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**Artificial Equilibrium Points for a Generalized Sail
in the Circular Restricted Three-Body Problem**

by

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Artificial Equilibrium Points for a Generalized Sail in the Circular Restricted Three-Body Problem

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Abstract This paper introduces a new approach to the study of artificial equilibrium points in the circular restricted three-body problem for propulsion systems with continuous and purely radial thrust. The propulsion system is described by means of a general mathematical model that encompasses the behavior of different systems like a solar sail, a magnetic sail and an electric sail. The proposed model is based on the choice of a coefficient related to the propulsion type and a performance parameter that quantifies the system technological complexity. The propulsion system is therefore referred to as generalized sail. The existence of artificial equilibrium points for a generalized sail is investigated. It is shown that three different families of equilibrium points exist, and their characteristic locus is described geometrically by varying the value of the performance parameter. The linear stability of the artificial points is also discussed.

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1 Introduction

The Lagrangian points of the Circular Restricted Three-Body Problem (CR3BP) are known to be the five positions in an orbital system rotating with the two massive bodies where a small object (e.g., a spacecraft), affected only by gravity (and centripetal forces), can theoretically be stationary relative to the two larger objects. These points are the only five positions in space where the small object, if placed there, would maintain its position relative to the two massive bodies. If however the object is equipped with a suitable propulsion system, capable of balancing the gravitational pull of the two massive bodies, other equilibrium points can be created allowing the third body to be stationary with respect to the first two bodies. According to Dusek (1965) these new points are usually referred to as Artificial Equilibrium (or Lagrangian) Points (AEPs).

The problem of describing the location of AEPs and of investigating their stability properties has been addressed by several authors. In particular Perezhogin (1976); Kunitsyn and Perezhogin (1978); Schuerman (1980); Simmons et al. (1985); Perezhogin and Tureshbaev (1989) investigated the effects of the thrust due to the radiation pressure on the CR3BP and showed that seven equilibrium points exist if one of the massive bodies is luminous. This problem is equivalent to consider a photonic (or solar) sail, whose thrust direction is fixed and radial with respect to the luminous body. Subsequently, different studies regarding the use of solar sails (McInnes et al., 1994; McInnes, 1999a; Baig and McInnes, 2008; Bookless and McInnes, 2008; Waters and McInnes, 2008) or low-thrust systems (Morimoto et al., 2007) have been carried out, which proved the existence of infinite equilibrium surfaces depending on the magnitude of the propulsive acceleration. However, only a subset of the potentially achievable AEPs turns out to be stable and, as such, could not be exploited by a spacecraft without the use of a suitable control system. The topology of such subset of stable AEPs is strictly dependent on the propulsion system type employed by the spacecraft. In fact, as was recently pointed out by Bombardelli and Peláez (2011), if the available propulsive acceleration is low, the stable AEPs are confined to a very restricted region around the classical Lagrange points.

In this paper a general approach to the CR3BP with a continuous and purely radial thrust (with respect to the more massive body) is discussed. The concept of *generalized sail* is introduced, in such a way that different continuous propulsion systems can be described within a single mathematical model,

depending on the choice of a coefficients that defines the propulsion type and a suitable performance parameter that quantifies the system technological complexity. This model allows one to investigate, in an unified framework, the position of the AEPs as a function of the performance parameter and to check their linear stability. In this context the assumption of purely radial thrust is particularly significant as it is representative of a situation in which the spacecraft attitude is maintained in a passive way (McInnes, 1998b, 2003; Quarta and Mengali, 2009, 2011a).

The paper is organized as follows. The spacecraft motion in a CR3BP is briefly discussed under the assumption that the propulsion system provides a radial thrust with respect to the more massive body. The corresponding equations of motion are derived in a dimensionless form to simplify the study of AEPs. The mathematical model of a generalized sail is then introduced and it is shown that the AEPs can be described in geometrical terms by varying the value of the sail's performance parameter. Three different families of equilibrium points, located on suitable loci, are shown to exist. Finally, the linear stability of the equilibrium points is discussed.

2 Mathematical Model

Consider the motion of a spacecraft under the gravitational effect of two celestial bodies P_1 and P_2 , with masses m_1 and m_2 (assume that $m_2 \leq m_1$). The two celestial bodies, which affect the motion of the spacecraft without being affected by the spacecraft themselves, cover a circular orbit about their center of mass C with a constant angular velocity $\boldsymbol{\omega}$ whose modulus is

$$\omega \triangleq \|\boldsymbol{\omega}\| = \sqrt{\frac{G(m_1 + m_2)}{l^3}} \quad (1)$$

where l is the distance between the attractors and G is the universal gravitational constant.

Let $\mathcal{T}(C; x, y, z)$ be a synodic reference frame with origin at C and unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}} \triangleq \boldsymbol{\omega}/\omega$. Referring to Fig. 1, the (x, y) plane coincides with the orbital plane of the attractors, and the x axis is in the direction from P_1 to P_2 .

At a generic time instant the angular position of x is defined by means of the true anomaly $\nu \geq 0$, which is measured anticlockwise from some reference direction belonging to the (x, y) plane. Let $\boldsymbol{\rho}_1$ ($\boldsymbol{\rho}_2$) be the dimensionless vector from P_1 (P_2) to the spacecraft, and let $\mu \triangleq m_2/(m_1 + m_2) \in (0, 0.5]$ be the

dimensionless mass of P_2 , which univocally locates the position of the two attractors along the x axis, see Fig. 1. Although in the Solar System the highest value of μ is about 10^{-3} (when P_1 is the Sun),

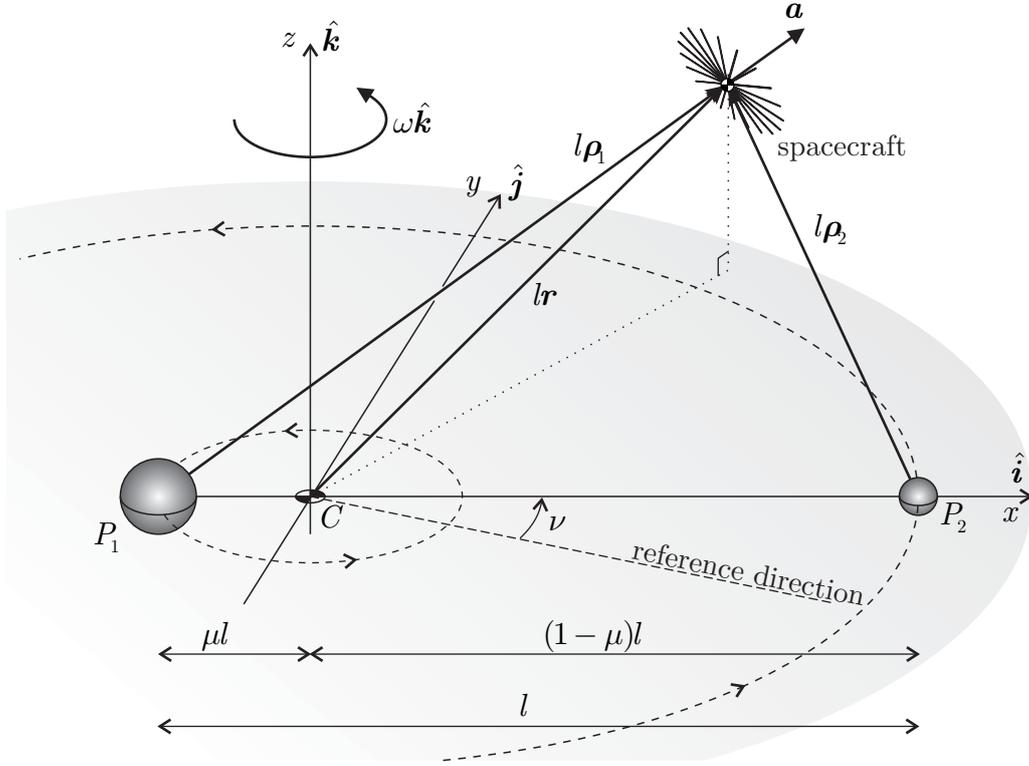


Fig. 1 Reference frame in the circular restricted three-body problem.

the range $(0, 0.5]$ for the parameter μ is used, in this work, to achieve fully general results with the proposed mathematical model. Moreover, the selected range is useful to obtain a comparison with the classical results of the existing literature (Simmons et al., 1985) where the same variation range for the dimensionless mass μ is adopted.

Assume that the spacecraft propulsive acceleration \mathbf{a} is parallel to $\boldsymbol{\rho}_1$ and is given by the following relationship

$$\mathbf{a} = f \frac{G m_1}{l^2} \hat{\boldsymbol{\rho}}_1 \quad (2)$$

where $\hat{\boldsymbol{\rho}}_1 = \boldsymbol{\rho}_1 / \|\boldsymbol{\rho}_1\|$, and the magnitude of the dimensionless acceleration f depends on the available propulsion system technology.

According to Battin (1987), the spacecraft equations of motion in the synodic frame are:

$$l\ddot{\mathbf{r}} + 2l\boldsymbol{\omega} \times \dot{\mathbf{r}} + l\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \frac{Gm_1}{l^2\rho_1^3}\boldsymbol{\rho}_1 + \frac{Gm_2}{l^2\rho_2^3}\boldsymbol{\rho}_2 = \mathbf{a} \quad (3)$$

where \mathbf{r} is the dimensionless vector that defines the spacecraft position with respect to C , $\rho_1 \triangleq \|\boldsymbol{\rho}_1\|$, $\rho_2 \triangleq \|\boldsymbol{\rho}_2\|$ and \mathbf{a} is given by Eq. (2). Bearing in mind that $\hat{\boldsymbol{\omega}} \equiv \hat{\mathbf{k}}$, the equations of motion in dimensionless form can be obtained by using the true anomaly ν as the independent variable. The result is

$$\mathbf{r}'' + 2\hat{\mathbf{k}} \times \mathbf{r}' + \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{r}) + \frac{1-\mu}{\rho_1^3}\boldsymbol{\rho}_1 + \frac{\mu}{\rho_2^3}\boldsymbol{\rho}_2 = \frac{f(1-\mu)}{\rho_1}\boldsymbol{\rho}_1 \quad (4)$$

where the prime symbol denotes a derivative taken with respect to ν . Note that, according to Fig. 1

$$\mathbf{r} = \boldsymbol{\rho}_1 - \mu\hat{\mathbf{i}} \quad , \quad \boldsymbol{\rho}_2 = \boldsymbol{\rho}_1 - \hat{\mathbf{i}} \quad (5)$$

As a result, Eq. (4) can be rearranged and written as a function of the vector $\boldsymbol{\rho}_1$ only.

2.1 Artificial Equilibrium Points

The positions of AEPs are obtained by substituting Eqs. (5) into Eq. (4) and enforcing the conditions $\mathbf{r}'' = 0$ and $\mathbf{r}' = 0$. The result can be expressed in compact form as

$$d_k\hat{\mathbf{k}} + d_i\hat{\mathbf{i}} = d_\rho\hat{\boldsymbol{\rho}}_1 \quad (6)$$

where

$$d_k \triangleq \boldsymbol{\rho}_1 \cdot \hat{\mathbf{k}} \quad , \quad d_i \triangleq \mu \left(1 - \frac{1}{\rho_2^3}\right) \quad , \quad d_\rho \triangleq \rho_1 - \frac{1-\mu}{\rho_1^2} - \frac{\mu\rho_1}{\rho_2^3} + f(1-\mu) \quad (7)$$

An analysis of Eq. (6) provides the loci of AEPs and the corresponding value of the dimensionless acceleration f required to maintain such equilibrium points. In what follows, the AEPs will be distinguished according to their different geometrical positions.

2.1.1 Triangular-type points

The triangular-type points are obtained by looking for the values of f , ρ_2 , and ρ_1 in correspondence of which the coefficients d_k , d_i , and d_ρ of Eq. (7) are all equal to zero, viz

$$\boldsymbol{\rho}_1 \cdot \hat{\mathbf{k}} = 0 \quad , \quad \rho_2 = 1 \quad , \quad f = \rho_1 \left(\frac{1}{\rho_1^3} - 1 \right) \quad (8)$$

As a result, the locus of AEPs is constituted by a circle, with center P_2 and radius l , belonging to the (x, y) plane (see the succeeding Fig. 4). For a given point on the locus, that is, for a given value of $\rho_1 \in (0, 2]$, the required value of f can be calculated using the third equation of (8). The name triangular type points comes from the fact that in the absence of any thrust ($f = 0$), Eq. (8) provides $\rho_1 \equiv \rho_2 = 1$, which corresponds to the position of the classical Lagrange triangular points $\rho_{1L_4} \equiv \rho_{2L_4} = 1$, and $\rho_{1L_5} \equiv \rho_{2L_5} = 1$.

2.1.2 Collinear points

Assume now that $d_k = 0$, but $d_i \neq 0$ and $d_\rho \neq 0$. From Eq. (6) $d_i \hat{\mathbf{i}} = d_\rho \hat{\boldsymbol{\rho}}_1$, which implies that $\hat{\boldsymbol{\rho}}_1$ and $\hat{\mathbf{i}}$ are parallel. Accordingly, the AEPs belong to the x axis and are such that

$$d_\rho = d_i \hat{\boldsymbol{\rho}}_1 \cdot \hat{\mathbf{i}} \quad (9)$$

The second equation of (5) states that $\rho_2 = \left| \rho_1 \hat{\boldsymbol{\rho}}_1 \cdot \hat{\mathbf{i}} - 1 \right|$. The latter relationship can be substituted into Eq. (6) to find the dimensionless acceleration required to obtain the collinear AEPs. After some algebraic manipulations the result is:

$$f = \frac{\mu}{(1-\mu) \hat{\boldsymbol{\rho}}_1 \cdot \hat{\mathbf{i}}} \left(1 + \frac{\rho_1 \hat{\boldsymbol{\rho}}_1 \cdot \hat{\mathbf{i}} - 1}{\left| \rho_1 \hat{\boldsymbol{\rho}}_1 \cdot \hat{\mathbf{i}} - 1 \right|^3} \right) + \frac{1}{\rho_1^2} - \frac{\rho_1}{1-\mu} \quad (10)$$

As in the previous case, the condition $f = 0$ in Eq. (10) provides the position of the three classical collinear Lagrange points ρ_{1L_1} , ρ_{1L_2} , and ρ_{1L_3} . Recall that L_1 is between the two attractors, L_2 is placed on the positive x axis beyond P_2 , and L_3 is on the left of P_1 .

2.1.3 Displaced points

If d_k , d_i , and d_ρ are all different from zero, Eq. (6) can be satisfied only provided that the AEPs belong to the (x, z) plane, that is, if $y = 0$. Such points will be referred to as displaced points, in analogy to the usual nomenclature for non-Keplerian orbits (Gong et al., 2009; Xu and Xu, 2008; Mengali and Quarta, 2009b), which are obtained by using low-thrust propulsion systems. A general definition of non-Keplerian orbits is provided in McKay et al. (2011). To define the position of displaced points, take the dot product of both sides of Eq. (6) by $\hat{\mathbf{k}}$ and by $\hat{\mathbf{i}}$. The result is:

$$\frac{\mu}{\rho_2^3} = \mu - \rho_1 \cdot \hat{\mathbf{i}} \quad (11)$$

$$f = \frac{1}{\rho_1^2} + \frac{\rho_1 (\mu - \rho_1 \cdot \hat{\mathbf{i}})}{1 - \mu} \quad (12)$$

where

$$\rho_1 \cdot \hat{\mathbf{i}} = x/l + \mu \quad (13)$$

$$\rho_2 = \sqrt{(x/l + \mu - 1)^2 + (z/l)^2} \quad (14)$$

Then substitute Eqs. (13) and (14) into (11) to get

$$z/l = \pm \sqrt{(-l\mu/x)^{2/3} - (x/l + \mu - 1)^2} \quad (15)$$

while Eq. (12) provides the corresponding value of f .

The locus of displaced AEPs is qualitatively drawn in Fig. 2. Note that the locus is symmetrical with respect to the x axis and the points are such that $x \in (-\mu, 0)$ or, equivalently, $\rho_2 > 1$, see Eq. (15). As expected, Eqs. (11) and (12) show that displaced points cannot be obtained if $f = 0$. Indeed, the right hand side of (12) is the sum of two strictly positive terms and, therefore, it cannot vanish.

2.2 Power radial thrust

So far the AEPs positions have been expressed as a function of the dimensionless acceleration f . Now the study is specialized to the situation in which f varies with the distance ρ_1 from the massive attractor

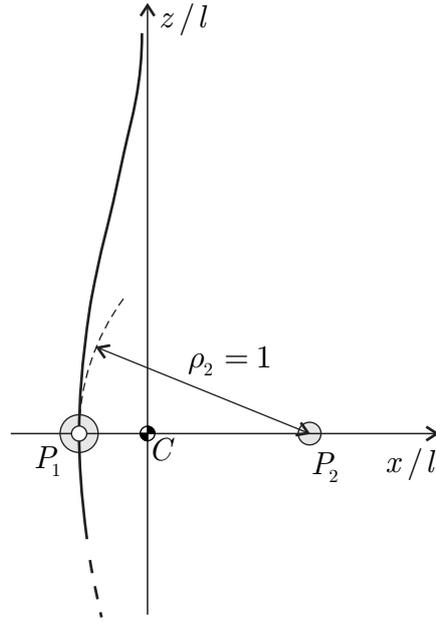


Fig. 2 Locus of displaced equilibrium points.

according to the following power law:

$$f = \frac{\beta}{\rho_1^\eta} \quad (16)$$

where η is a real, nonnegative, coefficient that depends on the propulsion system type, while β is a real parameter that quantifies the propulsion system performance. More precisely, for a given value of η , a decrease of $|\beta|$ implies a corresponding decrease in terms of propulsion system overall dimensions and mass (equivalently, for a given launch mass, the available payload mass increases).

Note that with the aid of Eq. (16) the equations of motion (3) can be written in compact form as

$$l \ddot{\mathbf{r}} + 2l \boldsymbol{\omega} \times \dot{\mathbf{r}} = \nabla J^* \quad (17)$$

where $J^* \triangleq J + \Psi$, and J is the sum of the gravitational and centrifugal potential (Battin, 1987), while Ψ represents the potential due to the propulsive acceleration, viz

$$\Psi = \frac{\beta}{1 - \eta} \frac{G m_1}{l^2} \frac{1}{\rho_1^{\eta-1}} \quad (18)$$

Paralleling Battin (1987), it is possible to obtain a generalization of the Jacobi integral for a radial propulsive thrust as

$$l^2 \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - 2J^* = \mathcal{C} \quad (19)$$

where \mathcal{C} is an integration constant.

A spacecraft equipped with a propulsion system capable of generating an acceleration in the form of Eq. (2), with f given by Eq. (16), will be referred to as generalized sail. The reason is that if the massive body m_1 coincides with the Sun, Eq. (16) encompasses different types of propulsion systems, either currently available or under development (Racca, 2003; Frisbee, 2003). This is possible by simply specializing the value of η . For example $\eta = 2$ describes the thrust provided by a solar sail (McInnes, 1999b) or a magnetic sail (Zubrin and Andrew, 1991), while $\eta \in [1, 7/6]$ corresponds to an electric sail model (Janhunen and Sandroos, 2007; Mengali et al., 2008; Janhunen, 2010). Finally $\eta = 0$ represents a constant propulsive acceleration, or a thrust that is independent of the distance from the massive attractor. Actually $\eta = 0$ is consistent with a situation in which the propulsion system is either a mini-magnetospheric plasma thruster (Winglee et al., 2000; Trask et al., 2004; Mengali and Quarta, 2006), or an electric thruster when the thrust is modulated such as to provide a constant thrust-to-mass ratio (Prussing and Coverstone, 1998; Mengali and Quarta, 2009a) and the power source is independent of the Sun-spacecraft distance, as in a nuclear-powered system. This happens, for example, when the power subsystem is represented by a radio-isotope thermoelectric generator (Hunt, 1993; Lyngvi et al., 2007). Note however that an electric thruster can also be characterized by a coefficient $\eta \neq 0$. In fact, the propulsive acceleration of a spacecraft of total mass m_s , equipped with an electric thruster of constant specific impulse I_{sp} , can be written as (Kechichian, 1995; Vadali et al., 2000; Mengali and Quarta, 2005)

$$\mathbf{a} = \frac{2\tau\sigma P}{m_s g_0 I_{sp}} \hat{\mathbf{t}} \quad (20)$$

where σ is the constant thruster efficiency, P is the engine maximum input power, $\tau \in [0, 1]$ is the dimensionless throttle parameter ($\tau = 1$ corresponding to the maximum thrust level for a given value of P), g_0 is the Earth's standard gravitational acceleration, and $\hat{\mathbf{t}}$ is the thrust unit vector.

In a solar-powered spacecraft, in which the electric power is supplied by solar arrays, the maximum input

power P is a function of the distance from the Sun (Sauer, 1978; Rayman and Williams, 2002), but also depends on the flight time due to the solar cells degradation (Richardson and Warren, 1971; Bourke, 1972; Saleh et al., 2002). If P_1 coincides with the Sun and the time degradation of solar cells is neglected, the maximum input power P can be described through a rational function of ρ_1 , that is, to a first order approximation (Sauer, 1978; Williams and Coverstone-Carroll, 1997)

$$P = \frac{P_r}{\rho_1^\eta} \quad (21)$$

where P_r is a reference value that usually coincides with the maximum power generated at 1 Astronomical Unit from the Sun, and η depends on the numerical coefficients that model the solar cells performance variation with the Sun-spacecraft distance. For example, using the numerical coefficients of Sauer (1978), the value of η is about 1.6. Note that in an ideal case, that is, when the dependence of P from ρ_1 is due only to the solar intensity radial variation, the exponent in Eq. (21) becomes $\eta = 2$, and an inverse square law of P with the Sun-spacecraft distance is obtained. In general, a reasonable assumption is that the value of η varies within a range depending on the solar array characteristics. Finally note that the case of nuclear-powered spacecraft ($\eta = 0$) is recovered from Eq. (21) by observing that, for such a power source, $P \simeq P_r$.

Also note that for an electric propulsion system the instantaneous spacecraft mass is a function of time t . In fact, under the assumption that the thruster cannot be switched off and introducing the propellant mass flow rate $\dot{m}_p > 0$, the instantaneous spacecraft mass is

$$m_s = m_{s_0} \left(1 - \frac{\dot{m}_p}{m_{s_0}} t \right) \quad (22)$$

where m_{s_0} is the initial mass. Substituting Eqs. (21) and (22) into Eq. (20) yields:

$$\mathbf{a} = \frac{2 \sigma P_r}{m_{s_0} g_0 I_{sp}} \left(\frac{\tau}{1 - \dot{m}_p t / m_{s_0}} \right) \frac{\hat{\mathbf{t}}}{\rho_1^\eta} \quad (23)$$

If the thrust level τ is modulated in such a way to counterbalance the mass loss, viz.

$$\tau = 1 - \frac{\dot{m}_p}{m_{s_0}} t \quad (24)$$

and the thrust is oriented parallel to $\hat{\rho}_1$ (that is $\hat{\mathbf{t}} \cdot \hat{\rho}_1 = \pm 1$), Eq. (23) becomes:

$$\mathbf{a} = \pm \frac{2 \sigma P_r}{m_{s_0} g_0 I_{sp} \rho_1^\eta} \hat{\rho}_1 \quad (25)$$

which is in the form of Eq. (2) provided that

$$f = \pm \frac{2 \sigma P_r l^2}{G m_1 m_{s_0} g_0 I_{sp} \rho_1^\eta} \quad (26)$$

Bearing in mind Eq. (16), the performance parameter β of a solar-powered spacecraft is related to the main vehicle's characteristics through the equation:

$$\beta = \pm \frac{2 \sigma P_r l^2}{G m_1 m_{s_0} g_0 I_{sp}} \quad (27)$$

Note that, unlike a solar sail or an electric sail, an electric thruster can orient the thrust in any space direction. In particular, the propulsive acceleration can be oriented toward the attractor P_1 (that is, $\hat{\mathbf{t}} \cdot \hat{\rho}_1 = -1$) to obtain a negative value of β .

However, the use of an electric thruster as a primary propulsion system introduces a constraint on the total mission duration, as an AEP can be maintained only for a limited time interval. In fact, if $m_p < m_{s_0}$ is the onboard propellant mass, the mission duration t_m is

$$t_m = \frac{m_p}{\dot{m}_p} \quad (28)$$

For example, assuming a propellant flow rate $\dot{m}_p = 2.3 \text{ mg/s}$, corresponding to the NASA's NSTAR ion thruster (Brophy, 2003, 2001), and a propellant mass $m_p = 100 \text{ kg}$, Eq. (28) states that an AEP can be maintained for about 500 days. Note however that the value of \dot{m}_p (and the mission duration t_m) actually depends on the thruster setting, and a constant value of the propellant flow rate should be thought of as a first order approximation only. An in-depth discussion of the complex interactions between the thruster mathematical model and the mission performances is beyond the scope of this work, and the interested reader is referred to Quarta and Mengali (2011b).

As previously stated, while η defines the propulsion system type, β quantifies the propulsion system performance. With reference to both solar sails and electric sails, β can be expressed as a function of the maximum propulsive acceleration a_c (referred to as characteristic acceleration) at a Sun-spacecraft distance equal to 1 Astronomical Unit. This is possible by combining Eqs. (2) and (16).

A brief discussion about the order of magnitude of the characteristic acceleration is now given. The only solar sails that have been successfully deployed in space so far, that is the JAXA's "IKAROS" (Higuchi and Ishimura, 2009) and the NASA's "NanoSail-D" (Johnson et al., 2011), have a characteristic acceleration of about 1-2 micro- g_0 (β is on the order of 10^{-3}). However such a value is well below the current technological capabilities. For example proposals for NASA's Millennium Space Technology 9 mission include solar sails that produce thrust on the order of 0.58 mm/s^2 to values as high as 1.70 mm/s^2 (Lichodzeijewski and Derbes, 2006). Moreover it is widely accepted (Macdonald et al., 2006; Ozimek et al., 2009) that for a near term solar sail technology a reasonable performance is $a_c = 0.5 - 1 \text{ mm/s}^2$ (β is on the order of $0.05 - 0.2$), especially when non-Keplerian orbits and AEPs are considered (West, 2008). A second generation of solar sails (Leipold et al., 1999) will probably have a higher characteristic acceleration. As a matter of fact, many studies exist with medium-high performance solar sails ($a_c = 1 - 3 \text{ mm/s}^2$, or β around $0.2 - 0.5$) and very-high performance solar sails ($a_c = 3 - 6 \text{ mm/s}^2$ or β on the order of $0.5 - 1$) either for Solar System escape missions or for the fulfilment of highly-Non-Keplerian orbits (Sauer, 1999; Forward, 1991; McInnes, 1998a; Vulpetti and Scaglione, 1999; Vulpetti, 1997; Dandouras et al., 2004; Mengali and Quarta, 2007; Macdonald et al., 2010).

As far as the electric sail is concerned, the latest simulations by Janhunen (Janhunen, 2009, 2011) show that the propulsive thrust of a spacecraft with an in-flight mass of 100 kg is about 1 N. Using these data, and according to the propulsion system's mathematical model discussed by Quarta and Mengali (2010), an electric sail is potentially capable of providing a characteristic acceleration of 2 mm/s^2 to a spacecraft with a total mass of 625 kg and a payload mass of about 215 kg. In any case, it is important to recall that the value of a_c is closely related to the value of the payload mass fraction for a given in-flight total mass. Therefore, the effective value of β can be calculated only after that the payload mass has been selected.

As a final remark, the succeeding discussion will be given in fully general terms, including values of η and β that do not correspond to any currently available propulsion system (this happens, for example, for $\eta > 2$). The reason is to get an effective method for easily comparing the performance of different propulsion devices in generating AEPs. In this context the use of η and β as design parameters provides a certain degree of flexibility in dealing with exotic/future thrusters. For example it is worth mentioning that until the definition of the electric sail concept (that is, until Janhunen (2004)), the only realistic value for a propellantless propulsion system was $\eta = 2$.

In what follows, for a given propulsion system type (that is, for a given value of η) the problem is to find the value of β required for an AEP to be placed at a prescribed distance ρ_1 from the massive attractor. The pairs (β, ρ_1) describe a locus, in the synodic reference frame, whose shape can be obtained for each of the three different families of AEPs. This matter is discussed in the next sections.

2.2.1 Triangular-type points

For triangular-type equilibrium points substitute Eq. (16) into Eq. (8) to get an expression of β as a function of the distance $\rho_1 \in (0, 2]$ and of the coefficient η . The result is

$$\beta = \rho_1^{\eta+1} \left(\frac{1}{\rho_1^3} - 1 \right) \quad (29)$$

whose shape is qualitatively shown in Fig. 3. For a fixed value of $\eta > 2$ the contour line $\beta = \beta(\rho_1, \eta)$ presents a maximum with coordinates $(\rho_{1\beta_{\max}}, \beta_{\max})$, where

$$\rho_{1\beta_{\max}} = \sqrt[3]{\frac{\eta-2}{\eta+1}} \quad , \quad \beta_{\max} = 3 \sqrt[3]{\frac{(\eta-2)^{\eta-2}}{(\eta+1)^{\eta+1}}} \quad (30)$$

In particular, when $\eta = 2$ and $\beta > 0$, that is, when the propulsion system is constituted by either a solar or a magnetic sail, in accordance with Simmons et al. (1985), Eq. (30) states that β takes its maximum value ($\beta = 1$) as ρ_1 tends to zero. Clearly such a condition is not physically acceptable because the corresponding AEP would coincide with the attractor P_1 . In the following those unattainable equilibrium points will be marked in the figures with a white circle.

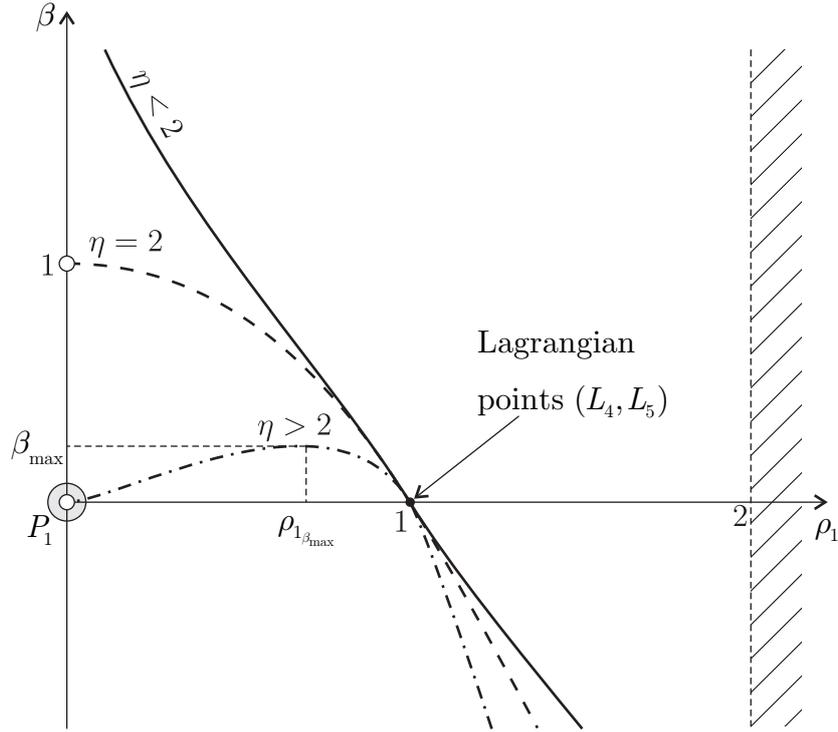


Fig. 3 Performance parameter β as a function of ρ_1 and η for triangular-type points.

The presence of a maximum value of the performance parameter β implies that, for a given value of $\beta > 0$, there exist either two AEPs (if $\eta \leq 2$) or four AEPs (if $\eta > 2$). These points are at symmetrical positions with respect to the x axis of the synodic reference frame, as is shown in Fig. 4. This figure illustrates the variation of $|\beta|$ on the locus of triangular-type points in the three cases $\eta < 2$, $\eta = 2$, and $\eta > 2$. Note that for a fixed distance $\rho_1 < 1$ (> 1), the parameter η must be increased (decreased) to reduce the value of $|\beta|$ necessary to maintain such an AEP.

2.2.2 Collinear points

In this case the relationship between the performance parameter β and the distance ρ_1 is obtained by substituting Eq. (16) into (10). To simplify the analysis of the function $\beta = \beta(\rho_1, \eta)$ it is useful to distinguish the collinear points according to their position with respect to the attractors, as shown in Fig. 5. The variation of β with ρ_1 for L_3 -type points (the points in which $\hat{\rho}_1 \cdot \hat{i} = -1$) is qualitatively shown in Fig. 6(a), while Fig. 6(b) illustrates the corresponding variation of the AEP position as a function of β . Similar to the previous triangular-type points, Fig. 6(a) shows that the contour curves $\beta = \beta(\rho_1, \eta > 2)$ take an absolute maximum whose value decreases as η is increased. Once again, when $\eta = 2$ the maximum

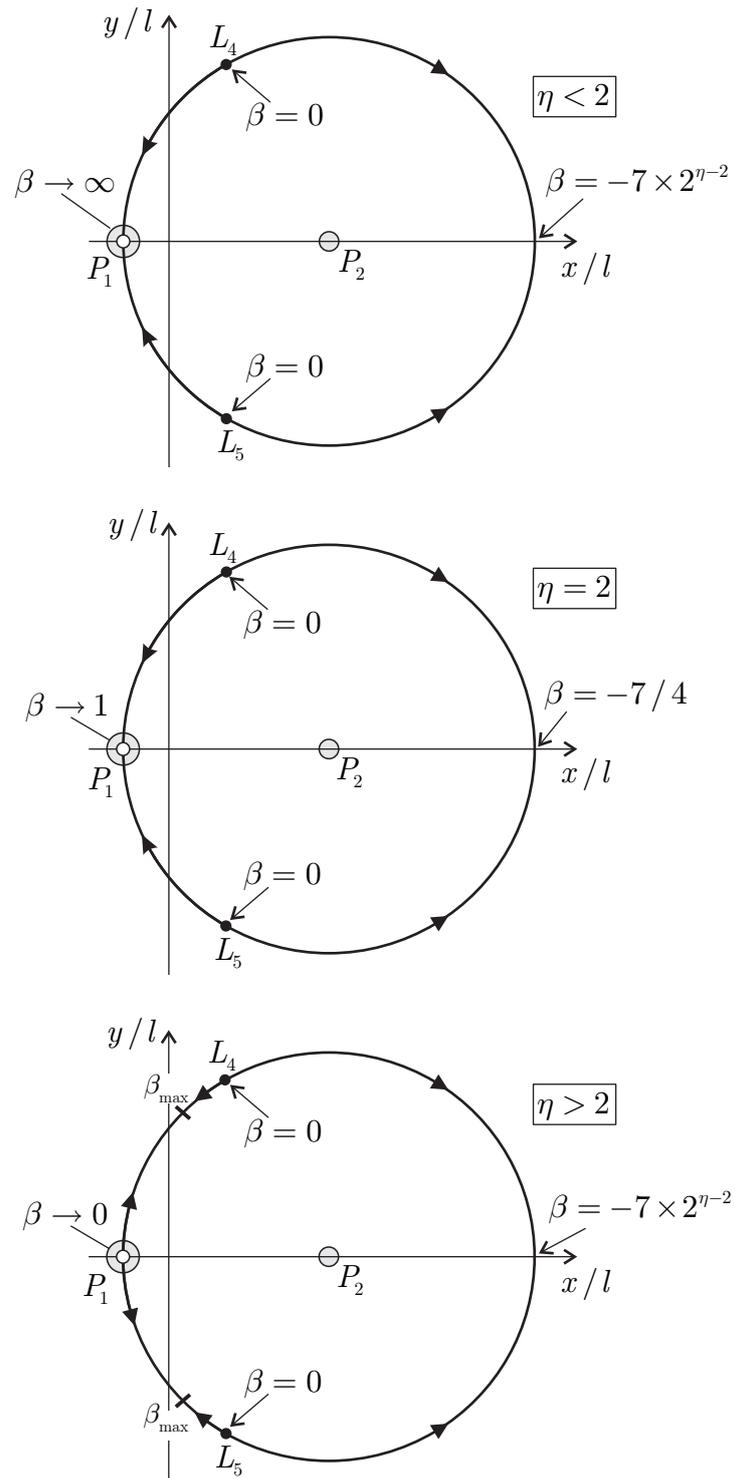


Fig. 4 Triangular-type equilibrium points locus (arrows indicate a $|\beta|$ increase).

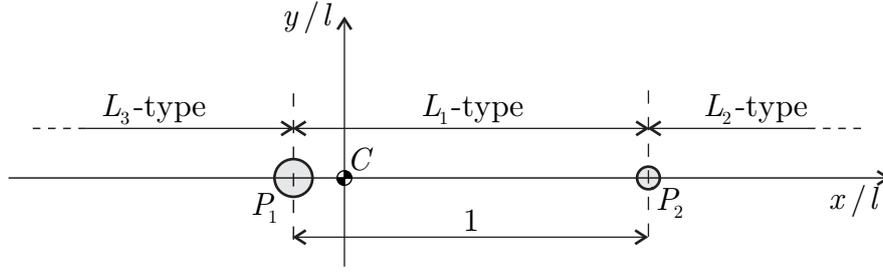
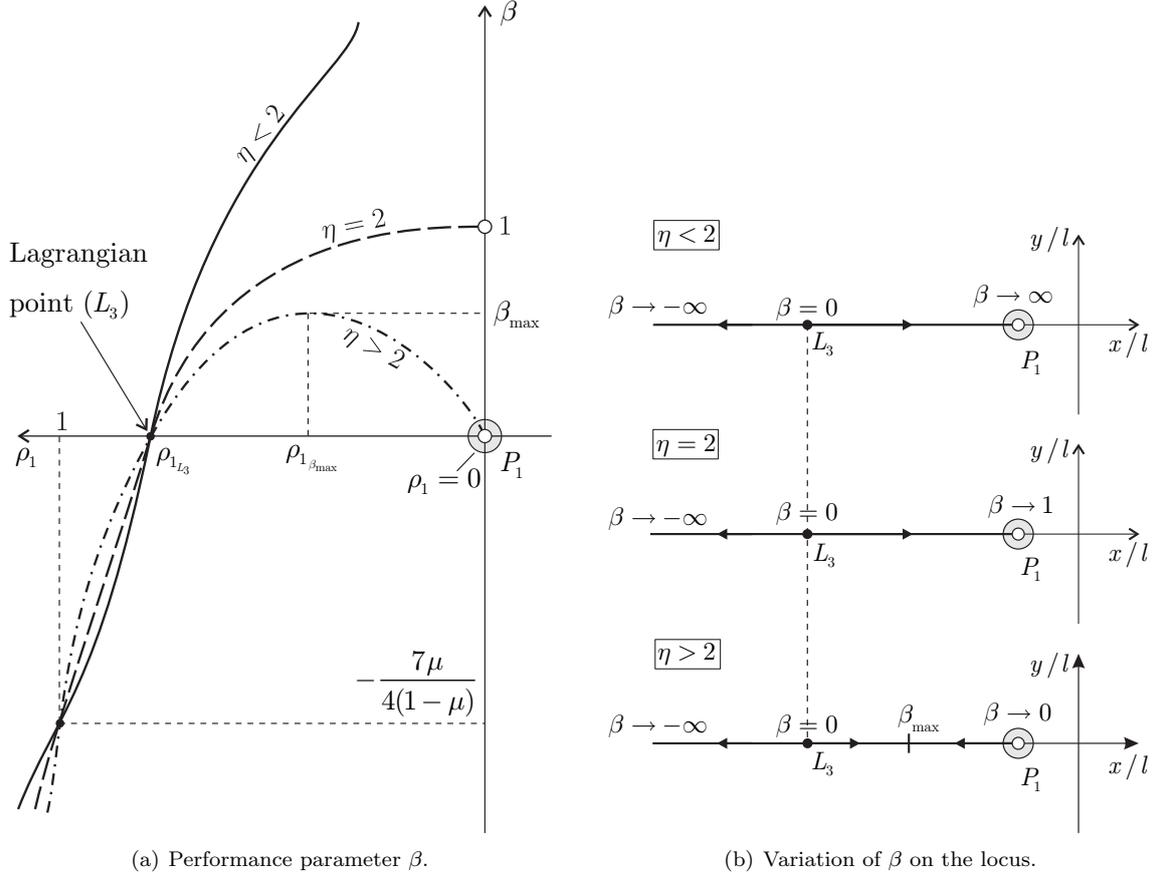


Fig. 5 Nomenclature of the collinear equilibrium points.



(a) Performance parameter β .

(b) Variation of β on the locus.

Fig. 6 Performance parameter as a function of ρ_1 and η for L_3 -type equilibrium points locus (arrows indicate a $|\beta|$ increase).

is not attainable because it would coincide with the attractor P_1 . If $\eta > 2$ and $\beta \in (0, \beta_{\max})$ two AEPs are obtained, whose position along the x axis is between P_1 and the classical Lagrange point L_3 . As the thrust tends to zero ($\beta = 0$), one AEP tends to coincide with L_3 and the other approaches P_1 . Moreover, if β is negative, that is, the propulsive acceleration is in the direction of the attractor P_1 , there exists a single AEP on the left of L_3 . Such a point can be theoretically placed at any distance from P_1 provided that a sufficient value of β is available. Similar conclusions can be drawn for L_1 -type points, that is, points in which $\hat{\rho}_1 \cdot \hat{i} = +1$ and $\rho_1 < 1$, see Figs. 7(a) and 7(b). In this case, η being the same, a negative value of β

corresponds to an AEP between the Lagrangian point L_1 and the attractor P_2 , while the condition $\beta > 0$ permits to move the AEP toward the massive attractor P_1 . Unlike the two previous cases an increase of η corresponds now to a decrease of $|\beta|$.

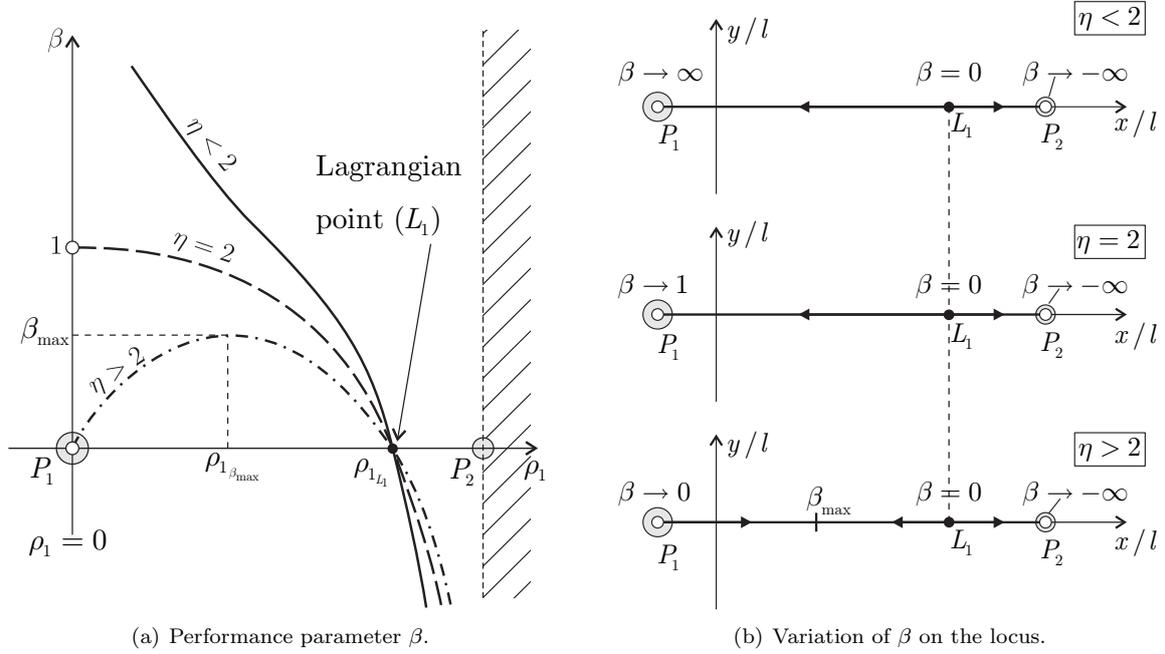


Fig. 7 Performance parameter as a function of ρ_1 and η for L_1 -type equilibrium points locus (arrows indicate a $|\beta|$ increase).

Finally, the contour curves corresponding to the L_2 -type points (when $\hat{\rho}_1 \cdot \hat{i} = +1$ and $\rho_1 > 1$) are illustrated in Fig. 8(a). Fig. 8(a) shows that there exists a suitable value of η , referred to as $\tilde{\eta}$, beyond which the contour curves $\beta = \beta(\rho_1, \eta > \tilde{\eta})$ display both a local minimum (β_{\min}) and a maximum (β_{\max}) at a distance $\rho_{1\beta_{\min}}$ and $\rho_{1\beta_{\max}}$ from the attractor P_1 , respectively. The value of $\tilde{\eta}$ is a function of the dimensionless mass of P_2 only and it decreases with μ , as is shown in Fig. 9. Note however that the presence of the above points of maximum and minimum is confined to high value of η , that is, $\tilde{\eta} > 9.5$, see Fig. 9.

2.2.3 Displaced points

For displaced AEPs that belong to the plane (x, z) of the synodic reference frame (except for the points on x axis, see Fig. 2), the expression for β is obtained by substituting Eq. (16) into (12):

$$\beta = \rho_1^{\eta-2} \left(1 + \frac{\mu}{1-\mu} \frac{\rho_1^3}{\rho_2^3} \right) \quad (31)$$

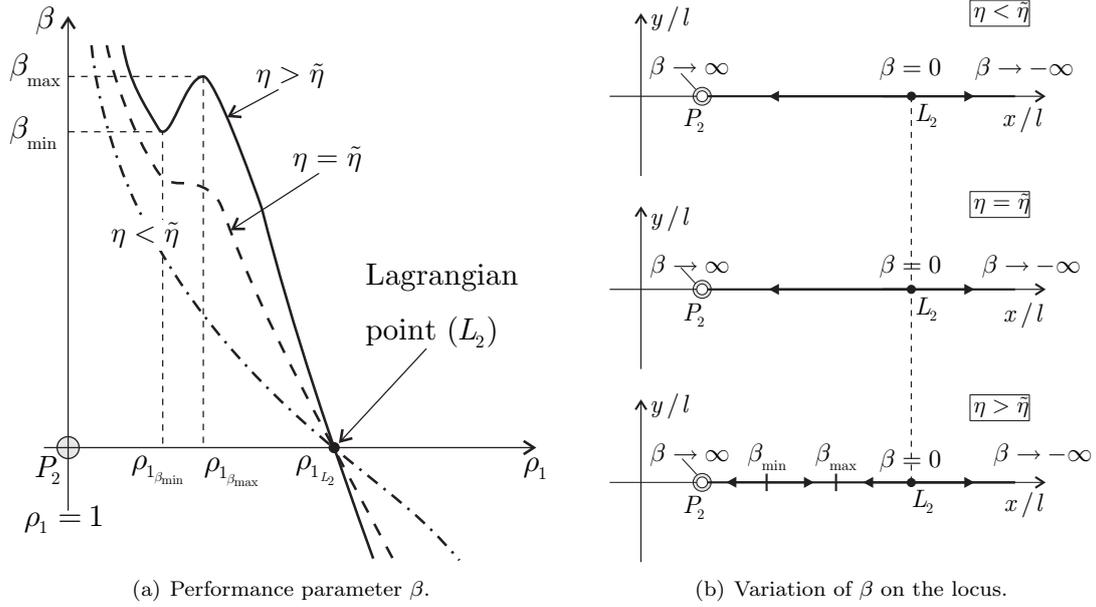


Fig. 8 Performance parameter as a function of ρ_1 and η for L_2 -type equilibrium points locus (arrows indicate a $|\beta|$ increase).

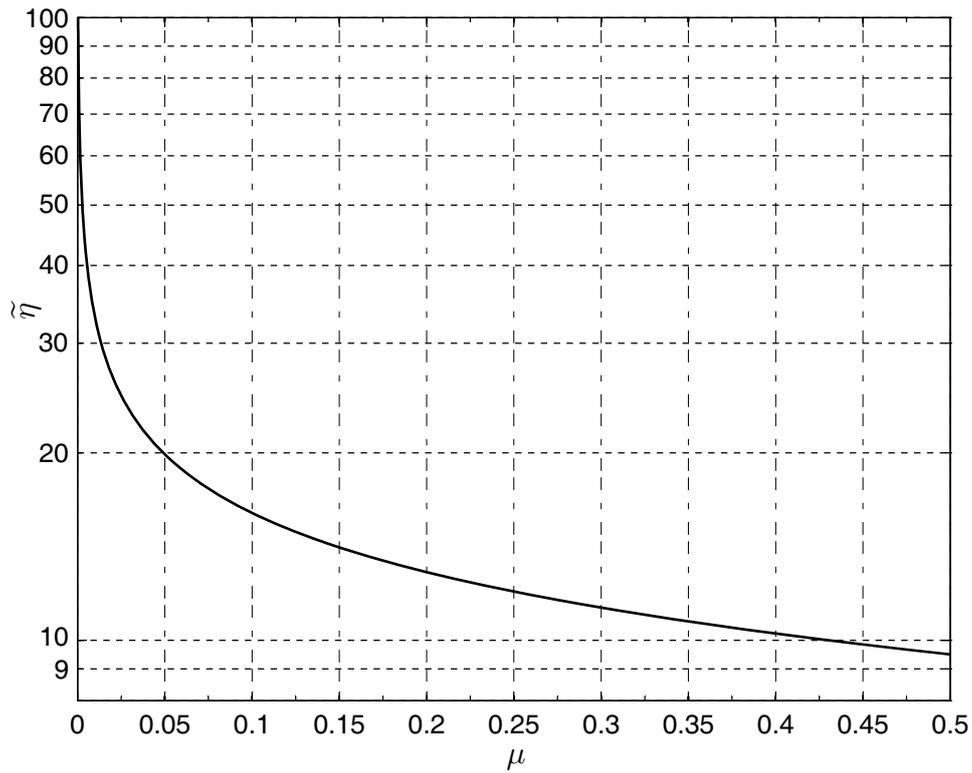


Fig. 9 Parameter $\tilde{\eta}$ as a function of μ for L_2 -type points.

where ρ_2 is given by Eq. (14). In particular for $\eta = 2$ the result obtained with Eq. (31) is in agreement with Kunitsyn and Perezhogin (1978) and Simmons et al. (1985). Note that in the plane (x, z) the distance ρ_1

is

$$\rho_1 = \sqrt{(x/l + \mu)^2 + (z/l)^2} \quad (32)$$

where z/l is obtained from Eq. (15). Accordingly, β is a function only of the dimensionless coordinate x/l . The function $\beta = \beta(x/l, \eta)$ is drawn in Fig. 10.

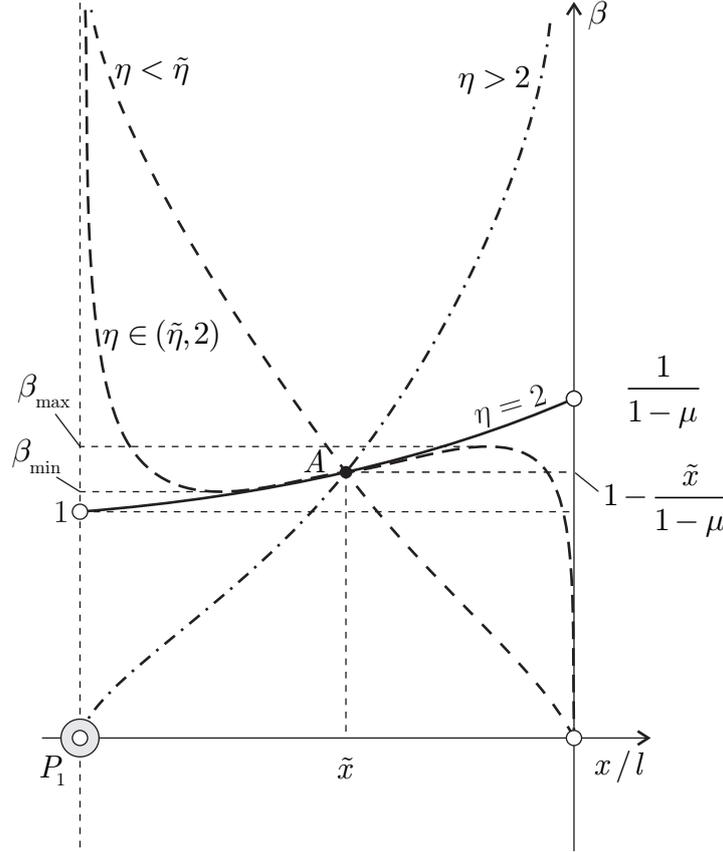


Fig. 10 Performance parameter β as a function of x/l and η for displaced points.

As for the L_2 -type points, it can be shown that a suitable value $\tilde{\eta} < 2$ exists, such that the function $\beta = \beta(x/l, \eta)$ presents two stationary points when $\eta \in (\tilde{\eta}, 2)$. Figure 11 shows how $\tilde{\eta}$ varies with μ .

When $\eta \leq \tilde{\eta}$ or $\eta > 2$, only two equilibrium points exist for a given value of β . On the other hand, when $\eta \in (\tilde{\eta}, 2)$, there is a range of $\beta \in [\beta_{\min}, \beta_{\max}]$, such that four or six equilibrium points are possible. The different kinds of AEP locus are illustrated in Fig. 12, where the point A corresponds to the condition $\rho_1 = 1$. Note that for a given $\rho_1 < 1$ (> 1) it is necessary to increase (decrease) η to decrease the performance parameter β . In particular, according to Kunitsyn and Perezhgin (1978), and Simmons et al. (1985), if $\eta = 2$ two equilibrium points exist only for β ranging in the interval $(1, 1/(1 - \mu))$.

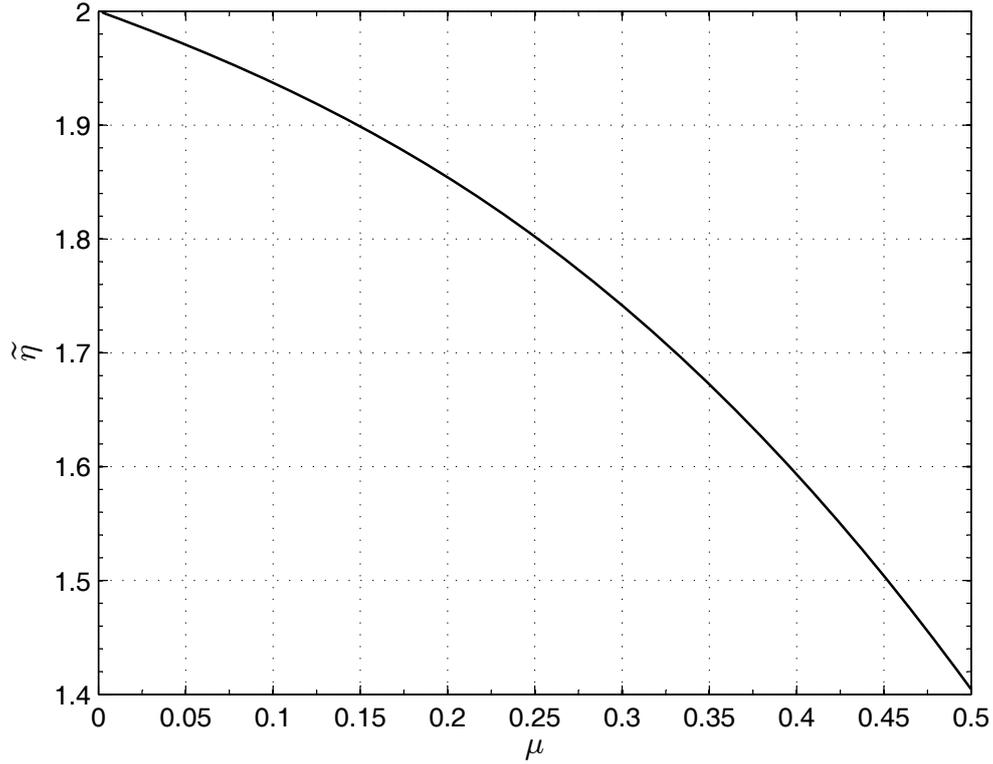


Fig. 11 Parameter $\langle \eta \rangle$ as a function of μ for displaced points.

3 Linear Stability Analysis

It is now possible to better characterize the spacecraft dynamical behavior by investigating its linear stability around the AEPs. Note that a linear analysis provides necessary conditions for stability and sufficient conditions for instability. To begin, introduce the second-order tensors \mathbf{K} and \mathbf{E} :

$$\mathbf{K} \triangleq \nabla \left[-\frac{1-\mu}{\rho_1^3} \boldsymbol{\rho}_1 - \frac{\mu}{\rho_2^3} \boldsymbol{\rho}_2 + \beta \frac{1-\mu}{\rho_1^{\eta+1}} \boldsymbol{\rho}_1 - \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{r}) \right] \Big|_{\mathbf{r}_0} \quad (33)$$

$$\hat{\mathbf{k}} \times \mathbf{r} = \mathbf{E} \cdot \mathbf{r} \quad (34)$$

where \mathbf{r}_0 is the dimensionless vector that defines the position of a generic AEP with respect to the center of mass of the synodic frame. The variational equation of motion is obtained by linearizing the nonlinear equation (4) [where f is given by Eq. (16)] around the equilibrium position \mathbf{r}_0 . To this end, use the transformation $\mathbf{r} = \mathbf{r}_0 + \delta \mathbf{r}$ and define a state vector $\mathbf{x} \triangleq [\delta \mathbf{r}, \delta \mathbf{r}']^T$. The linearized equation takes the form

$$[\mathbf{x}']_{\mathcal{T}} = \mathbb{M}[\mathbf{x}]_{\mathcal{T}} \quad (35)$$

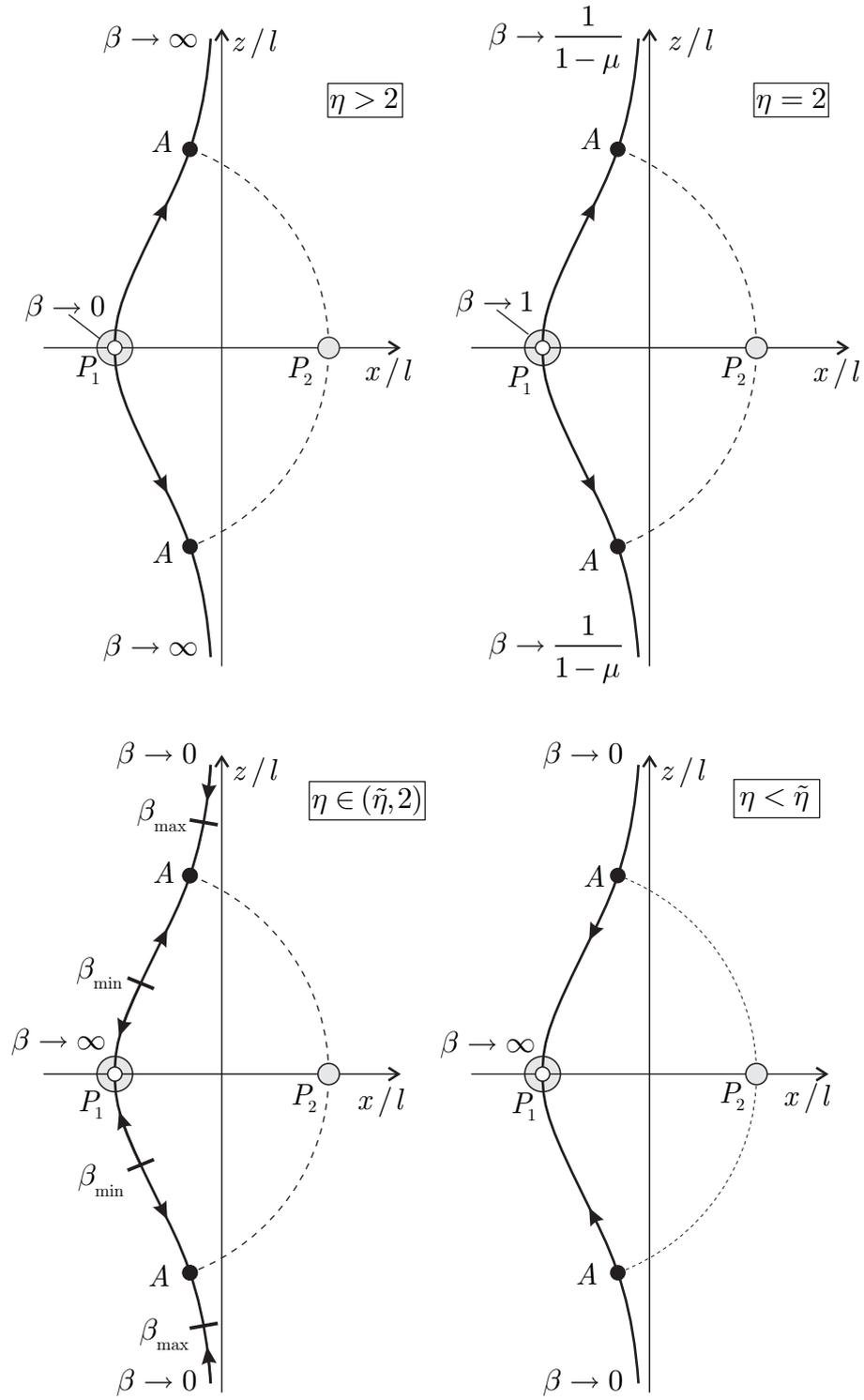


Fig. 12 Variation of β on the displaced points locus (arrows indicate a $|\beta|$ increase).

with

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{K}^T & -2\mathbf{E} \end{bmatrix} \quad (36)$$

where \mathbb{O} is a 3×3 zero matrix, \mathbb{I} is a 3×3 identity matrix, while matrices \mathbb{K} and \mathbb{E} contain the components of the second order tensors \mathbf{K} and \mathbf{E} in the synodic frame, that is

$$\mathbb{K} \triangleq [\mathbf{K}]_{\mathcal{T}} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \quad \mathbb{E} \triangleq [\mathbf{E}]_{\mathcal{T}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (37)$$

The value of all of matrix entries k_{ij} (with $(i, j) = 1, 2, 3$) is detailed in the appendix, as a function of ρ_1 (or ρ_2), η , and μ , for the three different families of AEPs.

If λ is the generic eigenvalue of \mathbb{M} , the characteristic equation of \mathbb{M} can be shown to be in the form

$$\lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \quad (38)$$

or

$$s^3 + as^2 + bs + c = 0 \quad (39)$$

where $s \triangleq \lambda^2$ and a , b , and c are real coefficients that depend on the entries k_{ij} . Note that if $\bar{\lambda}$ is a root of Eq. (38), then so is $-\bar{\lambda}$. For $\bar{\lambda}$ to be a stable eigenvalue, that is, a root with no positive real part, it is necessary that $\bar{\lambda}$ be both imaginary and a simple root of (38). Therefore, according to Abramowitz and Stegun (1965) and using the Descartes sign rule (Henrici, 1988), the system is marginally stable if and only if

$$\Delta_3 < 0 \cap a \geq 0 \cap b \geq 0 \cap c > 0 \quad (40)$$

where Δ_3 is the discriminant of the cubic equation (39), that is:

$$\Delta_3 \triangleq \frac{1}{4} \left(c + \frac{2a^3 - 9ab}{27} \right)^2 + \frac{1}{27} \left(b - \frac{a^2}{3} \right)^3 \quad (41)$$

As will be shown later, in some cases the characteristic equation can be factorized as

$$(s - \tilde{c}) \left(s^2 + \tilde{a}s + \tilde{b} \right) = 0 \quad (42)$$

with suitable values of \tilde{a} , \tilde{b} , and \tilde{c} , still depending on the entries k_{ij} . In such circumstances the necessary and sufficient conditions for the marginal stability become

$$\left(\tilde{a}^2 - 4\tilde{b}\right) > 0 \cap \tilde{a} \geq 0 \cap \tilde{b} > 0 \cap \tilde{c} < 0 \cap \left(\tilde{c}^2 + \tilde{a}\tilde{c} + \tilde{b}\right) \neq 0 \quad (43)$$

Because both the conditions (40) and (43) affect only the coefficients of the characteristic equation, the stability of the linearized system can be investigated without the need of calculating the roots of Eq. (38). Moreover, with the aid of the results in the appendix, it can be verified that for a given system of attractors the coefficients of the characteristic equation are a function only of η and ρ_1 (or ρ_2). Therefore, it is possible to draw the regions of stable AEPs for the different families of points, as is now discussed in detail.

3.1 Collinear points stability

The characteristic equation for collinear points is in the form of Eq. (42) with

$$\tilde{a} = 4 - k_{11} - k_{22} \quad , \quad \tilde{b} = k_{11} k_{22} \quad , \quad \tilde{c} = k_{33} \quad (44)$$

where k_{11} , k_{22} , and k_{33} are given in Eqs. (51)–(52) or (56)–(57). The stability conditions, provided by Eq. (43), must be specialized to the different families of AEPs.

For L_3 -type points, taking into account that $k_{33} < 0$ and $k_{22} < 0$, the stability conditions (43) reduce to

$$(4 - k_{11} - k_{22})^2 - 4 k_{11} k_{22} > 0 \cap k_{11} < 0 \quad (45)$$

where k_{22} and k_{11} are given by (52) and (54), respectively. First assume that $k_{11} < 0$. The left inequality (45) can be written as

$$(k_{11} - k_{22})^2 - 8 (k_{11} + k_{22}) + 16 > 0 \quad (46)$$

which is always fulfilled because $k_{22} < 0$. That means that L_3 -type points are stable provided that $k_{11} < 0$. From Eq. (54) the sign of k_{11} is a function of $\partial\beta/\partial\rho_1$. According to Fig. 6(a), one concludes that $k_{11} < 0$

if $\eta > 2 \cap \rho_1 < \rho_{1\beta_{\max}}$, where $\rho_{1\beta_{\max}}$ corresponds to the distance ρ_1 at which the function $\beta = \beta(\rho_1, \eta > 2)$ attains its maximum value. The variation of $\rho_{1\beta_{\max}}$ with η , for a given value of μ , is qualitatively drawn in Fig. 13(a) along with the stability region of L_3 -type points.

Similar considerations apply to L_1 -type points, and the corresponding stability region is shown in gray in Fig. 13(b). Note that for both Figs. 13(a) and 13(b) the boundary of the stability region is to be excluded as it corresponds to the condition $k_{11} = 0$. Figures 14(a) and 14(b) show the stability regions corresponding to $\mu = 0.01$. Finally, the stability conditions (43) for L_2 -type points can be simplified by

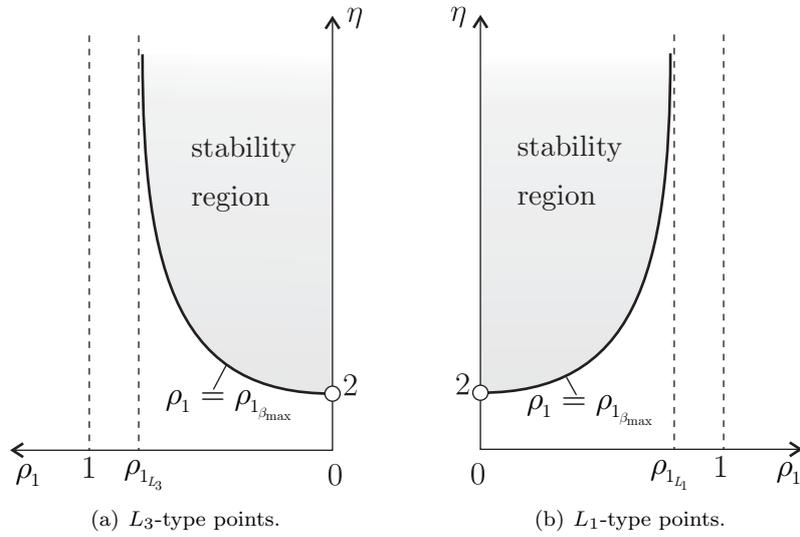


Fig. 13 Distance $\rho_{1\beta_{\max}}$ as a function of η (with μ given) for L_3 and L_1 type points.

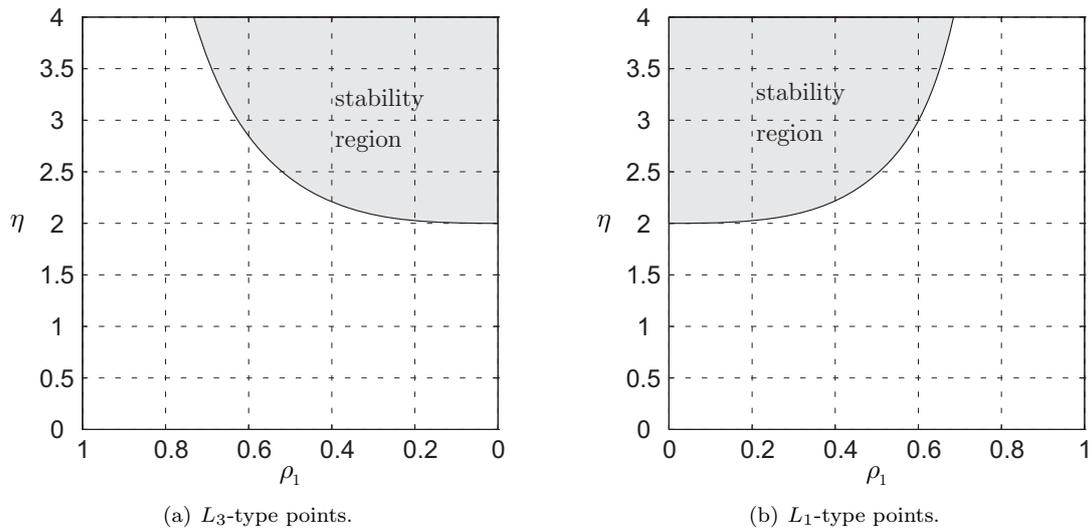


Fig. 14 Stability region with $\mu = 0.01$ for L_3 and L_1 type points.

taking into account the actual value of ρ_1 . In fact, when $\rho_1 < 2$ the stability conditions coincide with those shown in Eq. (45) in which k_{11} and k_{22} are now given by Eqs. (56) and (57). The study of the stability region proceeds as in case of L_3 -type points, and the corresponding graphical results are summarized in Fig. 15. Note that the value $\tilde{\eta}$ shown in Fig. 15 depends on μ and can be taken from Fig. 9.

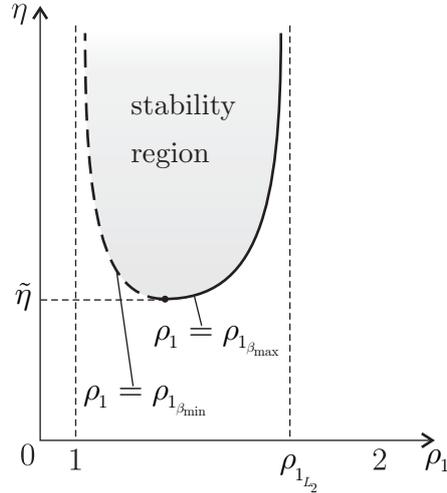


Fig. 15 Position of stationary values of β for L_2 -type points.

On the contrary, when $\rho_1 > 2$ the conditions (43) must be studied with a numerical approach. For a given value of μ the stability region (with the exclusion of the boundary line) can be drawn in the plane (ρ_1, η) . An example of this kind of study is shown in Fig. 16 where the region corresponding to $\rho_1 < 2$ is also displayed for the sake of completeness. Note that the equilibrium point corresponding to $\rho_1 = 2$ is unstable. In fact in such a case $\tilde{b} = 0$, see Eqs. (43).

Figure 16 shows that for $\eta = 0$ there exist stable points only provided that $\rho_1 > 2$. Note that the results of Fig. 16 do not match that of Morimoto et al. (2007) because of the different assumptions made on the thrust direction. In fact Morimoto et al. (2007) uses a constant thrust vector, that is, a vector with a constant modulus and a fixed direction. Accordingly, in Morimoto et al. (2007) the gradient of the propulsive acceleration is equal to zero. In this paper, instead, when $\eta = 0$ the thrust modulus is constant (see Eq. (16)), but its orientation is always radial, see Eq. (2). Therefore, when the system is perturbed, the gradient of the propulsive acceleration is different from zero, see Eq. (33).

The existence of the stability regions shown in Fig. 16 is confirmed by numerical simulations of the perturbed trajectory. For example, according to Fig. 16, the L_2 -type collinear point located at $\rho_1 = 1.5$

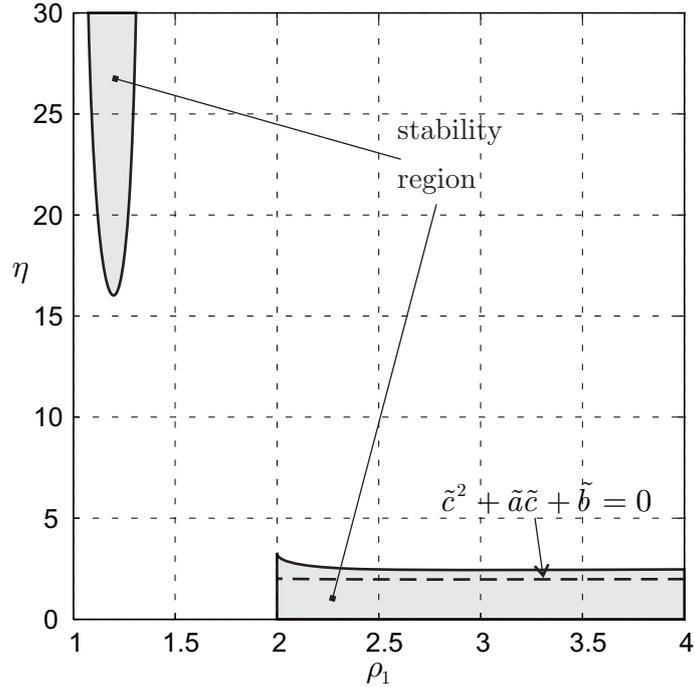


Fig. 16 Stability region for L_2 -type collinear points with $\mu = 0.1$ (the dashed line is excluded).

is unstable both for $\eta = 0$ and $\eta = 1$ (see Figs. 17(a) and 17(c)). On the contrary, assuming $\rho_1 = 2.5$, Figs. 17(b) and 17(d) prove that the perturbed motion around AEP is actually stable.

3.2 Triangular points stability

The characteristic equation for triangular type points is in the form of Eq. (42) with

$$\tilde{a} = 4 - k_{11} - k_{22} \quad , \quad \tilde{b} = k_{11}k_{22} - k_{12}^2 \quad , \quad \tilde{c} = k_{33} \quad (47)$$

where k_{11} , k_{22} , and k_{33} are given by Eqs. (60)–(62).

The conditions (43) (with the exception of $\tilde{c} < 0$, which is always satisfied) must be studied numerically to find the stability region in the plane (ρ_1, η) . An example is shown in Fig. 18 for some values of μ , including the critical value (Szebehely, 1967) $\mu^* \triangleq (1 - \sqrt{23/27})/2 \simeq 3.85209 \times 10^{-2}$, which defines the stability of the classic solution with $\rho_1 = 1$.

As long as $\mu \geq \mu^*$, the plane (ρ_1, η) contains two distinct stability regions, see Fig. 18(a), that exclude the case $\rho_1 = 1$. As μ is decreased, the two regions get closer and eventually they touch when $\mu = \mu^*$,

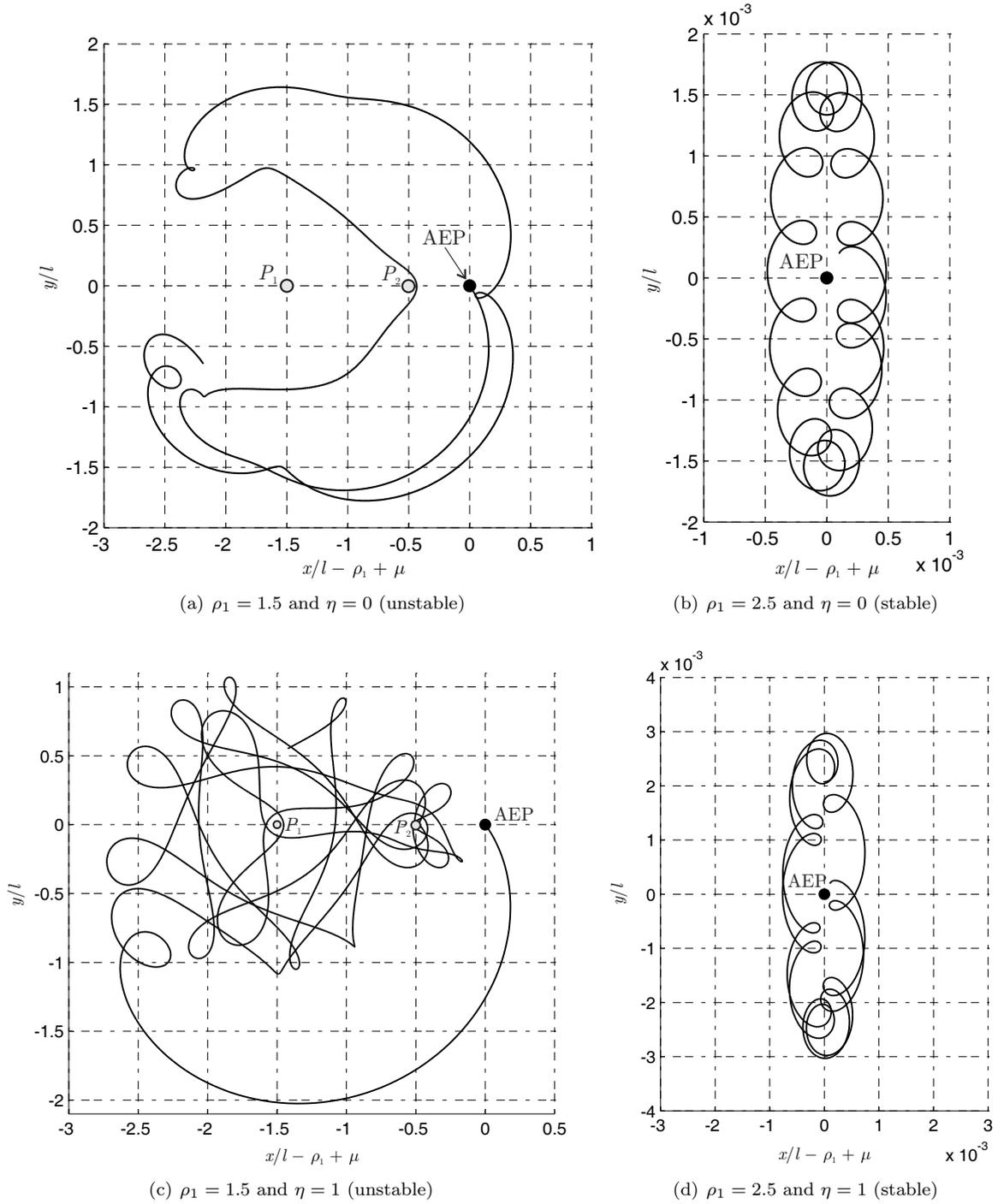


Fig. 17 Numerically integrated perturbed trajectory (ten-periods time span) around an L_2 -type collinear point with $\mu = 0.1$.

see Fig. 18(b). Finally, if $\mu < \mu^*$ there exists a single stability region in the plane (ρ_1, η) , as shown in Fig. 18(c). Note that, as in the previous case, the boundary line does not belong to the stability region.

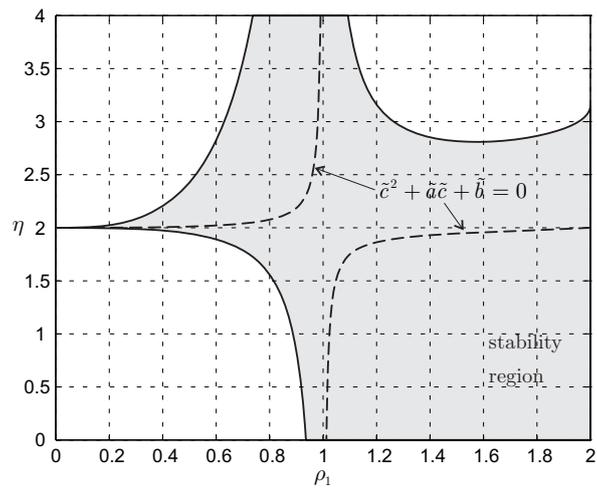
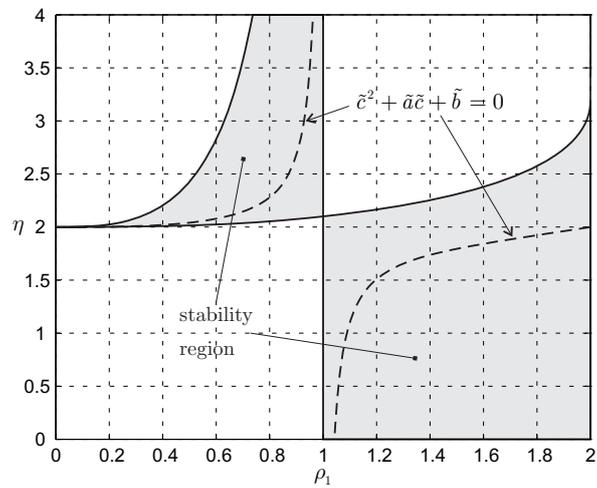
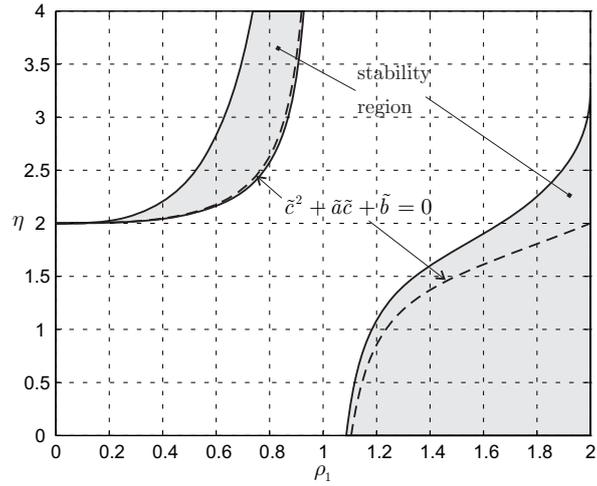


Fig. 18 Stability region for triangular points for different values of μ (the dashed lines are excluded).

3.3 Displaced points stability

The characteristic equation for displaced points is in the form of Eq. (39) in which, bearing in mind Eq. (68), the coefficients a , b , and c can be written as

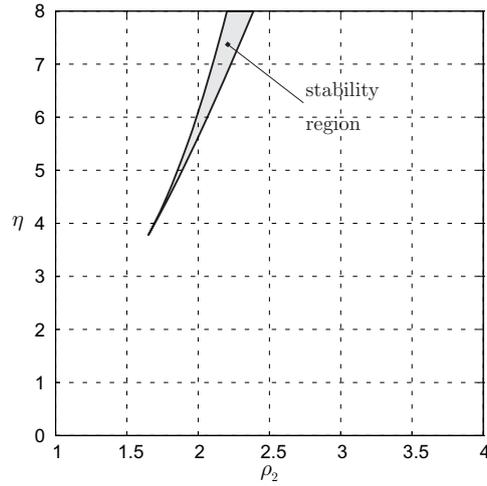
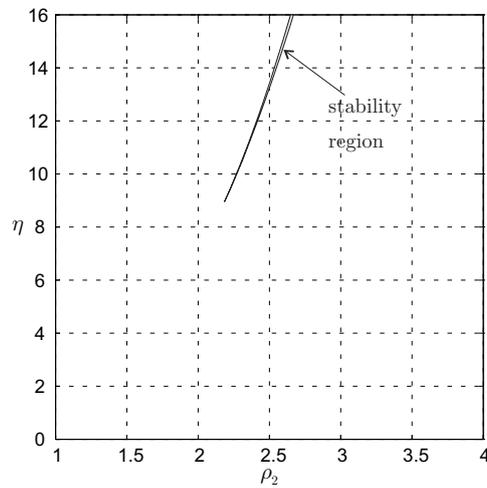
$$a = 3 - k_{11} - k_{33} \quad , \quad c = k_{13}^2 - k_{11} k_{33} \quad , \quad b = k_{11} - 3 k_{33} - c \quad (48)$$

where k_{11} , k_{33} , and k_{13} are obtained from Eqs. (67), (69) and (70).

The stability region must be investigated with a numerical approach by looking for the pairs ρ_2 and η that, for a given value of μ , satisfy the condition (40). The results are summarized in Fig. 19, in which the stability region does not include the boundary line.

4 Conclusions

Under the assumption of a spacecraft subject to a radial propulsive acceleration with respect to the massive attractor, a new approach has been discussed for the analysis of artificial equilibrium points in the circular restricted three body problem. The concept of generalized sail is capable of describing, by means of a unified mathematical model, the position and the linear stability of AEPs for different propulsion systems of practical interest. Within this new model the spacecraft propulsive acceleration depends on the available propulsion system technology by means of two parameters, η and β , one defining the propulsion system type, and the other that quantifies the propulsion system performance. While the paper contains an analysis about the order of magnitude of current and near future technological capabilities, the discussion is given in very general terms and, in particular, it includes values of η and β that do not correspond to any currently available propulsion system. The rationale is to provide a certain degree of flexibility in dealing with exotic/future thrusters and space missions. Likewise, although in the Solar System μ does not exceed about 10^{-3} , the paper includes a discussion in which μ ranges in the interval $(0, 0.5]$. The selected range is useful to obtain a comparison with classical results of the existing literature in which the same variation is assumed. The proposed model allows one to compare, either in an analytical or graphical form, the performance required by a given propulsion system to produce and maintain a prescribed AEP. Accordingly, it is possible to quickly obtain information about different propulsion systems and decide

(a) $\mu = 0.1$ (b) $\mu = 0.01$ **Fig. 19** Stability region for displaced points for different values of μ .

upon which one is better suited for a particular mission type. The proposed model can also be extended to deal with an elliptical restricted three body problem and with the more general situation in which the spacecraft thrust is not constrained to coincide with the radial direction. The latter aspects are currently under development.

A Entries of \mathbb{K} -matrix

Consider the second-order tensor \mathbf{K} defined by means of Eq. (33). The entries of the symmetric matrix \mathbb{K} , corresponding to the components of \mathbf{K} in the synodic frame $\mathcal{T}(C; x, y, z)$, are obtained observing that

$$\mathbb{K} = \left[\frac{1-\mu}{\rho_1^5} \left(3 [\boldsymbol{\rho}_1]_{\mathcal{T}} [\boldsymbol{\rho}_1]_{\mathcal{T}}^T - \rho_1^2 \mathbb{I} \right) + \frac{\mu}{\rho_2^5} \left(3 [\boldsymbol{\rho}_2]_{\mathcal{T}} [\boldsymbol{\rho}_2]_{\mathcal{T}}^T - \rho_2^2 \mathbb{I} \right) - \mathbb{E}^2 - \beta \frac{1-\mu}{\rho_1^{\eta+3}} \left[(\eta+1) [\boldsymbol{\rho}_1]_{\mathcal{T}} [\boldsymbol{\rho}_1]_{\mathcal{T}}^T - \rho_1^2 \mathbb{I} \right] \right] \Big|_{\mathbf{r}_0} \quad (49)$$

where \mathbb{E} is given by Eq. (37), and β takes that particular value for which \mathbf{r}_0 corresponds to an equilibrium position. As a result, the entries of \mathbb{K} take different expressions for the distinct families of AEPs.

A.1 Collinear points: L_3 -type

The expression for β is obtained by combining Eqs. (10) and (16), and

$$[\boldsymbol{\rho}_1]_{\mathcal{T}} = -[\rho_1 \ 0 \ 0]^T \quad , \quad [\boldsymbol{\rho}_2]_{\mathcal{T}} = -[(\rho_1 + 1) \ 0 \ 0]^T \quad (50)$$

Therefore, Eq. (49) gives the following results

$$k_{11} = 2 \frac{1-\mu}{\rho_1^3} + 2 \frac{\mu}{(\rho_1 + 1)^3} + 1 - \frac{\eta}{\rho_1} \left[\frac{1-\mu}{\rho_1^2} + \frac{\mu}{(\rho_1 + 1)^2} - \rho_1 - \mu \right] \quad (51)$$

$$k_{22} = k_{33} + 1 = \frac{\mu}{\rho_1} \left[\frac{1}{(\rho_1 + 1)^3} - 1 \right] \quad (52)$$

$$k_{12} = k_{21} = k_{23} = k_{32} = k_{13} = k_{31} = 0 \quad (53)$$

Note that Eq. (51) can also be rearranged into the following equivalent expression:

$$k_{11} = -\frac{\partial \beta}{\partial \rho_1} \left(\frac{1-\mu}{\rho_1^\eta} \right) \quad (54)$$

A.2 Collinear points: L_1 -type and L_2 -type

In this case

$$[\boldsymbol{\rho}_1]_{\mathcal{T}} = [\rho_1 \ 0 \ 0]^T \quad , \quad [\boldsymbol{\rho}_2]_{\mathcal{T}} = [(\rho_1 - 1) \ 0 \ 0]^T \quad (55)$$

and Eq. (49) provides

$$k_{11} = 2 \frac{1-\mu}{\rho_1^3} \pm 2 \frac{\mu}{(\rho_1-1)^3} + 1 - \frac{\eta}{\rho_1} \left[\frac{1-\mu}{\rho_1^2} \pm \frac{\mu}{(1-\rho_1)^2} - \rho_1 + \mu \right] \quad (56)$$

$$k_{22} = k_{33} + 1 = \frac{\mu}{\rho_1} \left[1 \pm \frac{1}{(1-\rho_1)^3} \right] \quad (57)$$

$$k_{12} = k_{21} = k_{23} = k_{32} = k_{13} = k_{31} = 0 \quad (58)$$

where the sign of the terms containing \pm is $+$ for L_2 -type points and $-$ for L_1 -type points.

A.3 Triangular-type points

The triangular-type points are characterized by means of the vectors

$$[\boldsymbol{\rho}_1]_{\mathcal{T}} = \begin{bmatrix} \rho_1^2/2 \\ \pm \sqrt{\rho_1^2 - \rho_1^4/4} \\ 0 \end{bmatrix}, \quad [\boldsymbol{\rho}_2]_{\mathcal{T}} = \begin{bmatrix} \rho_1^2/2 - 1 \\ \pm \sqrt{\rho_1^2 - \rho_1^4/4} \\ 0 \end{bmatrix} \quad (59)$$

Because β is given by Eq. (29), from Eq. (49) one obtains

$$k_{11} = \frac{1-\mu}{\rho_1^3} \left(\frac{3\rho_1^2}{4} - 1 \right) + \mu \left[3 \left(\frac{\rho_1^2}{2} - 1 \right)^2 - 1 \right] + 1 - (1-\mu) \left(\frac{1}{\rho_1^3} - 1 \right) \left(\frac{\eta+1}{4} \rho_1^2 - 1 \right) \quad (60)$$

$$k_{22} = \frac{1-\mu}{\rho_1^3} \left(2 - \frac{3\rho_1^2}{4} \right) + \mu \left(3\rho_1^2 - \frac{3\rho_1^4}{4} - 1 \right) + 1 - (1-\mu) \left(\frac{1}{\rho_1^3} - 1 \right) \left[(\eta+1) \left(1 - \frac{\rho_1^2}{4} \right) - 1 \right] \quad (61)$$

$$k_{33} = -1 \quad (62)$$

$$k_{12} = k_{21} = \pm \sqrt{\rho_1^2 - \frac{\rho_1^4}{4}} \left[\frac{3(1-\mu)}{2\rho_1^3} + 3\mu \left(\frac{\rho_1^2}{2} - 1 \right) - \frac{(1-\mu)(\eta+1)}{2} \left(\frac{1}{\rho_1^3} - 1 \right) \right] \quad (63)$$

$$k_{13} = k_{31} = k_{23} = k_{32} = 0 \quad (64)$$

where the sign of the terms containing \pm is $+$ for points with $y > 0$ and $-$ for points with $y < 0$.

A.4 Displaced points

In this last case one has

$$[\boldsymbol{\rho}_1]_{\mathcal{T}} = [\rho_{1x} \quad 0 \quad \rho_{1z}]^{\text{T}}, \quad [\boldsymbol{\rho}_2]_{\mathcal{T}} = [\rho_{2x} \quad 0 \quad \rho_{1z}]^{\text{T}} \quad (65)$$

where

$$\rho_{1_x} = \rho_{2_x} + 1 = \mu(1 - 1/\rho_2^3) \quad , \quad \rho_{1_z} = \pm \sqrt{\rho_2^2 - (-\mu/\rho_2^3 + \mu - 1)^2} \quad (66)$$

Substituting the expression (31) for β into Eq. (49) provides:

$$k_{11} = \frac{1-\mu}{\rho_1^5} (3\rho_{1_x}^2 - \rho_1^2) + \frac{\mu}{\rho_2^5} (3\rho_{2_x}^2 - \rho_2^2) + 1 - \frac{[(\eta+1)\rho_{1_x}^2 - \rho_1^2]}{\rho_1^2} \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) \quad (67)$$

$$k_{22} = 1 \quad (68)$$

$$k_{33} = \frac{1-\mu}{\rho_1^5} (3\rho_{1_z}^2 - \rho_1^2) + \frac{\mu}{\rho_2^5} (3\rho_{2_z}^2 - \rho_2^2) + 1 - \frac{[(\eta+1)\rho_{1_z}^2 - \rho_1^2]}{\rho_1^2} \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) \quad (69)$$

$$k_{13} = k_{31} = \rho_{1_z} \left[3 \frac{1-\mu}{\rho_1^5} \rho_{1_x} + \frac{3\mu\rho_{2_x}}{\rho_2^5} - \frac{(\eta+1)\rho_{1_x}}{\rho_1^2} \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) \right] \quad (70)$$

$$k_{12} = k_{21} = k_{23} = k_{32} = 0 \quad (71)$$

where $\rho_1 = \sqrt{\rho_2^2 - 1 + 2\mu(1 - 1/\rho_2^3)}$ and the sign of the terms containing \pm is $+$ for points with $z > 0$ and $-$ for points with $z < 0$.

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