12$ . Therefore,  $\mathcal{M}(\sigma) = \{\epsilon, 12^3, 12^3 1, 12^3 12\}$ .

The gray area in Figure 3 shows the support of  $\nabla \bar{g}_\sigma$ . We have only represented the domain  $\Omega^\sigma$ , which corresponds to the small area in dark gray in Figure 2.

In what follows, we will need a uniform bound on the cardinal of  $\mathcal{M}(\sigma)$ ,  $\sigma \in \mathcal{A}$  :

**Lemma 1** For all  $\sigma \in \mathcal{A}$ ,

$$\#\mathcal{M}(\sigma) \leq 4. \quad (43)$$

**Proof.** Take  $n > 0$  and  $\sigma \in \mathcal{A}_n$ . If  $n \leq 2$ , then the result is clear. Suppose  $n > 2$ , we first note that  $\sigma_{\uparrow n-1}, \sigma \in \mathcal{M}(\sigma)$  by point 2 in Remark 6. Let us look for elements of  $\mathcal{M}(\sigma)$  distinct from  $\sigma_{\uparrow n-1}$  and  $\sigma$ .

First assume that  $\sigma(n) = \sigma(n-1)$ , suppose for example  $\sigma(n) = 2$ . Then, any suffix  $\sigma'$  of  $\sigma$  such that  $\sigma' \in \mathcal{B}$  is of the form  $12^k$  with  $k \leq \mathfrak{m} + 1$ . If there were two of them, then one of them would be a suffix of the other, which is impossible. Therefore, in this case,  $\#\mathcal{M}(\sigma) \leq 3$ .

If  $\sigma(n) \neq \sigma(n-1)$ , then  $(\sigma(n-1), \sigma(n)) \in \mathcal{B}$ , which implies that  $\sigma_{\uparrow n-2} \in \mathcal{M}(\sigma)$ . Let us look for a string  $\sigma' \in \mathcal{M}(\sigma)$  such that  $\sigma' \in \mathcal{A}_k$  with  $k > 2$ . Suppose for example  $\sigma(n) = 2$ . Then  $\sigma'$  must be of the form  $12^{\mathfrak{m}+1}(12)^l$  or  $21^{\mathfrak{m}+1}(21)^l 2$ , for some  $l > 0$ . As for the previous case, were there two such strings, one of them would be a suffix of the other, which is impossible. Hence, in this case,  $\#\mathcal{M}(\sigma) \leq 4$ . ■

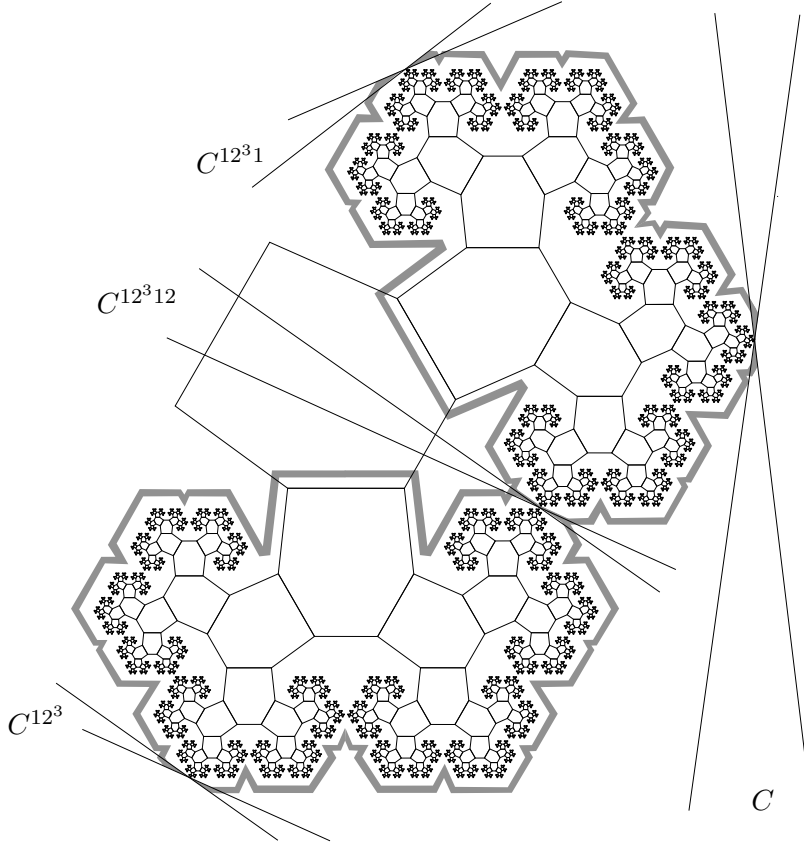


Figure 3: The support of  $\nabla \bar{g}_\sigma$  (represented by the gray area) for  $\theta = \frac{\pi}{3}$  and  $\sigma = 12^{m+1}12 = 12^3 12$ . In this case,  $\mathcal{M}(\sigma) = \{\epsilon, 12^3, 12^3 1, 12^3 12\}$ .

**Proposition 4** For  $\sigma \in \mathcal{A}_n$  and  $p < 2$ ,

$$\|\nabla \bar{g}_\sigma\|_{L^p(\mathbb{R}^2)}^p \simeq 2^{n(\frac{p}{2} + \frac{2-p}{d})} \quad (44)$$

**Proof.** First, we note that  $\|2^{\frac{n}{2}} \nabla(\bar{g}_0 \circ f_\sigma^{-1})\|_{L^p(\mathbb{R}^2)}^p = 2^{\frac{np}{2}} a^{n(2-p)} \|\nabla \bar{g}_0\|_{L^p(\mathbb{R}^2)}^p \simeq 2^{n(\frac{p}{2} + \frac{2-p}{d})}$ .

The other terms to consider are of the form  $\|2^{\frac{n}{2}} \bar{g}_0 \circ f_\sigma^{-1} \cdot \nabla(\psi \circ f_\tau^{-1})\|_{L^p(\mathbb{R}^2)}^p$ , where  $\tau \in \mathcal{M}(\sigma) \setminus \{\sigma\}$ . One has

$$\begin{aligned} \|2^{\frac{n}{2}} \bar{g}_0 \circ f_\sigma^{-1} \cdot \nabla(\psi \circ f_\tau^{-1})\|_{L^p(\mathbb{R}^2)}^p &\leq 2^{\frac{np}{2}} \|\nabla(\psi \circ f_\tau^{-1})\|_{L^p(f_\sigma(\text{supp } \bar{g}_0))}^p \\ &= 2^{\frac{np}{2}} a^{k(2-p)} \|\nabla \psi\|_{L^p(f_{\sigma'}(\text{supp } \bar{g}_0))}^p, \end{aligned} \quad (45)$$

where  $\sigma = \tau\sigma'$ ,  $\tau \in \mathcal{A}_k$  and  $\sigma' \in \mathcal{B}$  (see (41)). Denote  $R = \text{Diam supp } \bar{g}_0$ , note that  $\text{Diam } f_{\sigma'}(\text{supp } \bar{g}_0) = a^{n-k}R$ . Since  $\sigma' \in \mathcal{B}$ , one has  $A \in f_{\sigma'}(\text{supp } \bar{g}_0)$ . Switching to polar coordinates centered at the point  $A$ , we get

$$\|\nabla \psi\|_{L^p(f_{\sigma'}(\text{supp } \bar{g}_0))}^p \lesssim \int_{\Phi} \int_0^{a^{n-k}R} \frac{1}{r^p} r \, dr \, d\varphi \simeq \int_0^{a^{n-k}R} \frac{dr}{r^{p-1}} \simeq a^{(2-p)(n-k)},$$

where  $\Phi = (-\varphi_0, \varphi_0) \cup (\pi - \varphi_0, \pi + \varphi_0)$ . Together with (45), this achieves the proof since  $a^d = 2$ .

■



From Proposition 4, we deduce that  $\bar{g}_\sigma \in W^{1,p}(\mathbb{R}^2)$  for  $p < 2$ .

### 6.1.2 Geometrical results

The following geometrical results will be crucial in the proof of Theorem 6. The proofs of these results rely on simple but technical geometrical arguments and have been postponed to the Appendix for the ease of the reader.

We introduce the truncated cones  $S = C \cap \text{supp } \bar{g}_0$  and  $S^\tau = f_\tau(S)$  for  $\tau \in \mathcal{A}$ . Define  $\mathcal{S} = \bigcup_{\tau \in \mathcal{A}} S^\tau$  to be the union of these truncated cones. We also define  $F = \text{supp } \nabla \bar{g}_0 \setminus S$ , and  $F^\tau = f_\tau(F)$  for  $\tau \in \mathcal{A}$ .

**Remark 7** *Condition (35) implies by a simple geometric argument that  $S$  lies in the convex hull of the ramified domain  $\Omega$ .*

Note that, for  $\sigma \in \mathcal{A}$ ,

$$\text{supp } \nabla \bar{g}_\sigma \subset F^\sigma \cup \left( \bigcup_{\tau \in \mathcal{M}(\sigma)} S^\tau \right). \quad (46)$$

Proposition 5 below states a stronger version of condition (32).

**Proposition 5** *If  $\sigma, \tau \in \mathcal{A}$  and  $\sigma \neq \tau$ , then*

$$(\text{supp } \nabla \bar{g}_\sigma \setminus \mathcal{S}) \cap (\text{supp } \nabla \bar{g}_\tau \setminus \mathcal{S}) = \emptyset. \quad (47)$$

Proposition 6 below somehow justifies the definition of the cut-off functions  $\gamma_\tau^\sigma$ .

**Proposition 6** *If  $\sigma \in \mathcal{A}$  and  $\tau \in \mathcal{M}(\sigma) \setminus \{\sigma\}$ , then*

$$\bar{g}_\sigma = 2^{\frac{n}{2}} \bar{g}_0 \circ f_\sigma^{-1} \gamma_\tau^\sigma \text{ on } S^\tau. \quad (48)$$

Propositions 7 and 8 below describe the case when, for a given  $\tau \in \mathcal{A}$ ,  $\nabla(\bar{g}_0 \circ f_\sigma^{-1})$  or  $\nabla \bar{g}_\sigma$  are not identically zero on  $S^\tau$ .

**Proposition 7** *There exists a constant  $C > 0$  such that for all  $\tau \in \mathcal{A}$  and  $x \in S^\tau$ ,*

$$\#\{\sigma \in \mathcal{A}, \tau \in \mathcal{M}(\sigma), \nabla(\bar{g}_0 \circ f_\sigma^{-1})(x) \neq 0\} \leq C.$$

**Proposition 8** *If  $\sigma, \tau \in \mathcal{A}$  and  $\tau \notin \mathcal{M}(\sigma)$ , then  $\nabla \bar{g}_\sigma \equiv 0$  on  $S^\tau$ .*

In particular, if  $\nabla \bar{g}_\sigma \neq 0$  on  $S^\tau$ , then  $\tau$  is a prefix of  $\sigma$ .

### 6.1.3 Proof of point 1 in Theorem 6

We start by stating a lemma that will prove useful in the proof of Theorem 6.

**Lemma 2 (discrete Hardy inequality, [17], page 121, Lemma 3)** *If  $p \geq 1$ , for any  $\gamma > 0$  and  $a \in (0, 1)$ , there exists a constant  $C$  such that, for any sequence of positive real numbers  $(c_k)_{k \in \mathbb{N}}$ ,*

$$\sum_{n \in \mathbb{N}} a^{\gamma n} \left( \sum_{k \leq n} c_k \right)^p \leq C \sum_{n \in \mathbb{N}} a^{\gamma n} c_n^p. \quad (49)$$

**Proof of point 1.** We first suppose  $\langle v \rangle_{\Gamma^\infty} = \int_{\Gamma^\infty} v \, d\mu = 0$ . The function  $v \in J\text{Lip}(1 - \frac{2-d}{p}, p, p, 0; \Gamma^\infty)$  then reads  $v = \sum_n \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma g_\sigma$  where the  $\beta_\sigma$  are the coefficients of  $v$  in the Haar wavelet basis of  $\Gamma^\infty$ . We introduce the lifting  $\mathcal{E}v$  of  $v$  defined on  $\mathbb{R}^2$  by:

$$\mathcal{E}v = \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma \bar{g}_\sigma. \quad (50)$$

Recall that  $\mathcal{S}$  is the union of all the truncated cones  $S^\tau$ ,  $\tau \in \mathcal{A}$ . By Proposition 5,

$$\begin{aligned} \|\nabla(\mathcal{E}v)\|_{L^p(\mathbb{R}^2 \setminus \mathcal{S})}^p &= \int_{\mathbb{R}^2 \setminus \mathcal{S}} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma \nabla \bar{g}_\sigma(x) \right|^p dx = \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p \int_{\text{supp } \bar{g}_\sigma \setminus \mathcal{S}} |\nabla \bar{g}_\sigma(x)|^p dx \\ &\lesssim \sum_{n \in \mathbb{N}} 2^{n(\frac{p}{2} + \frac{p}{d} - \frac{2}{p})} \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p \\ &= \|v\|_{J\text{Lip}(1 - \frac{2-d}{p}, p, p, 0; \Gamma^\infty)}^p. \end{aligned}$$

On the other hand,

$$\|\nabla(\mathcal{E}v)\|_{L^p(\mathcal{S})}^p = \int_{\mathcal{S}} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma \nabla \bar{g}_\sigma(x) \right|^p dx = \sum_{\tau \in \mathcal{A}} \int_{S^\tau} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma \nabla \bar{g}_\sigma(x) \right|^p dx.$$

Take  $k \in \mathbb{N}$  and  $\tau \in \mathcal{A}_k$ . By Proposition 8, if  $\sigma \in \mathcal{A}_n$  ( $n \in \mathbb{N}$ ) is such that  $\nabla \bar{g}_\sigma \neq 0$  in  $S^\tau$ , then  $\tau \in \mathcal{M}(\sigma)$ . Therefore, by Proposition 6,  $\gamma_{\tau'}^\sigma \equiv 1$  in  $S^\tau$  for any  $\tau' \in \mathcal{M}(\sigma) \setminus \{\sigma\}$ , which implies that  $\bar{g}_\sigma$  coincides with  $2^{\frac{n}{2}} \bar{g}_0 \circ f_\sigma^{-1} \gamma_\tau^\sigma$  in  $S^\tau$ . Therefore,

$$\begin{aligned} \int_{S^\tau} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma \nabla \bar{g}_\sigma(x) \right|^p dx &= \int_{S^\tau} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n, \tau \in \mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \nabla ((\bar{g}_0 \circ f_\sigma^{-1}) \cdot \gamma_\tau^\sigma)(x) \right|^p dx \\ &\lesssim I_1^\tau + I_2^\tau, \end{aligned}$$

where

$$\begin{aligned} I_1^\tau &= \int_{S^\tau} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n, \tau \in \mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \nabla (\bar{g}_0 \circ f_\sigma^{-1})(x) \right|^p dx \\ I_2^\tau &= \int_{S^\tau} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n, \tau \in \mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(x) \nabla \gamma_\tau^\sigma(x) \right|^p dx. \end{aligned} \quad (51)$$

Let us first consider  $I_1^\tau$ . By Proposition 7, one has:

$$\begin{aligned} I_1^\tau &\leq C^{p-1} \sum_{n \in \mathbb{N}} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \tau \in \mathcal{M}(\sigma)}} 2^{\frac{np}{2}} |\beta_\sigma|^p \int_{S^\tau} |\nabla (\bar{g}_0 \circ f_\sigma^{-1})(x)|^p dx \\ &\leq C^{p-1} \sum_{n \in \mathbb{N}} 2^{\frac{np}{2}} a^{(2-p)n} \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p \int_{S^\tau} |\nabla \bar{g}_0(x)|^p dx, \end{aligned} \quad (52)$$

where  $C$  is the constant in Proposition 7. Therefore,

$$\begin{aligned} \sum_{\tau \in \mathcal{A}} I_1^\tau &\lesssim \sum_{n \in \mathbb{N}} 2^{\frac{np}{2}} a^{(2-p)n} \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p \sum_{\tau \in \mathcal{A}} \int_{S^\tau} |\nabla \bar{g}_0(x)|^p dx \\ &\leq \sum_{n \in \mathbb{N}} 2^{\frac{np}{2}} a^{(2-p)n} \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p \int_{\mathbb{R}^2} |\nabla \bar{g}_0(x)|^p dx \\ &\lesssim \|v\|_{J\text{Lip}(1 - \frac{2-d}{p}, p, p, 0; \Gamma^\infty)}^p. \end{aligned} \quad (53)$$

We are left with dealing with  $I_2^\tau$ . Denote  $\Phi = (-\varphi_0, \varphi_0) \cup (\pi - \varphi_0, \pi + \varphi_0)$ . We resort to a polar change of variables centered at the point  $f_\tau(A)$  such that the vertical half-line starting from the point  $f_\tau(A)$  and pointing up is given by  $\{\varphi = 0\}$ :

$$I_2^\tau \lesssim \int_0^\infty \int_\Phi \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n, \tau \in \mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(r e^{i\varphi}) \right|^p r^{1-p} d\varphi dr, \quad (54)$$

since for  $r > 0$  and  $\varphi \in \Phi$ ,  $|\nabla \gamma_\tau^\sigma(r e^{i\varphi})| \lesssim \frac{1}{r}$ .

Define  $R = \text{Diam supp } \bar{g}_0$ . If  $\tau \in \mathcal{M}(\sigma)$ , then  $f_\tau(A) \in \overline{\Omega^\sigma} \subset \text{supp } \bar{g}_0 \circ f_\sigma^{-1}$ , by the definition of  $\mathcal{M}(\sigma)$  in (41). Therefore, if  $\text{supp } \bar{g}_0 \circ f_\sigma^{-1} \cap C(f_\tau(A), r) \neq \emptyset$  where  $C(f_\tau(A), r)$  is the circle centered at  $f_\tau(A)$  with radius  $r$ , then  $r \leq a^n R$ , i.e.  $n \leq N_r$  where  $N_r = \frac{\log r/R}{\log a}$ . Hence

$$\begin{aligned} I_2^\tau &\lesssim \int_0^{a^k R} \int_\Phi \left| \sum_{n=k}^{[N_r]} \sum_{\sigma \in \mathcal{A}_n, \tau \in \mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(r e^{i\varphi}) \right|^p r^{1-p} d\varphi dr \\ &\lesssim \int_0^{a^k R} \left( \sum_{n=k}^{[N_r]} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \tau \in \mathcal{M}(\sigma)}} 2^{\frac{n}{2}} |\beta_\sigma| \right)^p r^{1-p} dr. \end{aligned}$$

Recall that if  $\tau \in \mathcal{M}(\sigma)$ , then  $\sigma$  is of the form  $\tau\sigma'$  with  $\sigma' \in \mathcal{B}$ . Consequently,

$$I_2^\tau \lesssim \int_0^{a^k R} \left( \sum_{n=0}^{[N_r^k]} \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p r^{1-p} dr,$$

where  $N_r^k = N_r - k = \frac{\log(r/(a^k R))}{\log a}$ . Therefore, the change of variable  $\rho = N_r^k$  yields:

$$\begin{aligned} I_2^\tau &\lesssim \int_0^\infty a^{(\rho+k)(2-p)} \left( \sum_{n=0}^{[\rho]} \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p d\rho \\ &\leq \sum_{m=0}^\infty a^{(m+k)(2-p)} \left( \sum_{n=0}^m \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p, \end{aligned}$$

since  $\rho \mapsto a^{(\rho+k)(2-p)} \left( \sum_{n=0}^{[\rho]} \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n}{2}} |\beta_{\tau\sigma'}| \right)$  is increasing. Then, by the Hardy inequality of Lemma 2,

$$I_2^\tau \lesssim \sum_{m=0}^\infty a^{(m+k)(2-p)} 2^{\frac{(m+k)p}{2}} \sum_{\sigma' \in \mathcal{B}} |\beta_{\tau\sigma'}|^p = \sum_{m=k}^\infty a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma \in \mathcal{A}_m \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p. \quad (55)$$

Therefore, we obtain:

$$\begin{aligned} \sum_{\tau \in \mathcal{A}} I_2^\tau &\lesssim \sum_{k \in \mathbb{N}} \sum_{\tau \in \mathcal{A}_k} \sum_{m=k}^\infty a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma \in \mathcal{A}_m \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p \\ &= \sum_{m \in \mathbb{N}} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma \in \mathcal{A}_m} \sum_{k=0}^m \sum_{\substack{\tau \in \mathcal{A}_k \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p \\ &\lesssim \sum_{m \in \mathbb{N}} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma \in \mathcal{A}_m} |\beta_\sigma|^p, \end{aligned} \quad (56)$$

since  $\#\mathcal{M}(\sigma) \leq 4$  by Lemma 1, which shows that  $\sum_{\tau \in \mathcal{A}} I_2^\tau \lesssim \|v\|_{JLip(1-\frac{2-d}{p}, p, 0; \Gamma^\infty)}^p$ .

Finally, if  $\langle v \rangle_{\Gamma^\infty} \neq 0$ , then we get the desired result by taking  $\mathcal{E}v = \langle v \rangle_{\Gamma^\infty} \chi + \mathcal{E}(v - \langle v \rangle_{\Gamma^\infty})$ . ■

## 6.2 Proof of point 2 in Theorem 6

We will proceed in the same manner as we did in the proof of point 1 in Theorem 6. The extensions of the Haar wavelets will differ since the set  $\Xi^\infty = f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty)$  is now infinite.

### 6.2.1 Lifting of the Haar wavelets

Define  $\kappa_1 = d(\text{conv}(\Omega^{11}), \mathcal{H})$ , where we recall that  $\mathcal{H}$  is the line tangent to the upper part of  $\Gamma^\infty$  (see Figure 2). It has been proved in [1] that  $\kappa_1 > 0$ . It has also been seen in [1] that there exists a constant  $\kappa_2$  such that for all  $\sigma \notin \mathcal{B}$  such that  $12^{m+1}, 21^{m+1}$  are not prefix of  $\sigma$ ,

$$d(\text{conv}(\Omega^\sigma), \Lambda) > \kappa_2. \quad (57)$$

Let  $A$  be the limit point of the string  $12^{m+1}(12)^\infty$  and  $B$  the limit point of  $12^{m+1}(21)^\infty$ , see figure 4. The points  $A$  and  $B$  are respectively the upper and lower ends of the set  $\Xi^\infty$ . Take  $\varphi_0 > 0$  such that:

1.  $\varphi_0 < \theta$ ,
2.  $\varphi_0$  is small enough so that the vertical open half-cones  $C_u$  centered at  $A$  and  $C_l$  centered at  $B$  with common angle  $\varphi_0$ , as in figure 4, do not intersect  $f_1(\Omega)$  or  $f_2(\Omega)$ ,
3.  $\varphi_0 < \min(\kappa_1/2R, \kappa_2/2R)$ , where  $R = \text{Diam } \Omega$ .

Condition 3 will be useful in the proof of the geometrical results below.

We introduce the points  $M_1 = f_{12^{m+1}12(21)^\infty}(O)$  and  $M_2 = f_{12^{m+1}21(12)^\infty}(O)$ . Call  $D$  the diamond-shaped intersection of the vertical open half-cones with respective vertices  $M_1$  and  $M_2$  and with common angle  $\varphi_0$ , as in figure 4. Call  $M_3$  and  $M_4$  the other two vertices of  $D$ , see figure 4.

Call  $D^0 = f_{12^{m+1}}^{-1}(D)$  and  $D^\eta = f_\eta(D^0)$  for  $\eta \in \mathcal{A}$ . Note that  $D = D^{12^{m+1}}$ . Similarly, we define  $M_i^\eta = f_\eta(f_{12^{m+1}}^{-1}(M_i))$  for  $i = 1, 2$  and  $\eta \in \mathcal{B}^+$ ; note that  $M_1^\eta$  and  $M_2^\eta$  are vertices of the diamond  $D^\eta$ . Write  $\mathcal{B}^+ = 12^{m+1}(12|21)^*$ , we also introduce the sets  $\mathcal{D} = \bigcup_{\eta \in \mathcal{B}^+} D^\eta$  and

$$C = \overline{C_u \cup C_l \cup \mathcal{D}}. \quad (58)$$

The set  $C$  corresponds to the gray area in the right part of Figure 4, and will play the part of the set  $C$  defined in paragraph 6.1.1 for the proof of point 1.

In view of Proposition 2, introduce the set  $\mathcal{B} \subset \mathcal{A}$  such that  $\sigma \in \mathcal{B}$  if and only if one of the two following conditions is satisfied:

- (i)  $\sigma$  is a prefix of  $12^m$  or  $21^m$ ,
- (ii)  $\sigma \in (12^{m+1}|21^{m+1})(12|21)^*(1|2|\varepsilon)$ ,

where the notations have been defined in paragraph 2.1.1, see also Example 1. The following result is analogous to (33) in the proof of point 1. The proof is postponed to the Appendix for the ease of the reader.

**Lemma 3** *There exists a constant  $c > 0$  such that for any  $\sigma \in \mathcal{A}_n$  ( $n \in \mathbb{N}$ ) such that  $\sigma \notin \mathcal{B}$ ,*

$$d(\Omega^\sigma, C) > ca^n, \quad (60)$$

As in paragraph 6.1.1, we introduce a smooth compactly supported function  $\chi$  on  $\mathbb{R}^2$  such that  $\chi = 1$  on a neighborhood of the closure of the ramified domain  $\Omega$ ,  $\chi$  is symmetric with respect to the axis  $\Lambda$ , and  $\chi$  satisfies the following condition that replaces condition (35):

$$d_2 < d(C \cap D^0, \Omega), \quad (61)$$

where  $d_2 = \sup\{d(x, \Omega), x \in \text{supp } \chi\}$ , along with conditions (36) and (37) with the same notations.

We consider a function  $\psi$  on  $\mathbb{R}^2$  valued in  $[0, 1]$  such that  $\psi \equiv 1$  on  $\{x_1 \leq 0\} \setminus C$ ,  $\psi \equiv 0$  on  $\{x_1 > 0\} \setminus (C_u \cup C_l)$ ,  $\psi$  is continuous on  $\overline{C} \setminus \{A, B\}$ , and  $\psi$  is constant on the lines through the point  $A$  in  $C_u$  and on the lines through the point  $B$  in  $C_l$ .

Take  $p \in (1, 2 - \frac{d}{2})$ . Consider a function  $\zeta \in W^{1,p}(D)$  valued in  $[0, 1]$  such that  $\zeta|_{[M_1 M_3]} = \zeta|_{[M_3 M_2]} = 1$ , and  $\zeta|_{[M_2 M_4]} = \zeta|_{[M_4 M_1]} = 0$  (such a function can be found since  $p < 2$ ). See figure 4 where we have only represented the part of the ramified domain around  $\Xi^\infty$ .

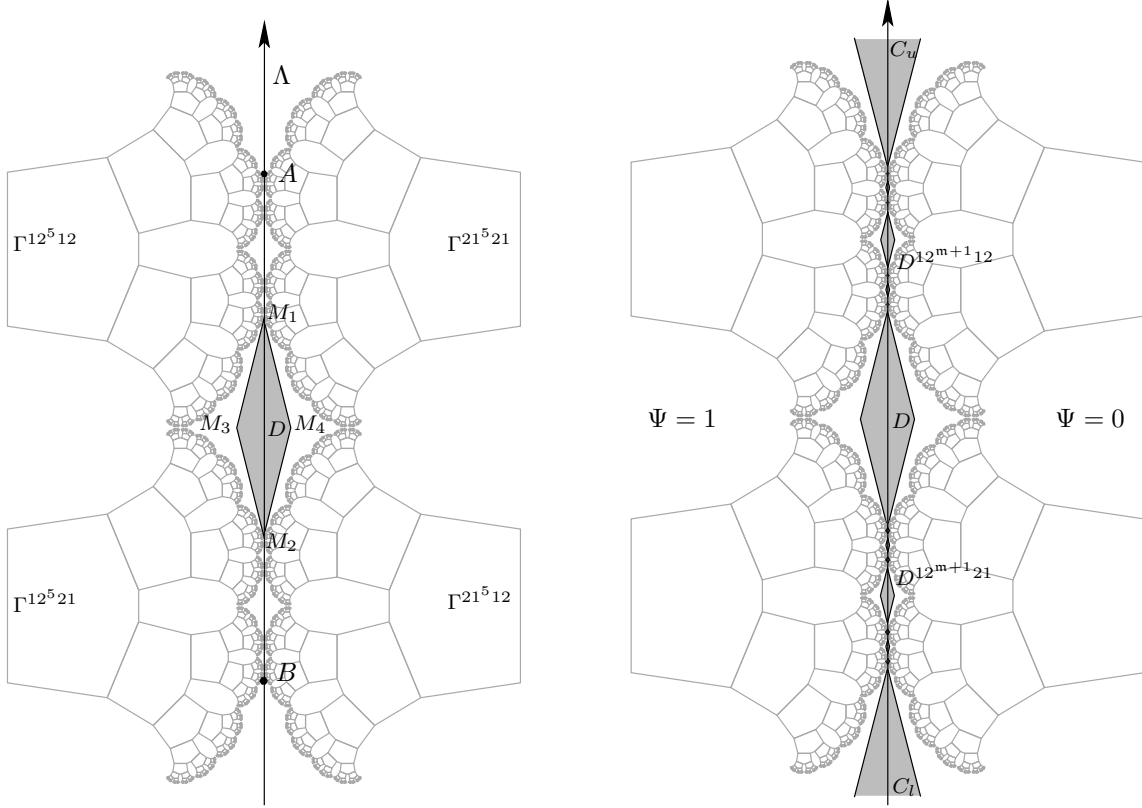


Figure 4: The functions  $\zeta$  and  $\Psi$  in the case  $\theta = \frac{\pi}{8}$  ( $m = 4$ ). Left: construction of the function  $\zeta$ , in the gray area lies the support of  $\zeta$ . Right: the gray area corresponds to the support of  $\nabla \Psi$ .

For  $\eta \in \mathcal{B}^+$ , define  $\zeta_\eta = \zeta \circ f_{12^{m+1}} \circ f_\eta^{-1} \in W^{1,p}(D^\eta)$ . We introduce the function

$$\Psi : x = (x_1, x_2) \in \mathbb{R}^2 \mapsto \begin{cases} \zeta_\eta(x) & \text{if } x \in D^\eta, \eta \in \mathcal{B}^+, \\ \psi(x) & \text{if } x \in C_u \cup C_l, \\ 1 & \text{if } x_1 \leq 0 \text{ and } x \notin C, \\ 0 & \text{if } x_1 > 0 \text{ and } x \notin C, \end{cases} \quad (62)$$

see Figure 4. This definition is unambiguous since the sets  $D^n$  are pairwise disjoint. Function  $\Psi$  will play the part of function  $\psi$  in paragraph 6.1.1. Note that  $\Psi$  is continuous on  $\mathbb{R}^2 \setminus \Xi^\infty$ . We start by defining the lifting of the Haar mother wavelet:

$$\bar{g}_0 = (\chi \circ f_1^{-1})\Psi - (\chi \circ f_2^{-1})(1 - \Psi). \quad (63)$$

One has:

$$\begin{aligned} \sum_{\eta \in \mathcal{B}^+} \int_{D^\eta} |\nabla \zeta_\eta|^p &\simeq \sum_{n \in \mathbb{N}} \sum_{\eta \in \mathcal{A}_n \cap \mathcal{B}^+} a^{n(2-p)} \int_{\mathbb{R}^2} |\nabla \zeta|^p \\ &\lesssim \sum_{n \in \mathbb{N}} 2^{\frac{n}{2}} a^{n(2-p)} \|\nabla \zeta\|_{L^p(\mathbb{R}^2)}^p, \end{aligned}$$

since  $\#\mathcal{A}_n \cap \mathcal{B}^+ \lesssim 2^{\frac{n}{2}}$ . Since  $2^n a^{2n(2-p)} = 2^{n(1 + \frac{2(p-2)}{d})}$  and  $p < 2 - \frac{d}{2}$ , the latter sum converges, and  $\Psi \in W_{loc}^{1,p}$ . Since the function  $\Psi$  is continuous on  $\mathbb{R}^2 \setminus \Xi^\infty$ , so is  $\bar{g}_0$ , which implies that  $\bar{g}_0 \in W^{1,p}(\mathbb{R}^2)$ .

Take  $n > 0$  and  $\sigma \in \mathcal{A}_n$ . For any prefix  $\tau \in \mathcal{A}_k$  ( $0 \leq k < n$ ) of  $\sigma$ , we define the cut-off function

$$\gamma_\tau^\sigma = (\mathbb{1}_{\sigma(k+1)=1}\Psi + \mathbb{1}_{\sigma(k+1)=2}(1 - \Psi)) \circ f_\tau^{-1}, \quad (64)$$

as we did in paragraph 6.1.1. For every  $\sigma \in \mathcal{A}$ , we introduce the set

$$\mathcal{M}(\sigma) = \{\tau \in \mathcal{A}, \sigma = \tau\sigma', \sigma' \in \mathcal{B}\}, \quad (65)$$

as in paragraph 6.1.1, where  $\mathcal{B}$  was defined in (59). Note that Lemma 1 is still true in that case. We can now define liftings  $\bar{g}_\sigma$  for the Haar wavelets: if  $n > 0$  and  $\sigma \in \mathcal{A}_n$ , introduce

$$\bar{g}_\sigma = \prod_{\tau \in \mathcal{M}(\sigma), \tau \neq \sigma} \gamma_\tau^\sigma \cdot 2^{\frac{n}{2}} \bar{g}_0 \circ f_\sigma^{-1}. \quad (66)$$

As in (44), there is a constant  $C$  such that for all  $\sigma \in \mathcal{A}$ ,

$$\int_{\mathbb{R}^2} |\nabla \bar{g}_\sigma|^p \leq C 2^{n(\frac{1}{2} + \frac{p-2}{d})} \int_{\mathbb{R}^2} |\nabla \bar{g}_0|^p. \quad (67)$$

As in paragraph 6.1.1, we introduce the sets  $S = C \cap \text{supp } \bar{g}_0$ ,  $C^\tau = f_\tau(C)$  and  $S^\tau = f_\tau(S)$  for  $\tau \in \mathcal{A}$ ,  $\mathcal{S} = \bigcup_{\tau \in \mathcal{A}} S^\tau$ . We also write  $S_u = C_u \cap \text{supp } \bar{g}_0$ ,  $S_l = C_l \cap \text{supp } \bar{g}_0$  and  $S_u^\tau = f_\tau(S_u)$ ,  $S_l^\tau = f_\tau(S_l)$  for  $\tau \in \mathcal{A}$ . Finally, we define  $F = \text{supp } \nabla \bar{g}_0 \setminus S$ , and  $F^\tau = f_\tau(F)$  for  $\tau \in \mathcal{A}$ .

**Remark 8** Condition (61) implies by a simple geometric argument that  $S \cap D^0 = \emptyset$ .

### 6.2.2 Geometrical results

We state some geometrical results that will be useful for the proof of point 2. As in paragraph 6.1.2, we postpone the proof of these results to the Appendix.

Propositions 5, 6 and 7 remain true in this case. Proposition 8 still holds in the following sense:

**Proposition 9** *If  $\sigma, \tau \in \mathcal{A}$  and  $\tau \notin \mathcal{M}(\sigma)$ , then  $\nabla \bar{g}_\sigma \equiv 0$  on  $S_u^\tau \cup S_l^\tau$ .*

**Proposition 10** *Take  $\tau \in \mathcal{A}$  and  $\eta \in \mathcal{B}^+$ . If  $\sigma \in \mathcal{A}$  is not a prefix of  $\tau\eta 12(21)^\infty$  or  $\tau\eta 21(12)^\infty$ , then  $\nabla \bar{g}_\sigma \equiv 0$  on  $f_\tau(D^\eta)$ .*

### 6.2.3 Proof of point 2 in Theorem 6

The proof that  $\|\nabla(\mathcal{E}v)\|_{L^p(\mathbb{R}^2 \setminus \mathcal{S})}^p \lesssim \|v\|_{JLip(1-\frac{2-d}{p}, p, p; 0; \Gamma^\infty)}^p$  is the same as in the proof of point 1. We are left with dealing with  $\|\nabla(\mathcal{E}v)\|_{L^p(\mathcal{S})}^p$ . Write

$$\|\nabla(\mathcal{E}v)\|_{L^p(\mathcal{S})}^p \lesssim \sum_{\tau \in \mathcal{A}} I_1^\tau + I_2^\tau, \quad (68)$$

where  $I_1^\tau$  and  $I_2^\tau$  are as in (51). Since Proposition 7 still holds, we can deal with  $\sum_{\tau \in \mathcal{A}} I_1^\tau$  as in the proof of point 1.

Since  $S^\tau = S_u^\tau \cup S_l^\tau \cup f_\tau(\mathcal{D})$ , the integration on  $S^\tau$  in  $I_2^\tau$  can be decomposed into integrals on  $S_u^\tau$ ,  $S_l^\tau$  and  $f_\tau(\mathcal{D})$ . The first two integrals can be dealt with exactly as in the proof of point 1, since Proposition 9 holds. We refer to the last one as  $J_2^\tau$ . For  $\eta \in \mathcal{B}^+$ , call  $\mathcal{B}_\eta = \{\sigma \in \mathcal{A}, \sigma \text{ is a prefix of } \eta 12(21)^\infty \text{ or } \eta 12(21)^\infty\}$ . By Proposition 10,  $\nabla \bar{g}_\sigma \equiv 0$  on  $f_\tau(D^\eta)$  if  $\sigma \notin \tau \mathcal{B}_\eta$ . Therefore,

$$J_2^\tau = \sum_{\eta \in \mathcal{B}^+} \int_{f_\tau(D^\eta)} \left| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n \cap \tau \mathcal{B}_\eta} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(x) \nabla \gamma_\tau^\sigma(x) \right|^p dx. \quad (69)$$

We split the integral over  $f_\tau(D^\eta)$  into two integrals over portions of cones with respective vertices  $f_\tau(M_1^\eta)$  and  $f_\tau(M_2^\eta)$ . As in (54), we express them in polar coordinates centered respectively at  $f_\tau(M_1^\eta)$  and  $f_\tau(M_2^\eta)$ . Call  $\ell$  the length of the sides of the diamond  $D^0$ . The length of the sides of  $f_\tau(D^\eta)$  is  $a^{k+l}\ell \leq a^{k+l}R$ , and we may take  $r \leq a^{k+l}R$  in the integrals.

As in the proof of point 1, we note that if  $\tau \in \mathcal{M}(\sigma)$  and  $\text{supp } \bar{g}_0 \circ f_\sigma^{-1} \cap C(f_\tau(M_i^\tau), r) \neq \emptyset$ ,  $i = 1, 2$ , then  $n \leq N_r$  where  $N_r = \frac{\log(r/R)}{\log a}$ . Therefore,

$$\begin{aligned} \left\| \sum_{n \in \mathbb{N}} \sum_{\sigma \in \mathcal{A}_n \cap \tau \mathcal{B}_\eta} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1} \cdot \nabla \gamma_\tau^\sigma \right\|_{L^p(f_\tau(D^\eta))}^p &\lesssim \int_0^{a^{k+l}R} \left( \sum_{n=k}^{[N_r]} \sum_{\sigma \in \mathcal{A}_n \cap \tau \mathcal{B}_\eta} 2^{\frac{n}{2}} |\beta_\sigma| \right)^p r^{1-p} dr \\ &= \int_0^{a^{k+l}R} \left( \sum_{n=0}^{[N_r]-k} \sum_{\sigma' \in \mathcal{A}_n \cap \mathcal{B}_\eta} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p r^{1-p} dr \\ &\lesssim \sum_{m=0}^{\infty} a^{(m+k+l)(2-p)} \left( \sum_{n=0}^m \sum_{\sigma' \in \mathcal{A}_n \cap \mathcal{B}_\eta} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p \\ &\lesssim a^{l(2-p)} \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma \in \mathcal{A}_n \cap \tau \mathcal{B}_\eta} |\beta_\sigma|^p, \end{aligned}$$

where we have proceeded exactly as in the proof of point 1, using the Hardy inequality of Lemma 2. Hence,

$$\begin{aligned} J_2^\tau &\lesssim \sum_{l=0}^{\infty} \sum_{\eta \in \mathcal{A}_l \cap \mathcal{B}^+} a^{l(2-p)} \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma \in \mathcal{A}_n \cap \tau \mathcal{B}_\eta} |\beta_\sigma|^p \\ &\lesssim \sum_{l=0}^{\infty} a^{l(2-p)} 2^{\frac{l}{2}} \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p, \end{aligned}$$

since  $\#\mathcal{A}_l \cap \mathcal{B}^+ \lesssim 2^{\frac{l}{2}}$  and  $\#\mathcal{A}_n \cap \tau\mathcal{B}_\eta = 2$ . Therefore, since  $a^{l(2-p)}2^{\frac{l}{2}} = a^{l(2-\frac{d}{2}-p)}$  and  $p < 2 - \frac{d}{2}$ ,

$$J_2^T \lesssim \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p. \quad (70)$$

We show that  $\sum_{\tau \in \mathcal{A}} J_2^T \lesssim \|v\|_{JLip(1-\frac{2-d}{p}, p, p; 0; \Gamma^\infty)}$  exactly as in the proof of point 1 (see (56)). ■

## 7 Proof of Theorem 7

Take  $\theta \in (0, \frac{\pi}{2})$ , and  $p \in (1, p_\theta^*)$ . We will construct a sequence of continuous linear operators  $\mathcal{F}_n : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^2)$  such that for all  $u \in W^{1,p}(\Omega)$ ,

$$\mathcal{F}_n u|_{Z^n} = u|_{Z^n}, \quad (71)$$

where  $Z^n = \bigcup \{Y^\sigma, \sigma \in \mathcal{A}_k, k \leq n\}$ . This implies that  $\ell^n(\mathcal{F}_n u) = \ell^n(u)$ .

It will be proved in Proposition 12 that the sequence  $(\mathcal{F}_n)_n$  converges in  $\mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^2))$  to a continuous linear operator  $\mathcal{F}$ , which will yield Theorem 7 since  $\mathcal{F}u|_\Omega = u$  for all  $u \in W^{1,p}(\Omega)$ . An immediate consequence is that  $\ell^\infty(\mathcal{F}u) = \ell^\infty(u)$ .

First, we introduce an extension operator from  $W^{1,p}(\Omega)$  to  $W^{1,p}(\widehat{\Omega})$ , where  $\widehat{\Omega}$  is a larger domain defined below and presented in Figure 5.

**The domain  $\widehat{\Omega}$**  Take  $\varepsilon < 2$  and write  $P'_1 = (-1 - \varepsilon, 0)$ , and  $P'_2 = (1 + \varepsilon, 0)$ . Define  $\widehat{Y}^0$  to be the open domain inside the closed polygonal line joining the points  $P'_1, P'_2, f_2(P'_2), f_2(P'_1), f_1(P'_2), f_1(P'_1), P'_1$  in this order. Let  $\widehat{K}^0$  be the closure of  $\widehat{Y}^0$ . We define the wider ramified domain  $\widehat{\Omega}$  to be:

$$\widehat{\Omega} = \text{Interior} \left( \widehat{K}^0 \cup \left( \bigcup_{\sigma \in \mathcal{A}} f_\sigma(\widehat{K}^0) \right) \right), \quad (72)$$

see figure 5. We can suppose  $\varepsilon > 0$  is small enough so that  $f_2(P'_1)$  and  $f_2(P'_2)$  have positive coordinates, the domain  $\widehat{Y}^0$  is convex, and Assumption 1 is satisfied.

We introduce the open domains  $\widehat{Y}^\sigma = f_\sigma(\widehat{Y}^0)$  for  $\sigma \in \mathcal{A}$ , along with their closure  $\widehat{K}^\sigma$ . We also write  $\widehat{\Omega}^\sigma = f_\sigma(\widehat{\Omega})$  for  $\sigma \in \mathcal{A}$ , and  $\widehat{\Omega}^n = \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\Omega}^\sigma$ .

We introduce the open domain  $Y_1^0$  inside the polygonal line joining the points  $P'_1, P_1, f_1(P_1), f_1(P'_1), P'_1$ , its symmetric  $Y_2^0$  with respect to the vertical axis  $\Lambda$ , and the open domain  $Y_3^0$  inside the polygonal line joining the points  $f_1(P_2), f_2(P_1), f_2(P'_1), f_1(P'_2), f_1(P_2)$ . For  $\sigma \in \mathcal{A}$  and  $i = 1, 2, 3$ , write  $Y_i^\sigma = f_\sigma(Y_i^0)$ .

**Proposition 11** *There exists a continuous extension operator  $\mathcal{G}$  from  $W^{1,p}(\Omega)$  to  $W^{1,p}(\widehat{\Omega})$ .*

**Proof.** We start by defining a mapping  $\xi_1$  in polar coordinates  $(r, \varphi)$  centered at the intersection  $M_1$  of the axes  $(P_1P_2)$  and  $(f_1(P_1)f_1(P_2))$  (see figure 6) and such that the half line  $[M_1P_1]$  is given by  $\varphi = 0$ . Write  $(x_1, 0)$  the euclidean coordinates of  $M_1$ . For every  $\varphi \in [0, \theta]$ , define  $r_\varphi$  to be the unique  $r > 0$  such that the point  $(x_1 + r \cos \varphi, r \sin \varphi)$  belongs to the segment  $[P_1f_1(P_1)]$ . Now define  $\xi_1 : Y_1^0 \rightarrow Y^0$  by:

$$\xi_1(x) = \begin{pmatrix} x_1 + r' \cos \varphi \\ r' \sin \varphi \end{pmatrix}, \quad (73)$$



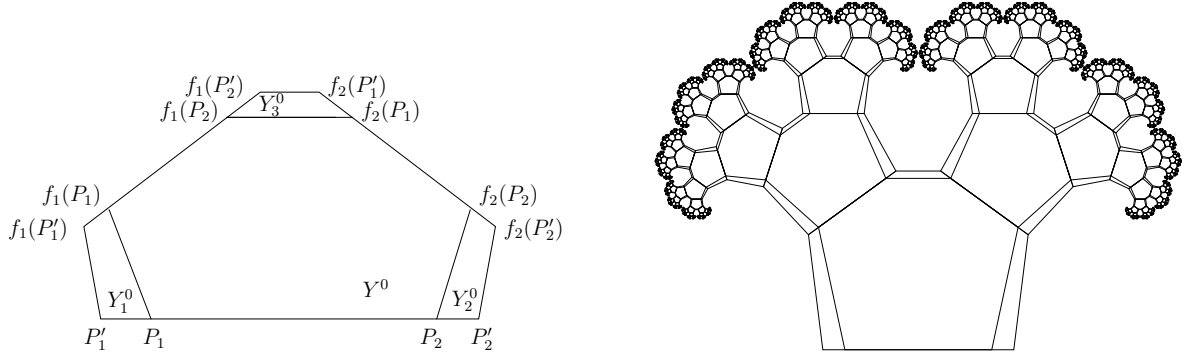


Figure 5: Left: First cells  $Y^0$  and  $\widehat{Y}^0$  of the ramified domains. Right: The ramified domains  $\Omega$  and  $\widehat{\Omega}$ .

where  $x = (x_1 + r \cos \varphi, r \sin \varphi)$  with  $\varphi \in [0, \theta]$ , and  $r' = 2r_\varphi - r$ . We define  $\xi_2 : Y_2^0 \rightarrow Y^0$  as the symmetric of  $\xi_1$  with respect to the axis  $\Lambda$ .

Similarly, we define the mapping  $\xi_3$  in polar coordinates  $(r, \varphi)$  centered at the intersection  $M_3$  of the axes  $f_1(P_1)f_1(P_2)$  and  $f_2(P_1)f_2(P_2)$  (see figure 6) and such that the half line  $[M_3O]$  is given by  $\varphi = 0$ . Write  $(0, y_3)$  the euclidean coordinates of  $M_3$ . For every  $\varphi \in [\theta - \frac{\pi}{2}, \frac{\pi}{2} - \theta]$ , define  $r_\varphi$  to be the unique  $r > 0$  such that the point  $(r \cos \varphi, y_3 + r \sin \varphi)$  belongs to the segment  $[f_1(P'_2)f_2(P'_1)]$ . Now define  $\xi_3 : Y_3^0 \rightarrow Y^0$  by:

$$\xi_3(x) = \begin{pmatrix} r' \cos \varphi \\ y_3 + r' \sin \varphi \end{pmatrix}, \quad (74)$$

where  $x = (r \cos \varphi, y_B + r \sin \varphi)$ , and  $r' = 2r_\varphi - r$ .

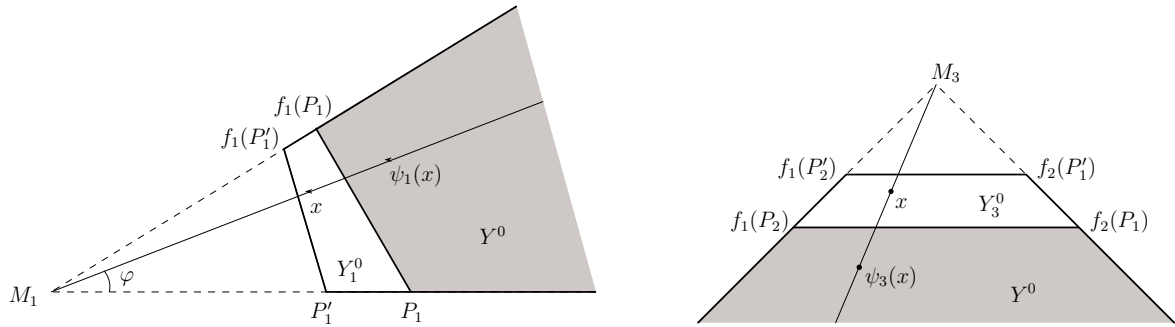


Figure 6: Left: Construction of the mapping  $\xi_1 : Y_1^0 \rightarrow Y^0$ . Right: Construction of the mapping  $\xi_3 : Y_3^0 \rightarrow Y^0$ .

We now define the mappings  $\xi_i^\sigma : Y_i^\sigma \rightarrow Y^\sigma$  for  $\sigma \in \mathcal{A}$  and  $i = 1, 2, 3$  by  $\xi_i^\sigma = f_\sigma \circ \xi_i \circ f_\sigma^{-1}$ .

**Remark 9** It is important to note that the functions  $\xi_i^\sigma$ ,  $\sigma \in \mathcal{A}$ ,  $i = 1, 2$  respect the self-similarity of the domain, in the sense that for each  $i \in \{1, 2\}$  and  $\sigma \in \mathcal{A}_n$  ( $n > 0$ ),  $\xi_i^\sigma = \xi_j^{\sigma|^{n-1}}$  on  $f_\sigma([P_i P'_i])$ , where  $j$  is the unique integer in  $\{1, 2, 3\}$  such that  $f_\sigma([P_i P'_i]) \subset Y_i^\sigma \cap Y_j^{\sigma|^{n-1}}$ .

We define the operator  $\mathcal{G} : W^{1,p}(\Omega) \rightarrow W^{1,p}(\widehat{\Omega})$  by putting  $\mathcal{G}u = u$  on  $\Omega$ , and  $\mathcal{G}u = u \circ \xi_i^\sigma$  on  $Y_i^\sigma$  for  $\sigma \in \mathcal{A}$  and  $i = 1, 2, 3$ .

Note that for all  $\sigma \in \mathcal{A}$  and  $i \in \{1, 2, 3\}$ ,

$$\int_{Y_i^\sigma} |\nabla \mathcal{G}u|^p \leq C \int_{Y^\sigma} |\nabla u|^p, \quad (75)$$

where the constant  $C$  is independent of  $i$  and  $\sigma$ . We deduce from Remark 9 that  $\mathcal{G}u \in W_{loc}^{1,p}(\widehat{\Omega})$ , which implies that  $\mathcal{G}u \in W^{1,p}(\widehat{\Omega})$ , and we deduce from (75) that  $\mathcal{G}$  is continuous. ■

**The extension operators  $\mathcal{F}_n$**  Let us now construct the sequence  $(\mathcal{F}_n)_n$ . Introduce a continuous function  $\chi \in W^{1,\infty}(\widehat{Y}^0)$  valued in  $[0, 1]$  such that  $\chi = 1$  on  $Y^0$ , the trace of  $\chi$  on the segments  $[P'_1 f_1(P'_1)]$ ,  $[P'_2 f_2(P'_2)]$  and  $[f_1(P'_2) f_2(P'_1)]$  is 0, and

$$\begin{aligned} \chi \circ f_1^{-1} &= \chi \text{ on } f_1([P'_1 P'_2]), \\ \chi \circ f_2^{-1} &= \chi \text{ on } f_2([P'_1 P'_2]). \end{aligned} \quad (76)$$

Condition (76) implies that a certain self-similar property is satisfied by  $\chi$ . Such a function  $\chi$  can be constructed in a similar manner as in the proof of Proposition 11.

Introduce a function  $\eta \in \mathcal{C}^0(\widehat{K}_0)$  with values in  $[0, 1]$  such that  $\eta = 1$  on  $\widehat{\Gamma}^0 = [P'_1 P'_2]$ , and  $\eta = 0$  on  $f_1(\widehat{\Gamma}^0) \cup f_2(\widehat{\Gamma}^0)$ .

For every  $n > 0$ , we define a function  $\rho_n$  on  $\mathbb{R}^2$  by:

$$\rho_n = \sum_{k=0}^n \sum_{\sigma \in \mathcal{A}_k} \chi \circ f_\sigma^{-1} \mathbb{1}_{\widehat{K}^\sigma} + \sum_{\sigma \in \mathcal{A}_{n+1}} \chi \eta \circ f_\sigma^{-1} \mathbb{1}_{\widehat{K}^\sigma}. \quad (77)$$

Note that  $\rho_n$  is continuous on  $\mathbb{R}^2$ .

Introduce the linear operators  $\mathcal{F}_n$  on  $W^{1,p}(\Omega)$  defined by:

$$\mathcal{F}_n u = \rho_n \mathcal{G}u + (1 - \rho_n) \mathcal{E} \ell^\infty(u), \quad \forall u \in W^{1,p}(\Omega). \quad (78)$$

Condition (76) implies that for all  $u \in W^{1,p}(\Omega)$ ,  $\mathcal{F}_n u \in W_{loc}^{1,p}(\mathbb{R}^2)$ , and therefore  $\mathcal{F}_n u \in W^{1,p}(\mathbb{R}^2)$ . Moreover, note that since  $\mathcal{G}$ ,  $\mathcal{E}$  and  $\ell^\infty$  are continuous, so are the operators  $\mathcal{F}_n$ .

**Proposition 12** *The sequence  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  converges pointwise to a continuous operator  $\mathcal{F} \in \mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^2))$  such that  $(\mathcal{F}u)|_\Omega = u$  for all  $u \in W^{1,p}(\Omega)$ .*

The proof of Proposition 12 will use the following Poincaré-Wirtinger inequality.

**Lemma 4** *If  $u \in W^{1,p}(\Omega)$ , there exists a constant  $C > 0$  such that:*

$$\int_{\widehat{Y}^0} |\mathcal{G}u(x) - \langle u \rangle_{\Gamma^0}|^p dx \leq C \int_{\widehat{Y}^0} |\nabla \mathcal{G}u(x)|^p dx. \quad (79)$$

We will also use the following result that is very similar to Theorem 11 in [7].

**Proposition 13** *For all real number  $\kappa$  satisfying  $(2a^2)^{p-1} < \kappa < 1$ , there exists a constant  $C > 0$  such that for all  $u \in W^{1,p}(\Omega)$ ,*

$$\|\ell^\infty(u) - \langle u \rangle_{\Gamma^0}\|_{L^p(\Gamma^\infty)}^p \leq C \sum_{i \in \mathbb{N}} \kappa^i \sum_{\tau \in \mathcal{A}_i} \|\nabla u\|_{L^p(Y^\tau)}^p. \quad (80)$$

**Proof.** See [7]. ■

**Proof of Proposition 12.** Take  $u \in W^{1,p}(\Omega)$ . Let us prove that  $(\mathcal{F}_n u)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $W^{1,p}(\mathbb{R}^2)$ . Take  $n, m \in \mathbb{N}$  with  $n < m$ . We denote  $\hat{u} = \mathcal{G}u$  and  $\bar{u} = \mathcal{E}l^\infty(u)$ . First, we note that

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathcal{F}_m u - \mathcal{F}_n u|^p &= \sum_{k=n+1}^{m+1} \int_{\hat{Y}^k} |\mathcal{F}_m u - \mathcal{F}_n u|^p \\ &= \sum_{k=n+1}^{m+1} \int_{\hat{Y}^k} |(\rho_n - \rho_m)(\hat{u} - \bar{u})|^p \\ &\leq \int_{\hat{\Omega}^n} |\hat{u} - \bar{u}|^p \xrightarrow{n \rightarrow \infty} 0, \end{aligned} \quad (81)$$

since  $\hat{u} - \bar{u}|_{\hat{\Omega}} \in L^p(\hat{\Omega})$ . On the other hand,

$$\int_{\mathbb{R}^2} |\nabla \mathcal{F}_m u - \nabla \mathcal{F}_n u|^p = \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} \int_{\hat{Y}^\sigma} |\nabla \mathcal{F}_m u - \nabla \mathcal{F}_n u|^p. \quad (82)$$

Take  $k \in \mathbb{N}$  such that  $n \leq k \leq m$  and  $\sigma \in \mathcal{A}_k$ . One has:

$$\begin{aligned} \int_{\hat{Y}^\sigma} |\nabla \mathcal{F}_m u - \nabla \mathcal{F}_n u|^p &= \int_{\hat{Y}^\sigma} |\nabla((\rho_m - \rho_n)(\hat{u} - \bar{u}))|^p \\ &\leq C \left( \int_{\hat{Y}^\sigma} |\nabla(\hat{u} - \bar{u})|^p + a^{-kq} \int_{\hat{Y}^\sigma} |\hat{u} - \bar{u}|^p \right), \end{aligned}$$

where  $C$  is a constant independent of  $\sigma$ , since  $\rho_m - \rho_n = \chi \circ f_\sigma^{-1}$  if  $k > n$ , and  $\rho_m - \rho_n = \chi \circ f_\sigma^{-1} - \chi \eta \circ f_\sigma^{-1}$  if  $k = n$ . Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla \mathcal{F}_m u - \nabla \mathcal{F}_n u|^p \\ &\lesssim \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} \int_{\hat{Y}^\sigma} |\nabla(\hat{u} - \bar{u})|^p + \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} a^{-kq} \int_{\hat{Y}^\sigma} |\hat{u} - \bar{u}|^p. \end{aligned} \quad (83)$$

To deal with the first term of the right hand side in (83), we note that:

$$\sum_{k=n}^m \sum_{\sigma \in \mathcal{A}_k} \int_{\hat{Y}^\sigma} |\nabla(\hat{u} - \bar{u})|^p = \|\nabla(\hat{u} - \bar{u})\|_{L^p(\hat{\Omega}^n)}^p \xrightarrow{n \rightarrow \infty} 0. \quad (84)$$

Therefore, we are left with considering the second term. For  $\sigma \in \mathcal{A}_k$ , one has:

$$\sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} a^{-pk} \int_{\hat{Y}^\sigma} |\hat{u} - \bar{u}|^p \lesssim S_1 + S_2,$$

where  $S_1 = \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} a^{-pk} \int_{\hat{Y}^\sigma} |\hat{u} - \langle u \rangle_{\Gamma^\sigma}|^p$  and  $S_2 = \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} a^{-pk} \int_{\hat{Y}^\sigma} |\bar{u} - \langle u \rangle_{\Gamma^\sigma}|^p$ .

By Lemma 4, one has:

$$S_1 \leq C \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} a^{(2-p)k} \int_{\hat{Y}^\sigma} |\nabla \hat{u}|^p \lesssim \int_{\hat{\Omega}_n} |\nabla \hat{u}|^p \xrightarrow{n \rightarrow \infty} 0. \quad (85)$$

We are left with considering  $S_2$ . Since for all  $\tau \in \mathcal{A}_n$  with  $n > k$ ,  $\bar{g}_\tau|_{\hat{Y}^\sigma} \equiv 0$ ,

$$\begin{aligned} \int_{\hat{Y}^\sigma} |\bar{u} - \langle u \rangle_{\Gamma^\sigma}|^p &= \int_{\hat{Y}^\sigma} |P_0 \ell^\infty(u) + \sum_{i \leq k} \sum_{\tau \in \mathcal{A}_i} \beta_\tau \bar{g}_\tau - \langle u \rangle_{\Gamma^\sigma}|^p \\ &\leq c \left( a^{2k} |P_0 \ell^\infty(u \circ f_\sigma) - \langle u \rangle_{\Gamma^\sigma}|^p + \int_{\hat{Y}^\sigma} |\beta_\sigma \bar{g}_\sigma|^p \right), \end{aligned} \quad (86)$$

where  $c$  is a positive constant independent of  $k$  and  $\sigma$ , since  $P_0 \ell^\infty(u) + \sum_{i < k} \sum_{\tau \in \mathcal{A}_i} \beta_\tau \bar{g}_\tau = P_0 \ell^\infty(u \circ f_\sigma)$  on  $Y^\sigma$ . This last result can be easily seen by writing the expansions of  $\ell^\infty(u \circ f_\sigma)$  and  $\ell^\infty(u)$  in the Haar wavelets basis of  $\Gamma^\infty$ , and observing that  $\ell^\infty(u \circ f_\sigma) = \ell^\infty(u) \circ f_\sigma$ . This yields that on  $\Gamma^{\infty, \sigma}$ ,

$$P_0 \ell^\infty(u) + \sum_{i < k} \sum_{\tau \in \mathcal{A}_i} \beta_\tau g_\tau = P_0 \ell^\infty(u \circ f_\sigma). \quad (87)$$

If  $i < k$  and  $\tau \in \mathcal{A}_i$ , then  $\bar{g}_\tau$  is constant on  $f_\tau(f_j(\Omega))$ ,  $j = 1, 2$ , therefore  $\bar{g}_\tau$  is constant on  $Y^\sigma$  and  $\bar{g}_\tau(Y^\sigma) = g_\tau(\Gamma^{\infty, \sigma})$ , hence the equality.

Therefore, one has:

$$\int_{\hat{Y}^\sigma} |\bar{u} - \langle u \rangle_{\Gamma^\sigma}|^p \leq c' a^{2k} \left( \int_{\Gamma^\infty} |\ell^\infty(u \circ f_\sigma) - \langle u \circ f_\sigma \rangle_{\Gamma^0}|^p d\mu + 2^{\frac{kp}{2}} |\beta_\sigma|^p \right), \quad (88)$$

where  $c'$  is a positive constant independent of  $k$  and  $\sigma$ . By Proposition 13, for all  $\kappa \in ](2a^2)^{p-1}, 1[$ ,

$$\begin{aligned} \int_{\Gamma^\infty} |\ell^\infty(u \circ f_\sigma) - \langle u \circ f_\sigma \rangle_{\Gamma^0}|^p d\mu &\leq C \sum_{i \geq 0} \kappa^i \sum_{\tau \in \mathcal{A}_i} \int_{Y^\tau} |\nabla(u \circ f_\sigma)|^p \\ &= C a^{(p-2)k} \sum_{i \geq 0} \kappa^i \sum_{\tau \in \mathcal{A}_i} \int_{Y^{\sigma\tau}} |\nabla u|^p \\ &= C a^{(p-2)k} \sum_{i \geq k} \kappa^{i-k} \sum_{\tau \in \mathcal{A}_i, \tau_{1k} = \sigma} \int_{Y^\tau} |\nabla u|^p. \end{aligned} \quad (89)$$

Therefore,

$$\begin{aligned} \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} a^{(2-p)k} \int_{\Gamma^\infty} |\ell^\infty(u \circ f_\sigma) - \langle u \circ f_\sigma \rangle_{\Gamma^0}|^p d\mu &\lesssim \sum_{k=n+1}^{m+1} \sum_{\sigma \in \mathcal{A}_k} \sum_{i \geq k} \kappa^{i-k} \sum_{\tau \in \mathcal{A}_i, \tau_{1k} = \sigma} \int_{Y^\tau} |\nabla u|^p \\ &= \sum_{i \geq n+1} \sum_{\tau \in \mathcal{A}_i} \sum_{k=n+1}^{\min(i, m+1)} \kappa^{i-k} \int_{Y^\tau} |\nabla u|^p \\ &\lesssim \sum_{i \geq n} \sum_{\tau \in \mathcal{A}_i} \int_{Y^\tau} |\nabla u|^p = \int_{\Omega^n} |\nabla u|^p. \end{aligned}$$

Then, (88) yields:

$$S_2 \lesssim \int_{\Omega^n} |\nabla u|^p + \sum_{k \geq n} \sum_{\sigma \in \mathcal{A}_k} a^{(2-p)k} 2^{\frac{kp}{2}} |\beta_\sigma|^p \xrightarrow{n \rightarrow \infty} 0. \quad (90)$$

Indeed, since  $\ell^\infty(u) \in J\text{Lip}(1 - \frac{2-d}{p}, p, p, 0; \Gamma^\infty)$ ,  $\sum_k 2^{\frac{kp}{2}} \sum_{\sigma \in \mathcal{A}_k} a^{(2-p)k} |\beta_\sigma|^p < \infty$ . This proves that the sequence  $(\mathcal{F}_n u)_n$  has a limit in  $W^{1,p}(\mathbb{R}^2)$ . We define  $\mathcal{F}u$  to be the latter limit;  $\mathcal{F}$  obviously defines a linear operator from  $W^{1,p}(\Omega)$  to  $W^{1,p}(\mathbb{R}^2)$ .

Therefore, Banach Steinhaus Theorem insures that  $\mathcal{F}$  is continuous. The fact that  $(\mathcal{F}u)|_\Omega = u$  is a direct consequence of (71). ■

## A Proof of the geometrical results

In this section, we prove the geometrical results stated in paragraphs 6.1.2 and 6.2.2.

### A.1 Proof of the geometrical results from paragraph 6.1.2

We start by stating and proving several geometrical lemmas.

**Lemma 5** *One has:*

$$\begin{aligned} d(\Omega \setminus Y^0, F) &\geq ad_1, \\ \sup\{d(x, \Omega \setminus Y^0), x \in \text{supp } \bar{g}_0\} &\leq ad_2, \end{aligned}$$

where  $d_1$  and  $d_2$  were defined in paragraph 6.1.1.

**Proof.**

- Since the function  $\chi$  is symmetric with respect to the axis  $\Lambda$ , so is  $F$ . Therefore, since  $f_1(\Omega) = \Omega \setminus Y^0 \cap \{x_1 \leq 0\}$ , we have  $d(\Omega \setminus Y^0, F) = d(f_1(\Omega), F \cap \{x_1 \leq 0\})$ . Note that  $F \cap \{x_1 \leq 0\} \subset f_1(\text{supp } \nabla \chi)$ . This implies that  $d(f_1(\Omega), F \cap \{x_1 \leq 0\}) \geq d(f_1(\Omega), f_1(\text{supp } \nabla \chi)) = ad_1$ , hence the result.
- Take  $x = (x_1, x_2) \in \text{supp } \bar{g}_0$ . If  $x_1 \leq 0$ , then by symmetry  $d(x, \Omega \setminus Y^0) = d(x, f_1(\Omega)) \leq \sup\{d(x, f_1(\Omega)), x \in f_1(\text{supp } \chi)\} = ad_2$  since  $\text{supp } \bar{g}_0 \cap \{(x_1, x_2) \in \mathbb{R}^2, x_1 \leq 0\} \subset f_1(\text{supp } \chi)$ . Similarly, if  $x_1 \geq 0$ , then  $d(x, \Omega \setminus Y^0) \leq ad_2$ . The result follows. ■

**Lemma 6** *If  $\sigma \notin \mathcal{B}$ , then*

$$\text{supp } \bar{g}_0 \circ f_\sigma^{-1} \cap C = \emptyset, \quad (91)$$

where  $\mathcal{B}$  was defined in (34).

**Proof.** If  $x \in C$ , then  $d(x, \Omega^\sigma \setminus Y^\sigma) \geq d(C, \Omega^\sigma) > ca^n$  since  $\sigma \notin \mathcal{B}$ , by (33). Lemma 5 and an argument of self-similarity imply that  $\sup\{d(x, \Omega^\sigma \setminus Y^\sigma), x \in \text{supp } \bar{g}_0 \circ f_\sigma^{-1}\} \leq a^{n+1}d_2 < ca^n$ , by condition (36). Therefore  $x \notin \text{supp } \bar{g}_0 \circ f_\sigma^{-1}$ . ■

**Lemma 7** *If  $\sigma \in \mathcal{A} \setminus \{\epsilon\}$  and  $\sigma(1) = 1$  (resp.  $\sigma(1) = 2$ ), then*

$$\text{supp } \bar{g}_\sigma \setminus S \subset \{(x_1, x_2) \in \mathbb{R}^2, x_1 < 0\} \quad (92)$$

(resp.  $\text{supp } \bar{g}_\sigma \setminus S \subset \{(x_1, x_2) \in \mathbb{R}^2, x_1 > 0\}$ ).

**Proof.** Suppose for example  $\sigma(1) = 1$ . If  $\sigma \notin \mathcal{B}$ , then by Lemma 6,  $\text{supp } \bar{g}_\sigma \cap C = \emptyset$ . Therefore,  $\text{supp } \bar{g}_\sigma$  lies in the left-hand connected component of  $\mathbb{R}^2 \setminus C$ , and  $\text{supp } \bar{g}_\sigma \subset \{(x_1, x_2) \in \mathbb{R}^2, x_1 < 0\}$ , hence the result.

If  $\sigma \in \mathcal{B}$ , then, by the definition of  $\bar{g}_\sigma$ ,  $\psi$  is a factor of  $\bar{g}_\sigma$ , which implies that  $\text{supp } \bar{g}_\sigma \setminus C \subset \{(x_1, x_2) \in \mathbb{R}^2, x_1 > 0\}$ . Since  $\text{supp } \bar{g}_\sigma \subset \text{supp } \bar{g}_0$ , one has  $\text{supp } \bar{g}_\sigma \setminus S = \text{supp } \bar{g}_\sigma \setminus C$ , hence the result. ■

**Lemma 8** *If  $\sigma, \tau \in \mathcal{A}$  and  $\sigma$  is a prefix of  $\tau$ , then*

$$F^\sigma \cap \text{supp } \bar{g}_\tau = \emptyset. \quad (93)$$

**Proof.** Suppose  $\sigma \in \mathcal{A}_n$  and  $\tau \in \mathcal{A}_k$ . Note that  $d(\Omega^\tau \setminus Y^\tau, F^\sigma) \geq d(\Omega^\sigma \setminus Y^\sigma, F^\sigma) \geq a^{n+1}d_1$  and  $\sup\{d(y, \Omega^\tau \setminus Y^\tau), y \in \text{supp } \bar{g}_\tau\} \leq a^{k+1}d_2$  by lemma 5. Since  $n < k$ , (37) yields:

$$d(\Omega^\tau \setminus Y^\tau, F^\sigma) > \sup\{d(y, \Omega^\tau \setminus Y^\tau), y \in \text{supp } \bar{g}_\tau\}.$$

Therefore, if  $x \in \text{supp } \bar{g}_\tau$ , then  $d(x, \Omega^\tau \setminus Y^\tau) < d(\Omega^\tau \setminus Y^\tau, F^\sigma)$  and  $x \notin F^\sigma$ . ■

**Lemma 9** *If  $n > 0$  and  $\sigma \in \mathcal{A}_n$ , then, for every  $k < n$ ,*

$$\text{supp } \bar{g}_\sigma \subset f_{\sigma_{|k}}(\text{supp } \bar{g}_{\sigma'}), \quad (94)$$

where  $\sigma = \sigma_{|k}\sigma'$ .

**Proof.** Take  $\tau' \in \mathcal{A}_i$ ,  $i \geq 0$  such that  $\tau' \in \mathcal{M}(\sigma')$  and  $\tau' \neq \sigma'$ . Therefore,  $\sigma_{|k}\tau'$  is a prefix of  $\sigma$ . We observe that:

$$\begin{aligned} \gamma_{\tau'}^{\sigma'} \circ f_{\sigma_{|k}}^{-1} &= (\mathbb{1}_{\sigma'(i+1)=1}\psi + \mathbb{1}_{\sigma'(i+1)=2}(1-\psi)) \circ f_{\sigma_{|k}\tau'}^{-1} \\ &= (\mathbb{1}_{\sigma(k+i+1)=1}\psi + \mathbb{1}_{\sigma(k+i+1)=2}(1-\psi)) \circ f_{\sigma_{|k}\tau'}^{-1} \\ &= \gamma_{\sigma_{|k}\tau'}^\sigma. \end{aligned}$$

Note that for any such  $\tau'$ ,  $\sigma_{|k}\tau' \in \mathcal{M}(\sigma)$  and  $\sigma_{|k}\tau' \neq \sigma$ . Therefore, since  $(\bar{g}_0 \circ f_{\sigma'}^{-1}) \circ f_{\sigma_{|k}}^{-1} = \bar{g}_0 \circ f_\sigma^{-1}$ ,

$$\begin{aligned} |\bar{g}_{\sigma'} \circ f_{\sigma_{|k}}^{-1}| &= 2^{\frac{n-k}{2}} |\bar{g}_0 \circ f_\sigma^{-1}| \prod_{\tau' \in \mathcal{M}(\sigma'), \tau' \neq \sigma'} \gamma_{\sigma_{|k}\tau'}^\sigma \\ &\geq 2^{\frac{n-k}{2}} |\bar{g}_0 \circ f_\sigma^{-1}| \prod_{\tau \in \mathcal{M}(\sigma), \tau \neq \sigma} \gamma_\tau^\sigma \\ &= 2^{\frac{-k}{2}} |\bar{g}_\sigma|, \end{aligned}$$

since for any prefix  $\tau \in \mathcal{A}$  of  $\sigma$ ,  $\gamma_\tau^\sigma \leq 1$ , which implies the result. ■

**Lemma 10** *If  $\sigma, \tau \in \mathcal{A}$  and  $\sigma \neq \tau$ , then  $S^\sigma \cap S^\tau = \emptyset$ .*

**Proof.** Let us first examine the case when  $\tau = \epsilon$ . Suppose for example  $\sigma(1) = 1$ . The set  $\Omega^\sigma$  lies in the convex set  $\mathcal{D}$  defined by  $\{x_1 < 0\} \setminus C$ . Therefore, its convex hull  $\text{conv}(\Omega^\sigma)$  also lies in  $\mathcal{D}$ . Since, by Remark 7,  $S^\sigma \subset \text{conv}(\Omega^\sigma)$ ,  $S^\sigma$  lies in  $\mathcal{D}$ , and  $S^\sigma \cap C = \emptyset$ , hence the result.

Let us consider the general case. First suppose that  $\tau$  is a prefix of  $\sigma$ : write  $\sigma = \tau\sigma'$ , where  $\sigma' \in \mathcal{A}$ . Then,  $S^\sigma \cap S^\tau = f_\tau(S^{\sigma'} \cap S) = \emptyset$  by the previous point.

If none of the strings  $\sigma, \tau$  is a prefix of the other, there is a string  $\eta \in \mathcal{A}$  such that  $\sigma = \eta\sigma'$  and  $\tau = \eta\tau'$  with  $\sigma'(1) \neq \tau'(1)$ . Since, by Lemma 7,  $S^{\sigma'}$  and  $S^{\tau'}$  lie in opposite sides of the axis  $\Lambda$  (given by  $\{x_1 = 0\}$ ),  $S^{\sigma'} \cap S^{\tau'} = \emptyset$ . Therefore,  $S^\sigma \cap S^\tau = f_\eta(S^{\sigma'} \cap S^{\tau'}) = \emptyset$ . ■

**Remark 10** *In particular, if  $\sigma \in \mathcal{A}$  and  $\sigma(1) = 1$  (resp.  $\sigma(1) = 2$ ), then  $S^\sigma \subset \{(x_1, x_2) \in \mathbb{R}^2, x_1 < 0\}$  (resp.  $S^\sigma \subset \{(x_1, x_2) \in \mathbb{R}^2, x_1 > 0\}$ ).*

**Proof of Proposition 5.** If one of the strings  $\sigma, \tau$  is a prefix of the other, then  $F^\sigma \cap F^\tau = \emptyset$  by Lemma 8. Since  $\text{supp } \nabla \bar{g}_\sigma \setminus \mathcal{S} \subset F^\sigma$  and  $\text{supp } \nabla \bar{g}_\tau \setminus \mathcal{S} \subset F^\tau$  by (46),  $(\text{supp } \nabla \bar{g}_\sigma \setminus \mathcal{S}) \cap (\text{supp } \nabla \bar{g}_\tau \setminus \mathcal{S}) = \emptyset$ .

Now suppose none of the sequences  $\sigma, \tau$  is a prefix of the other, *i.e.* there exists an integer  $k$

such that  $\sigma_{\uparrow k} = \tau_{\uparrow k}$  and  $\sigma(k+1) \neq \tau(k+1)$ . Then we can write  $\sigma = \sigma_{\uparrow k}\sigma'$  and  $\tau = \tau_{\uparrow k}\tau'$  for some  $\sigma', \tau' \in \mathcal{A}$ . By Lemma 7,  $(\text{supp } \bar{g}_{\sigma'} \setminus S) \cap (\text{supp } \bar{g}_{\tau'} \setminus S) = \emptyset$ , hence the result since  $\text{supp } \bar{g}_{\sigma} \subset f_{\sigma_{\uparrow k}}(\text{supp } \bar{g}_{\sigma'})$  and  $\text{supp } \bar{g}_{\tau} \subset f_{\sigma_{\uparrow k}}(\text{supp } \bar{g}_{\tau'})$  by Lemma 9. ■

**Proof of Proposition 6.** By the definition of  $\bar{g}_{\sigma}$  (see (42)), it is enough to show that if  $\tau' \in \mathcal{M}(\sigma) \setminus \{\sigma, \tau\}$ , then  $\gamma_{\tau'}^{\sigma} \equiv 1$  on  $S^{\tau} \cap \text{supp } \bar{g}_{\sigma}$ .

Take  $\tau' \in \mathcal{M}(\sigma) \setminus \{\sigma, \tau\}$ . Let us first prove that  $(S^{\tau} \cap \text{supp } \bar{g}_{\sigma}) \cap C^{\tau'} = \emptyset$ . Note that  $S^{\tau} \cap \text{supp } \bar{g}_{\sigma} \subset f_{\sigma}(\text{supp } \bar{g}_0) \subset f_{\tau'}(\text{supp } \bar{g}_0)$ , since  $\tau'$  is a prefix of  $\sigma$ . Therefore,  $(S^{\tau} \cap \text{supp } \bar{g}_{\sigma}) \cap C^{\tau'} \subset f_{\tau'}(\text{supp } \bar{g}_0) \cap C^{\tau'} = S^{\tau'}$ . The result follows since  $S^{\tau} \cap S^{\tau'} = \emptyset$  by Lemma 10. Since  $\nabla \gamma_{\tau'}^{\sigma} \equiv 0$  outside the cone  $C^{\tau'}$ ,  $\gamma_{\tau'}^{\sigma}$  is constant (and takes the value 0 or 1) on  $S^{\tau} \cap \text{supp } \bar{g}_{\sigma}$ .

Suppose  $\tau' \in \mathcal{A}_k$  ( $k \geq 0$ ), and  $\sigma(k+1) = 1$ . This implies that  $\Omega^{\sigma} \subset f_{\tau'}(\{x_1 > 0\})$ . Since  $\tau' \in \mathcal{M}(\sigma)$ , one has  $f_{\tau}(A) \in \overline{\Omega^{\sigma}}$  by the definition of  $\mathcal{M}(\sigma)$  (see (41)). Consequently,  $f_{\tau}(A) \in f_{\tau'}(\{x_1 < 0\} \setminus C)$ , since  $f_{\tau'}(A) \neq f_{\tau}(A)$ . Therefore, the connected set  $S^{\tau} \cap \text{supp } \bar{g}_{\sigma}$  lies in the connected component  $f_{\tau'}(\{x_1 < 0\} \setminus C)$  of  $\mathbb{R}^2 \setminus C^{\tau}$  by the above result, and  $S^{\tau} \cap \text{supp } \bar{g}_{\sigma} \subset f_{\tau'}(\{x_1 < 0\}) \setminus C^{\tau'}$ . Since by definition  $\gamma_{\tau'}^{\sigma} \equiv 1$  on  $f_{\tau'}(\{x_1 < 0\}) \setminus C^{\tau'}$ , we get the desired result. The case  $\sigma(k+1) = 2$  is similar. ■

**Proof of Proposition 7.** Write  $M = \{\sigma \in \mathcal{A}, \tau \in \mathcal{M}(\sigma), \nabla(\bar{g}_0 \circ f_{\sigma}^{-1})(x) \neq 0\}$ . We first note that if  $\sigma \in M$  and  $\sigma \neq \tau$ , then  $\text{supp } \nabla(\bar{g}_0 \circ f_{\sigma}^{-1}) \cap S^{\tau} = F^{\sigma} \cap S^{\tau}$  since  $S^{\sigma} \cap S^{\tau} = \emptyset$  by Lemma 10 (see (46)).

If  $\sigma, \sigma' \in M$ , write  $\sigma = \tau\eta$  and  $\sigma' = \tau\eta'$  where  $\eta, \eta' \in \mathcal{B}$ . Suppose that  $\eta, \eta'$  are prefixes of  $12^{m+1}(12)^{\infty}$  and distinct from  $\epsilon$ . Therefore, one of the two strings  $\eta, \eta'$  is a prefix of the other. Assume for example that  $\eta'$  is a prefix of  $\eta$ . Consequently,  $\sigma'$  is a prefix of  $\sigma$ , and Lemma 8 implies that  $\sigma = \sigma'$ , since  $x \in F^{\sigma'} \cap F^{\sigma} \subset F^{\sigma'} \cap \text{supp } (\bar{g}_0 \circ f_{\sigma}^{-1})$ . Therefore, there is at most one string  $\sigma \in M \setminus \{\tau\}$  such that  $\sigma$  is a prefix of  $12^{m+1}(12)^{\infty}$ . Similarly, there is at most one string  $\sigma \in M \setminus \{\tau\}$  such that  $\sigma$  is a prefix of  $21^{m+1}(21)^{\infty}$ . Since  $\tau \in M$ , there are at most three strings in  $M$ . ■

**Proof of Proposition 8.** For  $n, k \in \mathbb{N}$ , take  $\sigma \in \mathcal{A}_n$  and  $\tau \in \mathcal{A}_k$ .

1. First, we prove that if  $n < k$ , then  $\nabla \bar{g}_{\sigma} \equiv 0$  on  $S^{\tau}$ .

If  $\eta \in \mathcal{M}(\sigma)$  and  $\eta \in \mathcal{A}_m$ , then  $m \leq n < k$ , and  $\eta \neq \tau$ , which implies by Lemma 10 that  $S^{\tau} \cap S^{\eta} = \emptyset$ . Therefore, by (46),  $\text{supp } \nabla \bar{g}_{\sigma} \cap S^{\tau} \subset F^{\sigma}$ . If  $\sigma$  is a prefix of  $\tau$ , then Lemma 8 implies that  $\text{supp } \nabla \bar{g}_{\sigma} \cap S^{\tau} = \emptyset$  since  $S^{\tau} \subset \text{supp } \bar{g}_{\tau}$ .

If  $\sigma$  is not a prefix of  $\tau$ , then there exists  $m < n$  such that  $\sigma = \sigma_{\uparrow m}\sigma'$ ,  $\tau = \sigma_{\uparrow m}\tau'$  with  $\sigma', \tau' \in \mathcal{A} \setminus \{0\}$  and  $\sigma'(1) \neq \tau'(1)$ . Remark 10 and Lemma 7 imply that  $\text{supp } \bar{g}_{\sigma'} \setminus S$  and  $S^{\tau'}$  lie in opposite sides of the axis  $\Lambda$ . Since  $S \cap S^{\tau'} = \emptyset$  by Lemma 8,  $\text{supp } \bar{g}_{\sigma'} \cap S^{\tau'} = \emptyset$ . Therefore,  $\text{supp } \nabla \bar{g}_{\sigma} \cap S^{\tau} = \emptyset$ , since  $\text{supp } \bar{g}_{\sigma} \subset f_{\sigma_{\uparrow k}}(\text{supp } \bar{g}_{\sigma'})$  by Lemma 9.

2. Second, we prove that if  $n \geq k$  and  $\tau \notin \mathcal{M}(\sigma)$ , then  $\bar{g}_{\sigma} \equiv 0$  on  $S^{\tau}$ . This will achieve the proof of Proposition 8.

Suppose  $\tau \notin \mathcal{M}(\sigma)$ . If  $\tau$  is not a prefix of  $\sigma$ , then the same argument as above applies. If  $\tau$  is a prefix of  $\sigma$ , write  $\sigma = \tau\sigma'$ . Since  $\tau \notin \mathcal{M}(\sigma)$ ,  $\sigma' \notin \mathcal{B}$  and  $\text{supp } \bar{g}_{\sigma'} \cap S = \emptyset$  by Lemma 6. Lemma 9 yields the result as above. ■

## A.2 Proof of the geometrical results from paragraph 6.2

We start by giving a proof of Lemma 3. We introduce the following notations: call  $\mathcal{D}^0 = f_{12^{m+1}}^{-1}(\mathcal{D})$ ,  $C_u^0 = f_{12^{m+1}}^{-1}(C_u)$ ,  $C_l^0 = f_{12^{m+1}}^{-1}(C_l)$ , and  $C^0 = f_{12^{m+1}}^{-1}(C) = \mathcal{D}^0 \cup C_u^0 \cup C_l^0$ .

**Proof of Lemma 3.** First, we prove that  $d(\Omega^{11}, C^0) > 0$ . Note that  $\sup\{d(x, \mathcal{H}), x \in \mathcal{D}^0\} = \ell \sin \varphi_0$  where  $\ell$  is the length of the sides of the diamond  $D^0$ . Since  $d(\Omega^{11}, \mathcal{H}) \geq \kappa_1$ , condition 3 in §6.2.1 implies that  $d(\Omega^{11}, \mathcal{H}) > R\varphi_0 \geq \ell \sin \varphi_0$ . Therefore,  $d(\Omega^{11}, \mathcal{D}^0) > d(\Omega^{11}, \mathcal{H}) - \sup\{d(x, \mathcal{H}), x \in \mathcal{D}^0\} > \ell \sin \varphi_0$ . Since  $d(\Omega^{11}, C_u^0 \cup C_l^0) > 0$ , we deduce the result.

Take  $\sigma \in \mathcal{A}_n \setminus \mathcal{B}$ , suppose for example that  $\sigma(1) = 1$ . Suppose  $12^{m+1}$  is a prefix of  $\sigma$ . Therefore,  $\sigma$  can be written  $\sigma_{|k}\sigma'$  with  $\sigma_{|k} \in 12^{m+1}(12|21)^*$  and  $\sigma'(1) = \sigma'(2)$ . Suppose for example that  $\sigma'(1) = 1$ . Hence,  $\Omega^\sigma \subset \Omega^{\sigma_{|k}11}$ , and  $d(\Omega^\sigma, f_{\sigma_{|k}}(C^0)) \geq d(\Omega^{\sigma_{|k}11}, f_{\sigma_{|k}}(C^0)) = a^k d(\Omega^{11}, C^0)$ . Since  $C \subset f_{\sigma_{|k}}(C^0)$ , the result is proved in this case.

Suppose  $12^{m+1}$  is not a prefix of  $\sigma$ . We know that in this case,  $d(\Omega^\sigma, \Lambda) > \kappa_2 > 2R\varphi_0$  by (57) and condition 3 in §6.2.1. Since  $\sup\{d(x, \Lambda), x \in \mathcal{D}\} = a^{m+2}\ell \sin \varphi_0 < R\varphi_0$ , we deduce as above that  $d(\Omega^\sigma, \mathcal{D}) > \kappa_2 - R\varphi_0 > 0$ . It is easily seen that for every  $\tau \notin \mathcal{B}$ ,  $d(\Omega^\tau, C_u \cup C_l) > 0$ . Since  $\Omega^\sigma \subset \bigcup_{k=0}^m \Omega^{12^k1}$ , there exists a constant  $c_0$  independent of  $\sigma$  such that  $d(\Omega^\sigma, C_u \cup C_l) > c_0$ , which concludes the proof. ■

It is straightforward to check that, with the notations introduced in paragraph 6.2.1 and (60), Lemmas 5, 6, 7, 8 and 9 still hold in this case. We start by stating a preliminary lemma.

**Lemma 11** *If  $\eta \notin \mathcal{B}$  and  $\eta(1) = 1$  (resp.  $\eta(1) = 2$ ), then  $D^\eta \subset \{x_1 < 0\}$  (resp.  $D^\eta \subset \{x_1 > 0\}$ ).*

**Proof.** Take  $\eta \in \mathcal{A}_n \setminus \mathcal{B}$  and suppose  $\eta(1) = 1$ . Let us first show that  $d(\text{conv}(\Omega^\eta), \Lambda) > a^n R\varphi_0$ . First suppose that  $12^{m+1}$  is not a prefix of  $\eta$ , then (57) and condition 3 in §6.2.1 imply that  $d(\text{conv}(\Omega^\eta), \Lambda) > \kappa_2 > R\varphi_0$ .

If  $12^{m+1}$  is a prefix of  $\eta$ , then  $\eta = \eta_{|k}\eta'$  with  $\eta_{|k} \in 12^{m+1}(12|21)^*$  and  $\eta'(1) = \eta'(2)$ . For example, suppose  $\eta'(1) = 1$ . Since  $\text{conv}(\Omega^\eta) \subset f_{\eta_{|k}}(\text{conv}(\Omega^{11}))$ , we deduce that  $d(\text{conv}(\Omega^\eta), \Lambda) \geq d(f_{\eta_{|k}}(\text{conv}(\Omega^{11}), \Lambda) = a^k d(\text{conv}(\Omega^{11}), \mathcal{H}) > a^k R\varphi_0$  by condition 3 in §6.2.1.

Observe that  $d(D^\eta, \Lambda) \geq d(\text{conv}(\Omega^\eta), \Lambda) - \sup\{d(x, \text{conv}(\Omega^\eta)), x \in D^\eta\} > a^n R\varphi_0 - a^n \ell \sin \varphi_0 \geq 0$  by the last point, hence the result. ■

We now give the proof that Lemma 10 also remains true:

**Proof.** We first suppose that  $\tau = \epsilon$ . The proof that  $S^\sigma \cap S_u^0 = S^\sigma \cap S_l^0 = \emptyset$  is the same as in Lemma 10.

We are left with checking that  $S^\sigma \cap \mathcal{D} = \emptyset$ . Take  $\eta \in \mathcal{B}^+$ . If  $\eta$  is not a prefix of  $\sigma$ , then we can write  $\eta = \eta_{|k}\eta'$  and  $\sigma = \eta_{|k}\sigma'$  with  $k \geq 0$  and  $\sigma'(1) \neq \eta'(1)$ . Suppose for example that  $\sigma(1) = 1$ . Then  $S^{\sigma'} \subset \text{conv}(\Omega^{\sigma'}) \subset \{x_1 > 0\}$ . On the other hand, by Lemma 11, since  $\eta'(1) = 2$ ,  $D^{\eta'} \subset \{x_1 > 0\}$ , and  $S^\sigma \cap D^\eta = \emptyset$ .

If  $\eta = \sigma$ , then  $S^\sigma \cap D^\eta = \emptyset$  by Remark 8 and by self-similarity.

If  $\sigma = \eta\sigma'$  with  $\sigma' \in \mathcal{A} \setminus \{\epsilon\}$ , then  $S^\sigma \subset \text{conv}(\Omega^\sigma) \subset \text{conv}(\Omega^{\eta\sigma'(1)})$ . Since  $\text{conv}(f_1(\Omega)) \cap D^0 = \text{conv}(f_2(\Omega)) \cap D^0 = \emptyset$ , we deduce by self-similarity that  $\text{conv}(\Omega^{\eta\sigma'(1)}) \cap D^\eta = \emptyset$ , and  $S^\sigma \cap D^\eta = \emptyset$ .

In the case when  $\tau \neq \epsilon$ , we conclude in the same way as in Lemma 10. ■

Therefore, it is easily checked that Propositions 5, 6 and 7 hold.

**Proof of Proposition 10.** Take  $\sigma \in \mathcal{A}$  such that  $\sigma$  is not a prefix of  $\tau\eta 12(21)^\infty$  or  $\tau\eta 21(12)^\infty$ . In the first place, suppose  $\tau = \epsilon$ . We first note that if  $\sigma \notin \mathcal{B}$ , then Lemma 6 yields the result. In the following, we assume that  $\sigma \in \mathcal{B}$ .

We first examine the case when  $\eta$  is not a prefix of  $\sigma$ . By symmetry, we can assume that  $\sigma(1) = \eta(1) = 1$ . There exists an integer  $k > 0$  such that  $\eta = \eta_{|k}\eta'$ ,  $\sigma = \eta_{|k}\sigma'$  and  $\eta'(1) \neq \sigma'(1)$ . Suppose for example  $\sigma'(1) = 1$ . Since  $\sigma \in \mathcal{B}$ ,  $\sigma' \notin \mathcal{B}$ , and Lemma 6 implies that  $\text{supp } \nabla \bar{g}_{\sigma'} \subset \{x_1 < 0\}$ . Since  $\eta' \notin \mathcal{B}$  and  $\eta'(1) = 2$ , the set  $D^{\eta'}$  lies in  $\{x_1 > 0\}$  by Lemma 11, which implies that  $\text{supp } \nabla \bar{g}_{\sigma'} \cap D^{\eta'} = \emptyset$ . We conclude with Lemma 9.



In the case when  $\eta$  is a prefix of  $\sigma$ , the hypothesis we made on  $\sigma$  and  $\eta$  imply that  $\sigma$  has a prefix of the form  $\eta 12(21)^l 1$  or  $\eta 21(12)^l 2$  with  $l \geq 0$ . Suppose for example that the former is true, and write  $\sigma = \eta 12(21)^l \sigma'$ . Therefore, since  $\sigma' \notin \mathcal{B}$ , Lemma 6 implies that  $\text{supp } \nabla \bar{g}_\sigma$  lies above the horizontal axis  $f_{\eta 12(21)^l}(\Lambda)$  (recall that  $\Lambda$  is the axis given by  $\{x_1 = 0\}$ ). On the other hand, the limit point of  $\eta 12(21)^\infty$  lies below this axis. Since this point is the highest vertex of the set  $D^\eta$ , the result follows.

Let us now examine the case when  $\tau \neq \epsilon$ . If  $\sigma = \tau \sigma'$  with  $\sigma' \in \mathcal{A}$ , then  $\sigma'$  is not a prefix of  $\eta 12(12)^\infty$  or  $\eta 21(21)^\infty$ , and  $\text{supp } \nabla \bar{g}_{\sigma'} \cap D^\eta = \emptyset$  by what precedes, we conclude with Lemma 9. If  $\sigma = \tau_{\uparrow k} \sigma'$  and  $\tau = \tau_{\uparrow k} \tau'$  with  $k \geq 0$  and  $\sigma'(1) \neq \tau'(1)$  (for example  $\sigma'(1) = 1$ ), then  $\text{supp } \nabla \bar{g}_{\sigma'} \subset \{x_1 < 0\} \cup S^0$ . On the other hand,  $f_{\tau'}(D^\eta) \subset S^{\tau'}$ , which implies that  $f_{\tau'}(D^\eta) \cap \text{supp } \nabla \bar{g}_{\sigma'} = \emptyset$ . We can conclude with Lemma 9. ■

## References

- [1] Y. Achdou, T. Deheuvels, and N. Tchou. *JLip* versus Sobolev spaces on a class of self-similar fractal foliages. To appear in *Journal de Mathématiques Pures et Appliquées*.
- [2] Y. Achdou, C. Sabot, and N. Tchou. Diffusion and propagation problems in some ramified domains with a fractal boundary. *M2AN Math. Model. Numer. Anal.*, 40(4):623–652, 2006.
- [3] Y. Achdou, C. Sabot, and N. Tchou. A multiscale numerical method for Poisson problems in some ramified domains with a fractal boundary. *Multiscale Model. Simul.*, 5(3):828–860 (electronic), 2006.
- [4] Y. Achdou, C. Sabot, and N. Tchou. Transparent boundary conditions for the Helmholtz equation in some ramified domains with a fractal boundary. *J. Comput. Phys.*, 220(2):712–739, 2007.
- [5] Y. Achdou and N. Tchou. Comparison of traces for a class of ramified domains with self-similar fractal boundaries. In progress.
- [6] Y. Achdou and N. Tchou. Neumann conditions on fractal boundaries. *Asymptotic Analysis*, 53(1-2):61–82, 2007.
- [7] Y. Achdou and N. Tchou. Trace theorems for a class of ramified domains with self-similar fractal boundaries. *SIAM j. of Math. Anal.*, 42(4):828–860 (electronic), 2010.
- [8] R.A. Adams and J.J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [9] G. Berger. Eigenvalue distribution of elliptic operators of second order with Neumann boundary conditions in a snowflake domain. *Math. Nachr.*, 220:11–32, 2000.
- [10] T. Deheuvels. PhD dissertation, in progress.
- [11] K. Falconer. *Techniques in fractal geometry*. John Wiley & Sons Ltd., Chichester, 1997.
- [12] K.J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.
- [13] J.E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.
- [14] P.W. Jones. Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.*, 147(1-2):71–88, 1981.
- [15] A. Jonsson. Wavelets on fractals and Besov spaces. *J. Fourier Anal. Appl.*, 4(3):329–340, 1998.
- [16] A. Jonsson. Haar wavelets of higher order on fractals and regularity of functions. *J. Math. Anal. Appl.*, 290(1):86–104, 2004.
- [17] A. Jonsson and H. Wallin. Function spaces on subsets of  $\mathbf{R}^n$ . *Math. Rep.*, 2(1):xiv+221, 1984.
- [18] A. Jonsson and H. Wallin. The dual of Besov spaces on fractals. *Studia Math.*, 112(3):285–300, 1995.

- [19] J. Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.
- [20] P. Koskela. Extensions and imbeddings. *J. Funct. Anal.*, 159(2):369–383, 1998.
- [21] B.B. Mandelbrot and M. Frame. The canopy and shortest path in a self-contacting fractal tree. *Math. Intelligencer*, 21(2):18–27, 1999.
- [22] O. Martio and J. Sarvas. Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 4(2):383–401, 1979.
- [23] P.A.P. Moran. Additive functions of intervals and Hausdorff measure. *Proc. Cambridge Philos. Soc.*, 42:15–23, 1946.