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Marie-Françoise Bidaut-Véron, Nguyen Anh Dao

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Marie Françoise BIDAUT-VERON* Nguyen Anh DAO†

Abstract

Here we study the nonnegative solutions of the viscous Hamilton-Jacobi equation

$$u_t - \Delta u + |\nabla u|^q = 0$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, where $q > 1, T \in (0,\infty]$, and Ω is a smooth bounded domain of $\mathbb{R}^N$ containing 0, or $\Omega = \mathbb{R}^N$. We consider solutions with a possible singularity at point $(x,t) = (0,0)$. We show that if $q \geq q_* = (N+2)/(N+1)$ the singularity is removable. For $1 < q < q_*$, we prove the uniqueness of a very singular solution without condition as $|x| \to \infty$; we also show the existence and uniqueness of a very singular solution of the Dirichlet problem in $Q_{\Omega,\infty}$, when Ω is bounded. We give a complete description of the solutions in each case.

Keywords Viscous Hamilton-Jacobi equation; regularity; initial isolated singularity; removability; very singular solution

A.M.S. Subject Classification 35K15, 35K55, 35B33, 35B65, 35D30

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*Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 7350, Faculté des Sciences, 37200 Tours France. E-mail address:veronmf@univ-tours.fr

†Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 7350, Faculté des Sciences, 37200 Tours France. E-mail address:Anh.Nguyen@lmpt.univ-tours.fr

1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ containing 0, or $\Omega = \mathbb{R}^N$, and $\Omega_0 = \Omega \backslash \{0\}$. Here we consider the nonnegative solutions of the viscous parabolic Hamilton-Jacobi equation

$$u_t - \Delta u + |\nabla u|^q = 0 \tag{1.1}$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, where $q > 1$, with a possible singularity at point $(x,t) = (0,0)$, in the sense:

$$\lim_{t\to 0} \int_\Omega u(.,t)\varphi dx = 0, \qquad \forall \varphi \in C_c(\Omega_0), \tag{1.2}$$

which means *formally* that $u(x,0) = 0$ for $x \neq 0$.

Such a problem was first considered for the semi-linear equation with a lower term or order 0 :

$$u_t - \Delta u + |u|^{q-1}u = 0 \quad \text{ in } Q_{\Omega,T}, \tag{1.3}$$

with $q > 1$. In a well-known article of Brezis and Friedman [16], it was shown that the problem admits a critical value $q_c = (N+2)/N$. For any $q < q_c$, and any bounded Radon measure $u_0 \in \mathcal{M}_b(\Omega)$, there exists a unique solution of (1.3) with Dirichlet conditions on $\partial\Omega$ with initial data u_0, in the weak * sense:

$$\lim_{t\to 0} \int_\Omega u(.,t)\varphi dx = \int_\Omega \varphi du_0, \qquad \forall \varphi \in C_c(\Omega). \tag{1.4}$$

Moreover, from [17] and [21],there exists a *very singular solution* in $\mathbb{R}^N$, satisfying

$$\lim_{t\to 0} \int_{B_r} u(.,t)dx = \infty, \qquad \forall\ B_r \subset \Omega, \tag{1.5}$$

and it is the limit as $k \to \infty$ of the solutions with initial data $k\delta_0$, where δ_0 is the Dirac mass at 0; its uniqueness, obtained in [33], is also a consequence of the general results of [31]. For any $q \geqq q_c$, such solutions do not exist, and the singularity is *removable*, in other words any solution of (1.3), (1.2) satisfies $u \in C^{2,1}(\Omega \times [0,T))$ and $u(x,0) = 0$ in Ω, see again [16].

The problem was extended in various directions, where the Laplacian is replaced by the porous medium operator $\Delta(|u|^{m-1}u)$, see among them [35], [24], [25], [26],[27], [29], or the p-Laplacian $\Delta_p u$, see for example [22], [36], [23].

Concerning equation (1.1), up to now, the description was not yet complete. Here another critical value is involved:

$$q_* = \frac{N+2}{N+1}.$$

In the case $\Omega = \mathbb{R}^N$, we define a very singular solution (called VSS) in $Q_{\mathbb{R}^N,\infty}$ as any function $u \in L^1_{loc}(Q_{\mathbb{R}^N,\infty})$, such that $|\nabla u| \in L^q_{loc}(Q_{\mathbb{R}^N,\infty})$, satisfying equation (1.1) in $\mathcal{D}'(Q_{\mathbb{R}^N,\infty})$, and conditions

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\varphi dx = 0, \qquad \forall \varphi \in C_c(\mathbb{R}^N \setminus \{0\}). \tag{1.6}$$

$$\lim_{t\to 0} \int_{B_r} u(.,t)dx = \infty, \qquad \forall r > 0. \tag{1.7}$$

For $q \in (1, q_*)$, it was shown in [10] that, for any $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$, there exists a solution u with initial data u_0, unique in a suitable class, which was enlarged in [7]. The existence of a radial self-similar VSS U in $Q_{\mathbb{R}^N,\infty}$, unique in that class, was obtained in [39]; independently in [11], proved the existence of a VSS as a limit as $k \to \infty$ of the solutions with initial data $k\delta_0$. From [12], it is unique among (possibly nonradial) functions such that

$$\lim_{t\to 0} \int_{\mathbb{R}^N \setminus B_r} U(.,t)dx = 0, \qquad \forall r > 0, \tag{1.8}$$

$$U \in C^{2,1}\left(Q_{\mathbb{R}^N,\infty}\right) \cap C((0,\infty); L^1(\mathbb{R}^N)) \cap L^q_{loc}((0,\infty); W^{1,q}(\mathbb{R}^N)), \tag{1.9}$$

$$\sup_{t>0}(t^{N/2} \left\|u(.,t)\right\|_{L^\infty(\mathbb{R}^N)} + t^{(q(N+1)-N)/2q} \left\|\nabla(u^{(q-1)/q}(.,t))\right\|_{L^\infty(\mathbb{R}^N)} < \infty \tag{1.10}$$

If $q \geqq q_*$, it was proved in [11] that there is no solution u in $Q_{\mathbb{R}^N,T}$ with initial data δ_0, under the constraints

$$u \in C((0,T); L^1(\mathbb{R}^N)) \cap L^q((0,T); W^{1,q}(\mathbb{R}^N); \tag{1.11}$$

and the nonexistence of VSS was stated as an open problem.

In the case of the Dirichlet problem in $Q_{\Omega,T}$, with Ω bounded, similar results were obtained in [8]: for $q \in (1, q_*)$ and any $u_0 \in \mathcal{M}_b(\Omega)$, there exists a solution u such that

$$u \in C((0,T); L^1(\Omega)) \cap L^1((0,T); W^{1,1}_0(\Omega), \qquad |\nabla u|^q \in L^1\left(Q_{\Omega,T}\right), \tag{1.12}$$

satisfying (1.4) for any $\varphi \in C_b(\Omega)$, and unique in that class; for $q \geqq q_*$ there exists no solution in this class when u_0 is a Dirac mass; the existence or nonexistence of a VSS was not studied.

In this article we answer to these questions and complete the description of the solutions.

In Section 2 we introduce the notion of weak solutions and study their first properties. We extend some universal estimates of [19] for the Dirichlet problem. When $q \leqq 2$, we show that the solutions are smooth, improving some results of [12], see Theorems 2.12 and 2.13. We point out some particular singular solutions or supersolutions, fundamental in the sequel. We also give some trace results, in the footsteps of [31], and apply them to the solutions of (1.1), (1.2).

Our main result is the *removability* in the supercritical case $q \geqq q_*$, proved in Section 3, extending the results of [16] to equation (1.1).

Theorem 1.1 *Assume $q \geqq q_*$. Let Ω be any domain in $\mathbb{R}^N$. Let $u \in L^1_{loc}(Q_{\Omega,T})$, such that $|\nabla u| \in L^q_{loc}(Q_{\Omega,T})$, be any solution of problem*

$$(P_\Omega) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0 \quad \text{in } \mathcal{D}'(Q_{\Omega,T}), \\ \\ \lim_{t\to 0} \int_\Omega u(.,t)\varphi dx = 0, \qquad \forall \varphi \in C_c(\Omega_0), \end{cases}$$

Then the singularity is removable, in the following sense:

If $q \leqq 2$, then $u \in C(\Omega \times [0,T))$ and $u(x,0) = 0$, $\forall x \in \Omega$.

If $q > 2$, then u is locally bounded near 0, and for any domain $\omega \subset\subset \Omega$,

$$\lim_{t\to 0}(\sup_{Q_{\omega,t}} u) = 0.$$

Observe that our conclusions hold *without any condition as* $|x| \to \infty$ if $\Omega = \mathbb{R}^N$, or near $\partial\Omega$ when $\Omega \neq \mathbb{R}^N$. As a consequence, for $q \geqq q_*$,

(i) there exists *no VSS in* $Q_{\mathbb{R}^N,\infty}$ in the sense above.

(ii) there exists *no solution of* (P_Ω) *with a Dirac mass* at $(0,0)$, without assuming (1.11) or (1.12).

We give different proofs of Theorem 1.1 according to the values of q. For $q \leqq 2$, we take benefit of the regularity of the solutions shown in Section 2. When $q < 2$, we make use of supersolutions, and the difficult case is the critical one $q = q_*$. When $q \geqq 2$, our proof is based on a change of unknown, and on our trace results; the case $q > 2$ is the most delicate, because of the lack of regularity.

Besides, if $\Omega = \mathbb{R}^N$, we can show a *global removability, without condition at* ∞*:*

Theorem 1.2 *Under the assumptions of Theorem 1.1 with* $\Omega = \mathbb{R}^N$*, then*

$$u(x,t) \equiv 0, \qquad a.e.\ in\ \mathbb{R}^N, \quad for\ any\ t > 0.$$

In Section 4, we complete the study of the subcritical case $q < q_*$. Our main result in this range is the uniqueness of the VSS in $Q_{\mathbb{R}^N,\infty}$ *without any condition*:

Theorem 1.3 *Let* $q \in (1, q_*)$*. Then there exists a unique VSS in* $Q_{\mathbb{R}^N,\infty}$.

Moreover we give a complete description of the solutions:

Theorem 1.4 *Let* $q \in (1, q_*)$*. Let* $u \in L^1_{loc}(Q_{\mathbb{R}^N,\infty})$*, be any function such that* $|\nabla u| \in L^q_{loc}(Q_{\mathbb{R}^N,\infty})$*, solution of equation (1.1) in* $\mathcal{D}'(Q_{\mathbb{R}^N,\infty})$*, and satisfying (1.6). Then*

- *either (1.7) holds and* $u = U$*,*
- *or there exists* $k > 0$ *such that* $u(.,0) = k\delta_0$ *in the weak sense of* $\mathcal{M}_b(\mathbb{R}^N)$:

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\varphi dx = k\varphi(0), \qquad \forall \varphi \in C_b(\mathbb{R}^N), \tag{1.13}$$

and u *is the unique solution satisfying (1.13),*

- *or* $u \equiv 0$.

We also consider the Dirichlet problem in $Q_{\Omega,T}$ when Ω is bounded:

$$(D_{\Omega,T}) \begin{cases} u_t - \Delta u + |\nabla u|^q = 0 & \text{in } Q_{\Omega,T} \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty)\,. \end{cases} \tag{1.14}$$

We give a notion of VSS for this problem, generally nonradial, and show the parallel of Theorem 1.3:

Theorem 1.5 *Assume that* $q \in (1, q_*)$ *and* Ω *is a smooth bounded domain of* $\mathbb{R}^N$*. Then there exists a unique VSS of problem* $(D_{\Omega,\infty})$.

Finally we describe all the solutions as above.

In conclusion, q_* clearly appears as the upperbound for existence of solutions with an isolated singularity at time 0. We refer to [14] for the study of equation (1.1) or more general quasilinear parabolic equations with rough initial data, where we give new decay and uniqueness properties. The problem of removability of nonpunctual singularities will be the object of a further article.

2 Weak solutions and regularity

2.1 First properties of the weak solutions

We set $Q_{\Omega,s,\tau} = \Omega \times (s,\tau)$, for any domain $\Omega \subset \mathbb{R}^N$, any $-\infty \leqq s < \tau \leqq \infty$, thus $Q_{\Omega,T} = Q_{\Omega,0,T}$.

Definition 2.1 *For any function $\Phi \in L^1_{loc}(Q_{\Omega,T})$, we say that a function U is a weak solution (resp. subsolution, resp. supersolution) of equation*

$$U_t - \Delta U = \Phi \qquad \text{in } Q_{\Omega,T}, \tag{2.1}$$

if $U \in L^1_{loc}(Q_{\Omega,T})$ and, for any $\varphi \in \mathcal{D}^+(Q_{\Omega,T})$,

$$\int_0^T \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)dxdt = 0 \qquad (\textit{resp.} \leqq, \ \textit{resp.} \geqq).$$

In all the sequel we use regularization arguments by to deal with weak solutions:

Notation 2.2 *For any function $u \in L^1_{loc}(Q_{\Omega,T})$, we set*

$$u_\varepsilon = u * \varrho_\varepsilon,$$

where (ϱ_ε) is sequence of mollifiers in $(x,t) \in \mathbb{R}^{N+1}$. Then u_ε is well defined in $Q_{\Omega,s,\tau}$ for any domain $\omega \subset\subset \Omega$ and $0 < s < \tau < T$ and $\varepsilon > 0$ small enough.

Lemma 2.3 *Any solution (resp. subsolution) U of (2.1) such that $U \in C((0,T); L^1_{loc}(\Omega))$ satisfies also for any nonnegative $\varphi \in C^\infty_c(\Omega \times [0,T])$ and any $s, \tau \in (0,T)$,*

$$\int_\Omega U(.,\tau)\varphi(.,\tau)dx - \int_\Omega U(.,t)\varphi(.,t)dx - \int_s^\tau \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)dxdt = 0 \qquad (\textit{resp.} \leqq 0) \tag{2.2}$$

and for any nonnegative $\psi \in C^2_c(\Omega)$,

$$\int_\Omega U(.,\tau)\psi dx - \int_\Omega U(.,s)\psi dx - \int_s^\tau \int_\Omega (U\Delta\psi + \Phi\psi)dxdt = 0 \qquad (\textit{resp.} \leqq 0). \tag{2.3}$$

Proof. The regularization gives the equation $(U_\varepsilon)_t - \Delta U_\varepsilon = \Phi_\varepsilon$, and the relations (2.2), (2.3) hold for $U_\varepsilon, \Phi_\varepsilon$, and for U, Φ as $\varepsilon \to 0$. Indeed, $\int_\Omega U_\varepsilon(.,\tau)\varphi(.,\tau)dx$ converges to $\int_\Omega U(.,\tau)\varphi(.,\tau)dx$ for almost any τ, see for example [4], hence the relations hold for any s, τ by continuity. ∎

Next we make precise our notion of solution of equation (1.1).

Definition 2.4 *(i) We say that a nonnegative function u is a weak solution of equation (1.1) in $Q_{\Omega,T}$, if $u \in L^1_{loc}(Q_{\Omega,T}), |\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$, and u is a weak solution of the equation in the sense above:*

$$\int_0^T \int_\Omega (-u\varphi_t - u\Delta\varphi + |\nabla u|^q\varphi)dxdt = 0, \qquad \forall \varphi \in \mathcal{D}(Q_{\Omega,T}).$$

(ii) We say that u is a weak solution of the Dirichlet problem $(D_{\Omega,T})$ if it is a weak solution of (1.1) in $Q_{\Omega,T}$, such that

$$u \in L^1_{loc}((0,T); W^{1,1}_0(\Omega)) \cap C((0,T); L^1(\Omega)), \quad \textit{and } |\nabla u| \in L^q_{loc}((0,T); L^q(\Omega)).$$

We first observe that the regularization keeps the subsolutions, which allow to give local estimates:

Lemma 2.5 *Let u be a weak nonnegative subsolution of (1.1) in $Q_{\Omega,T}$. Let ω be any domain $\omega \subset\subset \Omega$ and $0 < s < \tau < T$. Then for ε small enough, u_ε is a subsolution of equation (1.1) in $Q_{\omega,s,\tau}$.*

Proof. The function u_ε satisfies

$$(u_\varepsilon)_t - \Delta u_\varepsilon + |\nabla u|^q * \varrho_\varepsilon \leqq 0,$$

in $\mathcal{D}'(Q_{\omega,s,\tau})$ for ε small enough. We find easily that

$$|\nabla u_\varepsilon|^q \leqq |\nabla u|^q * \varrho_\varepsilon \qquad \text{in } Q_{\omega,s,\tau}, \tag{2.4}$$

from the Hölder inequality, since ϱ_ε has a mass 1; thus $|\nabla u_\varepsilon|^q \in L^1_{loc}(Q_{\omega,s,\tau})$ and

$$(u_\varepsilon)_t - \Delta u_\varepsilon + |\nabla u_\varepsilon|^q \leqq 0. \tag{2.5}$$

■

Next we recall some well known properties:

Lemma 2.6 *Any weak nonnegative solution of equation (1.1) satisfies*

$$u \in L^\infty_{loc}(Q_{\Omega,T}), \qquad \nabla u \in L^2_{loc}(Q_{\Omega,T}), \qquad u \in C((0,T); L^r_{loc}(\Omega)), \quad \forall r \geqq 1. \tag{2.6}$$

As a consequence, it satisfies

(i) for any $\varphi \in C^1_c(Q_{\Omega,T})$,

$$\int_0^T \int_\Omega (-u\varphi_t + \nabla u.\nabla\varphi + |\nabla u|^q \varphi) dxdt = 0, \tag{2.7}$$

(ii) for any $s, \tau \in (0,T)$, and any $\varphi \in C^1((0,T); C^1_c(\Omega))$,

$$\int_\Omega u(.,\tau)\varphi(.,\tau)dx - \int_\Omega u(.,s)\varphi(.,s)dx + \int_s^\tau \int_\Omega (-u\varphi_t + \nabla u.\nabla\varphi + |\nabla u|^q\varphi)dxdt = 0 \tag{2.8}$$

(iii) for any $s, \tau \in (0,T)$, and any $\psi \in C^1_c(\Omega)$,

$$\int_\Omega u(.,\tau)\psi dx - \int_\Omega u(.,s)\psi dx + \int_s^\tau \int_\Omega (\nabla u.\nabla\psi + |\nabla u|^q\psi)dxdt = 0 \tag{2.9}$$

Proof. The function $u \in L^1_{loc}(Q_{\Omega,T})$ is nonnegative and subcaloric, then regularizing u by u_ε, we get $u \in L^\infty_{loc}(Q_{\Omega,T})$, see for example [16]. Otherwise for any domains $\omega \subset\subset \omega' \subset\subset \Omega$, taking $\psi \in C^1_c(\Omega)$ with support in ω' such that $\psi \equiv 1$ on ω, $\psi(\Omega) \subset [0,1]$, we find

$$\begin{aligned}
&\int_\Omega u_\varepsilon^2(.,\tau)\psi^2 dx - \int_\Omega u_\varepsilon^2(.,s)\psi^2 dx + \int_s^\tau \int_\Omega |\nabla u_\varepsilon|^2 \psi^2 dx \\
&\leqq 2\int_s^\tau \int_\Omega u_\varepsilon |\nabla u_\varepsilon| |\nabla\psi| dx \leqq \frac{1}{2}\int_s^\tau \int_\Omega |\nabla u_\varepsilon|^2 \psi^2 dx + 4\int_s^\tau \int_\Omega u_\varepsilon^2 |\nabla\psi|^2 dx;
\end{aligned}$$

hence $\nabla u \in L^2_{loc}(Q_{\Omega,T})$ from the Fatou Lemma, and

$$\|\nabla u\|_{L^2(Q_{\omega,s,\tau})} \leqq C(\|u(.,s)\|_{L^2(Q_{\omega',s,\tau})} + \|u\|_{L^2(Q_{\omega',s,\tau})}) \leqq C\,\|u\|_{L^\infty(Q_{\omega',s,\tau})}\,, \tag{2.10}$$

with $C = C(N,\omega,\omega')$. Then (2.7) holds for any $\varphi \in \mathcal{D}(Q_{\Omega,T})$. Moreover, since $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$, the function u lies in the set

$$E = \left\{ v \in L^2_{loc}((0,T);W^{1,2}_{loc}(\Omega)) : v_t \in L^2_{loc}((0,T);W^{-1,2}(\Omega)) + L^1_{loc}\left(Q_{\Omega,T}\right) \right\} \tag{2.11}$$

From a local version of [38, Theorem 1.1], we have $E \subset C((0,T);L^1_{loc}(\Omega))$. Then (2.8) and (2.9) follow. Moreover $u \in L^\infty_{loc}(Q_{\Omega,T})$, then $u \in C((0,T);L^r_{loc}(\Omega))$ for any $r > 1$. ∎

In the case of the Dirichlet problem $(D_{\Omega,T})$, the regularization does not provide estimates up to the boundary, thus we use another argument: the notion of *entropy solution* that we recall now. For any $k > 0$ and $r \in \mathbb{R}$, we define as usual $T_k(r) = \max(-k, \min(k,r))$ the truncation function, and $\Theta_k(r) = \int_0^r T_k(s)ds$.

Definition 2.7 *Let $s < \tau$, and $f \in L^1(Q_{\Omega,s,\tau})$ and $u_s \in L^1(\Omega)$. A function u is an entropy solution of the problem*

$$\left\{ \begin{array}{ccc} u_t - \Delta u & = f & \text{in } Q_{\Omega,s,\tau}, \\ u & = 0 & \text{on } (s,\tau)\times\partial\Omega, \\ u(.,s) & = u_s & \text{in } \Omega, \end{array} \right. \tag{2.12}$$

if $u \in C([s,\tau]\,;L^1(\Omega))$, and $T_k(u) \in L^2((s,\tau)\,;W^{1,2}_0(\Omega))$ for any $k > 0$, and

$$\begin{aligned} &\int_\Omega \Theta_k(u-\varphi)(.,\tau)dx + \int_s^\tau \langle \varphi_t, T_k(u-\varphi)\rangle dt + \int_s^\tau \int_\Omega \nabla u.\nabla T_k(u-\varphi)dxdt \\ &= \int_\Omega \Theta_k(u_s - \varphi(.,0))dx + \int_s^\tau \int_\Omega f T_k(u-\varphi)dxdt \end{aligned}$$

for any $\varphi \in L^2((s,\tau);W^{1,2}(\Omega)) \cap L^\infty\left(Q_{\Omega,\tau}\right)$ such that $\varphi_t \in L^2((s,\tau);W^{-1,2}(\Omega))$.

As a consequence, we identify three ways of defining solutions:

Lemma 2.8 *Let $0 \leqq s < \tau \leqq T$, and $f \in L^1(Q_{\Omega,s,\tau})$ and $u \in C([s,\tau)\,;L^1\left(\Omega\right))$, $u_s = u(s)$. Denoting by $e^{t\Delta}$ the semi-group of the heat equation with Dirichlet conditions acting on $L^1\left(\Omega\right)$, the three properties are equivalent:*

(i) $u \in L^1_{loc}((s,\tau);W^{1,1}_0\left(\Omega\right))$ and $u_t - \Delta u = f, \qquad$ in $\mathcal{D}'(Q_{\Omega,s,\tau})$,

(ii) u is an entropy solution of problem (2.12) in $Q_{\Omega,s,\tau}$,

(iii)

$$u(.,t) = e^{(t-s)\Delta}u_s + \int_s^t e^{(t-\sigma)\Delta} f(\sigma)d\sigma \qquad \text{in } L^1\left(\Omega\right), \quad \forall t \in [s,\tau]\,.$$

Proof. It follows from the existence and uniqueness of the solutions of (i) from [6, Lemma 3.4], as noticed in [8], and of the entropy solutions, see [3], [34]. ∎

We deduce properties of all the *bounded* solutions u of $(D_{\Omega,T})$:

Lemma 2.9 *Any nonnegative weak solution of problem $(D_{\Omega,T})$, such that $u \in L^{\infty}_{loc}((0,T);L^{\infty}(\Omega))$ satisfies $\nabla u \in L^{2}_{loc}(0,T);L^{2}(\Omega))$ and $u \in C((0,T);L^{r}(\Omega))$ for any $r \geqq 1$.*

Proof. Since $u \in C((0,T);L^{1}(\Omega))$, for any $0 < s < \tau < T$, u is an entropy solution on $[s,\tau]$ from Lemma 2.8. Since u is bounded, it follows that $u = T_k(u) \in L^{2}((s,\tau);W_0^{1,2}(\Omega))$, and

$$\int_{\Omega} u^2(.,\tau)dx - \int_{\Omega} u^2(.,s)dx + \int_s^{\tau}\int_{\Omega} |\nabla u|^2\,dx + \int_s^{\tau}\int_{\Omega} u|\nabla u|^q dxdt = 0;$$

and $u \in C((0,T);L^{r}(\Omega))$ as in Lemma 2.6. ∎

2.2 Estimates of the classical solutions of the Dirichlet problem

First recall some results on the Dirichlet problem in a bounded domain Ω with regular initial and boundary data

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = 0, \quad \text{in } Q_{\Omega,T}, \\ u = \varphi, \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \geqq 0. \end{cases} \tag{2.13}$$

If $\varphi \equiv 0$ and $u_0 \in C_0^1\left(\overline{\Omega}\right)$, it is well known that problem (2.13) admits a unique solution $u \in C^{2,1}\left(Q_{\Omega,\infty}\right) \cap C\left(\overline{\Omega} \times [0,\infty)\right)$ such that $|\nabla u| \in C\left(\overline{\Omega} \times [0,\infty)\right)$. For general $\varphi \in C(\partial\Omega \times [0,T])$, the same happens on $[0,T)$ if $u_0 \in C^1(\overline{\Omega})$, and $u_0(x) = \varphi(x,0)$ on $\partial\Omega$. If one only assumes $u_0 \in C(\overline{\Omega})$, there exist a unique solution $u \in C(\overline{\Omega} \times [0,T])$ in the viscosity sense, see [5], but $|\nabla u|$ may have a blow-up near $\partial\Omega$ when $q > 2$.

Some fundamental universal estimates have been obtained in [19]:

Theorem 2.10 ([19]) *Let Ω be any smooth bounded domain. Let $q > 1$, and $u_0 \in C_0\left(\overline{\Omega}\right)$ be Lipschitz continuous. Let u be the classical solution of (2.13) with $\varphi = 0$. Then there exist functions $B, D \in C((0,\infty))$ depending only of N, q, Ω, such that such that, for any $t \in (0,T)$,*

$$\|u(.,t)\|_{L^{\infty}(\Omega)} \leqq B(t)d(x,\partial\Omega), \tag{2.14}$$

$$\|\nabla u(.,t)\|_{L^{\infty}(\Omega)} \leqq D(t). \tag{2.15}$$

In the following Lemma, we extend and make precise estimate (2.14), with nonzero data on the lateral boundary:

Lemma 2.11 *Let Ω be any smooth bounded domain. Let $q > 1$. Let $u \in C(\overline{\Omega} \times (0,T)) \cap C^{2,1}(Q_{\Omega,T})$ be a nonnegative solution of equation (1.1) in $Q_{\Omega,T}$, bounded on $\partial\Omega \times (0,T)$. Then there is a constant $C = C(N,q,\Omega)$ such that for any $\forall t \in (0,T)$,*

$$\|u(.,t)\|_{L^{\infty}(\Omega)} \leqq C(1 + t^{-\frac{1}{q-1}})d(x,\partial\Omega) + \sup_{\partial\Omega\times(0,T)} u,. \tag{2.16}$$

Proof. Let $M = \sup_{\partial\Omega\times(0,T)} u$. We set $u_\delta = u - (M+\delta)$ for any $\delta > 0$. On $\partial\Omega \times (0,T)$, we have $u_{\delta,k} \leqq -\delta < 0$. Since $u_\delta(0)$ is continuous, there exists $\Omega_\delta \subset\subset \Omega$ such that $u_\delta(0) \leqq -\delta/2$ on $\Omega\backslash\Omega_\delta$. Then there exists a contant C_δ such that $u_\delta(0) \leqq C_\delta d(x,\partial\Omega)$. From [19], for any $z \in \partial\Omega$, there exists a function $b_z(x)$ such that, for some $k, K, A > 0$ depending on Ω, and for any $x \in \Omega$,

$$kd(x,\partial\Omega) \leqq \inf_{z\in\partial\Omega} b_z(x) \leqq Kd(x,\partial\Omega), \quad b_z(x) \leqq A, \quad k \leqq |\nabla b_z(x)| \leqq 1, \quad |\Delta b_z(x)| \leqq K.$$

Then for any $z \in \partial\Omega$, there exists a function w_z of the form $w_z(x,t) = J(t)b_z(x)$ such that w_z is a supersolution of equation (1.1), $w_z \geqq 0$ on $\partial\Omega$, and

$$\lim_{t\to 0} d(x,\partial\Omega)^{-1} w_z(x,t) = \infty$$

uniformly in Ω. Otherwise J can be chosen explicitly by $J(t) = C(\mathrm{Arctan}\, t)^{-1/(q-1)}$ with $C^{q-1} = k^{-q}(K\pi/2 + A/(q-1))$. Thus there exists $\tau_\delta > 0$ such that $w_z(x,\tau) \geqq u_\delta(0)$ for $\tau \leqq \tau_\delta$. Since u_δ is a solution of (1.1), the function $w_z(x,\tau+t) - u_\delta(x,t)$ is nonnegative from the comparison principle. Letting $\tau \to 0$, and then $\delta \to 0$ and finally taking the infimum over $z \in \partial\Omega$ leads to the estimate

$$u(x,t) \leqq M + KJ(t)d(x,\partial\Omega), \tag{2.17}$$

hence (2.16) follows with another constant $C > 0$. ∎

2.3 Regularity for $q \leqq 2$

First of all, we give a result of regularity $\mathcal{C}^{2,1}$ *for any weak solution* of equation (1.1) and for any $q \leqq 2$. Such a regularity was obtained in [12, Proposition 3.2] for the VSS when $q < q_*$, and the proof was valid up to $q = (N+4)/(N+2)$. We did not find a good reference in the literature under our weak assumptions, even if a priori estimates can be found in [30], and Hölderian properties in [4], [40]. Our proof is based on a bootstrap technique, starting from the fact that u is subcaloric.

We set $\mathcal{W}^{2,1,\rho}(Q_{\omega,s,\tau}) = \{u \in L^\rho(Q_{\omega,s,\tau}) : u_t, \nabla u, D^2u \in L^\rho(Q_{\omega,s,\tau})\}$, for any $0 \leqq s < \tau < T$ and $1 \leqq \rho \leqq \infty$. This space is endowed with its usual norm.

Theorem 2.12 *Let $1 < q \leqq 2$. Let Ω be any domain in $\mathbb{R}^N$. Suppose that u is a weak nonnegative solution of (1.1) in $Q_{\Omega,T}$.*

(i) Then $u \in \mathcal{C}^{2,1}(Q_{\Omega,T})$, and there exists $\gamma \in (0,1)$ such that for any smooth domains $\omega \subset\subset \omega' \subset\subset \Omega$, and $0 < s < \tau < T$

$$\|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{\omega,s,\tau})} \leqq C\Phi(\|u\|_{L^\infty(Q_{\omega',s/2,\tau})}), \tag{2.18}$$

where Φ is a continuous increasing function and $C = C(N,q,\omega,\omega',s,\tau)$.

(ii) As a consequence, for any sequence (u_n) of weak solutions of equation (1.1) in $Q_{\Omega,T}$, uniformly locally bounded, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$.

Proof. (i) • Case $q < 2$. We can write (2.6) under the form

$$u_t - \Delta u = f, \qquad f = -|\nabla u|^q,$$

and $f \in L^{q_1}_{loc}(Q_{\Omega,T})$, with $q_1 = 2/q \in (1,2)$. From (2.6), there holds $u, \nabla u, f \in L^{q_1}_{loc}(Q_{\Omega,T})$. Then $u \in \mathcal{W}^{2,1,q_1}_{loc}(Q_{\Omega,T})$, see [30, theorem IV.9.1]. Choosing ω'' such that $\omega \subset\subset \omega'' \subset\subset \omega'$and denoting $Q = Q_{\omega,s,\tau}$, $Q' = Q_{\omega',s/2,\tau}$, $Q'' = Q_{\omega'',3s/4,\tau}$, we deduce from (2.10) that

$$\begin{aligned}\|u\|_{\mathcal{W}^{2,1,q_1}(Q)} &\leqq C(\|f\|_{L^{q_1}(Q'')} + \|u\|_{L^{q_1}(Q'')}) \leqq C(\|\nabla u\|^q_{L^2(Q'')} + \|u\|_{L^\infty(Q')}) \\ &\leqq C(\|u\|^q_{L^\infty(Q')} + \|u\|_{L^\infty(Q')}),\end{aligned}$$

with $C = C(N, q, \omega, \omega', s, \tau)$. From the Gagliardo-Nirenberg inequality, there exists $c = c(N, q, \omega) > 0$ such that for almost any $t \in (0,T)$,

$$\|\nabla u(.,t)\|_{L^{2q_1}(\omega)} \leqq c\|u(t)\|^{1/2}_{W^{2,q_1}(\omega)}\|u(t)\|^{1/2}_{L^\infty(\omega)}.$$

Then by integration, $|\nabla u| \in L^{2q_1}_{loc}(Q)$, and

$$\|\nabla u\|_{L^{2q_1}(Q)} \leqq c\|u(t)\|^{1/2}_{W^{2,q_1}(Q)}\|u\|^{1/2}_{L^\infty(Q)} \leqq C_1\Phi_1(\|u\|_{L^\infty(Q')}), \tag{2.19}$$

with a new constant C_1 as above, where Φ_1 is a continuous increasing function. Thus $f \in L^{q_2}_{loc}(Q_{\Omega,T})$, with $q_2 = (2/q)^2 \in (q_1, 2q_1)$ and $u, \nabla u, f \in L^{q_2}_{loc}(Q_{\Omega,T})$, in turn $u \in \mathcal{W}^{2,1,q_2}_{loc}(Q_{\Omega,T})$. By induction we find that $u \in \mathcal{W}^{2,1,q_k}_{loc}(\Omega \times (0,T))$, with $q_k = q_1^k$, for any $k \geqq 1$, and

$$\|\nabla u\|_{L^{2q_k}(Q)} \leqq C_k\Phi_k(\|u\|_{L^\infty(Q')})$$

with C_k, Φ_k as above. Choosing any k so that $q_k > N+2$, we deduce that $|\nabla u| \in C^{\gamma,\gamma/2}(\omega \times (s,\tau))$ for any $\gamma \in (0,1)$, see [30, Lemma II.3.3]. Then f is locally Hölderian, thus $u \in C^{2+\gamma,1+\gamma/2}(Q_{\omega,s,\tau})$, and (2.18) holds.

• Case $q = 2$. We define Q and Q' as above, and regularize by u_ε in Q' for ε small enough. Since u is locally bounded, u_ε converges to u in $L^s(Q')$ for any $s \geqq 1$, and by extraction *a.e.* in Q. And u_ε satisfies the equation in Q'

$$(u_\varepsilon)_t - \Delta u_\varepsilon + |\nabla u|^2 * \varrho_\varepsilon = 0.$$

Defining the functions $z = 1 - e^{-u}$ in $Q_{\Omega,T}$, and $z^\varepsilon = 1 - e^{-u_\varepsilon}$ in Q', we obtain that

$$(z^\varepsilon)_t - \Delta(z^\varepsilon) + h_\varepsilon = 0,$$

where $h_\varepsilon = e^{-u_\varepsilon}\left(|\nabla u|^2 * \varrho_\varepsilon - |\nabla u_\varepsilon|^2\right) \geqq 0$ from (2.4). Then $|\nabla u|^2 * \varrho_\varepsilon$ converges to $|\nabla u|^2$ and $|\nabla u_\varepsilon|^2$ converges to $|\nabla u|^2$ in $L^1_{loc}(Q_{\Omega,T})$, thus h_ε tends to 0 in $L^1_{loc}(Q_{\Omega,T})$. As $\varepsilon \to 0$, z^ε converges to z in $L^s(Q)$ for any $s \geqq 1$, and z is a solution of the heat equation in $\mathcal{D}'(Q')$, hence also in $\mathcal{D}'(Q_{\Omega,T})$). Then $z \in C^\infty(Q_{\Omega,T})$, hence $\max_Q z < 1$, thus $u \in C^\infty(Q_{\Omega,T})$. And $\|z\|_{L^\infty(Q')} < 1 - e^{-\|u\|_{L^\infty(Q')}}$, then (2.18) follows from analogous estimates on z.

(ii) From the estimate (2.18), one can extract a diagonal subsequence, converging *a.e.* to a function u in $Q_{\Omega,T}$, and the convergence holds in $C^{2,1}_{loc}(Q_{\Omega,T})$. Then u is a weak solution of (1.1) in $Q_{\Omega,T}$. ∎

In the case of the Dirichlet problem we obtain a corresponding regularity result for the *bounded* solutions. Our proof can be compared to the proof of [8, Proposition 4.1] relative to the case $q < 1$.

Theorem 2.13 *Let $1 < q \leqq 2$. Let Ω be a smooth bounded domain. Let u be any weak nonnegative solution of problem $(D_{\Omega,T})$, such that $u \in L^\infty_{loc}((0,T)\,;L^\infty(\Omega))$.*

(i) Then u satisfies the local estimates of Theorem 2.12. Moreover, $u \in C^{1,0}(\overline{\Omega}\times(0,T))$ and there exists $\gamma \in (0,1)$ such that, for any $0 < s < \tau < T$,

$$\|u\|_{C(\overline{\Omega}\times[s,\tau])} + \|\nabla u\|_{C^{\gamma,\gamma/2}(\overline{\Omega}\times[s,\tau])} \leqq C\Phi(\|u\|_{L^\infty(Q_{\Omega,s/2,\tau})}) \tag{2.20}$$

where $C = C((N,q,\Omega,s,\tau,\gamma)$, and Φ is an increasing function.

(ii) For any sequence (u_n) of weak solutions of $(D_{\Omega,T})$ uniformly bounded in $L^\infty_{loc}((0,T);L^\infty(\Omega))$, one can extract a subsequence converging in $C^{1,0}_{loc}(\overline{\Omega}\times(0,T))$ to a weak solution u of $(D_{\Omega,T})$.

Proof. (i) • Case $q < 2$. From Lemma 2.9, we have $\nabla u \in L^2_{loc}(0,T);L^2(\Omega))$ and $u \in C((0,T);L^1(\Omega))$. Then $f = -|\nabla u|^q \in L^{q_1}_{loc}((0,t);L^{q_1}(\Omega))$. For any $0 < s < \tau < T$, and $t \in [s/2,\tau]$, we can write $u(.,t) = u_1(.,t) + u_2(.,t)$, from Lemma 2.8, where

$$u_1(.,t) = e^{(t-s/2)\Delta}u(\frac{s}{2}), \qquad u_2(.,t) = \int_{s/2}^t e^{(t-\sigma)\Delta}f(\sigma)d\sigma.$$

We get $u_1 \in C^\infty(\overline{Q_{\Omega,s,\tau}})$ from the regularizing effect of the heat equation, and $u_2 \in \mathcal{W}^{2,1,q_1}(Q_{\Omega,T})$, from [30, theorem IV.9.1]. As above, from the Gagliardo estimate, we get $f \in L^{q_2}_{loc}((0,t);L^{q_2}(\Omega))$, and by induction $|\nabla u| \in C^{\gamma,\gamma/2}(\overline{Q_{\Omega,s,\tau}})$ for some $\gamma \in (0,1)$, see [30, Lemma II.3.3]. The estimates follow as above.

• Case $q = 2$. From Theorem 2.12, u is smooth in $Q_{\Omega,T}$, and $z = 1 - e^{-u}$ is a solution of the heat equation, and $z \in C((0,T);L^1(\Omega))$. Then $z(.,t) = e^{(t-s/2)\Delta}z(s/2)$, thus $z \in C^\infty(\overline{Q_{\Omega,s,\tau}})$. This implies that $\max_{\overline{Q_{\Omega,s,\tau}}} z < 1$, thus $u \in C^\infty(\overline{Q_{\Omega,s,\tau}})$ and the estimates follow again.

(ii) It follows directly from (2.20). ∎

Remark 2.14 *As a consequence, in the case $q \leqq 2$, we find again the estimate (2.15) for the problem $(D_{\Omega,T})$ without using the Bernstein argument, and it is valid for any weak solution $u \in L^\infty_{loc}((0,T)\,;L^\infty(\Omega))$.*

2.4 Singular solutions or supersolutions

In the study some functions play a fundamental role. The first one was introduced in [10].

2.4.1 A stationary supersolution

Assume that $1 < q < 2$. Equation (1.1) admits a stationary solution whenever $N = 1$ or $N \geqq 2$, $1 < q < N/(N-1)$, defined by

$$s \in (0,\infty) \longmapsto \Gamma_N(s) = \gamma_{N,q}s^{-a}, \qquad a = \frac{2-q}{q-1}, \qquad \gamma_{N,q} = a^{-1}(a+2-N)^{1-q}.$$

Moreover in the range $1 < q < 2$, the function $\Gamma = \Gamma_1$ defined by

$$s \in (0,\infty) \longmapsto \Gamma(s) = \gamma_q s^{-a}, \qquad a = \frac{2-q}{q-1}, \qquad \gamma_q = \frac{(q-1)^{-a}}{2-q}, \tag{2.21}$$

is a radial supersolution of equation (1.1) for any N.

2.4.2 Large solutions

Here we recall a main result of [19] obtained as a consequence of the universal estimates.

Theorem 2.15 ([19]) *Let G be any smooth bounded domain, and $\eta > 0$ such that $B_\eta \subset\subset G$. Then for any $q > 1$, there exists a (unique) solution Y_η^G of the problem*

$$\begin{cases} (Y_\eta^G)_t - \Delta Y_\eta^G + |\nabla Y_\eta^G|^q = 0, & \text{in } Q_{G,\infty}, \\ Y_\eta^G = 0, & \text{on } \partial G \times (0,\infty), \\ Y_\eta^G(x,0) = \begin{cases} \infty & \text{if } x \in B_\eta, \\ 0 & \text{if not,} \end{cases} \end{cases} \tag{2.22}$$

which is uniformly Lipschitz continuous in $\overline{G}$ for t in compacts sets of $(0,\infty)$ and is a classical solution of the problem for $t > 0$, and satisfies the initial condition in the sense:

$$\lim_{t\to 0} \inf_{x\in K} Y_\eta^G(x,t) = \infty, \quad \forall K \text{ compact} \subset B_\eta; \qquad \lim_{t\to 0} \sup_{x\in K} Y_\eta^G(x,t) = 0, \quad \forall K \text{ compact} \subset \overline{G}\backslash\overline{B_\eta}. \tag{2.23}$$

And Y_η^G is the supremum of the solutions $y_{\varphi_{\eta,G}}$ with nonnegative initial data $\varphi_{\eta,G} \in C(G)$ such that $\varphi_{\eta,G} = 0$ on $G\backslash\overline{B_\eta}$.

A crucial point for existence was the construction of a supersolution for the problem in a ball:

Lemma 2.16 *For any ball $B_s \subset \mathbb{R}^N$ and any $\lambda > 0$, there exists a supersolution $w_{\lambda,s}$ of equation (1.1) in $B_s \times [0,\infty)$, such that*

$$w_{\lambda,s} = \infty \quad \text{on } \partial B_s \times [0,\infty), \qquad w_{\lambda,s} = \lambda e^{ct+1/\alpha_s(x)}, \quad c = c(\lambda) > 0,$$

where α_s is the solution of $-\Delta\alpha_s = 1$ in B_s and $\alpha_s = 0$ on ∂B_s.

2.5 Some trace results

First we extend a trace result of [32].

Lemma 2.17 *Let $U \in C((0,T); L^1_{loc}(\Omega))$ be any nonnegative weak solution of equation*

$$U_t - \Delta U = \Phi \tag{2.24}$$

in $Q_{\Omega,T}$, with $\Phi \in L^1_{loc}(Q_{\Omega,T})$.

(i) Assume that $\Phi \geqq -F$, where $F \in L^1_{loc}(\Omega \times [0,T))$. Then $U(.,t)$ converges weak to some Radon measure U_0 :*

$$\lim_{t\to 0} \int_\Omega U(.,t)\varphi dx = \int_\Omega \varphi dU_0, \qquad \forall \varphi \in C_c(\Omega).$$

Furthermore, $\Phi \in L^1_{loc}([0,T); L^1_{loc}(\Omega))$, and for any $\varphi \in C^2_c(\Omega \times [0,T))$,

$$-\int_0^T \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)dxdt = \int_\Omega \varphi(.,0)dU_0. \tag{2.25}$$

(ii) Assume that Φ has a constant sign. Then

$$\Phi \in L^1_{loc}([0,T); L^1_{loc}(\Omega)) \Longleftrightarrow U \in L^\infty_{loc}([0,T); L^1_{loc}(\Omega)). \tag{2.26}$$

Proof. (i) Let $\omega \subset\subset \omega' \subset\subset \Omega$ and $0 < s < \tau < T$. We approximate U by U_ε and set $\Phi + F = E \geqq 0$, so that for ε small enough,

$$(U_\varepsilon)_t - \Delta U_\varepsilon = E_\varepsilon - F_\varepsilon \qquad \text{in } Q_{\omega',s/2,\tau}.$$

Let ϕ_1 be a positive eigenfunction associated to the first eigenvalue λ_1 of $-\Delta$ in $W_0^{1,2}(\omega)$. Multiplying the equation by ϕ_1 and integrating on over ω, we get, for any $t \in (s/2, \tau)$,

$$\frac{d}{dt}\int_\omega U_\varepsilon(.,t)\phi_1 dx + \lambda_1 \int_\omega U_\varepsilon(.,t)\phi_1 dx = -\int_{\partial\omega} U_\varepsilon(.,t)\frac{\partial\phi_1}{\partial\nu}d\sigma + \int_\omega E_\varepsilon(.,t)\phi_1 dx - \int_\omega F_\varepsilon(.,t)\phi_1 dx.$$

We set

$$X(t) = \int_\omega U(.,t)\phi_1 dx, \qquad h(t) = e^{\lambda_1 t}X(t) - \int_t^\tau \int_\omega e^{\lambda_1 s}F(.,s)\phi_1 dx d\theta,$$
$$X_\varepsilon(t) = \int_\omega U_\varepsilon(.,t)\phi_1 dx, \qquad h_\varepsilon(t) = e^{\lambda_1 t}X_\varepsilon(t) - \int_t^\tau \int_\omega e^{\lambda_1 s}F_\varepsilon(.,s)\phi_1 dx d\theta.$$

Then h_ε is nondecreasing on $(s/2,\tau)$, and then $h_\varepsilon(\tau) \geqq h_\varepsilon(s)$. On the other hand, $X_\varepsilon(t)$ converges to $X(t)$ a.e. *in* $(0,T)$ as $\varepsilon \to 0$. Since $U \in C((0,T); L^1_{loc}(\Omega))$, we deduce that $h(\tau) \geqq h(s) + \int_s^\tau \int_\omega E(.,t)\phi_1 dx$. Thus h is nondecreasing on $(0,T)$. From the assumption on F, X has a limit as $t \to 0$, and $\Phi \in L^1_{loc}([0,T); L^1_{loc}(\Omega))$.Otherwise, for any nonnegative $\psi \in \mathcal{C}^2_c(\Omega)$, for any $t < \tau$, there holds

$$\int_\Omega U(.,\tau)\psi dx - \int_t^\tau \int_\Omega (U\Delta\psi + \Phi\psi)dxdt = \int_\Omega U(.,t)\psi dx \tag{2.27}$$

from (2.3). Thus $\int_\Omega U(.,t)\psi dx$ has a nonnegative limit $\mu(\psi)$ as $t \to 0$, and

$$\int_\Omega U(.,\tau)\psi dx - \int_0^\tau \int_\Omega (U\Delta\psi + \Phi\psi)dxdt = \mu(\psi)$$

Then μ is a nonnegative linear functional on $\mathcal{C}^2_c(\Omega)$, thus it extends in a unique way as a Radon measure u_0 on Ω. Finally for any $\varphi \in C^\infty_c(\Omega \times [0,T])$, we have

$$-\int_t^T \int_\Omega (U\varphi_t + U\Delta\varphi + \Phi\varphi)dxdt = \int_\Omega U(.,t)\varphi(.,t)dx.$$

Going to the limit as $t \to 0$, we deduce (2.25), since

$$\left|\int_\Omega U(.,t)(\varphi(.,t) - \varphi(.,0))dx\right| \leqq Ct \int_{\text{supp}\varphi} U(.,t)dx.$$

(ii) If $U \in L^\infty_{loc}([0,T); L^1_{loc}(\Omega))$, then $\int_t^\tau \int_\Omega \Phi\psi dxdt$ is bounded as $t \to 0$, and $\Phi \in L^1_{loc}([0,T); L^1_{loc}(\Omega))$ from the Fatou Lemma. The converse is a direct consequence of (i). ∎

We deduce a trace property for equation (1.1), inspired by the results of [31] for equation 1.3, see also [13]:

Proposition 2.18 *For any nonnegative weak solution u of (1.1) in $Q_{\Omega,T}$, the following conditions are equivalent:*

(i) $u \in L^{\infty}_{loc}(\,[0,T)\,;L^1_{loc}(\Omega))$,

(ii) $\nabla u \in L^q_{loc}(\Omega \times [0,T))$,

(iii) $u(.,t)$ *converges weak* to some nonnegative Radon measure* u_0 *in* Ω.
And then for any $\tau \in (0,T)$, *and any* $\varphi \in C^1_c(\Omega \times [0,T))$,

$$\int_\Omega u(.,\tau)\varphi dx + \int_0^\tau \int_\Omega (-u\varphi_t + \nabla u.\nabla\varphi - |\nabla u|^q \varphi)dxdt = \int_\Omega \varphi(.,0)du_0. \tag{2.28}$$

Remark 2.19 *If* $q \geqq 2$, *and* u *admits a Radon measure* u_0 *as a trace, in the sense of condition (iii), then necessarily*

$$u_0 \in L^1_{loc}(\Omega), \quad \text{and } u \in C\left([0,T)\,;L^1_{loc}(\Omega)\right).$$

Indeed condition (ii) implies that $u \in L^2_{loc}([0,T)\,;W^{1,2}_{loc}(\Omega))$, *and* $u_t \in L^2_{loc}((0,T);W^{-1,2}_{loc}(\Omega)) + L^1\left(Q_{\omega,T}\right)$, *then the conclusion holds from [38]. As a first consequence, there exists no weak solution of equation (1.1) with a a Dirac mass as initial data. This had been shown in [1, Theorem 2.2 and Remark 2.1] for the Dirichlet problem* $(D_{\Omega,T})$..

2.6 Behaviour of Solutions of (1.1), (1.2) in Ω_0

Next we come to problem (1.1), (1.2). In order to see what occurs at $t = 0$, we extend the solutions on $(-T,T)$ as in [16].

Proposition 2.20 *Let* u *be any weak solution of (1.1), (1.2). Then the function* $\overline{u}$ *defined a.e. in* $Q_{\Omega,-T,T}$ *by*

$$\overline{u}(x,t) = \begin{cases} u(x,t), & \text{if } (x,t) \in Q_{\Omega,T}, \\ 0 & \text{if } (x,t) \in Q_{\Omega,-T,0}, \end{cases} \tag{2.29}$$

is a weak solution of the equation (1.1) in $Q_{\Omega_0,-T,T}$. *If moreover*

$$\lim_{t\to 0}\int_\Omega u(.,t)\varphi dx = 0, \qquad \forall \varphi \in C_c(\Omega), \tag{2.30}$$

then $\overline{u}$ *is a weak solution of (1.1) in* $Q_{\Omega,-T,T}$.

Proof. By assumption, $u \in L^1_{loc}([0,T) \times \Omega_0)$, hence $\overline{u} \in L^1_{loc}(Q_{\Omega_0,-T,T})$. Then we can define $\nabla\overline{u} \in \mathcal{D}'(Q_{\Omega_0,-T,T})$ and for any $\varphi \in \mathcal{D}(Q_{\Omega_0,-T,T})$,

$$< \nabla\overline{u}, \varphi >= -\int_{-T}^{T}\int_\Omega \overline{u}\nabla\varphi dxdt = -\int_0^T\int_\Omega u\nabla\varphi dxdt.$$

For any $k \geqq 1$, we consider a function ζ_k on $[0,\infty)$ such that

$$\zeta_k(t) = \zeta(kt), \text{ where } \zeta \in C^\infty\left([0,\infty)\right), \quad \zeta([0,\infty)) \subset [0,1], \quad \zeta \equiv 0 \text{ in } [0,1], \quad \zeta \equiv 1 \text{ in } [2,\infty). \tag{2.31}$$

Since u is a weak solution of (1.1), there holds

$$-\int_0^T\int_\Omega u\nabla(\varphi\zeta_k)dxdt=\int_0^T\int_\Omega\varphi\zeta_k\nabla udxdt. \tag{2.32}$$

From (1.2), we see that $u\in L^\infty_{loc}(\,[0,T)\,;L^1_{loc}(\Omega_0))$, hence $|\nabla u|\in L^q_{loc}(\Omega_0\times[0,T))$, from Proposition 2.18. Then we can go to the limit in (2.32) as $k\to\infty$ from the Lebesgue theorem, hence

$$-\int_0^T\int_\Omega u\nabla\varphi dxdt=\int_0^T\int_\Omega\varphi\nabla udxdt.$$

Thus $\nabla\overline{u}\in L^q_{loc}(Q_{\Omega_0,-T,T})$ and $\nabla\overline{u}(x,t)=\chi_{(0,T)}\nabla u(x,t)$; hence also $\nabla\overline{u}\in L^2_{loc}(Q_{\Omega_0,-T,T})$ from Lemma 2.6, and for any $\varphi\in\mathcal{D}(Q_{\Omega_0,-T,T})$,

$$\int_{-T}^T\int_\Omega(-\overline{u}\varphi_t+\nabla\overline{u}.\nabla\varphi+|\nabla\overline{u}|^q\varphi)dxdt=\int_0^T\int_\Omega(-u\varphi_t+\nabla u.\nabla\varphi+|\nabla u|^q\varphi)dxdt. \tag{2.33}$$

Moreover

$$\begin{aligned}0&=\int_0^T\int_\Omega(-u(\varphi\zeta_k)_t+\nabla u.\nabla(\varphi\zeta_k)+|\nabla u|^q\varphi\zeta_kdxdt\\&=-\int_0^T\int_\Omega u\varphi(\zeta_k)_tdxdt+\int_0^T\int_\Omega(-u\varphi_t\zeta_k+\nabla u.\nabla(\varphi\zeta_k)+|\nabla u|^q\varphi\zeta_kdxdt.\end{aligned}$$

As $k\to\infty$, the first term in the right hand side tends to 0 from (1.2), since

$$\left|\int_0^T\int_\Omega u\varphi(\zeta_k)_tdxdt\right|\leqq Ck\int_{1/k}^{2/k}\int_\Omega u\varphi dxdt\leqq C\sup_{t\in[1/k,2/k]}\int_{\mathrm{supp}\varphi}u(.,t)dx, \tag{2.34}$$

and we can go to the limit in the second term, since $|\nabla u|\in L^q_{loc}(\Omega_0\times[0,T))$. Thus from (2.33), $\overline{u}$ is a weak solution of equation (1.1) in $Q_{\Omega_0,-T,T}$. If (2.30) holds, the same result holds in Ω instead of Ω_0. ∎

Corollary 2.21 *Assume $1<q\leqq 2$. Then any weak solution u of (1.1), (1.2) satisfies $u\in C^{2,1}(\Omega_0\times[0,T))$ and $u(x,0)=0,\ \forall x\in\Omega_0$.*

If (2.30) holds, then $u\in C^{2,1}(\Omega\times[0,T))$ and $u(x,0)=0,\ \forall x\in\Omega$.

Proof. It follows directly from From Proposition 2.20 and Theorem 2.12 applied to $\overline{u}$. ∎

3 The critical or supercritical case

3.1 Removability in the range $q_*<q<2$

For any $1<q<2$ we can compare the solutions with the function Γ defined at (2.21).

Lemma 3.1 *Let $1 < q < 2$. Let u be any nonnegative weak solution of (1.1) in $Q_{\Omega,T}$, satisfying (1.2).*
(i) Let $r > 0$ such that $B_r \subset \Omega$. Then there exists $\tau_1 > 0$ (depending on u, r) such that

$$0 \leqq u(x,t) \leqq \Gamma(|x|) \qquad \forall (x,t) \in Q_{B_r \setminus \{0\},\tau_1}. \tag{3.1}$$

(ii) If $\Omega = \mathbb{R}^N$, then

$$0 \leqq u(x,t) \leqq \Gamma(|x|) \qquad \forall (x,t) \in Q_{\mathbb{R}^N \setminus \{0\},\tau_1}. \tag{3.2}$$

Proof. (i) For any $\eta \in (0,r)$, we put $\Omega_\eta = B_r \setminus \overline{B_\eta}$, and we set $F_\eta(x) = \Gamma(|x| - \eta)$, for any $x \in \Omega_\eta$. We find

$$-\Delta F_\eta + |\nabla F_\eta|^q = \gamma_q a \frac{(N-1)}{|x|} (|x| - \eta)^{-(a+1)} \geqq 0,$$

thus F_η is a super-solution of (1.1) in $Q_{\Omega_\eta,\infty}$. From Theorem 2.12 and Proposition 2.20, $u \in \mathcal{C}^{2,1}(Q_{\Omega,T}) \cap C(\Omega_0 \times [0,T))$ and $u(.,0) = 0$. Then there exists $\tau_1 < T$ such that $\max_{\substack{t \in [0,\tau_1] \\ |x|=r}} u(t,x) < 1$, and u is bounded in $\overline{\Omega_\eta} \times [0,\tau_1]$. For any $\varepsilon > 0$ small enough, we have $u(x,t) \leqq F_\eta(x)$ on $\partial B_{\eta+\varepsilon} \times [0,\tau_1]$. From the comparison principle in $Q_{\Omega_{\eta+\varepsilon},\tau_1}$, we get $u(x,t) \leqq F_\eta(x)$ in $\Omega_\eta \times [0,\tau_1]$, as $\varepsilon \to 0$. As $\eta \to 0$, we deduce (3.1).

(ii) From Lemma 2.16, for any $x_0 \in \mathbb{R}^N \setminus B_2$, the function $x \mapsto w_{1,1}(x - x_0)$ is a supersolution of equation (1.1) in $Q_{B(x_0,1),\infty}$, then in particular $u(t,x_0) \leqq e^{c(1)t + 1/\alpha_1(0)}$, thus u bounded in $Q_{\mathbb{R}^N \setminus B_2,T}$. From the comparison principle in $\mathbb{R}^N \setminus \overline{B_\eta}$ for any $\eta \in (0,1)$, see [20], we find $u(x,t) \leqq F_\eta(x)$ in $Q_{\mathbb{R}^N \setminus \overline{B_\eta},T}$, hence (3.2) holds as $\eta \to 0$. ∎

As a direct consequence we get a simple proof of Theorem 1.1 in case $q_* < q < 2$:

Theorem 3.2 *Let $q_* < q < 2$. Suppose that u is a nonnegative weak solution of (1.1),(1.2). Then $u \in C(\Omega \times [0,T))$ and $u(x,0) = 0$, $\forall x \in \Omega$.*

Proof. The assumption $q_* < q$ is equivalent to $a < N$. Let $B_r \subset \Omega$ and τ_1 defined at Lemma 3.1; we find for any $t \in (0,\tau_1)$,

$$\int_{B_r} u(.,t)dx \leqq \int_{B_r} \Gamma(|x|)dx \leqq \frac{\gamma_q |\partial B_1| r^{N-a}}{N-a};$$

then $u \in L^\infty((0,\tau_1); L^1(B_r))$. Applying Proposition 2.18, $u(.,t)$ converges weak* to a measure μ on B_r :

$$\lim_{t \to 0} \int_{B_r} u(.,t)\psi dx = \int_{B_r} \psi d\mu, \qquad \forall \psi \in C_c(B_r).$$

From (1.2), μ is concentrated at 0 and then $\mu = k\delta_0$ for some $k \geqq 0$. Suppose that $k > 0$, we choose ψ_η such that $\psi_\eta(0) = 1$, $\psi_\eta(B_r) \subset [0,1]$, $\text{supp}\psi_\eta \subset B_\eta$, with $\eta \in (0,r)$ small enough such that $\gamma_q |\partial B_1| \eta^{N-a} \leqq (N-a)k/2$. For any $t \in (0,\tau_1)$, lemma 3.1 yields

$$\int_{B_r} u(.,t)\psi_\eta dx \leqq \int_{B\eta} \Gamma(|x|)dx \leqq \frac{k}{2}. \tag{3.3}$$

As t tends to 0 the left-hand side tends to k, which is a contradiction. Then $k = 0$, hence for any $\psi \in C_c^\infty(B_r)$,

$$\lim_{t\to 0} \int_{B_r} u(.,t)\psi dx = 0, \tag{3.4}$$

and we conclude from Corollary 2.21. ∎

3.2 Removability in the whole range $q_* \leqq q < 2$

The proof of Theorem 3.2 is not valid in the critical case $q = q_*$, since the function $x \longmapsto \Gamma(|x|) = \gamma_q |x|^{-N}$ is not integrable near 0. Then we use another argument of comparison with the large solutions constructed at Theorem 2.15, valid for any $1 < q < 2$:

Proposition 3.3 *Let $1 < q < 2$. Under the assumptions of Theorem 2.15 with $G = B_n$ ($n \geqq 1$) the functions $Y_\eta^{B_n}$ converge as $n \to \infty$ to a radial solution Y_η of problem*

$$\begin{cases} (Y_\eta)_t - \Delta Y_\eta + |\nabla Y_\eta|^q = 0, & \text{in } Q_\infty, \\ Y_\eta(x,0) = \begin{cases} \infty & \text{if } x \in B_\eta, \\ 0 & \text{if not.} \end{cases} \end{cases} \tag{3.5}$$

Then, as $\eta \to 0$, Y_η converges to a radial self-similar solution Y of equation (1.1) in $Q_{\mathbb{R}^N,\infty}$, such that

$$Y(x,t) \leqq \Gamma(|x|), \qquad \text{in } Q_\infty, \tag{3.6}$$

$$Y(x,t) \leqq C(1 + t^{-\frac{1}{q-1}}), \qquad \text{in } Q_\infty, \tag{3.7}$$

where $C = C(N,q)$, and

$$\lim_{t\to 0}(\sup_{|x|\geqq r} Y(x,t)) = 0. \tag{3.8}$$

If $q_* \leqq q < 2$, then $Y = 0$.

Proof. Let $\eta \in (0, 1/2)$. For any $n \geqq 1$, $Y_\eta^{B_n}$ is the supremum of the solutions $y_{\varphi_{\eta,B_n}}$; from the comparison principle, since $q < 2$,

$$y_{\varphi_{\eta,B_n}}(x,t) \leqq \Gamma(|x| - \eta) \qquad \text{in } (B_n \backslash \overline{B_\eta}) \times [0,\infty). \tag{3.9}$$

From Lemma 2.11 in $Q_{B_1,\infty}$, we obtain, for any $(x,t) \in \overline{B_1} \times (0,\infty)$

$$y_{\varphi_{\eta,B_n}}(x,t) \leqq C(1 + t^{-\frac{1}{q-1}}) + \gamma_q\{1-\eta)\}^{-\frac{2-q}{q-1}} \leqq C(1 + t^{-\frac{1}{q-1}}) + \gamma_q 2^{\frac{2-q}{q-1}}, \tag{3.10}$$

with $C = C(N,q)$. And for any $(x,t) \in (B_n \backslash \overline{B_1}) \times (0,\infty)$, we have

$$y_{\varphi_{\eta,B_n}}(x,t) \leqq \Gamma(|x| - \eta) \leqq \Gamma(1-\eta) \leqq \gamma_q 2^{\frac{2-q}{q-1}} \tag{3.11}$$

Then (3.10) holds in $B_n \times [0,\infty)$. The same majoration holds for $Y_\eta^{B_n}$: with a new $C = C(N,q)$,

$$Y_\eta^{B_n}(.,t) \leqq C(1 + t^{-\frac{1}{q-1}}), \qquad \text{in } Q_{B_n,\infty}.$$

Then we can go to the limit as $n \to \infty$, for fixed η. From Theorem 2.12 we can extract a (diagonal) subsequence converging in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,\infty})$ to a weak solution Y_η of equation (1.1). In fact the whole sequence converges, since $Y_\eta^{B_n} \leqq Y_\eta^{B_{n+1}}$ in $Q_{B_n,\infty}$. Then $Y_\eta = \sup Y_\eta^{B_n}$ satisfies

$$Y_\eta \leqq C(1 + t^{-\frac{1}{q-1}}), \qquad \text{in } Q_\infty, \tag{3.12}$$

and Y_η solves the problem (3.5) in the sense

$$\lim_{t\to 0}\inf_{x\in K} Y_\eta(x,t) = \infty, \quad \forall K \text{ compact} \subset B_\eta; \qquad \lim_{t\to 0}\sup_{x\in K} Y_\eta(x,t) = 0, \quad \forall K \text{ compact} \subset \mathbb{R}^N\backslash\overline{B_\eta}. \tag{3.13}$$

Indeed from Lemma 2.16, for any ball $B(x_0,s) \subset \mathbb{R}^N\backslash\overline{B_\eta}$, and any $\lambda > 0$, we have $Y_\eta^{B_n} \leqq w_{\lambda,s}(x - x_0)$ in $Q_{B(x_0,s),\infty}$ for any $n > |x_0|+|r|$; in turn $Y_\eta \leqq w_{\lambda,s}(x-x_0)$, hence $\lim_{t\to 0}\sup_{B(x_0,s/2)} Y_\eta(.,t) \leqq \lambda e^{1/\alpha(s/2)}$ for any $\lambda > 0$. Moreover (3.9) implies that

$$Y_\eta(x,t) \leqq \Gamma(|x| - \eta) \qquad \text{in } Q_{\mathbb{R}^N\backslash\overline{B_\eta},\infty}. \tag{3.14}$$

Then for any $r > \eta$, and any $p > r$,

$$\sup_{|x|\geqq r} Y_\eta(x,t) \leqq \sup_{x\in B_p\backslash\overline{B_\eta}} Y_\eta(x,t) + \sup_{x\in\mathbb{R}^N\backslash\overline{B_p}} Y_\eta(x,t) \leqq \sup_{x\in B_p\backslash\overline{B_\eta}} Y_\eta(x,t) + \Gamma(|p| - \eta)$$

then we find

$$\lim_{t\to 0}(\sup_{|x|\geqq r} Y_\eta(x,t)) = 0, \tag{3.15}$$

since $\lim_{r\to\infty}\Gamma(r) = 0$.

Next we let $\eta \to 0$: observing that $Y_\eta \leqq Y_{\eta'}$ for $\eta \leqq \eta'$, in the same way from Theorem 2.12, the function $Y = \inf_{\eta>0} Y_\eta$ is a weak solution of equation (1.1) in $Q_{\mathbb{R}^N,\infty}$, satisfying the estimates (3.6), (3.7), and (3.8) which implies in particular (1.7). Because of their uniqueness, all the functions $Y_\eta^{B_n}$ are radial, and satisfy the relation of similarity,

$$\kappa^a Y_\eta^{B_n}(\kappa x, \kappa^2 t) = Y_{\eta/\kappa}^{B_{n/\kappa}}(x,t), \qquad \forall\kappa > 0, \quad \forall (x,t) \in B_{n/k};$$

then Y is radial and self-similar.

Suppose $q \geqq q_*$ and $Y \not\equiv 0$; writing Y under the similar form $Y(x,t) = t^{-a/2} f(t^{-1/2}|x|)$, then from [39, Theorem 2.1], we find $\underline{\lim}_{r\to\infty} r^a f(r) > 0$, which contradicts (3.8); thus $Y \equiv 0$. ∎

Proposition 3.4 *Let $1 < q < 2$. Let Ω be any domain in $\mathbb{R}^N$. Let u be any weak solution of (1.1),(1.2) in $Q_{\Omega,T}$. Then for any $\tau \in (0,T)$ and any ball $B_r \subset\subset \Omega$, there holds*

$$u \leqq Y + \max_{\partial B_r\times[0,\tau]} u, \qquad \text{in } Q_{B_r,\tau}.$$

Moreover, if $\Omega = \mathbb{R}^N$, then

$$u \leqq Y, \qquad \text{in } Q_{\mathbb{R}^N,T} \tag{3.16}$$

and $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty}) \cap C((0,\infty); C^2_b(\mathbb{R}^N))$.

Proof. Let u be such a solution in $Q_{\Omega,T}$. Let $\tau \in (0,T)$, $B_r \subset\subset \Omega$, and $M_r = \max_{\partial B_r \times [0,\tau]} u$ and $\varepsilon > 0$ be fixed. From Corollary 2.21, $u \in C(\Omega_0 \times [0,T))$ and $u(x,0) = 0, \forall x \in \Omega_0$. Then for any $0 < \eta < r/2$, there is $\delta_\eta > 0$ such that

$$u(x,t) < \varepsilon, \qquad \text{for } \ \eta \leqq |x| \leqq r, \quad t \in (0,\delta_\eta). \tag{3.17}$$

Let $R > r$. Next, for any $\delta \in (0,\delta_\eta)$, we make a comparison in $Q_{B_r,\delta,\tau}$ between $u(x,t)$ and

$$y_{2\eta,R,\delta}(x,t) = Y_{2\eta}^{B_R}(x,t-\delta) + M_r + \varepsilon$$

as follows. On the parabolic boundary of $Q_{B_r,\delta,\tau}$, it is clear that $u \leqq y_{2\eta,\delta,R}$, since $u \leqq M_r$ on $\partial B_r \times [\delta,\tau]$, $u(x,\delta) \leqq \varepsilon$ for $x \in \overline{B_r} \backslash \overline{B_\eta}$, and $u(x,\delta) \leqq \infty = y_{2\eta,\delta,R}$, for $x \in \overline{B_\eta}$. And $y_{2\eta,R,\delta}$ converges to $+\infty$ uniformly on $\overline{B_\eta}$ as $t \to \delta$, and $u(.,\delta)$ is bounded on $\overline{B_\eta}$. Then, from the comparison principle,

$$u \leqq y_{2\eta,R,\delta}, \qquad \text{in } Q_{B_r,\delta,\tau}. \tag{3.18}$$

As δ tends to 0 in (3.18), and we get

$$u \leqq Y_{2\eta}^{B_R} + M_r + \varepsilon, \qquad \text{in } Q_{B_r,\tau}, \tag{3.19}$$

by the continuity of $Y_{2\eta}^{B_R}$ in $Q_{B_r,T}$. Since (3.19) holds for any $\eta < r/2$, and any $\varepsilon > 0$, we finally obtain

$$u \leqq Y + M_r, \qquad \text{in } Q_{B_r,\tau}.$$

Moreover if $\Omega = \mathbb{R}^N$, then $M_r \leqq \Gamma(r)$ from Lemma 3.1, and we get (3.16) by letting $r \to \infty$. Moreover $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty})$ from Theorem 2.12, then from (3.7), $u \in C_b(Q_{\mathbb{R}^N,\epsilon,\infty})$ for any $\epsilon > 0$, then from [20, Theorems 3 and 6], $u \in C((0,\infty); C_b^2(\mathbb{R}^N))$. ∎

As a direct consequence, we deduce a new proof of Theorem 1.1, valid in the range $q_* \leqq q < 2$:

Theorem 3.5 *Let $q_* \leqq q < 2$. Suppose that u is a nonnegative weak solution of (1.1),(1.2) in $Q_{\Omega,T}$.*

Then $u \in C(\Omega \times [0,T))$ and $u(x,0) = 0$, $\forall x \in \Omega$.

Proof. Since $q \geqq q_*$, we have $Y = 0$, from Proposition 3.3, thus u is bounded in $Q_{B_r,\tau}$ from Proposition 3.4. Then (3.4) still holds for any $\psi \in C_c^\infty(B_r)$, and we conclude again from Corollary 2.21. ∎

3.3 Removability for $q \geqq 2$

When $q > 2$, the regularity of the solutions of equation (1.1), in particular the continuity property, is not known up to now. It was shown recently in [18] that *if* a solution in the viscosity sense is continuous, then it is Hölderian. Then it is difficult to apply comparison theorems. Here we use the transformation $u \longmapsto z = 1 - e^{-u}$, which reduces classically equation (1.1) to the heat equation when $q = 2$, where we gain the fact that z is bounded. For $p > 2$, our proof requires regularization arguments.

Theorem 3.6 *Let $q \geqq 2$. Let u be any weak solution u of equation (1.1), (1.2), in $Q_{\Omega,T}$.*

(i) If $q = 2$, then $u \in C^\infty(\Omega \times [0,T))$, and $u(x,0) = 0, \quad \forall x \in \Omega$.

(ii) If $q > 2$, then u satisfies

$$\lim_{t\to 0} \int_\Omega u(.,t)\varphi dx = 0, \qquad \forall \varphi \in C_c(\Omega),$$

and $u \in C([0,T); L^r_{loc}(\Omega))$ for any $r \geqq 1$ and $u(.,0) = 0$ in the sense of $L^r_{loc}(\Omega)$. Moreover $u \in L^\infty(Q_{\omega,\tau})$ for any $\omega \subset\subset \Omega$, and $\tau \in (0,T)$, and

$$\lim_{t\to 0} \sup_{Q_{\omega,t}} u = 0.$$

Proof. Let us set

$$z = 1 - v, \qquad v = e^{-u}, \tag{3.20}$$

Notice that z is an increasing function of u and z takes its values in $[0,1]$.

(i) Case $q = 2$. From Theorem 2.12, u is a classical solution in $Q_{\Omega,T}$. Then z is a classical solution of the heat equation

$$z_t - \Delta z = 0$$

in $Q_{\Omega,T}$, and $z \in C(\Omega_0 \times [0,T))$ and $z(x,0) = 0$ for $x \neq 0$. From Lemma 2.17, z converges weak* to a Radon measure μ as $t \to 0$, necessarily concentrated at 0, from (1.2), since $z \leqq u$. Then $\mu = 0$, because z is bounded. As for u, defining the extension $\overline{z}$ of z by 0 for $t \in (-T,0)$, we find that $\overline{z}$ is a solution of heat equation in $Q_{\Omega,-T,T}$, then $\overline{z} \in C^\infty(Q_{\Omega,-T,T})$. Hence $\overline{z}$ is strictly locally bounded by 1, thus also $\overline{u} \in C^\infty(Q_{\Omega,-T,T})$, thus $u(0,0) = 0$, and the proof is done.

(ii) Case $q > 2$. We regularize equation (1.1) and obtain

$$(u_\varepsilon)_t - \Delta u_\varepsilon + (|\nabla u|^q)_\varepsilon = 0,$$

and we set $v^\varepsilon = e^{u_\varepsilon}$.Then v^ε satisfies the equation

$$v^\varepsilon_t - \Delta v^\varepsilon = v^\varepsilon \left((|\nabla u|^q)_\varepsilon - |\nabla u_\varepsilon|^2 \right).$$

Observe that v^ε is not the regularisation of v, but it has the same convergence properties. Going to the limit as $\varepsilon \to 0$, we obtain

$$v_t - \Delta v = v(|\nabla u|^q - |\nabla u|^2)$$

in $\mathcal{D}'(Q_{\Omega,T})$. Next we apply lemma 2.17 to v, with

$$\Phi = v[|\nabla u|^q - |\nabla u|^2] \in L^1_{loc}(Q_{\Omega,T}), \qquad F = -1,$$

since from the Young inequality, $\Phi \geqq -v \geqq -1$. Then $z(.,t)$ converges weak* to a Radon measure μ as $t \to 0$, and $\Phi \in L^1_{loc}(\Omega \times [0,T))$; and for any $\varphi \in C^2_c(\Omega \times [0,T))$ there holds

$$\int_0^T \int_\Omega z(\varphi_t + \Delta\varphi) dx dt = \int_0^T \int_\Omega \Phi\varphi dx dt + \int_\Omega \varphi(x,0) d\mu, \tag{3.21}$$

from (2.25). We claim that $\mu = 0$ and the extension of z by 0 for $t = 0$ satisfies

$$z \in C([0,T), L^1_{loc}(\Omega)).$$

Indeed, from assumption (1.2), $u(.,t)$converges to 0 in $L^1_{loc}(\Omega_0)$ as $t \to 0$, thus also $z(.,t)$. For any sequence (t_n) tending to 0, we can extract a (diagonal) subsequence such that $u(.,t_\nu)$ converges to 0, a.e. in Ω. Since z is bounded, it follows that $(z(.,t_\nu))$ converges to 0 in $L^1_{loc}(\Omega)$ from the Lebesgue theorem. And then $z(.,t)$ converges to 0 in $L^1_{loc}(\Omega)$ as $t \to 0$.

We still consider the extension $\overline{z}$ of z by 0 on for $t \in (-T, 0)$. For any $\phi \in \mathcal{D}^+(Q_{\Omega,-T,T})$, we have from (3.21),

$$\begin{aligned} -\int_{-T}^{T}\int_{\Omega} \overline{z}(\phi_t + \Delta\phi)dxdt &= -\int_0^T\int_\Omega z(\phi_t+\Delta\phi)dxdt = -\int_0^T\int_\Omega \Phi\varphi dxdt \\ &\leqq \int_0^T\int_\Omega (1-z)\varphi dxdt \leqq \int_{-T}^T\int_\Omega (1-\overline{z})\varphi dxdt. \end{aligned}$$

Then $\overline{z}$ is a subsolution of equation

$$w_t - \Delta w + w = 1 \tag{3.22}$$

in $\mathcal{D}'(Q_{\Omega,-T,T})$. Otherwise $\overline{u}$ is the weak solution of equation (1.1) in $Q_{\Omega_0,-T,T}$, then $\overline{u}$ is subcaloric. As a consequence, for any $\tau \in (0,T)$, and any ball $B_{2r} \subset\subset \Omega$, the function $\overline{u}$ is essentially bounded on $Q_{B_{2r}\setminus\overline{B_{r/2}},-\tau,\tau}$ by a constant $M_{r,\tau}$, and then $\overline{z} \leqq 1 - e^{-M_{r,\tau}} = m_{r,\tau} < 1$ on this set. For any $K > 0$ the function $y_K(t) = 1 - Ke^{-t}$ is a solution of equation (3.22). Taking $K = e^{-(M_{r,\tau}+\tau+1)}$, we can apply the comparison principle in $Q_{B_r,-\tau,\tau}$ to the regularisation $\overline{z}_\varepsilon$ of $\overline{z}$ for ε small enough, and deduce that $\overline{z} \leqq y_K$ a.e. in $Q_{B_r,-\tau,\tau}$, and then

$$\overline{z} \leqq 1 - e^{-(M_{r,\tau}+2\tau+1)} < 1 \qquad \text{in } Q_{B_r,-\tau,\tau}.$$

Hence $\overline{u} = -\ln(1-\overline{z})$ is essentially bounded in $Q_{B_r,-\tau,\tau}$. Finally $\overline{u} \in L^\infty_{loc}(Q_{\Omega,-T,T})$, from the subcaloricity, hence $u \in L^\infty_{loc}(Q_{\Omega,T})$.

Besides, for any $0 < s < t < \tau$, and any domain $\omega \subset\subset \Omega$,

$$|u(.,t) - u(.,s)| \leqq e^{\|\overline{u}\|_{L^\infty(Q_{\omega,-\tau,\tau})}} |z(.,t) - z(.,s)|;$$

then $u \in \mathcal{C}([0,T); L^1_{loc}(\Omega))$, and $u \in C([0,T); L^r_{loc}(\Omega))$, for any $r > 1$, since u is locally bounded.

Furthermore, for any ball $B(x_0, 2\rho) \subset \Omega$, and any $t \in \left(\rho^2 - T, T\right)$,

$$\sup_{B(x_0,\rho)\times(t-\rho^2,t))} \overline{u} \leqq C\rho^{-(N+2)} \int_{t-\rho^2}^t \int_{B(x_0,2\rho)} \overline{u} dxds,$$

where $C = C(N)$, see for example [28, Theorem 6.17]. Hence for any $t \in (0,\tau)$ and $\rho < T^{1/2}$, we find

$$\sup_{B(x_0,\rho)\times(0,t))} u \leqq C\rho^{-(N+2)} \int_0^t \int_{B(x_0,2\rho)} u dxds \leqq C\rho^{-(N+2)} t \left\|u\right\|_{L^\infty(Q_{B(x_0,2\rho),\tau})},$$

which achieves the proof. ∎

3.4 Global removability in $\mathbb{R}^N$

Next we show Theorem 1.2 relative to $\Omega = \mathbb{R}^N$. It is a consequence of Proposition 3.4 in case $1 < q < 2$. In fact the result is general, as shown below:

Proposition 3.7 *Let $q > 1$. Let u be any non-negative weak subsolution of equation (1.1) in $Q_{\mathbb{R}^N,T}$ such that $u \in C((0,T,L^1_{loc}(\mathbb{R}^N))$, and*

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(.,t)\psi dx = 0, \tag{3.23}$$

for any $\psi \in C_c\left(\mathbb{R}^N\right)$. Then $u \equiv 0$.

Proof. From Lemmas 2.3 and 2.6, since $u \in C((0,T,L^1_{loc}(\mathbb{R}^N))$, there holds

$$\int_{\mathbb{R}^N} u(.,t)\psi dx - \int_{\mathbb{R}^N} u(.,s)\psi dx + \int_s^\tau \int_{\mathbb{R}^N} (\nabla u.\nabla\psi + |\nabla u|^q\psi dxdt \leqq 0,$$

for any $\psi \in C_c^{2,+}(\mathbb{R}^N)$, and any $(s,t) \subset (0,T)$ .Taking $\psi = \xi^{q'}$ with $\xi \in \mathcal{D}^+(\mathbb{R}^N)$ and using Hölder inequality, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N} u(.,t)\psi dx - \int_{\mathbb{R}^N} u(.,s)\psi dx + \int_s^t \int_{\mathbb{R}^N} |\nabla u|^q\psi dxdt &\leqq q'(\int_s^t \int_{\mathbb{R}^N} |\nabla u|^q\psi dx)^{\frac{1}{q}}(\int_s^t \int_{\mathbb{R}^N} |\nabla\xi|^{q'}dx)^{\frac{1}{q'}}\\ &\leqq \frac{1}{2}\int_s^t \int_{\mathbb{R}^N} |\nabla u|^q\psi dx + C_q\int_s^t \int_{\mathbb{R}^N} |\nabla\xi|^{q'}dx\end{aligned}$$

with $C_q = (2(q-1))^{q'}$.We choose for any $R > r > 0$,

$$\xi(x) = \phi(\frac{|x|}{R}),\ \text{where } \phi([0,\infty)) \subset [0,1]\,,\quad \phi \equiv 1 \text{ in } [0,1]\,,\quad \phi \equiv 0 \text{ in } [2,\infty)\,,$$

and go to the limit as $s \to 0$ from (3.23). It follows that

$$\int_{B_r} u(.,t)dx + \frac{1}{2}\int_0^t \int_{B_r} |\nabla u|^q dxdt \leqq C_q t R^{N-q'}. \tag{3.24}$$

• First assume $q < N/(N-1)$; then $N - q' < 0$. Letting $R \to \infty$, we deduce that $\int_{B_r} u(.,t)dx = 0$, for any $r > 0$, thus $u \equiv 0$.

• Next assume $q \geqq N/(N-1)$. Then we fix some $k \in (1, N/(N-1))$; for any $\eta \in (0,1)$, there holds $\eta|\nabla u|^k \leqq \eta + |\nabla u|^q$, hence the function $w_\eta = \eta^{1/(k-1)}(u - \eta t)$ satisfies

$$(w_\eta)_t - \Delta w_\eta + |\nabla w_\eta|^k \leqq 0$$

in the weak sense. Thanks to Kato's inequality, see for example [33] or [6], we deduce that

$$(w_\eta^+)_t - \Delta w_\eta^+ + |\nabla w_\eta^+|^k \leqq 0, \tag{3.25}$$

in $\mathcal{D}'(\mathbb{Q}_{\mathbb{R}^N,T})$. Moreover $w_\eta \in C([0,T)\,, L^1_{loc}(\mathbb{R}^N))$, and, for any $r > 0$,

$$\lim_{t\to 0^+}\int_{B_r} w_\eta^+(.,t)dx = \eta^{-\frac{1}{k-1}}\lim_{t\to 0^+}\int_{B_r}(u(.,t) - \eta t)^+ dx = 0.$$

By the above proof, $w_\eta^+ \equiv 0$. Letting η tend to 0 we get again $u \equiv 0$. ∎

3.5 Behaviour of the approximating sequences

When q is critical or supercritical, a simple question is to know what can happen to a sequence of solutions with smooth initial data converging to the Dirac mass, and one can expect that that it converges to 0. We get more generally the following:

Theorem 3.8 *Assume that $q \geqq q_*$. Let (φ_ε) be any sequence in $\mathcal{D}^+\left(\mathbb{R}^N\right)$, with supp $\varphi_\varepsilon \in B_\varepsilon$. Then the sequence (u_ε) of solutions of (1.1) in $Q_{\mathbb{R}^N,\infty}$, with inital data φ_ε, converges to 0 in $C_{loc}(Q_{\mathbb{R}^N,\infty})$. In the same way, if Ω is bounded, the sequence $\left(u_\varepsilon^\Omega\right)$ of solutions of $(D_{\Omega,\infty}$, with initial data φ_ε, converges to 0 in $C_{loc}(\overline{\Omega} \times (0,\infty))$.*

Proof. Let $\varepsilon \in (0,1)$. Since $u_\varepsilon^\Omega \leqq u_\varepsilon$, we only need to prove the result in case $\Omega = \mathbb{R}^N$.

(i) Case $q < 2$. We use the function $Y_{2\varepsilon}$ defined at (3.5). There holds $u_\varepsilon \leqq Y_{2\varepsilon}$ from the comparison principle; and $Y_{2\varepsilon}$ converges to 0 in $C^1_{loc}(Q_{\mathbb{R}^N,\infty})$ from Proposition 3.3, then also u_ε.

(ii) Case $q \geqq 2$. Let us fix some k such that $q_* < k < 2$. As in the proof of Proposition 3.7, for any $\eta \in (0,1)$, $w_{\varepsilon,\eta} = \eta^{1/(k-1)}(u_\varepsilon - \eta t)$ satisfies

$$(w_{\varepsilon,\eta})_t - \Delta w_{\varepsilon,\eta} + |\nabla w_{\varepsilon,\eta}|^k \leqq 0 \tag{3.26}$$

in $\mathcal{D}'(Q_{\mathbb{R}^N,\infty})$, and $w_{\varepsilon,\eta} \in L^\infty_{loc}([0,\infty); L^\infty(\mathbb{R}^N))$. From the comparison principle we find that $w_{\varepsilon,\eta} \leqq v_\varepsilon$, where v_ε is the solution of equation (1.1) with q replaced by k and $v_\varepsilon(.,0) = \rho_\varepsilon$; hence $u_\varepsilon \leqq \eta t + \eta^{1/(k-1)}$. And (v_ε) converges to 0 in $C_{loc}(Q_{\mathbb{R}^N,\infty})$ from (i). Let $\mathcal{K} = [s,\tau] \times K$ be any compact in $Q_{\mathbb{R}^N,\infty}$. Then

$$\limsup \|u_\varepsilon\|_{L^\infty(\mathcal{K})} \leqq \eta\tau + \eta^{1/(k-1)} \limsup \|v_\varepsilon\|_{L^\infty(\mathcal{K})} = \eta\tau$$

for any η, then $\lim \|u_\varepsilon\|_{L^\infty(\mathcal{K})} = 0$. ∎

4 The subcritical case $1 < q < q_*$

We first recall the following results of [8, Theorem 3.2 and Proposition 5.1] for the Dirichlet problem.

Theorem 4.1 ([8]) *Let $1 < q < q_*$ and Ω be a smooth bounded domain. Then for any $u_0 \in \mathcal{M}_b(\Omega)$ and any $T \in (0,\infty]$ there exists a weak solution of problem $(D_{\Omega,\infty})$ such that $u(.,0) = u_0$ in the weak sense of $\mathcal{M}_b(\Omega)$:*

$$\lim_{t\to 0} \int_\Omega u(.,t)\varphi dx = \int_\Omega \varphi du_0, \qquad \forall \varphi \in C_b(\Omega), \tag{4.1}$$

and u is given equivalently by the semi-group formula

$$u(.,t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} |\nabla u(.,s)|^q (s) ds \qquad \text{in } L^1(\Omega), \tag{4.2}$$

where $e^{t\Delta}u_0$ is the unique weak solution w of the heat equation such that

$$\lim_{t\to 0} \int_\Omega w(.,t)\varphi dx = \int_\Omega \varphi du_0, \qquad \forall \varphi \in C_b(\Omega). \tag{4.3}$$

Moreover $u \in C^{2,1}(Q_{\Omega,\infty})$, and $u \in C\left(\overline{Q_{\Omega,\epsilon,\infty}}\right)$ for any $\epsilon > 0$. And u is the unique weak solution of problem $(D_{\Omega,T})$ for any $T \in (0,\infty)$.

This solution was obtained from the Banach fixed point theorem. The existence was also obtained by approximation in [1], from the pioneer results of [15]. Here we give a shorter proof of Theorem 4.1 when u_0 is nonnegative, and firm in details the convergence:

Proposition 4.2 *Suppose $1 < q < q_*$. Let $u_0 \in \mathcal{M}_b^+(\Omega)$, and $(u_{0,n})$ be any sequence of functions of $C_b^1(\overline{\Omega}) \cap C_0(\Omega)$ converging weak * to u_0, such that $\|u_{0,n}\|_{L^1(\Omega)} \leqq \|u_0\|_{\mathcal{M}_b(\Omega)}$. Let u_n be the classical solution of $(D_{\Omega,\infty})$ with initial data $u_{0,n}$.*

Then (u_n) converges in $C_{loc}^{2,1}(Q_{\Omega,\infty}) \cap C_{loc}^{1,0}(\overline{\Omega} \times (0,\infty))$ to a function $u \in L_{loc}^q([0,\infty); W_0^{1,q}(\Omega))$ and u is the unique solution of $(D_{\Omega,T})$, (4.1) for any $T > 0$. And u satisfies the estimates (2.16) and (2.15).

Proof. There holds

$$u_n(.,t) = e^{t\Delta} u_{0,n} - \int_0^t e^{(t-s)\Delta} \left|\nabla u_n(.,s)\right|^q (s) ds \qquad \text{in } L^1(\Omega).$$

From estimate (2.16) and Theorem 2.13, since $q < 2$, one can extract a subsequence, still denoted (u_n), converging in $C_{loc}^{2,1}(Q_{\Omega,\infty}) \cap C_{loc}^1(\overline{\Omega} \times (0,\infty))$ to a weak solution u of $(D_{\Omega,\infty})$. And

$$\int_\Omega u_n(.,t)dx + \int_0^t \int_\Omega \left|\nabla u_n(.,s)\right|^q (s) dxds - \int_0^t \int_{\partial\Omega} \frac{\partial u_n}{\partial \nu}(.,s) dxds = \int_\Omega u_{0,n} dx; \tag{4.4}$$

hence $|\nabla u_n|^q$ is bounded in $L^1(Q_{\Omega,\infty})$ by $\|u_0\|_{\mathcal{M}_b(\Omega)}$. Then from [6, Lemma 3.3], (u_n) is bounded in $L^\gamma((0,\tau), W_0^{1,\gamma}(\Omega))$ for any $\gamma \in [1, q_*)$. Thus $(|\nabla u_n|^q)$ converges to $|\nabla u|^q$ in $L_{loc}^1([0,\infty), L^1(\Omega))$, and $\left(e^{t\Delta} u_{0,n}\right)$ converges a.e. to $e^{t\Delta} u_0$, and u satisfies (4.2). Moreover u is the unique solution of $(D_{\Omega,T})$. Indeed let v be any other solution; taking $\gamma \in (q, q_*)$, there holds from [6, Lemma 3.3], with constants $C = C(\gamma, \Omega)$,

$$\begin{aligned}
\|\nabla(u-v)\|_{L^\gamma(Q_{\Omega,\tau})} &\leqq C \left\| |\nabla u|^q - |\nabla v|^q \right\|_{L^1(Q_{\Omega,\tau})} \\
&\leqq C(\|\nabla u\|_{L^q(Q_{\Omega,T})}^{q-1} + \|\nabla v\|_{L^q(Q_{\Omega,T})}^{q-1}) \|\nabla(u-v)\|_{L^q(Q_{\Omega,\tau})} \\
&\leqq C \|u_0\|_{\mathcal{M}_b(\Omega)} \|\nabla(u-v)\|_{L^\gamma(Q_{\Omega,\tau})} \tau^{\frac{\gamma-q}{\gamma q}},
\end{aligned}$$

hence $v = u$ on $(0,\tau)$ for $\tau \leqq C = C(\gamma, \Omega, u_0)$, and then on $(0,T)$. Then the whole sequence (u_n) converges to u. ∎

Remark 4.3 *Applying Proposition 4.2 on (ϵ, T) for $\epsilon > 0$, we deduce regularity results: any weak solution u of $(D_{\Omega,T})$ extends as a solution of the problem $(D_{\Omega,\infty})$, and $u \in C^{2,1}(Q_{\Omega,\infty})$, and $u \in C\left(\overline{Q_{\Omega,\epsilon,\infty}}\right)$ for any $\epsilon > 0$, and u satisfies the universal estimates (2.16) and (2.15). In turn $u \in C_{loc}^{1,0}(Q_{\Omega,\infty})$ from Theorem 2.13.*

Notation 4.4 *For any $k > 0$, we denote by $u^{k,\Omega}$ the above solution of $(D_{\Omega,\infty})$ with initial data $k\delta_0$.*

4.1 The case $\Omega = \mathbb{R}^N$

We first show that the function Y constructed at Proposition 3.3 is a VSS:

Lemma 4.5 *The function Y is a maximal V.S.S. in $Q_{\mathbb{R}^N,\infty}$, and coincides with the radial self-similar solution constructed in [39]. It satisfies*

$$\lim_{t\to 0}\int_{\mathbb{R}^N\setminus B_r} Y(.,t)dx = 0, \qquad \forall r>0. \tag{4.5}$$

Proof. Consider any ball B_p with $p \geqq 1$. We can approximate the function u^{k,B_p} by u_ε^{k,B_p}, solution with initial data $k\rho_\varepsilon$, where (ρ_ε) is a sequence of mollifiers with support in $B_\varepsilon \subset B_1$. For any $\eta \in (0,1)$, there holds $u_\varepsilon^{k,B_p} \leqq Y_\eta$ for $\varepsilon < \eta$. Then we find $u^{k,B_p} \leqq Y$. As a first consequence, $Y \neq 0$, and for any ball B_r such that $r<1$, taking $\varphi \in C_c(B_r)$ with values in $[0,1]$, such that $\varphi \equiv 1$ on $B_{r/2}$,

$$\underline{\lim}_{t\to 0}\int_{B_r} Y(.,t)dx \geqq \lim_{t\to 0}\int_{B_r} u^{k,B_p}(.,t)\varphi dx = k,$$

thus Y satisfies (1.7). From (3.15), Y is the unique radial self-similar VSS constructed in **??**. It satisfies (4.5), since $Y(x,t) = t^{-a/2}f(t^{-1/2}|x|)$, and $\lim_{r\to\infty} r^{a-N}e^{r^2/4}f(r) > 0$, from [39, Theorem 2.1], which implies (1.6). And Y is a maximal VSS, since Y is greater than any weak solution of (1.1), (1.2), from Proposition 3.4. ∎

In [11], a VSS U is constructed as the limit of a sequence of solutions u^k of (1.1) in $Q_{\mathbb{R}^N,\infty}$ with initial data $k\delta_0$, constructed in [10]. The proof is based on difficult estimates of the gradient obtained from from the Bernstein technique by derivation of equation, showing that U satisfies (1.8), (1.9) and (1.10); and is minimal in that class, from [12, Theorem 3.8]. Here we prove again the existence of the u^k and U in a very simple way:

Lemma 4.6 *(i) For any $k>0$ there exists a weak solution u^k of (1.1) in $Q_{\mathbb{R}^N,\infty}$, such that $u^k \in L^\infty((0,\infty);L^1(\mathbb{R}^N))$ and $\left|\nabla u^k\right| \in L^q(Q_{\mathbb{R}^N,\infty})$, with initial data $k\delta_0$, in the weak sense of $\mathcal{M}_b\left(\mathbb{R}^N\right)$*

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u^k(.,t)\psi dx = k\psi(0), \qquad \forall \psi \in C_b(\mathbb{R}^N); \tag{4.6}$$

and $u^k = \sup u^{k,B_p}$, where u^{k,B_p} is the solution of the Dirichlet problem $(D_{B_p,\infty})$ with initial data $k\delta_0$.

(ii) As $k\to\infty$, u^k converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,\infty})$ to a V.S.S U in $Q_{\mathbb{R}^N,\infty}$.

Proof. (i) Let $k>0$ be fixed. Consider again the sequence $\left(u^{k,B_p}\right)$. We have

$$u^{k,B_p}(.,t) \leqq Y(.,t) \leqq C(1+t^{-\frac{1}{q-1}}). \tag{4.7}$$

from Proposition 3.3. From Theorem 2.12 the sequence converges in $C^{2,1}_{loc}(Q_{\Omega,\infty})$ to a solution u^k of equation (1.1) in $Q_{\mathbb{R}^N,\infty}$, and $u^k \leqq Y$, thus u^k satisfies (1.6) from (3.8). Moreover for any $t>0$, from (4.2) and (4.3),

$$\int_{B_p} u^{k,B_p}(.,t)dx \leqq k, \qquad \lim_{t\to 0}\int_{B_p} u^{k,B_p}(.,t)dx = k.$$

Then from the Fatou Lemma,

$$\int_{\mathbb{R}^N} u^k(.,t)dx \leqq k.$$

In turn from Proposition 2.18, $u^k(.,t)$ converges weak* to a Radon measure μ, concentrated at 0, then $\mu = k'\delta_0$, $k' > 0$. Otherwise $u^{k,B_p} \leqq u^k$, then $\int_{B_p} u^{k,B_p}(.,t)dx \leqq \int_{\mathbb{R}^N} u^k(.,t)dx$, thus

$$k \leqq \liminf_{t\to 0} \int_{\mathbb{R}^N} u^k(.,t)dx;$$

then $\lim_{t\to 0} \int_{\mathbb{R}^N} u^k(.,t)dx = k$. Taking $\varphi_p \in \mathcal{D}^+(\mathbb{R}^N)$, with values in $[0,1]$, such that $\varphi_p = 1$ on B_p, we get

$$\int_{B_p} u^{k,B_p}(.,t)dx \leqq \int_{\mathbb{R}^N} u^k(.,t)\varphi_p dx \leqq \int_{\mathbb{R}^N} u^k(.,t)dx$$

hence $k' = k$; thus $u^k(.,t)$ converges weak * to $k\delta_0$ as $t \to 0$. In fact the convergence holds in the weak sense of $\mathcal{M}_b(\mathbb{R}^N)$. Indeed for any $\psi \in C_b^+(\mathbb{R}^N)$, using a function $\varphi \in C_c(\mathbb{R}^N)$ with values in $[0,1]$ such that $\varphi \equiv 1$ on a ball B_r, we can write

$$\int_{\mathbb{R}^N} u^k(.,t)\psi dx = \int_{\mathbb{R}^N} u^k(.,t)\psi\varphi dx + \int_{\mathbb{R}^N} u^k(.,t)\psi(1-\varphi)dx,$$

and

$$\int_{\mathbb{R}^N} u^k(.,t)\psi(1-\varphi)dx \leqq \|\psi\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N\setminus B_r} u^k(.,t)dx \leqq \|\psi\|_{L^\infty(\Omega)} \int_{\mathbb{R}^N\setminus B_r} Y(.,t)dx$$

and the right hand side tends to 0 from (4.5). From (4.4), we find

$$\left\| \left| \nabla u_\varepsilon^{k,B_p} \right|^q \right\|_{L^1(Q_{B_p,\infty})} \leqq k \left\|\rho_\varepsilon\right\|_{L^1(B_p)} = k,$$

hence $\left\| \left|\nabla u^{k,B_p}\right|^q \right\|_{L^1(Q_{B_p,\infty})} \leqq k$, and finally $\left\| \left|\nabla u^k\right|^q \right\|_{L^1(Q_{\mathbb{R}^N,\infty})} \leqq k$, from the convergence a.e. of the gradients.

(ii) From (4.7) or from Proposition (3.4), there holds

$$u^k(.,t) \leqq Y(.,t) \leqq C(1 + t^{-\frac{1}{q-1}}). \tag{4.8}$$

From Theorem 2.12, u^k converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,\infty})$ to a weak solution U of equation (1.1). Then $u^k \leqq U \leqq Y$, thus U satisfies (1.7) and (4.5) as Y. Hence U is a VSS in $Q_{\mathbb{R}^N,\infty}$. ∎

Next we prove the uniqueness of the VSS:

Proof of Theorem 1.3. Let us show that U is minimal VSS. Let u be any VSS in $Q_{\mathbb{R}^N,\infty}$. From Proposition 3.4, and (3.7), $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty}) \cap C((0,\infty); C_b^2(\mathbb{R}^N))$ and $u \leqq Y$. For fixed $k > 0$ and $p > 1$, one constructs a sequence of functions $u_{0,n}^k \in \mathcal{D}^+(\mathbb{R}^N)$ with support in B_1 such that

$$u_{0,n}^k \leqq u(.,\frac{1}{n}) \quad \text{in } \mathbb{R}^N, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} u_{0,n}^k dx = k.$$

Indeed $\|u(.,1/n)\|_{L^1(\mathbb{R}^N)}$ tends to ∞, then, for n large enough, there exists $s_{n,k} > 0$ such that $\left\|T_{s_{n,k}}(u)(.,1/n)\right\|_{L^1(\mathbb{R}^N)} = k$. And $\varepsilon_n = \|u(.,1/n)\|_{L^1(\mathbb{R}^N\setminus B_1)} + \|u(.,1/n)\|_{L^\infty(\mathbb{R}^N\setminus B_1)}$ tends to 0, from (4.5) and (3.8). Then $v_n^k = (T_{s_{n,k}}(u)(.,1/n) - 2\varepsilon_n)^+$ has a compact support in B_1, and we can take for $u_{0,n}^k$ a suitable regularization of v_n^k. Let us call u_n^{k,B_p} the solution of $(D_{B_p,\infty})$ with initial data $u_{0,n}^k$. Then we obtain that $u_n^{k,B_p}(.,t) \leqq u(.,t+1/n)$ from the comparison principle. As $n \to \infty$, $u_{0,n}^k$ converges to $k\delta_0$ weakly in $\mathcal{M}_b(B_p)$, since for any $\psi \in C_b^+(B_p)$, and any $r \in (0,1)$,

$$\left|\int_{B_p} u_{0,n}^k \psi dx - k\psi(0)\right| \leqq \psi(0)\left|\int_{B_p}(u_{0,n}^k - k)dx\right| + 2\,\|\psi\|_{L^\infty(\mathbb{B}_p)}\int_{\mathbb{R}^N\setminus B_r} u(.,\frac{1}{n})dx + \sup_{B_r}|\psi-\psi(0)|\int_{\mathbb{R}^N} u_{0,n}^k dx.$$

Then u_n^{k,B_p} converges to u^{k,B_p} from Proposition 4.2, and $u^{k,B_p} \leqq u$. From Lemma 4.6, we get $u^k \leqq u \leqq Y$. As $k \to \infty$, we deduce that $U \leqq u \leqq Y$. Moreover U is radial and self-similar, then $U = Y = u$ from [39]. ∎

Finally we describe all the solutions:

Proof of Theorem 1.4. Let u be any weak solution of (1.1), (1.6). Either (1.7) holds, then $u = Y$. Or there exists a ball B_r such that $\int_{B_r} u(.,t)dx$ stays bounded as $t \to 0$. Then $u \in L^\infty_{loc}([0,T); L^1_{loc}(\mathbb{R}^N))$, from Corollary 2.21. From Proposition 2.18, $u(.,t)$ converges weak* to a measure μ as $t \to 0$. Then μ is concentrated at 0 from (1.6), hence the exists $k \geqq 0$ such that $\mu = k\delta_0$, and (1.13) holds as in Lemma 4.6, since $u \leqq Y$. If $k = 0$, then $u \equiv 0$ from Theorem 1.2.

Next we show the uniqueness, namely that $u = u^k$ constructed at Lemma 4.6. *Here only* we use the gradient estimates obtained by the Bernstein technique. We have $u \in C((0,\infty); C_b^2(\mathbb{R}^N))$ from Proposition (3.4), and $u \in L^\infty((0,\infty); L^1(\mathbb{R}^N))$ from (3.2) or (4.5) thus $u \in C((0,\infty); L^1(\mathbb{R}^N))$. From [10], [9], for any $\epsilon > 0$, and any $t \geqq \epsilon$, we have the semi-group formula

$$u(.,t) = e^{(t-\epsilon)\Delta}u(.,\epsilon) - \int_\epsilon^t e^{(t-s)\Delta}\left|\nabla u\right|^q(s)ds \qquad \text{in } L^1(\mathbb{R}^N), \tag{4.9}$$

and there exists $C(q)$ such that for any $t > 0$,

$$|\nabla u(.,t)|^q \leqq C(q)(t-\epsilon)^{-1}u(.,t).$$

Going to the limit as $\epsilon \to 0$ we deduce from (1.10), since $u \leqq Y$,

$$\|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C(q)t^{-1/q}\,\|Y(.,t)\|^{1/q}_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-(N+2)/2q}$$

where $C = C(N,q)$. From (1.13) and (4.9) there holds $|\nabla u|^q \in L^1_{loc}([0,\infty); L^1(\mathbb{R}^N))$. Otherwise $e^{(t-\epsilon)\Delta}u(x,\epsilon)$ converges to kg in $\mathcal{C}_b'(\mathbb{R}^N)$, where g is the heat kernel, then

$$u(.,t) = kg - \int_0^t e^{(t-s)\Delta}\left|\nabla u\right|^q(s)ds \qquad \text{in } \mathcal{C}_b'(\mathbb{R}^N).$$

Then

$$(u-u^k)(.,t) = -\int_0^t e^{(t-s)\Delta}(|\nabla u|^q - \left|\nabla u^k\right|^q)(s)ds \qquad \text{in } L^1(\mathbb{R}^N),$$

$$\left\|\nabla(u-u^k)(.,t)\right\|_{L^q(\mathbb{R}^N)} \leqq \int_0^t \left\|e^{(t-s)\Delta}\right\|_{L^1(\mathbb{R}^N)} \left\||\nabla u(.,s)|^q - \left|\nabla u^k(.,s)\right|^q\right\|_{L^q(\mathbb{R}^N)} ds$$
$$\leqq C\int_0^t (t-s)^{-1/2} s^{-(q-1)(N+2)/2q} \left\|\nabla(u-u^k)(.,s)\right\|_{L^q(\mathbb{R}^N)} ds.$$

Thus $\nabla(u-u^k)(.,t)=0$ in $L^q\left(\mathbb{R}^N\right)$, from the singular Gronwall lemma, valid since $q<\frac{N+2}{N+1}$; hence $u=u^k$. ∎

Remark 4.7 *This uniqueness result is a special case of a general one given for measure data in [14, Theorem 3.27].*

4.2 The Dirichlet problem $(D_{\Omega,\infty})$

Here Ω is bounded, and we consider the weak solutions of the problem $(D_{\Omega,\infty})$ such that

$$\lim_{t\to 0}\int_\Omega u(.,t)\varphi dx = 0, \qquad \forall \varphi \in C_c(\overline{\Omega}\setminus\{0\}). \tag{4.10}$$

First, we give regularity properties of these solutions.

Lemma 4.8 *Any weak solution u of $(D_{\Omega,\infty})$, (4.10), in $Q_{\Omega,\infty}$ satisfies*

$$u \in C^{1,0}(\overline{\Omega}\setminus\{0\}\times[0,\infty)) \cap C^{1,0}(\overline{\Omega}\times(0,\infty)) \cap C^{2,1}\left(Q_{\Omega,\infty}\right).$$

Proof. We know that $u\in C^{1,0}(\overline{\Omega}\times(0,\infty))\cap C^{2,1}\left(Q_{\Omega,\infty}\right)$, see Remark 4.3. Moreover $u\in C^{2,1}(\Omega_0\times[0,\infty))$ and $u(x,0)=0$, $\forall x\in\Omega_0$, from Corollary 2.21. Let $B_\eta \subset\subset \Omega$ be fixed, and $\Omega_\eta=\Omega\setminus\overline{B_\eta}$. Then $u\in C^1\left(\partial B_\eta\times[0,\infty)\right)$, thus for any $T\in(0,\infty)$, there exists $C_\tau>0$ such that $u(.,t)\leqq C_\tau t$ on $\partial B_\eta\times[0,T)$. Then the function $w=u-C_\tau t$ solves

$$w_t-\Delta w = -|\nabla u|^q - C_\tau \qquad \text{in } \mathcal{D}'\left(Q_{\Omega_\eta,T}\right),$$

then $w^+\in C((0,T);L^1\left(\Omega_\eta\right)\cap L^1_{loc}((0,T);W_0^{1,1}\left(\Omega_\eta\right))$, and

$$w_t^+-\Delta w^+ \leqq 0 \qquad \text{in } \mathcal{D}'\left(Q_{\Omega_\eta,T}\right)$$

from the Kato inequality. Moreover, from assumption (4.10), $w^+\in L^\infty((0,T);L^1\left(\Omega_\eta\right))$ and $w^+(.,t)$ converges to 0 in the weak sense of $\mathcal{M}_b\left(\Omega_\eta\right)$. As a consequence, $w\leqq 0$, from [6, Lemma 3.4]; thus $u(.,t)\leqq C_T t$ in $\Omega_{\eta,T}$. Then the function $\overline{u}$ defined by (2.29) is bounded in $Q_{\Omega_\eta,\tau}$. Hence $\overline{u}\in C^{1,0}(\overline{\Omega_\eta}\times(-T,T))$ from Theorem 2.13, thus $u\in C^{1,0}(\overline{\Omega}\setminus\{0\}\times[0,\infty))$. ∎

Definition 4.9 *Let $T\in(0,\infty]$. We call VSS in $Q_{\Omega,T}$ any weak solution u of the Dirichlet problem $(D_{\Omega,T})$, (4.10), such that*

$$\lim_{t\to 0}\int_{B_r} u(.,t)dx=\infty, \qquad \forall B_r\subset\Omega. \tag{4.11}$$

Remark 4.10 *From Remark 4.3, any VSS in $Q_{\Omega,T}$ extends as a VSS in $Q_{\Omega,\infty}$, and satisfies (2.16) and (2.15).*

Next we prove the existence and uniqueness of the VSS. Our proof is based on the uniqueness of the VSS in $\mathbb{R}^N$, and does not use the uniqueness of the function u^k.

Proof of Theorem 1.5. *(i) Existence of a minimal VSS.* For any $k > 0$ we consider the solution $u^{k,\Omega}$ of $(D_{\Omega,\infty})$ with initial data $k\delta_0$. By regularization as in Lemma 4.6, we obtain that $u^{k,\Omega} \leqq Y$. The sequence $\left(u^{k,\Omega}\right)$ is nondecreasing. From estimate (2.16) and Theorem 2.13, $\left(u^{k,\Omega}\right)$ converges in $C^{2,1}_{loc}(Q_{\Omega,\infty}) \cap C^{1,0}_{loc}(\overline{\Omega} \times (0,\infty))$ to a weak solution U^Ω of $(D_{\Omega,\infty})$, and then $U^\Omega \leqq Y$. Hence U^Ω satisfies (4.11), and (4.10) from (4.5), thus U^Ω is a VSS in Ω. Next we show that U^Ω is minimal. Consider any VSS u in $Q_{\Omega,\infty}$. Let $k > 0$ be fixed. As in the proof of Theorem 1.3, one constructs a sequence $u_n^{k,\Omega}$of solutions of $(D_{\Omega,\infty})$ with initial data functions $u_{0,n}^{k,\Omega} \in \mathcal{D}(\Omega)$ such that

$$0 \leqq u_{0,n}^{k,\Omega} \leqq u(.,\frac{1}{n}) \quad \text{in } \Omega, \qquad \lim_{n\to\infty} \int_\Omega u_{0,n}^{k,\Omega} dx = k.$$

We still find $u_{n,p}^k(.,t) \leqq u(.,t+1/n)$ from the comparison principle, valid from Lemma 4.8. As $n \to \infty$, $u_{0,n}^{k,\Omega}$ converges to $k\delta_0$ weakly in $\mathcal{M}_b(\Omega)$, then $u_n^{k,\Omega}$ converges to $u^{k,\Omega}$ from Proposition 4.2. Then $u^{k,\Omega} \leqq u$ for any $k > 0$, thus $U^\Omega \leqq u$.

(ii) Existence of a maximal VSS. For any ball $B_\eta \subset\subset \Omega$, we consider the function Y_η^Ω defined at Theorem 2.15. Consider again any VSS u in Ω, and follow the proof of Proposition 3.4, replacing B_r by Ω. Let $\varepsilon > 0$ be fixed. From Lemma 4.8, for any ball $B_\eta \subset\subset \Omega$, setting $\Omega_\eta = \Omega \backslash \overline{B_\eta}$ there is $\delta_\eta > 0$ such that

$$u(x,t) < \varepsilon, \qquad \text{in } Q_{\Omega_\eta,\delta_\eta} \tag{4.12}$$

Next, for any $\delta \in (0,\delta_\eta)$, from the comparison principle in $Q_{\Omega,\delta,\tau}$ we deduce that

$$u(x,t) \leqq Y_{2\eta}^\Omega(x,t-\delta) + \varepsilon \qquad \text{in } Q_{\Omega,\delta,\tau}.$$

As δ tends to 0, and then $\varepsilon \to 0$, we deduce that $u \leqq Y_{2\eta}^\Omega$ in $Q_{\Omega,\infty}$. We observe that $Y_\eta^\Omega \leqq Y_{\eta'}^\Omega$ for any $\eta \leqq \eta'$. From the estimates (2.16) and Theorem 2.12, Y_η^Ω converges in $C^{1,0}_{loc}(\overline{\Omega} \times (0,\infty))$ to a classical solution Y^Ω of $(D_{\Omega,\infty})$, and $u \leqq Y^\Omega$. Moreover Y^Ω satisfies (4.11), since $Y^\Omega \geqq U$, and (4.10) since $Y^\Omega \leqq Y$, then Y^Ω is a maximal VSS in Ω.

(iii) Uniqueness. For fixed $k > 0$, we intend to compare $u^{k,\Omega}$ with u^k, by approximation. Let $0 < \eta < r$ be fixed such that $B_r \subset\subset \Omega$. Consider again the function Y_η defined by (3.5). Let $\delta > 0$ be fixed. From (3.15), there exists $\tau_\delta > 0$ such that $\sup_{(\mathbb{R}^N \backslash B_r)\times[0,\tau_\delta]} Y_\eta \leqq \delta$. Let (ρ_ε) be a sequence of mollifiers with support in $B_\varepsilon \subset B_\eta$. Let $u_\varepsilon^{k,\Omega}$ be the solution of $(D_{\Omega,\infty})$ in $Q_{\Omega,\infty}$ with initial data $k\rho_\varepsilon$. For any $p > 1$ such that $\Omega \subset B_p$, let u_ε^{k,B_p} be the solution of $(D_{B_p,\infty})$ with the same initial data. By definition of $Y_\eta^{B_p}$ and Y_η, there holds $u_\varepsilon^{k,B_p} \leqq Y_\eta^{B_p} \leqq Y_\eta$, hence $\sup_{\partial\Omega\times[0,\tau_\delta]} u_\varepsilon^{k,B_p} \leqq \delta$. Applying the comparison principle to the smooth functions $u_\varepsilon^{k,\Omega}$ and u_ε^{k,B_p} in $\overline{\Omega} \times [0,\infty)$, we obtain that

$$u_\varepsilon^{k,B_p} \leqq u_\varepsilon^{k,\Omega} + \delta \qquad \text{in } \overline{\Omega} \times [0,\tau_\delta]\,.$$

Going to the limit as $\varepsilon \to 0$ from Proposition 4.2 and then as $p \to \infty$ from Lemma 4.6, we obtain that

$$u^k \leqq u^{k,\Omega} + \delta \qquad \text{in } \overline{\Omega} \times (0,\tau_\delta]\,;$$

and going to the limit as $k \to \infty$, we find

$$U \leqq U^\Omega + \delta \qquad \text{in } \overline{\Omega} \times (0, \tau_\delta]\,.$$

The function $W^\Omega = Y^\Omega - U^\Omega \in C^{1,0}(\overline{\Omega}\setminus\{0\} \times [0,\infty)) \cap C^{1,0}(\overline{\Omega} \times (0,\infty))$ from Lemma (4.8), and $W^\Omega = 0$ on $\partial\Omega \times [0,\infty)$. Since $Y^\Omega \leqq Y = U$, then $W^\Omega \leqq \delta$ in $\overline{\Omega} \times (0, \tau_\delta]$. Thus $W^\Omega(.,t)$ converges uniformly to 0 as $t \to 0$. Then for any $\varepsilon > 0$, $W^\Omega - \varepsilon$ cannot have an extremal point in $Q_{\Omega,\infty}$, thus $W^\Omega \leqq \varepsilon$, hence $Y^\Omega = U^\Omega$. ∎

Finally we describe all the solutions as in the case of $\mathbb{R}^N$:

Theorem 4.11 *Let u be any weak solution of $(D_{\Omega,\infty})$, (4.10). Then either $u = U^\Omega$, or there exists $k > 0$ such that $u = u^{k,\Omega}$, or $u \equiv 0$.*

Proof. Either $u = Y^\Omega$, or there exists a ball B_r such that $\int_{B_r} u(.,t)dx$ stays bounded as $t \to 0$. Then from (4.10), $u \in L^\infty_{loc}([0,\infty); L^1(\Omega))$. From Proposition 2.18, $u(.,t)$ converges weak* to a measure μ as $t \to 0$, concentrated at $\{0\}$ from (4.10). Hence the exists $k \geqq 0$ such that $\mu = k\delta_0$, thus

$$\lim_{t\to 0} \int_\Omega u(.,t)\varphi dx = k\varphi(.,0), \qquad \forall \varphi \in C_c(\Omega),$$

and it holds for any $\varphi \in C_b(\Omega)$, from (4.10). If $k > 0$, then $u = u^{k,\Omega}$ from uniqueness, see Proposition 4.2. If $k = 0$, then $u \equiv 0$ from Theorem 1.2. ∎

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