



**HAL**  
open science

# Stability of the determination of the surface impedance of an obstacle from the scattering amplitude

Mourad Bellassoued, Mourad Choulli, Aymen Jbalia

► **To cite this version:**

Mourad Bellassoued, Mourad Choulli, Aymen Jbalia. Stability of the determination of the surface impedance of an obstacle from the scattering amplitude. 2012. hal-00659032v2

**HAL Id: hal-00659032**

**<https://hal.science/hal-00659032v2>**

Preprint submitted on 26 Mar 2012 (v2), last revised 16 Dec 2014 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STABILITY OF THE DETERMINATION OF THE SURFACE IMPEDANCE OF AN OBSTACLE FROM THE SCATTERING AMPLITUDE

†MOURAD BELLAÏOUE, ‡MOURAD CHOULLI, AND §AYMEN JBALIA

ABSTRACT. We prove a stability estimate of logarithmic type for the inverse problem consisting in the determination of the surface impedance of an obstacle from the scattering amplitude. We present a simple and direct proof which is essentially based on an elliptic Carleman inequality.

**Key words :** stability estimate of logarithmic type, surface impedance of an obstacle, scattering amplitude, elliptic Carleman inequality.

**AMS subject classifications :** 35R30.

## CONTENTS

1.	Introduction	1
2.	An elliptic Carleman inequality	3
3.	A quantitative estimate of continuation from Cauchy data	7
4.	Lower bound for solutions of elliptic equations	10
5.	Proof of the stability theorem	15
	Appendix A.	17
	References	18

## 1. INTRODUCTION

Let  $D$  be a bounded subset of  $\mathbb{R}^3$ . For simplicity, even if it is not always necessary, we assume that  $D$  is of class  $C^\infty$ . As usual, we denote by  $u^i$  the incident plan wave :  $u^i(x) = e^{ikx \cdot \omega}$ , where  $k > 0$  is the wave number and  $\omega \in \mathbb{S}^2$  is the direction of propagation. A simplest model of the scattering problem for the impedance boundary condition is to find the total field  $u = u^i + u^s$ ,  $u^s$  is the scattered field, such that

$$(1.1) \quad \begin{cases} (\Delta + k^2)u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \partial_\nu u + i\lambda(x)u = 0 & \text{on } \partial D, \\ \lim_{r \rightarrow \infty} r(\partial_r u^s - iku^s) = 0, & r = |x|. \end{cases}$$

Here,  $\lambda$  is the surface impedance of the obstacle  $D$ . The last condition in (1.1) is called the Sommerfeld radiation condition. This condition guarantees that the scattered wave is outgoing.

**Theorem 1.1.** *Assume that  $\lambda \in C(D)$  and  $\lambda \geq 0$ . Then the scattering problem (1.1) has a unique solution  $u \in C(\mathbb{R}^3 \setminus D) \cap C^2(\mathbb{R}^3 \setminus \overline{D})^1$ . Moreover, for any  $M > 0$ , there exists a constant  $C > 0$ , depending only on  $M$ , such that*

$$(1.2) \quad \|u\|_{C(\mathbb{R}^3 \setminus D)} \leq C \text{ for any } \lambda \in C(D), 0 \leq \lambda \leq M,$$

and for any closed subset  $K$  of  $\mathbb{R}^3 \setminus \overline{D}$ ,  $\alpha \in \mathbb{N}^3$ , there exists a constant  $\tilde{C}$ , that can depend only on  $K$ ,  $\alpha$  and  $M$ , such that

$$(1.3) \quad \|\partial^\alpha u\|_{C(K)} \leq \tilde{C} \text{ for any } \lambda \in C(D), 0 \leq \lambda \leq M.$$

---

<sup>1</sup>Note that, with this regularity, Theorem 2.2 in [CK] implies that the solution of (1.1) is analytic in  $\mathbb{R}^3 \setminus \overline{D}$ .

The proof of the existence and the uniqueness part is similar to that of Theorem 3.10 in [CK] (Neumann boundary condition) with slight modifications. For sake of completeness, we give a sketch of this proof in Appendix A. In this appendix we give also the proof of estimates (1.2) and (1.3).

In order to give a regularity result of the solution of (1.1), we need to recall the definition of a boundary vector space introduced in [Ch]. Let  $m$  be a positive integer,  $s \in \mathbb{R}$  and  $1 \leq r \leq \infty$ . We consider the vector space

$$B_{s,r}(\mathbb{R}^m) = \{w \in S'(\mathbb{R}^m); (1 + |\xi|^2)^{s/2} \widehat{w} \in L^r(\mathbb{R}^m)\},$$

where  $S'(\mathbb{R}^m)$  is the space of temperate distributions on  $\mathbb{R}^m$  and  $\widehat{w}$  is the Fourier transform of  $w$ . Equipped with the norm

$$\|w\|_{B_{s,r}(\mathbb{R}^m)} = \|(1 + |\xi|^2)^{s/2} \widehat{w}\|_{L^r(\mathbb{R}^m)},$$

$B_{s,r}(\mathbb{R}^m)$  is a Banach space (it is noted that  $B_{s,2}(\mathbb{R}^m)$  is merely the Sobolev space  $H^s(\mathbb{R}^m)$ ). Using local charts and partition of unity, we construct  $B_{s,r}(\partial D)$  from  $B_{s,r}(\mathbb{R}^2)$  in the same way as  $H^s(\partial D)$  is built from  $H^s(\mathbb{R}^2)$ .

The space  $B_{s,1}(\partial D)$  is very useful because from Theorem 2.1 in [Ch] we know that the multiplication by a function from  $B_{s,1}(\partial D)$  defines a bounded operator from  $H^s(\partial D)$  into itself.

We proceed similarly to the proof of Theorem 2.3 in [Ch] to prove the  $H^\ell$ -regularity of the solutions of (1.1). We obtain with the help of the usual elliptic  $H^\ell$ -regularity (e.g. [LM]) and estimate (1.3) the following theorem.

**Theorem 1.2.** *Let  $\tilde{\omega} \supset \supset D$  be a bounded  $C^\infty$  open subset of  $\mathbb{R}^3$ ,  $\ell \geq 0$  an integer and set  $\omega = \tilde{\omega} \setminus D$ . If  $\lambda \in B_{\ell+1/2,1}(\partial D) \cap C(\partial D)$ , then  $u$ , the solution of the scattering problem (1.1), belongs to  $H^{2+\ell}(\omega)$ . In addition, for any  $M > 0$ , there exists a constant  $C > 0$ , depending only on  $M$  and  $\ell$ , such that*

$$(1.4) \quad \|u\|_{H^{2+\ell}(\omega)} \leq C \text{ for any } \lambda \in C(D), 0 \leq \lambda \leq M \text{ and } \|\lambda\|_{B_{\ell+1/2,1}(\partial D)} \leq M.$$

Since  $u^s$  is a radiating solution to the Helmholtz equation, it follows from Theorem 2.5 in [CK] that it has the asymptotic behavior of an outgoing spherical wave:

$$u^s(x) = \frac{e^{ik|x|}}{|x|} \left[ u_\infty(\widehat{x}) + O\left(\frac{1}{|x|}\right) \right], \quad |x| \rightarrow \infty,$$

uniformly in all directions  $\widehat{x} = x/|x|$ . The function  $u_\infty$  defined on  $\mathbb{S}^2$  is known as the scattering amplitude or the far field pattern.

In the present paper, we investigate the stability issue of the inverse problem consisting in the determination of the surface impedance coefficient  $\lambda$  from the scattering amplitude  $u_\infty$ . Before stating our main theorem, we need to introduce some geometric assumptions on the domain.

First, assume that  $\Omega = \mathbb{R}^3 \setminus D$  has the following uniform exterior sphere-interior cone property.

**(GA1)** There exist  $\rho > 0$  and  $\theta \in ]0, \pi/2[$  with the property that for all  $\tilde{x} \in \partial\Omega$ , we find  $x' \in \mathbb{R}^3 \setminus \overline{\Omega}$  such that  $B(x', \rho) \cap D = \emptyset$ ,  $B(x', \rho) \cap \overline{D} = \{\tilde{x}\}$  and

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^3; (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega, \text{ where } \xi = \frac{\tilde{x} - x'}{|\tilde{x} - x'|}.$$

Set  $\mathcal{B}(\tilde{x}, r) = B(x', \rho + r)$  and make the following second geometric assumption on  $\Omega$ .

**(GA2)** there exist  $C > 0$ ,  $0 < \kappa < 1$  and  $0 < r_0$  such that for all  $\tilde{x} \in \partial\Omega$  and  $0 < r \leq r_0$ ,

$$\mathcal{B}(\tilde{x}, r) \cap \partial\Omega \subset B(\tilde{x}, Cr^\kappa) \cap \partial\Omega.$$

Fix  $s > 0$ ,  $0 < \rho < s$  and let  $x' = (0_{\mathbb{R}^2}, -\rho) \in \mathbb{R}^3$ . A straightforward computation leads

$$B(x', \rho + r) \cap B((0_{\mathbb{R}^2}, -s), s) \subset B\left(0_{\mathbb{R}^3}, \left(\frac{2\rho + r}{1 - \rho}\right) r\right).$$

From this simple observation, we deduce that if  $D$  has the uniform interior sphere property (or equivalently  $\Omega$  has the uniform exterior sphere property) and, under a rigid transform,  $\tilde{x} = 0$  and  $\partial D$  is

represented near 0 by the graph  $\{x = (y, x_3); y \in V, x_3 = \psi(y)\}$ , where  $V$  is a neighborhood of  $0_{\mathbb{R}^2}$ ,  $\psi$  is such that  $\psi(0) = 0$  and  $\psi \leq 0^2$ , then there exist  $C > 0$  and  $r_0 > 0$  such that

$$\mathcal{B}(\tilde{x}, r) \cap \partial\Omega \subset B(\tilde{x}, Cr) \cap \partial\Omega, \quad 0 < r \leq r_0.$$

Therefore, **(GA2)** is satisfied for instance if  $D$  is a  $C^2$  convex bounded subset of  $\mathbb{R}^3$  (and more generally of  $\mathbb{R}^n$ ). Note that **(GA1)** is automatically satisfied when  $D$  is convex.

**Theorem 1.3.** *Let  $M > 0$  and  $0 < \alpha \leq 1$ , there exist  $C > 0$ ,  $\epsilon > 0$ , and  $\sigma > 0$  such that for all  $\lambda, \tilde{\lambda} \in B_{3/2,1}(\partial D) \cap C^\alpha(\partial D)$  satisfying<sup>3</sup>*

$$\|\lambda\|_{B_{3/2,1}(\partial D) \cap C^\alpha(\partial D)} + \|\tilde{\lambda}\|_{B_{3/2,1}(\partial D) \cap C^\alpha(\partial D)} \leq M$$

and

$$\|\lambda - \tilde{\lambda}\|_{C(\partial D)} \leq \epsilon,$$

we have

$$\|\lambda - \tilde{\lambda}\|_{C(\partial D)} \leq C \left| \ln \left( \frac{\ln |\ln \delta|^2}{|\ln \delta|} \right) \right|^{-\sigma}.$$

where,  $\delta = \|u_\infty(\lambda) - u_\infty(\tilde{\lambda})\|_{L^2(\mathbb{S}^2)}$ .

Our result can be seen as an extension of an earlier result by C. Labreuche [La] corresponding to the case where the impedance  $\lambda$  is analytic. Another result of the same kind as ours was proved by E. Sincich [Si] by a different method. In comparison with what have been done previously, we develop in the present work a simple method which is, in some sense, a direct method. Similarly to [La], our proof is based on a lower bound for the  $L^2$ -norm of the solution of the scattering problem in any ball around a boundary point. The crucial step in the proof consists in establishing the dependence of the lower bound on the radius of each ball. Instead of a lower bound of the solution of the scattering problem, the approach used in [Si] uses the boundary version of the so-called  $A_p$  weights.

We make some geometric assumptions that seem somehow restrictive. We choose to make these assumptions for a better presentation and because the proofs are more simple. We believe that some of these geometric assumptions can be relaxed.

The main ingredient in our method is a Carleman inequality with constants depending on an explicit way on data. This is done in section 2. An intermediate result consisting in a quantitative estimate of continuation from Cauchy data is proved section 3. This result is then used in section 4 to derive an appropriate lower bound for the  $L^2$ -norm of the solution of the scattering problem in any ball around a boundary point. The results in sections 2 to 4 are given in an arbitrary dimension greater or equal to two. The last section is devoted to the proof of the stability estimate for our inverse scattering problem.

One can see that these results can be adapted to other problems such as the problem of recovering the corrosion coefficient appearing in some usual corrosion detection problems.

## 2. AN ELLIPTIC CARLEMAN INEQUALITY

Let  $\Omega$  be a Lipschitz bounded domain of  $\mathbb{R}^n$  ( $n \geq 2$ ), with boundary  $\Gamma$ , and let  $0 \leq \psi \in C^2(\overline{\Omega})$  be such that

$$m = \min \left( 1, \min_{\overline{\Omega}} |\nabla \psi| \right) > 0.$$

Fix  $M \geq \max \left( \|\psi\|_{C^2(\overline{\Omega})}, 1 \right)$ , where

$$\|\psi\|_{C^2(\overline{\Omega})} = \sum_{|\alpha| \leq 2} \|\partial^\alpha \psi\|_{C(\overline{\Omega})},$$

and set  $\varphi = e^{\lambda\psi}$ ,  $\lambda > 0$ .

<sup>2</sup>Note that no regularity assumption is required for  $\psi$

<sup>3</sup>Here,  $\|\cdot\|_{B_{3/2,1}(\partial D) \cap C^\alpha(\partial D)} = \|\cdot\|_{B_{3/2,1}(\partial D)} + \|\cdot\|_{C^\alpha(\partial D)}$ .

**Proposition 2.1.** *Let  $v \in H^2(\Omega)$ . Then*

$$\begin{aligned} & \int_{\Omega} e^{2\tau\varphi} (m^4\lambda^4\tau^3\varphi^3v^2 + m^2\lambda^2\tau\varphi|\nabla v|^2) dx \\ & \leq 8 \int_{\Omega} e^{2\tau\varphi} (\Delta v)^2 dx + 48 \int_{\Gamma} e^{2\tau\varphi} (M^3\lambda^3\tau^3\varphi^3v^2 + M\lambda\tau\varphi|\nabla v|^2) d\sigma, \\ & \lambda \geq 6M^3/m^4, \tau \geq 88M^6/m^4. \end{aligned}$$

**Proof.** Set  $\Phi = e^{-\tau\varphi}$ . Then straightforward computations give

$$\begin{aligned} \nabla\Phi &= -\lambda\tau\varphi\Phi\nabla\psi \\ \Delta\Phi &= \lambda^2\tau^2\varphi^2\Phi|\nabla\psi|^2 - \lambda^2\tau\varphi\Phi|\nabla\psi|^2 - \lambda\tau\varphi\Phi\Delta\psi. \end{aligned}$$

Let  $w \in H^2(\Omega)$ . Then from formulae above, we deduce

$$Lw = [\Phi^{-1}\Delta\Phi]w = L_1w + L_2w + cw.$$

Here,

$$\begin{aligned} L_1w &= aw + \Delta w, \\ L_2w &= B \cdot \nabla w + bw, \end{aligned}$$

with

$$\begin{aligned} a &= a(x, \lambda, \tau) = \lambda^2\tau^2\varphi^2|\nabla\psi|^2, \\ B &= B(x, \lambda, \tau) = -2\lambda\tau\varphi\nabla\psi, \\ b &= b(x, \lambda, \tau) = -2\lambda^2\tau\varphi|\nabla\psi|^2, \\ c &= c(x, \lambda, \tau) = -\lambda\tau\varphi\Delta\psi + \lambda^2\tau\varphi|\nabla\psi|^2. \end{aligned}$$

We have

$$(2.1) \quad \int_{\Omega} awB \cdot \nabla w dx = \frac{1}{2} \int_{\Omega} aB \cdot \nabla w^2 dx = -\frac{1}{2} \int_{\Omega} \operatorname{div}(aB)w^2 dx + \frac{1}{2} \int_{\Gamma} aB \cdot \nu w^2 d\sigma$$

and

$$\begin{aligned} (2.2) \quad \int_{\Omega} \Delta w B \cdot \nabla w dx &= - \int_{\Omega} \nabla w \cdot \nabla(B \cdot \nabla w) dx + \int_{\Gamma} B \cdot \nabla w \nabla w \cdot \nu d\sigma \\ &= - \int_{\Omega} B' \nabla w \cdot \nabla w dx - \int_{\Omega} \nabla^2 w B \cdot \nabla w dx + \int_{\Gamma} B \cdot \nabla w \nabla w \cdot \nu d\sigma. \end{aligned}$$

Here,  $B' = (\partial_i B_j)$  is the jacobian matrix of  $B$  and  $\nabla^2 w = (\partial_{ij}^2 w)$  is the hessian matrix of  $w$ .

But,

$$\int_{\Omega} B_i \partial_{ij}^2 w \partial_j w dx = - \int_{\Omega} \partial_j w B_i \partial_{ij}^2 w dx - \int_{\Omega} \partial_i B_i (\partial_j w)^2 dx + \int_{\Gamma} B_i (\partial_j w)^2 \nu_i d\sigma.$$

Therefore,

$$(2.3) \quad \int_{\Omega} \nabla^2 w B \cdot \nabla w dx = -\frac{1}{2} \int_{\Omega} \operatorname{div}(B)|\nabla w|^2 dx + \frac{1}{2} \int_{\Gamma} |\nabla w|^2 B \cdot \nu d\sigma.$$

It follows from (2.2) and (2.3),

$$(2.4) \quad \int_{\Omega} \Delta w B \cdot \nabla w dx = \int_{\Omega} \left[ -B' + \frac{1}{2} \operatorname{div}(B)I \right] \nabla w \cdot \nabla w dx + \int_{\Gamma} B \cdot \nabla w \nabla w \cdot \nu d\sigma - \frac{1}{2} \int_{\Gamma} |\nabla w|^2 B \cdot \nu d\sigma.$$

As before, an integration by parts gives

$$\int_{\Omega} \Delta w b w dx = - \int_{\Omega} b |\nabla w|^2 dx - \int_{\Omega} w \nabla b \cdot \nabla w dx + \int_{\Gamma} b w \nabla w \cdot \nu d\sigma.$$

Then, using the following inequality

$$- \int_{\Omega} w \nabla b \cdot \nabla w dx \geq - \int_{\Omega} (\lambda^2\varphi)^{-1} |\nabla b|^2 w^2 dx - \int_{\Omega} \lambda^2\varphi |\nabla w|^2 dx,$$

we obtain

$$(2.5) \quad \int_{\Omega} \Delta w b w dx \geq - \int_{\Omega} (b + \lambda^2 \varphi) |\nabla w|^2 dx - \int_{\Omega} (\lambda^2 \varphi)^{-1} |\nabla b|^2 w^2 dx + \int_{\Gamma} b w \nabla w \cdot \nu d\sigma.$$

Now a combination of (2.1), (2.4) and (2.5) leads

$$(2.6) \quad \int_{\Omega} L_1 w L_2 w dx - \int_{\Omega} c^2 w^2 dx \geq \int_{\Omega} f w^2 dx + \int_{\Omega} F \nabla w \cdot \nabla w dx + \int_{\Gamma} g(w) d\sigma,$$

where,

$$\begin{aligned} f &= -\frac{1}{2} \operatorname{div}(aB) + ab - (\lambda^2 \varphi)^{-1} |\nabla b|^2 - c^2, \\ F &= -B' + \left( \frac{1}{2} \operatorname{div}(B) - b - \lambda^2 \varphi \right) I, \\ g(w) &= \frac{1}{2} a w^2 B \cdot \nu - \frac{1}{2} |\nabla w|^2 B \cdot \nu + B \cdot \nabla w \nabla w \cdot \nu + b w \nabla w \cdot \nu. \end{aligned}$$

From the following elementary inequality  $(s - t)^2 \geq s^2/2 - t^2$ ,  $s, t > 0$ , we obtain

$$\|Lw\|_2^2 \geq (\|L_1 w + L_2 w\|_2 - \|cw\|_2)^2 \geq \frac{1}{2} \|L_1 w + L_2 w\|_2^2 - \|cw\|_2^2 \geq \int_{\Omega} L_1 w L_2 w dx - \int_{\Omega} c^2 w^2 dx.$$

This and (2.6) imply

$$(2.7) \quad \|Lw\|_2^2 \geq \int_{\Omega} f w^2 dx + \int_{\Omega} F \nabla w \cdot \nabla w dx + \int_{\Gamma} g(w) d\sigma.$$

By a straightforward computation, we prove

$$-\frac{1}{2} \operatorname{div}(aB) = \lambda^3 \tau^3 \operatorname{div}(\varphi^3 |\nabla \psi|^2 \nabla \psi) = \lambda^3 \tau^3 (3\lambda \varphi^3 |\nabla \psi|^4 + \varphi^2 \operatorname{div}(|\nabla \psi|^2 \nabla \psi)).$$

Therefore,

$$-\frac{1}{2} \operatorname{div}(aB) + ab = \lambda^3 \tau^3 (\lambda \varphi^3 |\nabla \psi|^4 + \varphi^3 \operatorname{div}(|\nabla \psi|^2 \nabla \psi)).$$

Hence,

$$(2.8) \quad -\frac{1}{2} \operatorname{div}(aB) + ab \geq \lambda^3 \tau^3 \varphi^3 (\lambda m^4 - 3M^3).$$

From now, we assume that  $\lambda \geq 1$  and  $\tau \geq 1$ . Using  $-c^2 \geq -4\lambda^4 \tau^2 \varphi^2 M^4$ , we deduce from (2.8)

$$(2.9) \quad -\frac{1}{2} \operatorname{div}(aB) + ab - c^2 \geq \lambda^3 \tau^3 \varphi^3 (\lambda m^4 - 3M^3) - 12\lambda^4 \tau^2 \varphi^2 M^4.$$

Next, we estimate  $|\nabla b|^2$ . We have

$$\nabla b = -2\lambda^2 \tau \nabla(\varphi |\nabla \psi|^2) = -2\lambda^2 \tau (\lambda \varphi |\nabla \psi|^2 |\nabla \psi + \varphi \nabla |\nabla \psi|^2).$$

Consequently,

$$-|\nabla b|^2 \geq -10\lambda^6 \tau^2 \varphi^2 M^6.$$

This and (2.9) imply

$$-\frac{1}{2} \operatorname{div}(aB) + ab - c^2 - (\lambda^2 \varphi)^{-1} |\nabla b|^2 \geq \lambda^3 \tau^3 \varphi^3 (\lambda m^4 - 3M^3) - 22\lambda^4 \tau^2 \varphi^3 M^6.$$

That is,

$$f \geq \lambda^3 \tau^3 \varphi^3 (\lambda m^4 - 3M^3) - 22\lambda^4 \tau^2 \varphi^3 M^6.$$

Then,

$$f \geq \frac{1}{2} \lambda^4 \tau^3 \varphi^3 m^4 - 22\lambda^4 \tau^2 \varphi^3 M^6 = \lambda^4 \tau^2 \varphi^3 \left( \frac{1}{2} \tau m^4 - 22M^6 \right), \quad \lambda \geq 6M^3/m^4.$$

Hence,

$$(2.10) \quad f \geq \frac{1}{4} \lambda^4 \tau^3 \varphi^3 m^4, \quad \lambda \geq 6M^3/m^4, \quad \tau \geq 88M^6/m^4.$$

We have

$$(2.11) \quad -B' \xi \cdot \xi = 2\lambda \tau (\lambda \varphi |\nabla \psi \cdot \xi|^2 + \varphi \nabla^2 \psi \xi \cdot \xi) \geq -2\lambda \tau \varphi M |\xi|^2, \quad \xi \in \mathbb{R}^n.$$

On the other hand,

$$\begin{aligned} \frac{1}{2} \operatorname{div}(B) - b - \lambda^2 \varphi &= -\lambda^2 \tau \varphi |\nabla \psi|^2 - \lambda \varphi \Delta \psi + 2\lambda^2 \tau \varphi |\nabla \psi|^2 - \lambda^2 \varphi \\ &= \lambda^2 \tau \varphi |\nabla \psi|^2 - \lambda \varphi \tau \Delta \psi - \lambda^2 \varphi \\ &\geq \lambda^2 \tau \varphi m^2 - \lambda \varphi \tau M - \lambda^2 \varphi, \quad \lambda \geq 4M/m^2. \end{aligned}$$

A combination of this estimate and (2.11) implies

$$(2.12) \quad F\xi \cdot \xi \geq \frac{1}{4} \lambda^2 \tau \varphi m^2, \quad \lambda \geq 6M/m^2, \quad \tau \geq 4/m^2, \quad \xi \in \mathbb{R}^n, \quad |\xi| = 1.$$

For  $g(w)$ , we first note that

$$|bw \nabla w \cdot \nu| = \sqrt{\lambda |\nabla \psi| |b|} |w| \sqrt{(\lambda |\nabla \psi|)^{-1} |b|} |\nabla w \cdot \nu| \leq \lambda |\nabla \psi| |b| w^2 + (\lambda |\nabla \psi|)^{-1} |b| |\nabla w|^2.$$

From this inequality, we easily deduce

$$(2.13) \quad |g(w)| \leq 2(M^3 \lambda^3 \tau^3 \varphi^3 w^2 + M \lambda \tau \varphi |\nabla w|^2).$$

Finally, (2.7), (2.10), (2.12) and (2.13) yield

$$(2.14) \quad \begin{aligned} &\int_{\Omega} (m^4 \lambda^4 \tau^3 \varphi^3 w^2 + m^2 \lambda^2 \tau \varphi |\nabla w|^2) dx \\ &\leq 4 \int_{\Omega} (Lw)^2 dx + 8 \int_{\Gamma} (M^3 \lambda^3 \tau^3 \varphi^3 w^2 + M \lambda \tau \varphi |\nabla w|^2) d\sigma, \\ &\hspace{15em} \lambda \geq 6M^4/m^4, \quad \tau \geq 88M^6/m^4. \end{aligned}$$

Let us now apply this inequality to  $w = \Phi^{-1}v$ ,  $v \in H^2(\Omega)$ . We have

$$\begin{aligned} |\nabla w|^2 &= |\Phi^{-1} \nabla v - \Phi^{-2} v \nabla \Phi|^2 \geq \frac{1}{2} \Phi^{-2} |\nabla v|^2 - \Phi^{-2} |\Phi^{-1} \nabla \Phi|^2 v^2 \\ &\geq \frac{1}{2} \Phi^{-2} |\nabla v|^2 - \Phi^{-2} \lambda^2 \tau^2 \varphi^2 M^2 v^2. \\ &\geq \frac{1}{2} \Phi^{-2} |\nabla v|^2 - \Phi^{-2} \lambda^2 \tau^2 \varphi^3 M^2 v^2. \end{aligned}$$

Hence,

$$\begin{aligned} &2 \int_{\Omega} (m^4 \lambda^4 \tau^3 \varphi^3 w^2 + m^2 \lambda^2 \tau \varphi |\nabla w|^2) dx \\ &\geq \int_{\Omega} \Phi^{-2} ([2m^4 \lambda^4 \tau^3 - 2\lambda^2 \tau^2 M^2] \varphi^3 v^2 + m^2 \lambda^2 \tau \varphi |\nabla v|^2) dx. \end{aligned}$$

But,

$$2m^4 \lambda^4 \tau^3 - 2\lambda^2 \tau^2 M^2 = m^4 \lambda^4 \tau^3 + m^4 \lambda^4 \tau^3 - 2\lambda^2 \tau^2 M^2 = m^4 \lambda^4 \tau^3 + \lambda^2 \tau^2 (\lambda^2 \tau - M^2).$$

Therefore,

$$2m^4 \lambda^4 \tau^3 - 2\lambda^2 \tau^2 M^2 \geq m^4 \lambda^4 \tau^3, \quad \lambda \geq 6M^3/m^4, \quad \tau \geq 88M^6/m^4.$$

With these inequalities in view, we easily deduce from (2.14),

$$\begin{aligned} &\int_{\Omega} \Phi^{-2} (m^4 \lambda^4 \tau^3 \varphi^3 v^2 + m^2 \lambda^2 \tau \varphi |\nabla v|^2) dx \\ &\leq 8 \int_{\Omega} \Phi^{-2} (\Delta v)^2 + 48 \int_{\Gamma} \Phi^{-2} (M^3 \lambda^3 \tau^3 \varphi^3 v^2 + M \lambda \tau \varphi |\nabla v|^2) d\sigma, \\ &\hspace{15em} \lambda \geq 6M^3/m^4, \quad \tau \geq 88M^6/m^4, \end{aligned}$$

which is the desired inequality.  $\square$

Let  $P$  be a partial differential operator of the form

$$P = \Delta w + A \cdot \nabla + a,$$

where  $A \in L^\infty(\Omega)^n$ ,  $a \in L^\infty(\Omega)$ .

Fix  $\Lambda > 0$  satisfying

$$\Lambda \geq 4 \max \left( \|A\|_{L^\infty(\Omega)^n}^2, \|a\|_{L^\infty(\Omega)}^2 \right).$$

Then a straightforward computation shows

$$(\Delta w)^2 \leq (Pw)^2 + \Lambda (w^2 + |\nabla w|^2) \text{ in } \Omega, \quad \text{for any } w \in H^2(\Omega).$$

Using this inequality, we obtain as an immediate consequence of the previous proposition the following corollary.

**Corollary 2.1.** *For any  $v \in H^2(\Omega)$ , we have*

$$\begin{aligned} & \int_{\Omega} e^{2\tau\varphi} (m^4 \lambda^4 \tau^3 \varphi^3 v^2 + m^2 \lambda^2 \tau \varphi |\nabla v|^2) dx \\ & \leq 32 \int_{\Omega} e^{2\tau\varphi} (Pv)^2 dx + 96 \int_{\Gamma} e^{2\tau\varphi} (M^3 \lambda^3 \tau^3 \varphi^3 v^2 + M \lambda \tau \varphi |\nabla v|^2) d\sigma, \end{aligned}$$

for any

$$\lambda \geq 6M^3/m^4, \quad \tau \geq \max(88M^6, 16\Lambda)/m^4$$

or

$$\lambda \geq \max(6M^3, 16\Lambda)/m^4, \quad \tau \geq 88M^6/m^4.$$

We shall need also the following consequence of Proposition 2.1.

**Corollary 2.2.** *Let  $\tilde{\Lambda}$  be given. Then for any  $v \in H^2(\Omega)$  satisfying*

$$(\Delta v)^2 \leq \tilde{\Lambda} (v^2 + |\nabla v|^2) \text{ in } \Omega,$$

we have

$$\int_{\Omega} e^{2\tau\varphi} (m^4 \lambda^4 \tau^3 \varphi^3 v^2 + m^2 \lambda^2 \tau \varphi |\nabla v|^2) dx \leq 96 \int_{\Gamma} e^{2\tau\varphi} (M^3 \lambda^3 \tau^3 \varphi^3 v^2 + M \lambda \tau \varphi |\nabla v|^2) d\sigma,$$

for any

$$\lambda \geq 6M^3/m^4, \quad \tau \geq \max(88M^6, 16\tilde{\Lambda})/m^4$$

or

$$\lambda \geq \max(6M^3, 16\tilde{\Lambda})/m^4, \quad \tau \geq 88M^6/m^4.$$

### 3. A QUANTITATIVE ESTIMATE OF CONTINUATION FROM CAUCHY DATA

Let  $\Omega$  be as in the previous section. That is a bounded Lipschitz domain of  $\mathbb{R}^n$  with boundary  $\Gamma$ . Let  $\Gamma_0$  be a closed subset of  $\Gamma$  having nonempty interior. We assume that  $\Omega$  has the uniform exterior sphere property at any point of  $\Gamma_0$ :

**(GA0)** there exists  $\rho > 0$  with the property that, for all  $\tilde{x} \in \Gamma_0$ , we find  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$  such that  $B(x_0, \rho) \cap \Omega = \emptyset$  and  $B(x_0, \rho) \cap \overline{\Omega} = \{\tilde{x}\}$ .

We shall use the following notations

$$\mathcal{B}(\tilde{x}, r_1) = B(x_0, \rho + r_1) \quad \mathcal{B}(\tilde{x}, r_1, r_2) = \mathcal{B}(\tilde{x}, r_2) \setminus \overline{\mathcal{B}(\tilde{x}, r_1)}, \quad \mathcal{B} = \mathcal{B}(\tilde{x}, d),$$

where  $d = \text{diam}(\Gamma_0)$ .

Henceforth,  $P$  is an operator with bounded coefficients of the form

$$P = \Delta + A \cdot \nabla + a.$$

Set

$$\Lambda = 4 \max \left( \|A\|_{L^\infty(\Omega)^n}, \|a\|_{L^\infty(\Omega)} \right).$$

**Lemma 3.1.** *There exist two constants  $C > 0$  and  $0 < \gamma < 1$  with the property that, for any  $0 < r \leq d$  and any  $u \in H^2(\Omega)$  satisfying  $Pu = 0$  in  $\Omega$ , we have the following estimate*

$$Cr^2 \|u\|_{L^2(\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega)} \leq \|u\|_{H^1(\Omega)}^{1-\gamma} \left( \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} \right)^\gamma.$$



**Proof.** Pick  $\tilde{x} \in \Gamma_0$ . Let  $x_0$  be as in **(GA0)** and

$$\psi(x) = \psi_{\tilde{x}}(x) = \ln((\rho + d)^2 / |x - x_0|^2).$$

Then

$$|\nabla \psi(x)| = \frac{2}{|x - x_0|} \geq \frac{2}{\rho + d} = m', \quad x \in \bar{\mathcal{B}}.$$

Set  $m = \min(1, m')$  and

$$M = \max_{\tilde{y} \in \Gamma_0} \left( 1, \sum_{|\alpha| \leq 2} \|\partial^\alpha \psi_{\tilde{y}}\|_{C(\bar{\mathcal{B}})} \right).$$

Let  $\chi \in C_c^\infty(\mathcal{B}(\tilde{x}, r))$ ,  $\chi = 1$  on  $\mathcal{B}(\tilde{x}, \frac{3r}{4})$  and  $|\partial^\alpha \chi| \leq Kr^{-|\alpha|}$ ,  $|\alpha| \leq 2$ , where  $K$  is a constant independent on  $r$ .

Let  $u \in H^2(\Omega)$  satisfying  $Pu = 0$  in  $\Omega$ . We apply Corollary 2.1 to  $v = \chi u$ . For  $\lambda = \max(6M^3, 16\Lambda)/m^4$  and  $\tau \geq \tau_0 = 88M^6/m^4$ , we obtain

$$(3.1) \quad C \int_{\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega} e^{2\tau\varphi} u^2 dx \leq \int_{\mathcal{B}(\tilde{x}, r) \cap \Omega} e^{2\tau\varphi} (Qu)^2 dx + \frac{1}{r^2} \int_{\mathcal{B}(\tilde{x}, r) \cap \Gamma} e^{2\tau\varphi} (u^2 + |\nabla u|^2) d\sigma.$$

Here and in the sequel,  $C$  is a generic constant independent on  $r$  and

$$Qu = 2\nabla \chi \cdot \nabla u + \Delta \chi u + A \cdot \nabla \chi u.$$

Using the properties of  $\chi$ , we easily prove

$$\int_{\mathcal{B}(\tilde{x}, r) \cap \Omega} e^{2\tau\varphi} (Qu)^2 dx \leq \frac{C}{r^4} \int_{\mathcal{B}(\tilde{x}, \frac{3r}{4}, r) \cap \Omega} e^{2\tau\varphi} (u^2 + |\nabla u|^2) dx.$$

Therefore, (3.1) implies

$$Cr^4 \int_{\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega} e^{2\tau\varphi} u^2 dx \leq \int_{\mathcal{B}(\tilde{x}, \frac{3r}{4}, r) \cap \Omega} e^{2\tau\varphi} (u^2 + |\nabla u|^2) dx + \int_{\mathcal{B}(\tilde{x}, r) \cap \Gamma} e^{2\tau\varphi} (u^2 + |\nabla u|^2) d\sigma.$$

We have

$$\varphi = e^{\lambda \ln((\rho+d)^2 / |x-x_0|^2)} = \frac{(\rho+d)^{2\lambda}}{|x-x_0|^{2\lambda}}.$$

Consequently,

$$(3.2) \quad Cr^4 e^{2\tau\varphi_0} \int_{\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega} u^2 dx \leq e^{2\tau\varphi_1} \int_{\mathcal{B}(\tilde{x}, \frac{3r}{4}, r) \cap \Omega} (u^2 + |\nabla u|^2) dx + e^{2\tau\varphi_2} \int_{\mathcal{B}(\tilde{x}, r) \cap \Gamma} (u^2 + |\nabla u|^2) d\sigma,$$

where,

$$\varphi_0 = \frac{(\rho+d)^{2\lambda}}{(\rho+\frac{r}{2})^{2\lambda}}, \quad \varphi_1 = \frac{(\rho+d)^{2\lambda}}{(\rho+\frac{3r}{4})^{2\lambda}}, \quad \varphi_2 = \frac{(\rho+d)^{2\lambda}}{\rho^{2\lambda}}.$$

By the mean value theorem, for some  $\theta \in ]0, 1[$ ,

$$\begin{aligned} \varphi_0 - \varphi_1 &= \left( \left( \rho + \frac{r}{2} \right) - \left( \rho + \frac{3r}{4} \right) \right) \frac{-2\lambda(\rho+d)^{2\lambda}}{(\theta(\rho+\frac{r}{2}) + (1-\theta)(\rho+\frac{3r}{4}))^{2\lambda+1}} \\ &\geq \frac{\lambda r}{2} \frac{(\rho+d)^{2\lambda}}{(\rho+\frac{3r}{4})^{2\lambda+1}} \\ &\geq \frac{\lambda r}{2} \frac{(\rho+d)^{2\lambda}}{(\rho+\frac{3d}{4})^{2\lambda+1}} = \alpha r \end{aligned}$$

with

$$\alpha = \frac{\lambda(\rho+d)^{2\lambda}}{2(\rho+\frac{3d}{4})^{2\lambda+1}}.$$

Similarly, we prove

$$\varphi_2 - \varphi_1 \leq \beta r,$$

with

$$\beta = \frac{\lambda(\rho + d)^{2\lambda}}{\rho^{2\lambda+1}}.$$

We obtain from (3.2),

$$Cr^4 \int_{\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega} u^2 dx \leq e^{-\alpha r \tau} \int_{\mathcal{B}(\tilde{x}, \frac{3r}{4}, r) \cap \Omega} (u^2 + |\nabla u|^2) dx + e^{\beta r \tau} \int_{\mathcal{B}(\tilde{x}, r) \cap \Gamma} (u^2 + |\nabla u|^2) d\sigma.$$

In particular,

$$(3.3) \quad Cr^4 \int_{\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega} u^2 dx \leq e^{-\alpha r \tau} \int_{\Omega} (u^2 + |\nabla u|^2) dx + e^{\beta r \tau} \int_{\mathcal{B}(\tilde{x}, r) \cap \Gamma} (u^2 + |\nabla u|^2) d\sigma.$$

Let us introduce the following temporary notations

$$\begin{aligned} A &= \int_{\Omega} (u^2 + |\nabla u|^2) dx, \\ I &= \int_{\mathcal{B}(\tilde{x}, r) \cap \Gamma} (u^2 + |\nabla u|^2) d\sigma, \\ J &= Cr^4 \int_{\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega} u^2 dx. \end{aligned}$$

Then, (3.3) becomes

$$(3.4) \quad J \leq e^{-\alpha r \tau} A + e^{\beta r \tau} I.$$

Let

$$\tau_1 = \frac{\ln(A/I)}{\alpha r + \beta r}.$$

If  $\tau_1 \geq \tau_0$ , then  $\tau = \tau_1$  in (3.4) yields

$$(3.5) \quad J \leq A^{\frac{\alpha r}{\alpha r + \beta r}} I^{\frac{\beta r}{\alpha r + \beta r}} = A^{\frac{\alpha}{\alpha + \beta}} I^{\frac{\beta}{\alpha + \beta}}.$$

If  $\tau_1 < \tau_0$ , then  $A < e^{\tau_0(\alpha + \beta)r} I \leq e^{\tau_0(\alpha + \beta)d} I$ . Since

$$J = Cr^4 \int_{\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega} u^2 dx \leq Cd^2 A,$$

we have,

$$(3.6) \quad J \leq CI = CI^{\frac{\alpha}{\alpha + \beta}} I^{\frac{\beta}{\alpha + \beta}} \leq CA^{\frac{\alpha}{\alpha + \beta}} I^{\frac{\beta}{\alpha + \beta}}.$$

Hence, in any case, one of estimates (3.5) and (3.6) holds. That is, in terms of our original notations,

$$Cr^2 \|u\|_{L^2(\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega)} \leq \|u\|_{H^1(\Omega)}^{1-\gamma} (\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)})^\gamma,$$

with  $\gamma = \frac{\beta}{\alpha + \beta}$ . The proof is then complete.  $\square$

**Corollary 3.1.** *There exist two constants  $C > 0$  and  $0 < \gamma < 1$  with the property that, for any  $0 < r \leq d$  and any  $u \in H^2(\Omega)$  satisfying  $Pu = 0$ , we have the following estimates*

$$(3.7) \quad Cr^2 \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, \frac{r}{4}) \cap \Omega)} \leq \|u\|_{H^2(\Omega)}^{1-\gamma/2} (\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)})^{\gamma/2}.$$

and

$$(3.8) \quad Cr^2 \|u\|_{H^1(\mathcal{B}(\tilde{x}, \frac{r}{4}) \cap \Omega)} \leq \|u\|_{H^2(\Omega)}^{1-\gamma/2} (\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)})^{\gamma/2}.$$

**Proof.** Pick  $\chi \in C_c^\infty(\mathcal{B}(\tilde{x}, \frac{r}{2}))$  satisfying  $\chi = 1$  in  $\overline{\mathcal{B}(\tilde{x}, \frac{r}{4})}$  and  $|\partial^\alpha \chi| \leq Kr^{-|\alpha|}$ ,  $|\alpha| \leq 2$ , where  $K$  is a constant independent on  $r$ . Let  $u \in H^2(\Omega)$  satisfying  $Pu = 0$ . From the usual interpolation inequalities, there exists a constant  $c = c(\Omega) > 0$  such that

$$\|\nabla(\chi u)\|_{L^2(\Omega)} \leq c \| \chi u \|_{L^2(\Omega)}^{1/2} \| \chi u \|_{H^2(\Omega)}^{1/2}.$$

Hence,

$$(3.9) \quad \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, \frac{r}{4}) \cap \Omega)} \leq cr^{-1} \|u\|_{L^2(\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2}.$$

On the other hand, it follows from Lemma 3.1

$$(3.10) \quad Cr^2 \|u\|_{L^2(\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega)} \leq \|u\|_{H^2(\Omega)}^{1-\gamma} (\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)})^\gamma.$$

Therefore, (3.7) is a consequence of (3.9) and (3.10).

Next, as the trace mapping

$$w \in H^2 \rightarrow (w, \nabla w) \in L^2(\Gamma)^{n+1}$$

is bounded and

$$\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} \leq \|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)},$$

we have

$$\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} \leq K' \|u\|_{H^2(\Omega)}.$$

Here  $K'$  is a constant independent on  $r$ .

This estimate in (3.10) yields

$$(3.11) \quad Cr^2 \|u\|_{L^2(\mathcal{B}(\tilde{x}, \frac{r}{2}) \cap \Omega)} \leq \|u\|_{H^2(\Omega)}^{1-\gamma/2} (\|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma)})^{\gamma/2}.$$

We complete the proof by noting that (3.8) follows from a combination of (3.7) and (3.11).  $\square$

#### 4. LOWER BOUND FOR SOLUTIONS OF ELLIPTIC EQUATIONS

As in the previous section,  $\Omega$  is a bounded Lipschitz domain of  $\mathbb{R}^n$ , with boundary  $\Gamma$ , and  $P$  is an elliptic operator of the form

$$P = \Delta + A \cdot \nabla + a,$$

with  $A \in L^\infty(\Omega)^n$  and  $a \in L^\infty(\Omega)$ . Let

$$\Lambda = 4 \max(\|A\|_{L^\infty(\Omega)^n}^2 + \|a\|_{L^\infty(\Omega)}^2).$$

We start with a three sphere inequality. Set  $B(i) = B(0, i)$ ,  $i = 1, 2, 3$  and  $r_0 = \frac{1}{3} \text{diam}(\Omega)$ . Let  $y \in \Omega$  and  $u \in H^1(B(y, 3r))$ , where  $0 < r < \frac{1}{3} \text{dist}(y, \Gamma) (\leq r_0)$ . If

$$v(x) = u(rx + y), \quad x \in B(3),$$

a simple change of variables leads to the following inequalities

$$(4.1) \quad c_* r^{1-n/2} \|u\|_{H^1(B(y, ir))} \leq \|v\|_{H^1(B(i))} \leq c^* r^{-n/2} \|u\|_{H^1(B(y, ir))}.$$

Here,

$$c_* = \min(1, r_0), \quad c^* = \max(1, r_0).$$

In addition, if  $u$  satisfies  $Pu = 0$  in  $B(y, 3r)$ , then a straightforward computation yields

$$(\Delta v)^2 \leq \tilde{\Lambda}(v^2 + |\nabla v|^2) \text{ in } B(3),$$

where,  $\tilde{\Lambda} = \Lambda r_0^2 \max(1, r_0^2)$ .

We apply Corollary 2.2 to  $w = \chi v$ , where  $\chi \in C_c^\infty(U)$ ,  $\chi = 1$  in  $K$ , with

$$U = \{x \in \mathbb{R}^n; 1/2 < |x| < 3\}, \quad K = \{x \in \mathbb{R}^n; 1 \leq |x| \leq 5/2\}.$$

Similarly to the previous section, we prove the following interpolation inequality in which the constant  $C > 0$  and  $0 < \alpha < 1$  depend only on  $\Lambda$  and  $r_0$ .

$$(4.2) \quad \|v\|_{H^1(B(2))} \leq C \|v\|_{H^1(B(1))}^\alpha \|v\|_{H^1(B(3))}^{1-\alpha}.$$

The following lemma is a consequence of (4.1) and (4.2).

**Lemma 4.1.** *There exist  $C > 0$  and  $0 < \alpha < 1$ , depending only on  $\Lambda > 0$  and  $r_0 > 0$  such that, for all  $u \in H^2(\Omega)$  satisfying  $Pu = 0$  in  $\Omega$ ,  $y \in \Omega$  and  $0 < r \leq \frac{1}{3} \text{dist}(y, \Gamma)$ ,*

$$r \|u\|_{H^1(B(y, 2r))} \leq C \|u\|_{H^1(B(y, r))}^\alpha \|u\|_{H^1(B(y, 3r))}^{1-\alpha}.$$

Next, let  $\Omega_0 = \mathbb{R}^n \setminus K$ , where  $K$  is a compact subset of  $\mathbb{R}^n$  with nonempty interior. For simplicity, we assume that  $0 \in \overset{\circ}{K}$ . We also make the assumption that  $\Omega_0$  has the uniform exterior sphere-interior cone property **(GA1)**. Recall that this geometric assumption is the following one.

**(GA1)** There exist  $\rho > 0$  and  $\theta \in ]0, \pi/2[$  with the property that, for all  $\tilde{x} \in \partial\Omega_0$ , we find  $x' \in \mathbb{R}^n \setminus \overline{\Omega_0}$  such that  $B(x', \rho) \cap \Omega_0 = \emptyset$ ,  $B(x', \rho) \cap \overline{\Omega_0} = \{\tilde{x}\}$  and

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega_0, \text{ where } \xi = \frac{\tilde{x} - x'}{|\tilde{x} - x'|}.$$

Fix  $R > 4 \sup_K |x|$  and set  $\Omega = \Omega_0 \cap B(0, R)$ .

**Theorem 4.1.** *Let  $M > 0$ ,  $\tau > 0$  and  $\Lambda_0 > 0$  be given. There exist  $C > 0$ ,  $\eta > 0$  and  $r^* > 0$  such that for all  $u \in H^{5/2}(\Omega)$  satisfying*

$$\begin{cases} Pu = 0 & \text{in } \Omega, \\ |\partial_\nu u| \leq \Lambda_0 |u| & \text{on } \Gamma_0, \\ |u| \geq \tau & \text{in } \Omega \cap \{x \in \mathbb{R}^n; |x| \geq R/12\}, \\ \|u\|_{H^{5/2}(\Omega)} \leq M, \end{cases}$$

$\tilde{x} \in \Gamma_0 = \partial\Omega_0$  and  $0 < r \leq \text{diam}(\Gamma_0)$ , we have,

$$e^{-\frac{C}{r^\eta}} \leq \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma_0)}.$$

**Proof.** Let  $\tilde{x} \in \Gamma_0$  and  $x' \in \mathbb{R}^n \setminus \overline{\Omega}$  such that  $B(x', \rho) \cap \Omega = \emptyset$ ,  $B(x', \rho) \cap \overline{\Omega} = \{\tilde{x}\}$ . Set, for  $0 < r < \text{diam}(\Gamma_0)$ ,

$$x'' = \tilde{x} + r\xi, \quad x_0 = \frac{\tilde{x} + x''}{2}.$$

Clearly,  $B(x_0, r/2) \subset \mathcal{B}(\tilde{x}, r) \cap \Omega$  and, for  $d_0 = |x_0 - \tilde{x}|$ ,  $\rho_0 = (d_0 \sin \theta)/3$ ,  $B(x_0, 3\rho_0) \subset \mathcal{C}(\tilde{x})$ .

By induction in  $k$ , we construct a sequence of balls  $(B(x_k, 3\rho_k))$ , contained in  $\mathcal{C}(\tilde{x})$ , as follows

$$\begin{cases} x_{k+1} = x_k + \alpha_k \xi, \\ \rho_{k+1} = \mu \rho_k, \\ d_{k+1} = \mu d_k, \end{cases}$$

where,

$$d_k = |x_k - \tilde{x}|, \quad \rho_k = c d_k, \quad \alpha_k = (1 - \mu) d_k,$$

with

$$c = \frac{\sin \theta}{3}, \quad \mu = \frac{3 + 2 \sin \theta}{3 + \sin \theta}.$$

Note that our construction of these balls guaranties that

$$(4.3) \quad B(x_{k+1}, \rho_{k+1}) \subset B(x_k, 2\rho_k).$$

Let us denote by  $N$  the smallest integer such that  $d_N \geq R/8$ . Since  $d_N = \mu^N \frac{r}{2}$ ,

$$(4.4) \quad \frac{\ln \frac{R}{4r}}{\ln \mu} \leq N < \frac{\ln \frac{R}{4r}}{\ln \mu} + 1$$

or equivalently,

$$N = \left\lceil \frac{\ln \frac{R}{4r}}{\ln \mu} \right\rceil.$$

If  $0 \leq k \leq N$ , then

$$|x_k| + 3\rho_k \leq |\tilde{x}| + d_N + \sin \theta d_N \leq R/4 + \mu R/4 \leq 3R/4.$$

Here, we used that  $1 < \mu < 2$  and  $d_N = \mu \mu^{N-1} d_0 < \mu \frac{R}{8}$ .

Also, for  $x \in B(x_N, \rho_N)$ ,

$$|x| \geq |x_N| - \rho_N \geq d_N - \frac{\sin \theta}{3} d_N \geq \frac{2}{3} d_N \geq \frac{R}{12}.$$

In other words,

$$(4.5) \quad B(x_k, 3\rho_k) \subset \Omega, \quad 0 \leq k \leq N \text{ and } B(x_N, \rho_N) \subset \Omega \cap \{x \in \mathbb{R}^n; |x| \geq R/12\}.$$

We obtain by applying Lemma 4.1,

$$\rho_0 \|u\|_{H^1(B(x_0, 2\rho_0))} \leq CM^{1-\alpha} \|u\|_{H^1(B(x_0, \rho_0))}^\alpha.$$

But from (4.3),  $B(x_1, \rho_1) \subset B(x_0, 2\rho_0)$ . Therefore

$$(4.6) \quad \rho_0 \|u\|_{H^1(B(x_1, \rho_1))} \leq CM^{1-\alpha} \|u\|_{H^1(B(x_0, \rho_0))}^\alpha.$$

Set

$$I_k = \|u\|_{H^1(B(x_k, \rho_k))}.$$

Then (4.6) can be rewritten as follows

$$I_1 \leq \frac{C}{\rho_0} M^{1-\alpha} I_0^\alpha.$$

Using an induction in  $k$ , we prove

$$I_k \leq \frac{C^{1+\alpha+\dots+\alpha^{k-1}}}{\rho_{k-1}\rho_{k-2}^\alpha \dots \rho_0^{\alpha^{k-1}}} I_0^{\alpha^k} M^{(1-\alpha)(1+\alpha+\dots+\alpha^{k-1})}.$$

From the inequality

$$\rho_{k-1}\rho_{k-2}^\alpha \dots \rho_0^{\alpha^{k-1}} = \mu^m \rho_0^{\frac{1-\alpha^k}{1-\alpha}} \geq \rho_0^{\frac{1-\alpha^k}{1-\alpha}}, \quad \text{with } m = \sum_{j=0}^{k-2} (k-1-j)\alpha^j,$$

it follows

$$(4.7) \quad I_k \leq \left(\frac{C}{\rho_0}\right)^{\frac{1-\alpha^k}{1-\alpha}} M^{1-\alpha^k} I_0^{\alpha^k}.$$

We have

$$\frac{C}{\rho_0} = \frac{6C}{(\sin \theta)r}.$$

Hence, we find  $r^* > 0$  such that

$$\frac{6C}{(\sin \theta)r} \geq 1, \quad \text{if } 0 < r \leq r^*.$$

From now we assume that  $0 < r \leq r^*$ . We derive from (4.7),

$$(4.8) \quad I_N \leq \widetilde{M} \left(\frac{C}{r}\right)^\beta I_0^{\alpha^N},$$

where,

$$\beta = \frac{1}{1-\alpha}, \quad \widetilde{M} = \max(1, M).$$

Now as  $|u| \geq \tau$  in  $B(x_N, \rho_N)$ , we have

$$I_N \geq \tau |\mathbb{S}^{n-1}|^{1/2} \rho_N^{n/2} = \tau |\mathbb{S}^{n-1}|^{1/2} (\mu^N \rho_0)^{n/2}$$

and since  $\mu > 1$ , we deduce,

$$(4.9) \quad I_N \geq Cr^{n/2}.$$

A combination of (4.8) and (4.9) leads

$$Cr^\gamma \leq I_0^{\alpha^N}, \quad \text{with } \gamma = n/2 + \beta.$$

That is,

$$(4.10) \quad (Cr)^{\gamma/\alpha^N} \leq I_0.$$

By (4.4), we have

$$\frac{1}{\alpha^N} = e^{N|\ln \alpha|} < e^{|\ln \alpha|(\ln R + 4|\ln r| + 1)}.$$

Therefore, reducing  $r^*$  if necessary,

$$\frac{1}{\alpha^N} = e^{N|\ln \alpha|} < e^{6|\ln \alpha||\ln r|} = \frac{1}{r^s}, \quad \text{with } s = 6|\ln \alpha|.$$

Reducing once again  $r^*$  if necessary, we assume that  $Cr < 1$  in (4.10) (for any  $0 < r \leq r^*$ ). Then,

$$(Cr)^{\gamma/\alpha^N} \geq (Cr)^{\gamma/r^s} = e^{-\frac{\gamma}{r^s} \ln(\frac{1}{Cr})} \geq r^{-\frac{\gamma}{Cr^s+1}}.$$

This and (4.10) imply, where  $\eta = s + 1$ ,

$$e^{-\frac{C}{r^\eta}} \leq \|u\|_{H^1(B(x_0, \rho_0))} \leq \|u\|_{H^1(\mathcal{B}(\tilde{x}, r) \cap \Omega)}.$$

Combined with (3.8), this estimate yields

$$(4.11) \quad e^{-\frac{C}{r^\eta}} \leq \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma_0)} + \|\nabla u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma_0)}.$$

According to our assumption,

$$|\nabla u|^2 = (\partial_\nu u)^2 + |\nabla_\tau u|^2 \leq \max(1, \Lambda)(u^2 + |\nabla_\tau u|^2).$$

Hence, (4.11) implies

$$(4.12) \quad e^{-\frac{C}{r^\eta}} \leq \|u\|_{H^1(\mathcal{B}(\tilde{x}, r) \cap \Gamma_0)}.$$

We now estimate the  $H^1$  norm in the right hand of the previous inequality by an  $L^2$  norm with the help of an interpolation inequality. To this end, let  $\chi \in C_c^\infty(\mathcal{B}(\tilde{x}, 2r))$  satisfying  $\chi = 1$  in  $\mathcal{B}(\tilde{x}, r)$  and  $|\partial^\alpha \chi| \leq Cr^{-|\alpha|}$ ,  $|\alpha| \leq 2$ .

From classical interpolation inequalities, it follows

$$\begin{aligned} \|u\|_{H^1(\mathcal{B}(\tilde{x}, r) \cap \Gamma_0)} &\leq \|\chi u\|_{H^1(\Gamma_0)} \leq C \|\chi u\|_{H^2(\Gamma_0)}^{1/2} \|\chi u\|_{L^2(\Gamma_0)}^{1/2} \\ &\leq Cr^{-1} \|u\|_{H^2(\Gamma_0)}^{1/2} \|u\|_{L^2(\mathcal{B}(\tilde{x}, 2r) \cap \Gamma_0)}^{1/2}. \end{aligned}$$

On the other hand, by classical trace theorems, we have

$$\|u\|_{H^2(\Gamma_0)} \leq \|u\|_{H^{5/2}(\Omega)} \leq M.$$

The last two estimates together with (4.12) lead to the desired inequality.  $\square$

In the sequel, we assume that  $\Omega_0$  possesses **(GA1)** and

**(GA2)** there exist  $C > 0$ ,  $0 < \kappa < 1$  and  $0 < r_0$  such that, for all  $\tilde{x} \in \Gamma_0$  and  $0 < r \leq r_0$ ,

$$\mathcal{B}(\tilde{x}, r) \cap \Gamma_0 \subset B(\tilde{x}, Cr^\kappa) \cap \Gamma_0.$$

Under this new geometric assumption, we deduce from Theorem 3.1 the following corollary.

**Corollary 4.1.** *Let  $M > 0$ ,  $\tau > 0$  and  $\Lambda_0 > 0$  be given. There exist  $C > 0$ ,  $\eta > 0$  and  $r^* > 0$  such that, for all  $\tilde{x} \in \Gamma_0 = \partial\Omega_0$ ,  $0 < r \leq r_0$  and  $u \in H^{5/2}(\Omega)$  satisfying*

$$\begin{cases} Pu = 0 & \text{in } \Omega, \\ |\partial_\nu u| \leq \Lambda_0 |u| & \text{on } \Gamma_0, \\ |u| \geq \tau & \text{in } \Omega \cap \{x \in \mathbb{R}^n; |x| \geq R/12\}, \\ \|u\|_{H^{5/2}(\Omega)} \leq M, \end{cases}$$

we have,

$$e^{-\frac{C}{r^\eta}} \leq \|u\|_{L^2(\mathcal{B}(\tilde{x}, r) \cap \Gamma_0)}.$$

Next, we derive a result on which is based our stability estimate for the inverse problem consisting in the determination of the surface impedance of an obstacle in terms of boundary Cauchy data. Recall that  $f \in C^\alpha(\Gamma_0)$  if there exists  $L \geq 0$  such that

$$(4.13) \quad |f(x) - f(x')| \leq L|x - y|^\alpha, \quad x, x' \in \Gamma_0.$$

We denote by  $[f]_\alpha$  the infimum of  $L$ 's for which (4.13) is satisfied.

**Proposition 4.1.** *Let  $M > 0$ ,  $\tau > 0$ ,  $0 < \alpha \leq 1$  and  $\Lambda_0 > 0$  be given. There exist  $C > 0$ ,  $\epsilon > 0$  and  $\sigma > 0$  such that, for all  $u \in H^{5/2}(\Omega)$  satisfying*

$$(4.14) \quad \begin{cases} Pu = 0 & \text{in } \Omega, \\ |\partial_\nu u| \leq \Lambda_0 |u| & \text{on } \Gamma_0, \\ |u| \geq \tau & \text{in } \Omega \cap \{x \in \mathbb{R}^n; |x| \geq R/12\}, \\ \|u\|_{H^{5/2}(\Omega)} \leq M \end{cases}$$

and, for all  $f \in C^\alpha(\Gamma_0)$  satisfying  $[f]_\alpha \leq M$ ,  $\|f\|_{L^\infty(\Gamma_0)} \leq \epsilon$ ,

$$\|f\|_{L^\infty(\Gamma_0)} \leq \frac{C}{|\ln [\|fu\|_{L^\infty(\Gamma_0)}]^\sigma|}.$$

The following lemma will be used in the proof of Proposition 4.1. Hereafter,  $r^*$  is the same as in Corollary 4.1.

**Lemma 4.2.** *There exist  $\delta^*$  such that, for all  $u$  as in Corollary 4.1 satisfying  $u \in C(\Gamma_0)$ ,  $\tilde{x} \in \Gamma_0$  and  $0 < \delta \leq \delta^*$ ,*

$$\{x \in B(\tilde{x}, r^*) \cap \Gamma_0; |u(x)| \geq \delta\} \neq \emptyset.$$

**Proof.** Otherwise, we find a sequence  $(\delta_k)$ ,  $0 < \delta_k \leq \frac{1}{k}$ ,  $(u_k)$  satisfying the assumptions of Corollary 4.1 with  $u_k \in C(\Gamma_0)$ , for each  $k$ , and  $(\tilde{x}_k)$  in  $\Gamma_0$  such that,

$$\{x \in B(\tilde{x}_k, r^*) \cap \Gamma_0; |u_k(x)| \geq \delta_k\} = \emptyset.$$

In particular,

$$|u_k| \leq \frac{1}{k} \text{ in } B(\tilde{x}_k, r^*) \cap \Gamma_0.$$

Therefore, we have, by applying Corollary 4.1,

$$e^{-\frac{C}{(\delta_k)^\sigma}} \leq \frac{1}{k} |B(\tilde{x}_k, r^*) \cap \Gamma_0| \leq \frac{1}{k} |\Gamma_0|, \text{ for all } k \geq 1,$$

which is impossible. This leads to the desired contradiction and proves the lemma.  $\square$

**Proof of Proposition 4.1.** Let  $\delta^*$  be as in the previous lemma,  $0 < \delta < \delta^*$ ,  $u \in H^{5/2}(\Omega)$  satisfying (3.14) and  $f \in C^\alpha(\Gamma_0)$ .

Let  $\tilde{x} \in \Gamma_0$ . If  $|u(\tilde{x})| \geq \delta$  then

$$(4.15) \quad |f(\tilde{x})| \leq \frac{1}{\delta} |f(\tilde{x})u(\tilde{x})|.$$

Let  $\tilde{x} \in \Gamma_0$  such that  $|u(\tilde{x})| < \delta$  and set

$$r = \sup\{0 < \rho; |u| < \delta \text{ on } B(\tilde{x}, \rho) \cap \Gamma_0\}.$$

From Lemma 4.2, we know that

$$\{x \in B(\tilde{x}, r^*) \cap \Gamma_0; |u(x)| \geq \delta\} \neq \emptyset.$$

Hence,  $r \leq r^*$  and

$$\partial B(\tilde{x}, r) \cap \{x \in B(\tilde{x}, r^*) \cap \Gamma_0; |u(x)| \geq \delta\} \neq \emptyset.$$

Let  $\hat{x} \in \partial B(\tilde{x}, r)$  be such that  $|u(\hat{x})| \geq \delta$ . We have,

$$|f(\tilde{x})| \leq |f(\tilde{x}) - f(\hat{x})| + |f(\hat{x})| \leq [f]_\alpha |\tilde{x} - \hat{x}|^\alpha + \frac{1}{\delta} |f(\hat{x})u(\hat{x})|$$

and then,

$$|f(\tilde{x})| \leq |f(\tilde{x}) - f(\hat{x})| + |f(\hat{x})| \leq Mr^\alpha + \frac{1}{\delta} |f(\hat{x})u(\hat{x})|.$$

This and (4.15) show

$$(4.16) \quad \|f\|_{L^\infty(\Gamma_0)} \leq Mr^\alpha + \frac{1}{\delta} \|fu\|_{L^\infty(\Gamma_0)}.$$

Since  $|u| \leq \delta$  in  $B(\tilde{x}, r) \cap \Gamma_0$ , Corollary 4.1 implies

$$e^{-\frac{C}{r^\sigma}} \leq \delta |B(\tilde{x}, r) \cap \Gamma_0| \leq \delta |\Gamma_0|$$

or equivalently,

$$r \leq \frac{C}{|\ln \delta|^\sigma}, \text{ with } \sigma = 1/\eta.$$

Hence, (4.16) yields

$$(4.17) \quad \|f\|_{L^\infty(\Gamma_0)} \leq \frac{C}{|\ln \delta|^\sigma} + \frac{1}{\delta} \|fu\|_{L^\infty(\Gamma_0)}, \quad 0 < \delta \leq \delta^*.$$

Set  $\delta = e^{-s}$ . Then, (4.17) takes the form

$$(4.18) \quad \|f\|_{L^\infty(\Gamma_0)} \leq \frac{C}{s^\sigma} + e^s \|fu\|_{L^\infty(\Gamma_0)}, \quad s \geq s^* = |\ln \delta^*|.$$

We use the temporary notation  $N = \|fu\|_{L^\infty(\Gamma_0)}$ . The function  $s \rightarrow \frac{C}{s^\sigma} + Ne^s$  attains its minimum at  $\widehat{s}$  satisfying

$$-\frac{\sigma C}{\widehat{s}^{\sigma+1}} + Ne^{\widehat{s}} = 0.$$

Using the elementary inequality  $s^\lambda \leq e^{\lambda s}$ ,  $s \geq 1$ ,  $\lambda > 0$ , we obtain,

$$\frac{C}{N} = \widehat{s}^{\sigma+1} e^{-\widehat{s}} \leq e^{(\sigma+2)\widehat{s}} \quad \text{if } \widehat{s} \geq 1.$$

That is,

$$(4.19) \quad \frac{1}{\sigma+2} \ln \frac{C}{N} \leq \widehat{s} \quad \text{if } \widehat{s} \geq 1.$$

But

$$\ln \frac{C}{N} \geq \ln \frac{C}{M \|f\|_{L^\infty(\Gamma_0)}}.$$

Therefore, there exists  $\epsilon > 0$  (independent on  $u$  and  $f$ ) such that  $\widehat{s} \geq \max(1, s^*)$ , provided that  $\|f\|_{L^\infty(\Gamma_0)} \leq \epsilon$ . When this last condition is satisfied, we can take  $s = \widehat{s}$  in (4.19). Taking into account (4.18),  $e^{\widehat{s}} = \frac{\sigma C}{N \widehat{s}^{\sigma+1}}$  and the fact that  $\frac{1}{\widehat{s}^{\sigma+1}} \leq \frac{1}{\widehat{s}^\sigma}$ , we easily obtain

$$\|f\|_{L^\infty(\Gamma_0)} \leq \frac{C}{|\ln [\|fu\|_{L^\infty(\Gamma_0)}]|^\sigma},$$

which is the expected inequality.  $\square$

## 5. PROOF OF THE STABILITY THEOREM

In this section we prove Theorem 1.3. The solution of (1.1) corresponding to  $\lambda$  is denoted by  $u(\lambda)$ . Set  $u^s(\lambda) = u(\lambda) - u^i$ .

We start with the following Lemma.

**Lemma 5.1.** *Let  $M > 0$  be given,  $\lambda \in C(\partial D)$ ,  $0 \leq \lambda \leq M$ . Then there exists  $R > 0$ , depending only on  $M$  and  $D$ , such that  $D \subset \subset B(R)$  and*

$$(5.1) \quad |u(\lambda)| \geq 1/2, \quad |x| \geq R.$$

**Proof.** Since

$$|u(\lambda)(x)| = |u^i(x) + u^s(\lambda)(x)| \geq 1 - |u^s(\lambda)(x)|,$$

(5.1) will follow from

$$(5.2) \quad |u^s(\lambda)(x)| \leq 1/2, \quad |x| \geq R.$$

From Green's formula of Theorem 2.4 in [CK], we have,

$$u^s(\lambda)(x) = \int_{\partial D} \left[ \partial_{\nu(y)} \Phi(x, y) u^s(\lambda)(y) - \partial_\nu u^s(\lambda)(y) \Phi(x, y) \right] ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D},$$

where,

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y.$$

Then, (1.2) and the fact that

$$\partial_\nu u^s(\lambda) = -i\lambda u^s(\lambda) - (\partial_\nu u^i + i\lambda u^i) \quad \text{on } \partial D$$

imply

$$|u^s(\lambda)(x)| \leq C \max_{y \in \overline{D}} [|\partial_{\nu(y)} \Phi(x, y)| + |\Phi(x, y)|], \quad x \in \mathbb{R}^3 \setminus \overline{D}, \quad |x| \geq R.$$

A straightforward computation shows that the right hand of the last inequality tends to zero when  $R$  goes to infinity. Then, (5.2) follows.  $\square$



**Proof of Theorem 1.3.** Fix  $R$  as in Lemma 5.1 and set  $\omega = B(3R + 1)$ ,

$$v = u(\lambda) - u(\tilde{\lambda}) = u^s(\lambda) - u^s(\tilde{\lambda}).$$

Recall that by estimate (1.4), we have,

$$(5.3) \quad \|v\|_{H^3(\omega)} \leq C.$$

Here and henceforth,  $C$  is a generic constant that can depend only on  $M$  and  $R$ .

Let  $\omega_0$  be an open subset contained in  $\omega \setminus B(3R)$ . Since  $H^3(\omega) \subset C^{1,1/2}(\bar{\omega})$ , we can apply both Propositions 1 and 2 in [BD]<sup>4</sup>. An usual argument consisting in minimizing the right hand side of estimates in Propositions 1 and 2 in [BD], with respect to the small parameter  $\epsilon$ , leads to the following inequality

$$(5.4) \quad \|v\|_{C^1(\partial D)} \leq \frac{C}{|\ln [\|v\|_{H^1(\omega_0)}]|^\kappa} \quad \text{if } \|v\|_{H^1(\omega_0)} \leq \eta,$$

where, the constants  $C$ ,  $\kappa$  and  $\eta$  can depend only on  $M$  and  $R$ . Using the interpolation inequality  $\|v\|_{H^1(\omega_0)} \leq c\|v\|_{L^2(\omega_0)}^{1/2}\|v\|_{H^2(\omega_0)}^{1/2}$ , (5.3) and (5.4), we obtain,

$$(5.5) \quad \|v\|_{C^1(\partial D)} \leq \frac{C}{|\ln [\|v\|_{L^2(\omega_0)}]|^\kappa} \quad \text{if } \|v\|_{L^2(\omega_0)} \leq \eta,$$

We have by applying Lemma 6.1.2 in [Is]<sup>5</sup>, where  $\delta = \|u_\infty(\lambda) - u_\infty(\tilde{\lambda})\|_{L^2(\mathbb{S}^2)}$ ,

$$\|v\|_{C^1(\partial D)} \leq \frac{C}{|\ln [\delta^{\theta(\delta)}]|^\kappa} \quad \text{if } \delta \leq \delta_0,$$

for some constant  $\delta_0 > 0$ . Here,

$$\theta(\delta) = 1/(1 + \ln(|\ln \delta| + e)).$$

Therefore, reducing  $\delta_0$  if necessary,

$$(5.6) \quad \|v\|_{C^1(\partial D)} \leq C \left[ \frac{\ln |\ln \delta|^2}{|\ln \delta|} \right]^\kappa \quad \text{if } \delta \leq \delta_0.$$

From the estimate in Proposition 4.1, we have

$$(5.7) \quad \|\lambda - \tilde{\lambda}\|_{C(\partial D)} \leq \frac{C}{\left| \ln \left[ \|(\lambda - \tilde{\lambda})u(\lambda)\|_{C(\partial D)} \right] \right|^\sigma},$$

if  $\|\lambda - \tilde{\lambda}\|_{C(\partial D)} \leq \epsilon$ , for some  $\epsilon > 0$ .

Or,

$$(\lambda - \tilde{\lambda})u(\lambda) = \tilde{\lambda}v - \partial_\nu v.$$

Hence,

$$\|(\lambda - \tilde{\lambda})u(\lambda)\|_{C(\partial D)} \leq \max(1, M)\|v\|_{C^1(\partial D)}.$$

A combination of this last estimate, (5.6) and (5.7) yields

$$\|\lambda - \tilde{\lambda}\|_{C(\partial D)} \leq C \left| \ln \left( \frac{\ln |\ln \delta|^2}{|\ln \delta|} \right) \right|^{-\sigma} \quad \text{if } \delta \leq \delta_0.$$

To complete the proof, observe that the condition  $\delta \leq \delta_0$  is satisfied if  $\|(\lambda - \tilde{\lambda})\|_{C(\partial D)} \leq \tilde{\epsilon}$ , for some  $\tilde{\epsilon}$ , because  $\lambda \rightarrow u_\infty(\lambda)$  is continuous from the set  $\{h \in C(\partial D); \Im h = 0 \text{ and } h \geq 0\}$ , endowed with the topology of  $C(\partial D)$  into  $L^2(\mathbb{S}^2)$  (see Appendix A).  $\square$

<sup>4</sup>These two propositions are proved by similar tools to that we used in the proof of Theorem 4.1 and the main ingredient is an elliptic Carleman inequality.

<sup>5</sup>This result is due to I. Bushuyev [Bu].

## APPENDIX A

**Sketch of the proof of Theorem 1.1.** For the reader convenience, we kept the notations of [CK]. Let us first recall that the fundamental solution of the Helmholtz equation  $(\Delta + k^2)u = 0$ , with positive wave number  $k$ , is given as follows

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y.$$

We consider the single- and double-layer operators  $S$  and  $K$ , given by

$$\begin{aligned} (S\varphi)(x) &= 2 \int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D, \\ (K\varphi)(x) &= 2 \int_{\partial D} \partial_{\nu(y)}\Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D, \end{aligned}$$

and the normal derivative operators  $K'$  and  $T$ , given by

$$\begin{aligned} (K'\varphi)(x) &= 2 \int_{\partial D} \partial_{\nu(x)}\Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D, \\ (T\varphi)(x) &= 2\partial_{\nu(x)} \int_{\partial D} \partial_{\nu(y)}\Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D. \end{aligned}$$

As for the exterior Neumann problem, our problem is reduced to find a radiating solution  $u \in C(\mathbb{R}^3 \setminus D) \cap C^2(\mathbb{R}^3 \setminus \overline{D})$  of the Helmholtz equation

$$(A.1) \quad (\Delta + k^2)u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}$$

satisfying the boundary condition

$$(A.2) \quad \partial_\nu u + i\lambda(x)u = g \quad \text{on } \partial D.$$

Similarly to the Neumann case, we seek a solution in the form

$$(A.3) \quad u(x) = \int_{\partial D} [\Phi(x, y)\varphi(y) + i\eta\partial_{\nu(y)}\Phi(x, y)(S_0^2\varphi)(y)] ds(y), \quad x \notin \partial D,$$

with a continuous density  $\varphi$  and a real coupling parameter  $\eta \neq 0$ . The operator  $S_0$  is the single-layer operator in the potential theoretic limit  $k = 0$ . (Note that  $S_0$  plays the role of a smoothing operator. We refer to [CK] for more explanations.)

Next if  $M_{i\lambda}$  is the multiplication operator by  $i\lambda$ , we will use the fact that  $I + M_{i\lambda}$  is invertible. This fact is a simple consequence of the assumption that  $\lambda$  is real valued.

By the results in Theorem 3.1 in [CK], we easily prove that  $u$  is a solution of (A.3) provided that the density  $\varphi$  is the a solution of the equation

$$(A.4) \quad \varphi - (I + M_{i\lambda})^{-1} [K' + i\eta TS_0^2 + M_{i\lambda}(S + K)] \varphi = -2(I + M_{i\lambda})^{-1} g.$$

From Theorem 3.4 in [CK], we know that the operator  $(I + M_{i\lambda})^{-1} [K' + i\eta TS_0^2 + M_{i\lambda}(S + K)]$  is compact, so the Riesz-Fredholm theory is available in the space  $X = C(\partial D)$ . The proof will be complete if we prove that the equation (A.4) with  $g = 0$  has only  $\varphi = 0$  as a solution.

Let  $u_+ = u|_{\overline{D}}$  and  $u_- = u|_{\mathbb{R}^3 \setminus D}$ . Then,  $g = 0$  implies that  $u_-$  is such that

$$\partial_\nu u_- + i\lambda(x)u_- = 0, \quad \text{on } \partial D.$$

Therefore,

$$\Im \left( \int_{\partial D} u_- \partial_\nu \overline{u_-} ds \right) = \int_{\partial D} \lambda |u_-|^2 ds \geq 0.$$

We deduce from Theorem 2.12 in [CK] that  $u_- = 0$ . This and the transmission conditions in Theorem 3.1 in [CK] yield

$$u_+ = i\eta S_0^2 \varphi, \quad \partial_\nu u_+ = \varphi \quad \text{on } \partial D.$$

Then, a simple application of Green's formula leads

$$i\eta \int_{\partial D} |S_0^2 \varphi|^2 = i\eta \int_{\partial D} \varphi S_0^2 \overline{\varphi} ds = \int_{\partial D} \overline{u_-} \partial_\nu u_- ds = \int_D [\nabla u|^2 - k^2 |u|^2] dx,$$

whence  $S_0\varphi = 0$  on  $\partial D$  follows. The single-layer potential  $w$  with density  $\varphi$  and wave number  $k$  is continuous throughout  $\mathbb{R}^3$  and vanishes on  $\partial D$  and at infinity. Therefore, by maximum-minimum principle for harmonic functions, we have  $w = 0$  in  $\mathbb{R}^3$  and the jump conditions in Theorem 3.1 in [CK] yield  $\varphi = 0$ .

Next, we prove estimate (1.2) for the solutions of (A.1)-(A.2). This will imply that (1.2) is also valid for the solutions of (1.1). To this end, we introduce the following set

$$E = \{M \geq 0; \exists C(M) > 0 \text{ s.t. } \|u(\lambda)\| \leq C(M)\|g\|, \forall g, \lambda \in C(\partial D), 0 \leq \lambda \leq M\}.$$

Here and henceforth,  $u(\lambda)$  is the solution of (A.1)-(A.2), corresponding to  $\lambda$ , and

$$\|u(\lambda)\| = \|u(\lambda)\|_{C^\infty(\mathbb{R}^3 \setminus D)}, \quad \|g\| = \|g\|_{C(\partial D)}.$$

It follows from Theorem 3.10 in [CK] that  $0 \in E$  (corresponding to Neumann boundary condition). Let  $M \in E$  and  $g, \lambda \in C(\partial D)$ ,  $0 \leq \lambda \leq M + \epsilon$ . Since

$$\partial_\nu u(\lambda) + \frac{M\lambda}{M + \epsilon} u(\lambda) = -\frac{\epsilon\lambda}{M + \epsilon} u(\lambda) + g,$$

we have

$$\|u(\lambda)\| \leq C(M) (\epsilon\|u(\lambda)\| + \|g\|).$$

Therefore,

$$\|u(\lambda)\| \leq \frac{C(M)}{1 - C(M)\epsilon} \|g\| \text{ if } \epsilon < 1/C(M).$$

Noting that  $[0, M] \subset E$ , we deduce that  $E$  is an open subset of  $[0, +\infty[$ .

Now, let  $M_k$  be a sequence in  $E$ ,  $M_k \rightarrow M$ . Let  $g, \lambda \in C(\partial D)$ ,  $\|g\| = 1$  and  $0 \leq \lambda \leq M$ . Let  $\lambda_k = M_k\lambda/M$ . In view of (4.3) and (A.4), we easily deduce that  $\lambda \rightarrow u(\lambda)$  is continuous from the set  $\{h \in C(\partial D); \Im h = 0 \text{ and } h \geq 0\}$ , endowed with the topology of  $C(\partial D)$ , into  $C(\mathbb{R}^3 \setminus D)$ . Hence, there exists a positive integer  $k_0$  such that  $\|u(\lambda) - u(\lambda_{k_0})\| \leq 1$ . Consequently,

$$(A.5) \quad \|u(\lambda)\| \leq \|u(\lambda) - u(\lambda_{k_0})\| + \|u(\lambda_{k_0})\| \leq 1 + C(M_{k_0}).$$

Therefore, for any  $g \in C(\partial D)$ ,  $g \neq 0$ ,

$$(A.6) \quad \|u(\lambda)\| \leq (1 + C(M_{k_0}))\|g\|.$$

We note that (A.5) is trivially satisfied when  $g = 0$ , because in this case  $u(\lambda) = u(\lambda_{k_0}) = 0$ . In conclusion,  $E$  is a closed subset of  $[0, +\infty[$ .

We proved that  $E$  is a nonempty interval which is at the same time closed and open in  $[0, +\infty[$ . This implies immediately that  $E = [0, +\infty[$ .

With the help of Theorem 3.10 in [CK], we proceed similarly as previously to prove estimate (1.3).

□

## REFERENCES

- [BD] LAURENT BOURGEOIS AND JÉRÉMI DARDÉ, *About stability and regularization of ill-posed elliptic Cauchy problems: the case of Lipschitz domains*, Appl. Anal. 89 (11) (2010), 1745-1768.
- [Bu] I. BUSHUYEV, *Stability of recovering the near-field wave from the scattering amplitude*, Inverse Problems 12 (1996) 859-867.
- [Ch] M. CHOULLI, *Stability estimates for an inverse elliptic problem*, J. Inverse Ill-Posed Probl. 10 (6) (2002), 601-610.
- [CK] D. COLTON AND R. KRESS, *Inverse acoustic and electromagnetic scattering theory*, Springer Verlag, Berlin, 1998.
- [Is] V. ISAKOV, *Inverse problems for partial differential equations*, Springer Verlag, Berlin, 1998.
- [La] C. LABREUCHE, *Stability of the recovery of surface impedances in inverse scattering*, J. Math. Anal. Appl. 231 (1) (1999), 161-176.
- [LM] J.-L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes et applications*, Vol. I, Dunod, Paris, 1968.
- [Si] E. SINCICH, *Stable determination of the surface impedance of an obstacle by far field measurements*, SIAM J. Math. Anal. 38 (2) (2006) 434-451.

†FACULTÉ DES SCIENCES DE BIZERTE, DÉPARTEMENT DES MATHÉMATIQUES, 7021 JARZOUNA BIZERTE, TUNISIE  
*E-mail address:* [mourad.bellassoued@fsb.rnu.tn](mailto:mourad.bellassoued@fsb.rnu.tn)

‡LMAM, UMR 7122, UNIVERSITÉ DE LORRAINE ET CNRS, ILE DU SAULCY, 57045 METZ CEDEX 1, FRANCE  
*E-mail address:* [mourad.choulli@univ-lorraine.fr](mailto:mourad.choulli@univ-lorraine.fr)

§LMAM, UMR 7122, UNIVERSITÉ DE LORRAINE ET CNRS, ILE DU SAULCY, 57045 METZ CEDEX 1, FRANCE, ET,  
FACULTÉ DES SCIENCES DE BIZERTE, DÉPARTEMENT DES MATHÉMATIQUES, 7021 JARZOUNA BIZERTE, TUNISIE  
*E-mail address:* [jbalia.aymen@yahoo.fr](mailto:jbalia.aymen@yahoo.fr)