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# Rate of convergence to self-similarity for the fragmentation equation in $L^1$ spaces

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## Abstract

In a recent result by the authors [1] it was proved that solutions of the self-similar fragmentation equation converge to equilibrium exponentially fast. This was done by showing a spectral gap in weighted  $L^2$  spaces of the operator defining the time evolution. In the present work we prove that there is also a spectral gap in weighted  $L^1$  spaces, thus extending exponential convergence to a larger set of initial conditions. The main tool is an extension result in [4].

## 1 Introduction

In a recent paper [1] we have studied the speed of convergence to equilibrium for solutions of equations involving the fragmentation operator and first-order differential terms. In this paper we will focus on the case of self-similar fragmentation given by

$$\partial_t g_t(x) = -x\partial_x g_t(x) - 2g_t(x) + \mathcal{L}g_t(x) \quad (1.1a)$$

$$g_0(x) = g_{in}(x) \quad (x > 0). \quad (1.1b)$$

Here the unknown is a function  $g_t(x)$  depending on time  $t \geq 0$  and on size  $x > 0$ , which represents a density of units (usually particles, cells or polymers) of size  $x$  at time  $t$ , and  $g_{in}$  is an initial condition. The *fragmentation operator*  $\mathcal{L}$  acts on a function  $g = g(x)$  as

$$\mathcal{L}g(x) := \mathcal{L}_+g(x) - B(x)g(x), \quad (1.2)$$

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where the positive part  $\mathcal{L}_+$  is given by

$$\mathcal{L}_+g(x) := \int_x^\infty b(y, x)g(y) dy. \quad (1.3)$$

The coefficient  $b(y, x)$ , defined for  $y > x > 0$ , is the *fragmentation coefficient*, and  $B(x)$  is the *total fragmentation rate* of particles of size  $x > 0$ . It is obtained from  $b$  through

$$B(x) := \int_0^x \frac{y}{x} b(x, y) dy \quad (x > 0). \quad (1.4)$$

We refer to [1, 5, 7, 2, 6, 8] for a motivation of (1.1) in several applications and a general survey of the mathematical literature related to it.

We call  $T$  the operator on the right hand side of (1.1a), this is,

$$Tg(x) := -x\partial_x g(x) - 2g(x) + \mathcal{L}g(x) \quad (x > 0), \quad (1.5)$$

acting on a (sufficiently regular) function  $g$  defined on  $(0, +\infty)$ . Notice that, even though  $g$  is a one-variable function, we still denote its derivative as  $\partial_x g$  in order to be consistent with the notation in (1.1). The results in [1] show that  $T$  has a spectral gap in the space  $L^2(xG^{-1})$ , where  $G$  is the unique stationary solution of (1.1) with  $\int xG = 1$ . In the rest of this paper  $G$  will represent this solution, called the *self-similar profile*. Proofs of existence of the profile  $G$  and some estimates are given in [3, 6, 2], and additional bounds are given in [1].

The main result in [1] is a study of the long time behavior of (1.1): by means of an inequality relating the quadratic entropy and its dissipation rate, exponential convergence is obtained in  $L^2(xG^{-1})$ . Using the results in [4] this is further extended to the space  $L^2(x + x^k)$  for a sufficiently large exponent  $k$ . In this way one obtains convergence in a strong norm, but correspondingly has to impose more on the initial condition than just having finite mass.

The purpose of this work is to prove that  $T$  has a spectral gap in the larger spaces  $L^1(x^m + x^M)$ , where  $1/2 < m < 1 < M$  are suitable exponents. This extension is an example of application of the results in [4]. The interest of this concerning the asymptotic behavior of (1.1) is that it shows exponential convergence is valid for more general initial conditions (any function in  $L^1(x^m + x^M)$ ).

**Assumptions on the fragmentation coefficient** In order to use the results in [1] we will make the following hypotheses on the fragmentation coefficient  $b$ :

**Hypothesis 1.1.** *For all  $x > 0$ ,  $b(x, \cdot)$  is a nonnegative measure on the interval  $[0, x]$ . Also, for all  $\psi \in \mathcal{C}_0([0, +\infty))$ , the function  $x \mapsto \int_{[0, x]} b(x, y)\psi(y) dy$  is measurable.*

**Hypothesis 1.2.** *There exists  $\kappa > 1$  such that*

$$\int_0^x b(x, y) dy = \kappa B(x) \quad (x > 0). \quad (1.6)$$

**Hypothesis 1.3.** *There exists  $0 < B_m < B_M$  satisfying*

$$2B_m x^{\gamma-1} \leq b(x, y) \leq 2B_M x^{\gamma-1} \quad (0 < y < x) \quad (1.7)$$

for some  $0 < \gamma < 2$ .

This implies the following useful bound, as remarked in [1, Corollary 6.4]:

**Lemma 1.4.** *Consider a fragmentation coefficient  $b$  satisfying Hypotheses 1.1–1.3. There exists a strictly decreasing function  $k \mapsto p_k$  for  $k \geq 0$  with  $\lim_{k \rightarrow +\infty} p_k = 0$ ,*

$$p_k > 1 \text{ for } k \in [0, 1), \quad p_1 = 1, \quad 0 < p_k < 1 \text{ for } k > 1, \quad (1.8)$$

and such that

$$\int_0^x y^k b(x, y) dy \leq p_k x^k B(x) \quad (x > 0, k > 0). \quad (1.9)$$

**Main results** The main result of the present work is a spectral gap of  $T$  on weighted  $L^1$  spaces.

**Theorem 1.5.** *Assume hypotheses 1.1–1.3. For any  $1/2 < m < 1$  there exists  $1 < M < 2$  such that the operator (1.5) has a spectral gap in the space  $X := L^1(x^m + x^M)$ . More precisely, there exists  $\alpha > 0$  and a constant  $C \geq 1$  such that, for all  $g_{in} \in X$  with  $\int x g_{in} = 1$*

$$\|g_t - G\|_X \leq C e^{-\alpha t} \|g_{in} - G\|_X \quad (t \geq 0).$$

## 2 Preliminaries

In this section we gather some known results from previous works.

### 2.1 Previous results on the spectral gap of $T$

A result like Theorem 1.5 was proved in [1], but in the  $L^2$  space with weight  $x G^{-1}$ . This is summarized in the following theorem:

**Theorem 2.6** ([1]). *Assume Hypotheses 1.1–1.3, and consider  $G$  the self-similar profile with  $\int x G = 1$ . The operator  $T$  given by (1.5) has a spectral gap in the space  $H = L^2(x G^{-1})$ .*

*More precisely, there exists  $\beta > 0$  such that for any  $g_{in} \in H$  with  $\int x g = 1$  the solution  $g \in C([0, \infty); L^1(x dx))$  to equation (1.1) satisfies*

$$\|g_t - G\|_H \leq e^{-\beta t} \|g_{in} - G\|_H \quad (t \geq 0).$$

## 2.2 Bounds for the self-similar profile

We recall the following result from [1, Theorem 3.1]:

**Theorem 2.7.** *Assume Hypotheses 1.1–1.3 on the fragmentation coefficient  $b$ , and call  $\Lambda(x) := \int_0^x \frac{B(s)}{s} ds$ . Let  $G$  be the self-similar profile with  $\int x G = 1$ .*

*For any  $\delta > 0$  and any  $a \in (0, B_m/B_M)$ ,  $a' \in (1, +\infty)$  there exist constants  $C' = C'(a', \delta)$ ,  $C = C(a) > 0$  such that*

$$C' e^{-a'\Lambda(x)} \leq G(x) \leq C e^{-a\Lambda(x)} \quad \text{for } x > 0. \quad (2.10)$$

**Remark 2.8.** *In the case  $b(x, y) = 2x^{\gamma-1}$  (so  $B(x) = x^\gamma$ ), the profile  $G$  has the explicit expression  $G(x) = e^{-\frac{x^\gamma}{\gamma}}$  for  $\gamma > 0$ . This motivates the choice of  $e^{-a\Lambda(x)}$  as functions for comparison. For a general  $b(x, y)$  no explicit form is available.*

*Proof.* Everything but the lower bound of  $G$  for small  $x$  is proved in [1, Section 3]. For the lower bound, we calculate as follows:

$$\partial_x \left( x^2 e^{\Lambda(x)} G(x) \right) = x e^{\Lambda(x)} \int_x^\infty b(y, x) G(y) dy \quad (x > 0), \quad (2.11)$$

which implies that  $x^2 e^{\Lambda(x)} G(x)$  is a nondecreasing function. Hence, it must have a limit as  $x \rightarrow 0$ , and this limit must be 0 since we know  $x G(x)$  is integrable. Then, integrating (2.11), and for  $0 < z < 1$ ,

$$\begin{aligned} z^2 e^{\Lambda(z)} G(z) &= \int_0^z x e^{\Lambda(x)} \int_x^\infty b(y, x) G(y) dy dx \\ &= \int_0^\infty G(y) \int_0^{\min\{z, y\}} b(y, x) x e^{\Lambda(x)} dx dy \\ &\geq 2B_m \int_0^\infty y^{\gamma-1} G(y) \int_0^{\min\{z, y\}} x dx dy \\ &= B_m \int_0^\infty y^{\gamma-1} G(y) (\min\{z, y\})^2 dy \\ &\geq B_m z^2 \int_z^\infty y^{\gamma-1} G(y) dy \\ &\geq B_m z^2 \int_1^\infty y^{\gamma-1} G(y) dy = Cz^2 \quad (0 < z < 1). \end{aligned} \quad (2.12)$$

Notice that the number  $\int_1^\infty y^{\gamma-1} G(y) dy$  is strictly positive, as the profile  $G$  is strictly positive everywhere (see [2, 3, 1]). This proves the lower bound on  $G(x)$  for  $0 < x < 1$ , and completes the proof.  $\square$

## 2.3 A general spectral gap extension result

Our proof is based on the following result from [4], which was already used in [1] for an extension to an  $L^2$  space with a polynomial weight:

**Theorem 2.9.** Consider a Hilbert space  $H$  and a Banach space  $X$  such that  $H \subset X$  and  $H$  is dense in  $X$ . Consider two unbounded closed operators with dense domain  $T$  on  $H$ ,  $\Lambda$  on  $X$  such that  $\Lambda|_H = T$ . On  $H$  assume that

1. There is  $G \in H$  such that  $TG = 0$  with  $\|G\|_H = 1$ ;
2. Defining  $\psi(f) := \langle f, G \rangle_H$ , the space  $H_0 := \{f \in H; \psi(f) = 0\}$  is invariant under the action of  $T$ .
3.  $T - a$  is dissipative on  $H_0$  for some  $a < 0$ , in the sense that

$$\forall g \in D(T) \cap H_0 \quad ((T - a)g, g)_H \leq 0,$$

where  $D(T)$  denotes the domain of  $T$  in  $H$ .

4.  $T$  generates a semigroup  $e^{tT}$  on  $H$ ;

Assume furthermore on  $X$  that

5. there exists a continuous linear form  $\Psi : X \rightarrow \mathbb{R}$  such that  $\Psi|_H = \psi$ ;  
and  $\Lambda$  decomposes as  $\Lambda = \mathcal{A} + \mathcal{B}$  with
6.  $\mathcal{A}$  is a bounded operator from  $X$  to  $H$ ;
7.  $\mathcal{B}$  is a closed unbounded operator on  $X$  (with same domain as  $D(\Lambda)$  the domain of  $\Lambda$ ) and is  $a$ -dissipative; this is, there exists a constant  $C \geq 1$  such that

$$\forall t \geq 0, \forall g \in X \text{ with } \Psi(g) = 0, \quad \|e^{t\mathcal{B}}g\|_X \leq C\|g\|_X e^{at}. \quad (2.13)$$

Then, for any  $a' \in (a, 0)$  there exists  $C_{a'} \geq 1$  such that

$$\forall t \geq 0, \forall g \in X, \quad \|e^{t\Lambda}g - \Psi(g)G\|_X \leq C_{a'}\|g - \Psi(g)G\|_X e^{a't}.$$

### 3 Proof of the main theorem

The proof consists is an application of Theorem 2.9. For this, we consider the Hilbert space  $H := L^2(xG^{-1}(x))$ , where  $G$  is the unique self-similar profile with  $\int G = 1$ , and define  $\psi(g) := \int xg$ . Due to our previous results [1] we know that  $T$  and  $\psi$  satisfy points 1–4 of Theorem 2.9.

As the larger space we take  $X = L^1(x^m + x^M)$ , with  $1/2 < m < 1 < M$ , to be precised later. Observe that, due to the bounds on  $G$  from Theorem 2.7,

$$\begin{aligned} \|g\|_X &= \int_0^\infty (x^m + x^M)|g(x)| dx \\ &\leq \left( \int_0^\infty g(x)^2 \frac{x}{G(x)} dx \right)^{1/2} \left( \int_0^\infty (x^{m-\frac{1}{2}} + x^{M-\frac{1}{2}})^2 G(x) dx \right)^{1/2} = C\|g\|_H, \end{aligned}$$

and hence  $H \subseteq X$ . Similarly,

$$\int_0^\infty x|g(x)| dx \leq \int_0^\infty (x^m + x^M)|g(x)| dx,$$

which allows us to define  $\Psi : X \rightarrow \mathbb{R}$ ,  $\Psi(g) := \int xg$ , and proves that  $\Psi$  is continuous on  $X$ . Obviously  $\Psi|_H = \psi$ , so point 5 of Theorem 2.9 is also satisfied.

Consider  $\Lambda$  the unbounded operator on  $X$  given by the same expression (1.5) (with domain a suitable dense subspace of  $X$  which makes  $\Lambda$  a closed operator). To prove the remaining points 6 and 7 we use the following splitting of  $\Lambda$ , taking real numbers  $0 < \delta < R$  to be chosen later:

$$\begin{aligned} \mathcal{A}g(x) &:= \mathcal{L}^{+,s}g(x) := \int_x^\infty b_{R,\delta}(y,x)g(y) dy \\ &= \mathbf{1}_{x \leq R} \int_x^\infty \mathbf{1}_{y \geq \delta} b(y,x)g(y) dy, \end{aligned} \quad (3.14)$$

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{B}g := \Lambda g - \mathcal{A}g, \quad (3.15)$$

where we denote  $b_{R,\delta}(x,y) := b(x,y) \mathbf{1}_{x \geq \delta} \mathbf{1}_{y \leq R}$ . We define

$$\begin{aligned} \mathcal{L}^{+,r}g &:= \mathcal{L}^+g - \mathcal{L}^{+,s}g \\ &= \int_x^\infty b(y,x)(1 - \mathbf{1}_{y \geq \delta} \mathbf{1}_{x \leq R})g(y) dy \\ &= \int_x^\infty b(y,x) \mathbf{1}_{y \leq \delta} g(y) dy + \int_x^\infty b(y,x) \mathbf{1}_{y \geq \delta} \mathbf{1}_{x \geq R} g(y) dy \\ &=: \mathcal{L}_1^{+,r}g + \mathcal{L}_2^{+,r}g \end{aligned}$$

so we may write  $\mathcal{B}$  as

$$\mathcal{B}g = -2g - x\partial_x g - Bg + \mathcal{L}_1^{+,r}g + \mathcal{L}_2^{+,r}g. \quad (3.16)$$

First, let us prove that  $\mathcal{A}$  is bounded from  $X$  to  $H$ . We compute

$$\begin{aligned} \|\mathcal{A}g\|_H^2 &= \int_0^\infty x(\mathcal{L}^{+,s}g)^2 G(x)^{-1} dx \\ &\leq (2B_M)^2 \left( \sup_{[0,R]} xG(x)^{-1} \right) \int_0^R \left( \int_{\max(x,\delta)}^\infty y^{\gamma-1}g(y) dy \right)^2 dx \\ &\leq C_R \left( \int_\delta^\infty y^{\gamma-1}g(y) dy \right)^2 \leq C_{R,\delta} \left( \int_0^\infty yg(y) dy \right)^2 \leq C_{R,\delta} \|g\|_X^2, \end{aligned}$$

which shows  $\mathcal{A} : X \rightarrow H$  is a bounded operator. Notice that we have used here the lower bound  $G(x) \geq Cx$  for  $x$  small, proved in Theorem 2.7.

Then, let us prove that one can choose  $0 < \delta < R$  appropriately so that  $\mathcal{B}$  is  $\alpha$ -dissipative for some  $\alpha < 0$ . It is enough to prove that, for  $g$  in the domain of  $\Lambda$  (the same as the domain of  $\mathcal{B}$ ),

$$\int_0^\infty \text{sign}(g(x)) \mathcal{B}g(x) (x^m + x^M) dx \leq \alpha \|g\|_X. \quad (3.17)$$

To show this we calculate as follows for any  $k > 0$ , using (3.16):

$$\begin{aligned} \int_0^\infty \text{sign}(g(x)) \mathcal{B}g(x) x^k dx &\leq (k-1) \int_0^\infty x^k |g| dx \\ &\quad - \int_0^\infty B(x) x^k |g| dx + \int_0^\infty |\mathcal{L}_1^{+,r} g| x^k dx + \int_0^\infty |\mathcal{L}_2^{+,r} g| x^k dx, \end{aligned} \quad (3.18)$$

where the first term is obtained from the terms  $-2g - \partial_x g$  through an integration by parts. We give separately some bounds on the last two terms in (3.18). On one hand, we have

$$\begin{aligned} \int_0^\infty |\mathcal{L}_1^{+,r} g| x^k dx &\leq \int_0^\infty x^k \int_x^\infty b(y, x) \mathbf{1}_{y \leq \delta} |g(y)| dy dx \\ &\leq \int_0^\delta |g(y)| \left( \int_0^y x^k b(y, x) dx \right) dy \\ &\leq 2B_M \int_0^\delta |g(y)| B(y) y^k dy \\ &\leq p_k B_m \delta^\gamma \int_0^\delta |g(y)| y^k dy, \end{aligned} \quad (3.19)$$

where we have used (1.9). On the other hand, and again due to (1.9),

$$\begin{aligned} \int_0^\infty |\mathcal{L}_2^{+,r} g| x^k dx &\leq \int_0^\infty x^k \int_x^\infty b(y, x) \mathbf{1}_{x \geq R} \mathbf{1}_{y \geq \delta} |g(y)| dy dx \\ &\leq \int_0^\infty x^k \int_x^\infty b(y, x) \mathbf{1}_{x \geq R} \mathbf{1}_{y \geq R} |g(y)| dy dx \\ &\leq \int_R^\infty |g(y)| \left( \int_R^y x^k b(y, x) dx \right) dy \\ &\leq p_k \int_R^\infty |g(y)| y^k B(y) dy. \end{aligned} \quad (3.20)$$

Hence, from (3.18) and the bounds (3.19)–(3.20) we obtain

$$\begin{aligned} \int_0^\infty \mathcal{B}g(x) \text{sign}(g(x)) (x^m + x^M) dx &\leq (m-1) \int_0^\infty x^m |g| dx + (M-1) \int_0^\infty x^M |g| dx \\ &\quad - \int_0^\infty B(x) (x^m + x^M) |g| dx \\ &\quad + p_m B_m \delta^\gamma \int_0^\delta x^m |g(x)| dx + p_m \int_R^\infty x^m B(x) |g(x)| dx \\ &\quad + p_M B_m \delta^\gamma \int_0^\delta x^M |g(x)| dx + p_M \int_R^\infty x^M B(x) |g(x)| dx. \end{aligned} \quad (3.21)$$

We have to choose  $1/2 < m < 1 < M < 2$  so that this is bounded by  $-C\|g\|_X$  for some positive constant  $C$ . First, fix any  $m$  with  $1/2 < m < 1$ , and take  $0 < \delta < 1$  small enough such that

$$p_m B_m \delta^\gamma < \frac{1-m}{4}, \quad B_m \delta^\gamma < \frac{1-m}{4}.$$

(Which can be done due to  $\gamma > 0$ .) Then, as  $p_M < 1$  and  $x^M < x^m$  for  $x < \delta < 1$ ,

$$\begin{aligned} & \int_0^\infty \mathcal{B}g(x) \operatorname{sign}(g(x))(x^m + x^M) dx \\ & \leq -\frac{1-m}{2} \int_0^\infty x^m |g| dx + (M-1) \int_0^\infty x^M |g| dx \\ & \quad - \int_0^\infty B(x)(x^m + x^M) |g| dx \\ & \quad + p_m \int_R^\infty x^m B(x) |g(x)| dx + p_M \int_R^\infty x^M B(x) |g(x)| dx. \end{aligned} \quad (3.22)$$

Now, take  $R_0 > 0$  such that  $B(x) > 2 > M$  for  $x \geq R_0$ . Then, choose  $1 < M < 2$  such that  $(M-1)x^M < \frac{1-m}{4}x^m$  for  $0 < x < R_0$ . Then whatever  $R$  is we have from (3.21):

$$\begin{aligned} & \int_0^\infty \mathcal{B}g(x) \operatorname{sign}(g(x))(x^m + x^M) dx \\ & \leq -\frac{1-m}{4} \int_0^{R_0} x^m |g| dx - \int_{R_0}^R x^M |g| dx \\ & \quad - \int_R^\infty (B(x) - M + 1)x^M |g| dx \\ & \quad + p_m \int_R^\infty x^m B(x) |g(x)| dx + p_M \int_R^\infty x^M B(x) |g(x)| dx. \end{aligned} \quad (3.23)$$

Finally, choose  $R > 1$  such that

$$-(B(x)(1-p_M) - M + 1)x^M + p_m x^m \leq -x^M \quad \text{for } x > R.$$

With this, and continuing from (3.23),

$$\begin{aligned} & \int_0^\infty \mathcal{B}g(x) \operatorname{sign}(g(x))(x^m + x^M) dx \\ & \leq -\frac{1-m}{4} \int_0^{R_0} x^m |g| dx - \int_{R_0}^\infty x^M |g| dx \\ & \leq -C\|g\|_X, \end{aligned} \quad (3.24)$$

for some number  $C = C(m, M, R_0) > 0$ . This shows point that  $\mathcal{B}$  is dissipative with constant  $-C$ , and hence point 7 of Theorem 2.9 holds with  $a = -C$ . A

direct application of Theorem 2.9 then proves our result, Theorem 1.5, with  $\alpha := \min\{\beta, C\}$  (where  $\beta$  is the one appearing in Theorem 2.6).

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## References

- [1] M. J. Cáceres, J. A. Cañizo, and S. Mischler. Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations. *Journal de Mathématiques Pures et Appliquées*, to appear, 2011 (preprint arXiv:1010.546).
- [2] M. Doumic-Jauffret and P. Gabriel. Eigenelements of a general aggregation-fragmentation model. *Mathematical Models and Methods in the Applied Sciences*, 20(5):757–783, 2010.
- [3] M. Escobedo, S. Mischler, and M. Rodríguez Ricard. On self-similarity and stationary problem for fragmentation and coagulation models. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(1):99–125, 2005.
- [4] M. P. Gualdani, S. Mischler, and C. Mouhot. Factorization for non-symmetric operators and exponential H-theorem. *Preprint*, Jun 2010.
- [5] J. A. J. Metz and O. Diekmann. *The Dynamics of Physiologically Structured Populations*, volume 68 of *Lecture notes in Biomathematics*. Springer, 1st edition, August 1986.
- [6] P. Michel. Existence of a solution to the cell division eigenproblem. *Mathematical Models and Methods in Applied Sciences*, 16(1 supp):1125–1153, July 2006.
- [7] B. Perthame. *Transport equations in biology*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.
- [8] B. Perthame and L. Ryzhik. Exponential decay for the fragmentation or cell-division equation. *Journal of Differential Equations*, 210(1):155–177, March 2005.