



HAL
open science

Definable envelopes in groups with simple theory

Cédric Milliet

► **To cite this version:**

| Cédric Milliet. Definable envelopes in groups with simple theory. 2012. hal-00657716v1

HAL Id: hal-00657716

<https://hal.science/hal-00657716v1>

Preprint submitted on 9 Jan 2012 (v1), last revised 15 Jan 2016 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DEFINABLE ENVELOPES IN GROUPS WITH SIMPLE THEORY

CÉDRIC MILLIET

ABSTRACT. Let G be a group with simple theory. For any nilpotent subgroup N of class n , there is a definable nilpotent group E of class at most $2n$ finitely many translates of which cover N . The group E is definable with parameters in N . If S a soluble subgroup of G of derived length l , there is a definable soluble group F of derived length at most $2l$ finitely many translates of which cover S . The group F is definable with parameters in S .

1. INTRODUCTION

When studying a group, a model theorist focuses on sets which are definable by formulae. It happens regularly that in a definable group G , he finds a subgroup H of particular interest, having a given property P such as abelianity, nilpotency, solubility etc. He tries then to find a definable group containing H also having property P . We call an *envelope* of H any group containing H . Finding a definable envelope of H with P is possible in some cases :

Let us call *MC* any group satisfying the minimum condition on centralisers : every strictly monotone chain of centralisers has finite length in an *MC* group. As stable groups are *MC*, an abelian subgroup of a stable group is contained in a definable abelian group (the centre of its centralizer for instance). Poizat [11] showed that if G is stable, every nilpotent subgroup of G is contained in a definable nilpotent subgroup of same nilpotency class, and every soluble subgroup of G is contained in a definable soluble one of same derived length. If G is a dependent group, for any subgroup H of G which is abelian (respectively nilpotent of class n , or normal and soluble of derived length n), Shelah [13] and Aldama [1] have found a definable group E containing H which is also abelian (resp. nilpotent of class n , or normal and soluble of derived length n). The parameters needed to define E may be in a saturated extension of the ambient group though. In a recent paper, Altinel and Baginski have extended one of Poizat's result to the class of *MC* groups, showing that any nilpotent subgroup of an *MC* group is enveloped by a definable nilpotent one of same nilpotency class [2].

Another important class of groups extending stable ones is the class of group having a simple theory. The previous results however do not hold in general if G has merely a simple theory. For instance if G is an infinite extra-special p -group, *i.e.* if every g in G has order p , if the centre of G is cyclic of order p and equals G' . Such a group has a simple theory (actually supersimple of *SU*-rank 1, see Annexe). If $p > 2$, it is nilpotent of class 2, and obviously has infinite abelian

2010 *Mathematics Subject Classification.* 03C45, 03C60, 20F16, 20F18, 20F19, 20F24.

Thanks to Tuna Altinel for reminding the author that the problem tackled in this paper had been left partially opened in his PhD thesis. Thanks to Frank Wagner for reading and commenting the paper, and to Zoé Chatzidakis, Mark Sapir and Alain Valette for their help towards establishing the Annexe on extraspecial p -groups.

subgroups. If G had an infinite definable abelian subgroup, that abelian group would have SU -rank 1, hence would be of finite index in G , a contradiction. So every of its definable abelian subgroups is finite (this is shown in [9] using group theoretic methods). However, if G is any group with simple theory, it has been shown in [6] that an abelian subgroup of G is always contained in a definable group H having a normal finite subgroup N such that H/N is abelian (we call such a group H *finite-by-abelian*). A group is FC if every of its elements has finitely many conjugates. Finite-by-abelian groups are FC groups. When we were looking for the narrowest possible definable group which envelope an abelian, nilpotent or soluble subgroup of a group having simple theory, it turned out that the problem was conceptually simpler in a more general setting involving FC , FC -nilpotent and FC -soluble groups instead of abelian, nilpotent and soluble ones.

2. PRELIMINARIES ON FC -NILPOTENCY AND FC -SOLUBILITY

A group G is an FC -group if every of its conjugacy classes is finite, or equivalently if the centraliser of any element in G has finite index in G . Let G be any group and H a subgroup of G . Following Haimo in [5], we may define

- the FC -centraliser of H in G :

$$FC_G(H) = \{g \in G : H/C_H(g) \text{ is finite}\}$$

If N is a normal subgroup of H , we extend the definition by putting

$$FC_G(H/N) = \{g \in G : H/C_H(\{gN\}) \text{ is finite}\}$$

- the FC -centre of G :

$$FC(G) = FC_G(G)$$

- the n th FC -centre of G by the following induction on n :

$$FC_0(G) = \{1\} \text{ and } FC_{n+1}(G) = FC_G(G/FC_n(G))$$

- the FC -normaliser of H in G :

$$FN_G(H) = \{g \in G : H^g/H \cap H^g \text{ and } H/H \cap H^g \text{ are finite}\}$$

These are all subgroups of G . The chain $FC_1(G) \leq \dots \leq FC_n(G)$ is an ascending chain of characteristic subgroups of G (in particular, $FC_n(G)$ is normal in G and its inductive definition makes sense).

Lemma 2.1. *Let G be a group, g an element of this group, H a subgroup of G , N a subgroup of H and n any natural number.*

- (1) H normalises $FC_G(H)$.
- (2) If N and H are normal in G , then $C_G(\{gN\})$ is a subgroup of $C_G(\{gH\})$.
- (3) $H \cap FC_n(G)$ is a subgroup of $FC_n(H)$
- (4) If N is a finite normal subgroup of H , then $FC_G(H/N)$ equals $FC_G(H)$.

Proof. Let h be in H . As $C_H(g^h)$ equals $C_H(g)^h$, if $H/C_H(g)$ is finite then so is $H/C_H(g^h)$. It follows that H normalises $FC_G(H)$.

If h centralises $\{gN\}$, then g^h is in N , hence in H so h centralises $\{gH\}$.

If h is in $H \cap FC_n(G)$, then $G/C_G(h)$ is finite. As $H/C_H(h)$ embeds in $G/C_G(h)$, the element h belongs to $FC(H)$ as well. Now suppose that $FC_n(G) \cap H$ is a subgroup of $FC_n(H)$ and let h be in $FC_{n+1}(G) \cap H$. It follows that $G/C_G(\{hFC_n(G)\})$ is finite, so $H/C_H(\{hFC_n(G)\})$ is finite. As h is in H , the group $C_H(\{hFC_n(G)\})$

equals $C_H(\{h(FC_n(G) \cap H)\})$. By induction hypothesis and point 2, the group $H/C_H(\{hFC_n(H)\})$ must be finite which provides that h is in $FC_{n+1}(H)$.

By point 2, the group $FC_G(H)$ is a subgroup of $FC_G(H/N)$. If x is in the latter, there is a group F of finite index in H which centralises the finite set $\{xN\}$. So x^F is finite, and so is x^H . \square

Definition 2.2 ((Haimo [5])). A group is *FC-nilpotent* if one of the following equivalent facts holds :

- (1) There is an *FC*-central series of finite length, i.e. a sequence of normal subgroups of G

$$\{1\} = H_0 \leq H_1 \leq \dots \leq H_n = G$$

such that H_{i+1}/H_i is in the *FC*-centre of G/H_i for every i in $\{0, \dots, n-1\}$.

- (2) The sequence of iterated *FC*-centres ends on G after n steps. We call the *FC-class* of G , or simply its *class*, the least such n .

Definition 2.3 ((Duguid, McLain [3])). A group G is *FC-soluble* if there exists a normal *FC-series* of finite length, i.e. a finite sequence of normal subgroups G_0, G_1, \dots, G_n of G such that

$$G_0 = G \supseteq G_1 \supseteq \dots \supseteq G_n = \{1\}$$

and such that G_i/G_{i+1} is an *FC*-group for all i . We call the least such natural number n the *FC-solubility class* of G , or its *class*.

Remark 2.4. For a group G , the requirement to have a finite abelian series or a finite abelian series whose members are normal in G are equivalent. This may not be true anymore in the case of an *FC*-series. We have modified the original definition in [3] and we do require here that the *FC*-series of G consist of subgroups which are normal in G .

FC-nilpotent groups are *FC*-soluble. *FC*-nilpotent group of class 1 and *FC*-soluble group of class 1 coincide with *FC*-groups. Finite groups and nilpotent ones are *FC*-nilpotent. Soluble groups are *FC*-soluble. Recall from [7, Theorem 3.1] :

Theorem 2.5 ((Neumann)). *Let G be an *FC*-group whose conjugacy classes are finite and bounded. Then the derived group G' is finite.*

A group homomorphism f from G onto H maps a finite conjugacy G -class to a finite conjugacy H -class, and that the preimage by f of a finite set is finite providing that f has finite kernel. It follows that a direct image of an *FC*-group by a group homomorphism is an *FC*-group, as well as is a preimage of an *FC*-group by group homomorphism having finite kernel. As a corollary, *FC*-nilpotency (and *FC*-solubility) is preserved under direct images of group homomorphisms, and under preimages of group homomorphisms having finite kernel. In particular, subgroups and quotient groups of *FC*-nilpotent (resp. *FC*-soluble) groups are *FC*-nilpotent (resp. *FC*-soluble).

Proposition 2.6. *Let G be any group.*

- (1) *If G is *FC*-nilpotent and infinite, then $FC(G)$ is infinite.*
- (2) *$FC_n(G)$ is *FC*-nilpotent of class at most n .*

Proof. If $FC(G)$ is finite, then $FC_2(G)$ equals $FC(G)$ by Lemma 2.1.4, so G is finite.

The group $FC_n(G)$ is a subgroup of $FC_n(FC_n(G))$ by Lemma 2.1.3. \square

3. GROUPS WITH SIMPLE THEORY

For more details about this section, we refer the curious reader to [12, Shelah], [8, Pillay] and [15, 16, Wagner]. In a given theory, let $f(x)$ and $\phi(x, y)$ be any formula. Let k be a natural number. We define the $D_{\phi, k}$ -rank of f inductively by :

- $D(f, \phi, k) \geq 0$ if f is consistent.
- $D(f, \phi, k) \geq n + 1$ if there is a sequence a_0, a_1, \dots such that $D(f(x) \wedge \phi(x, a_i), \phi, k)$ is at least n for all i , and the formulae $\phi(x, a_0), \phi(x, a_1), \dots$ are k -inconsistent.

If X is a set defined by a formula f , we may write $D(X, \phi, k)$ for $D(f, \phi, k)$. Note that $D(f, \phi, k)$ takes three arguments. A basic induction shows that it is increasing in the first one (with respect to logical implication), decreasing in the second, and increasing in the third (with respect to the natural order on \mathbf{N}). The following lemma is easy to verify.

Lemma 3.1. *Let a be a finite tuple of parameter. For any formula $f(x, a)$, the sentence " $D(f(x, a), \phi, k)$ is at least n " is a type-definable condition on a .*

Definition 3.2 ((Shelah [12])). A complete theory is *simple* if the $D_{\phi, k}$ -rank of any of its formulae is a natural number, for every formula ϕ and natural number k . A structure is *simple* if its first order theory is so.

Lemma 3.3 ((Shelah [12])). *Let X and Y be two definable subsets of some structure. The rank $D(X \cup Y, \phi, k)$ is the maximum of $D(X, \phi, k)$ and $D(Y, \phi, k)$.*

Proof. Basic induction on $D(X \cup Y, \phi, k)$. □

If $\phi(x, y)$ is any formula, and g a function symbol. We write $g^{-1}\phi(x, y)$ for the formula $(\exists z)(x = g(z) \wedge \phi(z, y))$. In a group, we write $z\phi(x, y)$ or simply $z\phi$ for the formula $\phi(z^{-1}x, y)$ where z is thought of as a new parameter variable of arity 1. We take the opportunity to stress on the following crucial lemma that the author could not find anywhere in the litterature :

Lemma 3.4. *In any structure, let X and Y be two definable subsets, and g a definable map from X to Y .*

- (1) *If g is surjective, then $D(X, g^{-1}\phi, k) \geq D(Y, \phi, k)$.*
- (2) *If g has fibres of size at most n , then $D(X, g^{-1}\phi, k) \leq D(Y, \phi, kn)$.*

Proof. We proceed by induction on $D(Y, \phi, k)$. If Y is consistent, so is X . If $D(Y, \phi, k)$ is at least $n + 1$, there are formulae $\phi(x, a_1), \phi(x, a_2), \dots$ which are k -inconsistent, with each $Y \cap \phi(x, a_i)$ having $D_{\phi, k}$ -rank at least n . By induction hypothesis, their preimages by g witness that X has D -rank at least $n + 1$.

For point 2, there is no harm in assuming that g is onto. If X is consistent, then so is Y . We go on inductively and suppose that $D(X, g^{-1}\phi, k)$ is at least $m + 1$. This provides us with k -inconsistent sets X_0, X_1, \dots defined by $\phi(g(x), a_0), \dots$ with $D(X \wedge X_i, g^{-1}\phi, k)$ at least m . By induction hypothesis, as g is onto, $D(Y \wedge g(X_i), \phi, kn)$ is at least m . Let I be a subset of \mathbf{N} of cardinal kn and suppose there is some y in $\bigcap_{i \in I} g(X_i)$. Let x_1, \dots, x_p be the list of all preimages of y by g . Every x_i belongs to say n_i distinct sets among $(X_i)_{i \in I}$, and we also have $n_1 + \dots + n_p \geq |I|$. As p is at most n , at least one n_i must be greater or equal than k , a contradiction with X_0, X_1, \dots being k -inconsistent. This shows that the sets $g(X_0), g(X_1), \dots$ are kn -inconsistent, and that $D(Y, \phi, kn)$ is at least $m + 1$. □

It is worth mentioning that one could weaken the definability assumption on g in the previous lemma and only require that images and preimages by g of uniformly definable sets be uniformly definable. In particular, one could take g to be an automorphism of the structure. As our proofs on groups heavily relies on passing to quotient groups, we recall that simplicity is preserved when modding out by a definable equivalence relation.

Corollary 3.5 ((Shelah)). *Let M be a simple structure and E a definable equivalence relation on M . Then the disjoint union of M and M/E is a simple structure. (the language considered is the language on M extended by a predicate for M/E and a function for the canonical surjection from M to M/E).*

Proof. Note that as E is definable in M , the new language does not induce new formulae on M , so M is simple in the extended language. By Lemma 3.3, it is enough to show that M/E is simple. If $\phi(x, y)$ is any formula, we write $\phi_E(x, y)$ for the formula $(\exists z)(xEz \wedge \phi(z, y))$. By Lemma 3.4.1 applied to the canonical surjection from M to M/E , the rank $D(M, \phi, k)$ is at least $D(M/E, \phi_E, k)$, so M/E is simple. \square

Proposition 3.6 ((Pillay [16, Lemma 4.1.15])). *Let G be a group with simple theory, and let H be a definable subgroup of G . The index G/H is finite if and only if H and G have the same $D_{z\phi, k}$ -rank for every natural number k and formula $\phi(x, y)$.*

Proof. Every coset of H has the same $D_{z\phi, k}$ -rank by Lemma 3.4. If G is covered by finitely many cosets of H , then G and H have same $D_{z\phi, k}$ -rank by Lemma 3.3. Reciprocally, if G is covered by infinitely many (pairwise disjoint) cosets of H , we take for $\phi(x, y)$ the formula defining H and obviously have that $D(G, z\phi, 2)$ is at least $D(H, z\phi, 2) + 1$. \square

Theorem 3.7 ((Wagner [16, Theorem 4.2.12])). *In a group with simple theory, let $f(x, y)$ be a fixed formula and let H_1, H_2, \dots be a family of subgroups defined respectively by formulae $f(x, a_1), f(x, a_2), \dots$. If G_1, G_2, \dots is a descending chain of finite intersections of H_i , there exists a natural number n such that G_m has finite index in G_n for all $m \geq n$.*

Two subgroups of a given group G are *commensurable* if the index of their intersection is finite in both of them. Commensurability is an equivalence relation on the set of subgroups of G . A family of subgroups of G is *uniformly commensurable* if they are pairwise commensurable and if the correspondent family of finite indexes is bounded by some natural number. The following result appears in [14].

Theorem 3.8 ((Schlichting)). *Let G be a group and \mathfrak{H} a family of uniformly commensurable subgroups. There exists a subgroup N of G commensurable with members of \mathfrak{H} and invariant under the action of the automorphisms group of G stabilising the family \mathfrak{H} setwise. The inclusions $\bigcap_{H \in \mathfrak{H}} H \subset N \subset \mathfrak{H}^4$ hold. Moreover, N is a finite extension of a finite intersection of elements in \mathfrak{H} . In particular, if \mathfrak{H} consists of definable groups then N is also definable.*

4. DEFINABLE ENVELOPES

Proposition 4.1. *In a group G with simple theory, let H be a definable subgroup (with parameters in A). The FC -centraliser, FC -normaliser and iterated FC -centers of H are type-definable subgroups (with parameters in A). If G is sufficiently saturated, then those groups are definable.*

Proof. We prove the proposition for $FC(H)$ only. From Lemma 3.1 and Proposition 3.6, it follows immediately that $FC(H)$ is a type definable group. If the structure is saturated, one can identify type-definable sets over A and consistent types over A . On the one hand, $FC(H)$ is closed for the topology generated by formulae over A . On the other hand, the condition " $H/C_H(g)$ equals n " is an A -definable condition in g , so $FC(H)$ is a union of open sets. By compactness, finitely many of these open sets must cover $FC(H)$. \square

Theorem 4.2. *Let G be a group with simple theory and N a subgroup of G . If N is FC -nilpotent of class n , then it is contained in a definable FC -nilpotent group of class n .*

Proof. We may replace the ambient group G by a saturated extension and assume without loss of generality that G is sufficiently saturated. By Proposition 4.1, the FC -centralisers, FC -normaliser and iterated FC -centres of definable subgroups of G are definable. We build inductively a decreasing chain of definable subgroups $B_1 \geq \dots \geq B_n$ such that for every i in $\{1, \dots, n\}$

- (1) B_i contains N .
- (2) $FC_i(B_i)$ contains $FC_i(N)$.

Let a_1, \dots, a_p be elements in $FC(N)$ such that $C_G(a_1, \dots, a_p, a)$ has finite index in $C_G(a_1, \dots, a_p, a)$ for any a in $FC(N)$. They do exist by Theorem 3.7. We call C_1 the group $C_G(a_1, \dots, a_p)$. Note that $N/C_N(a_i)$ is finite for every a_i , so $N/N \cap C_1$ is also finite. For every g in N , the group C_1^g equals $C_G(a_1^g, \dots, a_p^g)$. As N normalises $FC(N)$, the groups C_1^g and C_1 are commensurable by minimality of C_1 . It follows that the FC -normaliser of C_1 contains N . As it is definable, the family of N -conjugates of C_1 must be uniformly commensurable. We may apply Theorem 3.8 to the set $\{C_1^g : g \in N\}$ and find some definable D_1 commensurable with C_1 which is normalised by N . As $N/N \cap C_1$ is finite, $N/N \cap D_1$ is finite too, so the group $N \cdot D_1$ is a finite union of cosets of D_1 , hence definable. We define B_1 to be $N \cdot D_1$, and claim that its FC -centre contains $FC(N)$: if g is in $B_1 \setminus FC(B_1)$, then $B_1/B_1 \cap C_G(g)$ is infinite. But B_1 and C_1 are commensurable so $C_1/C_1 \cap C_G(g)$ is infinite too and g cannot be in $FC(N)$ by minimality of C_1 . This complete the first step.

Now assume that B_1, \dots, B_{k-1} are build. The group B_{k-1} normalises $FC_{k-1}(B_{k-1})$ by Lemma 2.1.1 so $B_{k-1}/FC_{k-1}(B_{k-1})$ is a group. Note that the disjoint union of G and $B_{k-1}/FC_{k-1}(B_{k-1})$ is a simple structure by Corollary 3.5 so we may apply the chain condition in G taking parameters in $B_{k-1}/FC_{k-1}(B_{k-1})$. For any b in B_{k-1} , we write \bar{b} for the class of b in the quotient group $B_{k-1}/FC_{k-1}(B_{k-1})$. Let b_1, \dots, b_m be elements of $FC_k(N)$ such that $C_{B_{k-1}}(\bar{b}_1, \dots, \bar{b}_m, \bar{b})$ has finite index in $C_{B_{k-1}}(\bar{b}_1, \dots, \bar{b}_m)$ for any b in $FC_k(N)$. Let us call C_k the group $C_{B_{k-1}}(\bar{b}_1, \dots, \bar{b}_m)$. The quotient $N/C_N(\{b_i FC_{k-1}(N)\})$ is finite for every b_i , and $FC_{k-1}(B_{k-1})$ contains $FC_{k-1}(N)$ by induction hypothesis so $N/C_N(\{b_i FC_{k-1}(B_{k-1})\})$ is also finite by Lemma 2.1 2. It follows that $N/C_N(\bar{b}_1, \dots, \bar{b}_m)$ hence $N/N \cap C_k$ is finite.

We may once again show that the groups in $\{C_k^g : g \in N\}$ are uniformly commensurable, apply Theorem 3.8 to the set of N -conjugates of C_k and find some definable group D_k normalised by N and commensurable with C_k . We call B_k the definable group $N \cdot D_k$, which is a subgroup of B_{k-1} . To finish the proof, we just need to show that $FC_k(B_k)$ contains $FC_k(N)$. If g is in $B_k \setminus FC_k(B_k)$, then $B_k/B_k \cap C_G(\{gFC_{k-1}(B_k)\})$ is infinite. As B_k is a subgroup of B_{k-1} , Lemma 2.1.3 yields

$$FC_{k-1}(B_{k-1}) \cap B_k \leq FC_{k-1}(B_k)$$

So the group $B_k/B_k \cap C_G(\{g(FC_{k-1}(B_{k-1}) \cap B_k)\})$ is infinite too. Note that as g is in B_k , we have

$$B_k \cap C_G(\{g(FC_{k-1}(B_{k-1}) \cap B_k)\}) = B_k \cap C_G(\{gFC_{k-1}(B_{k-1})\}) = B_k \cap C_G(\bar{g})$$

It follows that $B_k/B_k \cap C_G(\bar{g})$ is infinite. As B_k and C_k are commensurable, the quotient $C_k/C_k \cap C_G(\bar{g})$ is infinite too, so g is not in $FC_k(N)$ by minimality of C_k . This completes the inductive construction of the groups B_i . It follows that $FC_n(B_n)$ is a definable FC -nilpotent subgroup of class at most n which contains $FC_n(N)$, hence N . \square

Remark 4.3. It may be desirable to have a closer look at the parameters necessary to define the definable envelope of N in the previous proof. Note that we had to go to a saturated extension of the ambient group. However, the only places where we may have used parameters from this monster model is when applying Schlichting's Theorem, the first time being to the set $\{C_1^g : g \in N\}$ where C_1 is N -definable which provides a finite extension D_1 of a N -definable group. But the set $\{C_1^g : g \in N\}$ is stabilised setwise (and even pointwise) by the automorphisms of the structure fixing N pointwise. By Schlichting's Theorem again, the group D_1 is N -invariant, hence N -definable. It follows that every definable group considered in the proof is in fact definable with parameters in N : once we have found our definable envelope in the monster model, we can compute it in the original group.

In [16] Wagner has defined a notion of "almost" nilpotent group suitable for groups hyperdefinable in a simple theory and showed that the hyperdefinable connected component of a hyperdefinable almost nilpotent group of class n is nilpotent of class at most $2n$. If G is a definable group, we write G_A^0 for the intersection of all A -definable subgroups of G of finite index.

Proposition 4.4. *In a group with simple theory, if H is an A -definable FC -nilpotent subgroup of class n , then H_A^0 is nilpotent of class at most $2n$.*

Proof. We follow the proof of [16, Proposition 4.4.10.3]. \square

Corollary 4.5. *In a group with simple theory, let N be a nilpotent subgroup of class n . There is a definable (with parameters in N) nilpotent group of class at most $2n$ finitely many translates of which cover N .*

Proof. We may assume that the ambient group is sufficiently saturated. By Theorem 4.2, there is H an N -definable FC -nilpotent group of class n containing N . By Proposition 4.4, the group H_N^0 is nilpotent of class at most $2n$. This is witnessed by the formula saying that the commutator of every $2n+1$ elements is trivial. As H_N^0 is the intersection of N -definable groups, it follows by compactness and saturation that there is an N -definable group of finite index in H which is nilpotent of class at most $2n$. \square

Corollary 4.6. *In a group with simple theory, let N be a nilpotent normal subgroup of class n . There is a definable normal nilpotent group of class at most $3n$ containing N .*

Proof. Corollary 4.5 provides an N -definable nilpotent group H such that the quotient $N/N \cap H$ is finite. It follows that $\bigcap_{g \in N} H^g$ is N -definable, nilpotent of class at most $2n$ (as a subgroup of H) and normalised by N . Let us call it H^N . By Fitting's Lemma, the group $N \cdot H^N$ is nilpotent of class at most $n + 2n$. Note that $N \cdot H^N$ is a finite extension of H^N , hence N -definable. The group $(N \cdot H^N)^G$ is as desired. \square

Corollary 4.7. *In a group simple with simple theory, if there is an infinite nilpotent subgroup, there is an infinite definable finite-by-abelian subgroup.*

Proof. We may assume that the group is saturated. By Corollary 4.5, there is an infinite definable FC -nilpotent group. Its FC -centre is definable by Proposition 4.1, hence finite-by-abelian by Theorem 2.5, and infinite by Proposition 2.6. \square

We go on with the soluble case. It was shown laboriously in [6, Corollary 5.12] that a soluble subgroup of class n in a group with simple theory is enveloped by a definable FC -soluble group of class $2n - 1$. We give a better version here.

Theorem 4.8. *Let G be a group with simple theory, and let S be a subgroup of G . If S is FC -soluble of class n , then it is contained in a definable FC -soluble group of class n the members of whose FC series are definable subgroups.*

Proof. We may again assume that the ambient group is sufficiently saturated so that the FC -normalisers, FC -centers and FC -centralisers of definable groups are definable. Let $S \supseteq S_1 \supseteq \dots \supseteq S_n = \{1\}$ be an FC -series for S . Recall that every S_i is normal in S . We set Z_0 equal to the trivial group and inductively build an ascending chain of definable subgroups $Z_1 \trianglelefteq Z_2 \trianglelefteq \dots \trianglelefteq Z_n$ such that for every i in $\{1, \dots, n\}$

- (1) Z_i is normal in Z_n
- (2) Z_i contains S_{n-i}
- (3) S normalises Z_i
- (4) Z_i/Z_{i-1} is an FC group.

By Theorem 3.7, let a_1, \dots, a_p be elements in S_{n-1} such that $C_G(a_1, \dots, a_p, a)$ has finite index in $C_G(a_1, \dots, a_p)$ for all a in S_{n-1} . As S_{n-1} is an FC group, the group $S_{n-1}/S_{n-1} \cap C_1$ is finite. Let g be in S . As g normalises S_{n-1} , the group C_1^g is a centraliser of elements in S_{n-1} , hence is commensurable with C_1 by minimality of C_1 . As before, the set $\{C_1^g : g \in S\}$ consists of uniformly commensurable groups. By Theorem 3.8, there is a definable group B_1 commensurable to C_1 and normalised by S . The group $B_1 \cdot S_{n-1}$ is definable and commensurable to B_1 . Let us call it D_1 . Let Z_1 be its FC -centre, and N_1 the normaliser of Z_1 . We show that Z_1 contains S_{n-1} : if g is in $D_1 \setminus FC(D_1)$, then $D_1/D_1 \cap C_G(g)$ is infinite, and so is $C_1/C_1 \cap C_G(g)$. By minimality of C_1 , the element g is not in S_{n-1} .

Now assume that Z_1, \dots, Z_k are build and let N_i be the normaliser of Z_i for each i . Note that $N_1 \cap \dots \cap N_k$ contains S . For any b in $N_G(Z_k)$ we write \bar{b} for the class of b in the quotient group $N_G(Z_k)/Z_k$. Let b_1, \dots, b_p be a maximal set of elements in S_{n-k-1} such that $N_1 \cap \dots \cap N_k \cap C_G(\bar{b}_1, \dots, \bar{b}_p)$ is minimal up to finite index. Let us call C_{k+1} the group $N_1 \cap \dots \cap N_k \cap C_G(b_1, \dots, b_p)$. By induction

hypothesis, the group Z_k contains S_{n-k} so $Z_k S_{n-k-1} Z_k$ is an FC group. It follows that $(Z_k S_{n-k-1} / Z_k) \cap (C_{k+1} / Z_k)$ has finite index in $Z_k S_{n-k-1} / Z_k$. If g is in S , it normalises S_{n-k-1} so C_{k+1}^g and C_{k+1} are commensurable by minimality of C_{k+1} . We apply Theorem 3.8 to the set $\{C_{k+1}^g : g \in S\}$ and find a definable group B_{k+1} normalised by S and commensurable with C_{k+1} . The group B_{k+1} is a subgroup of $N_1 \cap \dots \cap N_k$. Let D_{k+1} be the group $B_{k+1} S_{n-k-1} / Z_k$. It is a finite extension of a definable group hence definable, and commensurable with C_{k+1} / Z_k . Let Z_{k+1} be the preimage in G of the FC -centre of D_{k+1} . We claim that Z_{k+1} contains S_{n-k-1} : if g is in $(B_{k+1} S_{n-k-1}) \setminus Z_{k+1}$, then $(B_{k+1} S_{n-k-1}) / (B_{k+1} S_{n-k-1}) \cap C_G(\bar{g})$ is infinite, so $C_{k+1} / C_{k+1} \cap C_G(\bar{g})$ is also infinite, hence g is cannot be in S_{n-k-1} . This completes the induction. \square

Corollary 4.9. *In a group with simple theory, a definable FC -soluble group has an FC -series whose members are definable subgroups.*

Proof. Let G be this group. We apply Theorem 4.8 to G inside G . \square

Note that [16, Proposition 4.4.10.3] shows in particular that a definable FC -nilpotent group in a simple theory has a nilpotent definable subgroup of finite index. For definable FC -soluble groups, we have the following :

Corollary 4.10. *In a simple theory, let G be a definable (with parameters in A) FC -soluble group of class n . Then G has a definable (with parameters in A) subgroup of finite index which is soluble of class at most $2n$.*

Proof. Without loss of generality, we may suppose that the ambient group is sufficiently saturated. Let $G_0 \supseteq G_1 \supseteq \dots \supseteq G_n$ be an FC series for G with every G_i normal in G . By Corollary 4.9, we may assume that the subgroups G_i are definable. As each quotient G_i / G_{i+1} is FC , it follows by Proposition 4.1 and saturation that the centraliser of every \bar{g} in G_i / G_{i+1} has finite bounded index in G_i / G_{i+1} . By Theorem 2.5, the quotient group $[G_i, G_i] / G_{i+1}$ is finite. Let G^0 stand for the intersection of the A -definable subgroups of G of finite index. As G normalises the finite set $[G_i, G_i] / G_{i+1}$, the centraliser of $[G_i, G_i] / G_{i+1}$ has finite index in G . As it is A -definable, it must contain G^0 . Thus we have

$$[[G_i, G_i], G^0] \leq G_{i+1}$$

With i equals 0, this shows that $(G^0)^{(2)}$ is a subgroup of G_1 . Let us show inductively on k , that $(G^0)^{(2k)}$ is a subgroup of G_k . Assume that this is done until the step p . Then we have

$$G^{(2p+2)} = [[(G^0)^{(2p)}, (G^0)^{(2p)}], [(G^0)^{(2p)}, (G^0)^{(2p)}]] \leq [[G_p, G_p], G^0] \leq G_{p+1}$$

This shows that the derived group $(G^0)^{(2n)}$ is trivial. By compactness, we find an A -definable subgroup of finite index in G which is soluble of class at most $2n$. \square

Corollary 4.11. *In a group with simple theory, let H be a soluble subgroup of class n . There is a definable (with parameters in H) soluble group of class at most $2n$ finitely many translates of which cover H .*

Proof. By Theorem 4.8 and Corollary 4.9, there is K a definable FC -soluble group of class n containing H with a definable FC series. By Corollary 4.10, the group K has a definable subgroup of finite index which is soluble of class at most $2n$. \square

Corollary 4.12. *In a group with simple theory, let H be a normal soluble subgroup of class n . There is a definable (with parameters in H) normal soluble group of class at most $3n$ containing H .*

Proof. Corollary 4.11 provides a definable soluble K with $H/H \cap K$ finite. It follows that $\bigcap_{h \in H} K^h$ is definable, soluble of class at most $2n$ (as a subgroup of K) and normalised by H . Call it K^H . Being the product of two normal soluble subgroups, the group $H \cdot K^H$ is also soluble, of class at most $n + 2n$. The group $(H \cdot K^H)^G$ is as desired. \square

ANNEXE. ON INFINITE EXTRA-SPECIAL p -GROUPS

It seems to be well known among model theorists that infinite extra-special p -groups are supersimple of SU -rank 1. As we could not find any reference for that, we provide a proof here.

Definition 4.13. For a natural number $p > 2$, a group G is an *infinite extra-special p -group* if G is infinite, if $g^p = 1$ for every g in G , if the centre of G is cyclic of order p and equals G' .

Note that this axiomatisation is expressible in first order logic and gives a complete theory according to [4, Felgner]. We call V an infinite vector space over a finite field F equipped with a non-degenerate skew-symmetric bilinear form $[\ , \]$.

Lemma 4.14. *Let $(\lambda_1, \dots, \lambda_n)$ be in F^n . If a_1, \dots, a_n are linearly independent in V , then the solution set of $\bigwedge_{i \in \{1, \dots, n\}} [x, a_i] = \lambda_i$ is an affine subspace of V whose underlying vector space has codimension n in V .*

Proof. As the bilinear form is non-degenerate, the linear forms mapping x to $[x, a_i]$ are linearly independent. \square

Lemma 4.15. *The theory of V is complete and eliminates quantifiers.*

Proof. If T is the theory of V , let A and B be two models of T and σ a local isomorphism between a_1, \dots, a_n and b_1, \dots, b_n . If a_{n+1} is in A , we show that σ can be extended to a local isomorphism of domain $\{a_1, \dots, a_{n+1}\}$. This is enough by [10, Fraïssé, théorème 2.02]. There are scalars $\lambda_1, \dots, \lambda_n$ such that $[a_{n+1}, a_i] = \lambda_i$ for all $i \leq n$. We may replace a_1, \dots, a_n by a basis of $\text{Span}(a_1, \dots, a_n)$ and assume without loss of generality that a_1, \dots, a_n are linearly independent. It follows that b_1, \dots, b_n also are linearly independent. If a_{n+1} belongs to $\text{Span}(a_1, \dots, a_n)$, then $a_{n+1} = \lambda_1 a_1 + \dots + \lambda_n a_n$ and we choose $\sigma(a_{n+1}) = \lambda_1 b_1 + \dots + \lambda_n b_n$. If not, by Lemma 4.14 we can choose some b_{n+1} in B satisfying $\bigwedge_{i \in \{1, \dots, n\}} [x, b_i] = \lambda_i$ and put $\sigma(a_{n+1}) = b_{n+1}$. \square

Corollary 4.16. *V is \aleph_0 -categorical.*

Proof. The same proof produces a back and forth between any two countable structures having same theory as V . \square

Lemma 4.17. *V is supersimple of SU -rank 1.*

Proof. By Lemma 4.15, any non algebraic formula is a finite disjunction of formulae $\phi(x, a, \lambda)$ of the form $"[x, a] = \lambda"$. If $[x, a] = \lambda$ k -divides over A for some $k \geq 2$, there is an A -indiscernible sequence a_1, a_2, \dots such that the conjunction $\bigwedge_{i \in \{1, \dots, k\}} \phi(x, a_i, \lambda_i)$ is inconsistent. By Lemma 4.14 this means that a_1, \dots, a_k

are linearly dependent over F . By indiscernability, as F is finite, the type of a is algebraic over A , a contradiction. Hence the only forking formulae are algebraic ones. It follows that non-algebraic types can only fork once. \square

Lemma 4.18. *Let V be an infinite vector space over the finite field \mathbf{F}_p with p elements, equipped with a skew-symmetric bilinear non-degenerate form. Then V interprets an infinite extra-special p -groups.*

Proof. One defines G as follows. G is the set of all pairs $V \times \mathbf{F}_p$ with product

$$(u, a) * (v, b) = (u + v, a + b + [u, v])$$

\square

Note that reciprocally, an infinite extra-special p -groups interprets an infinite vector space over \mathbf{F}_p , equipped with a skew-symmetric bilinear non-degenerate form.

Corollary 4.19. *An infinite extra-special p -groups is supersimple of SU -rank 1 (and \aleph_0 -categorical).*

\aleph_0 -categoricity of infinite extra-special p -groups has been first established in [4, Felgner].

REFERENCES

- [1] Ricardo de Aldama, ‘A result on definable groups without the independence property’, to appear in the *Bulletin of Symbolic Logic*.
- [2] Tuna Altinel and Paul Baginski, ‘Nilpotence in groups with bounded chains of centralizers’, preprint.
- [3] A.M. Duiguid and D.H. McLain, ‘ FC -nilpotent and FC -soluble groups’, *Proceedings of the Cambridge Philosophical Society* 52 (1956) 391–398.
- [4] Ulrich Felgner, ‘On \aleph_0 -categorical extra-special p -groups’, *Logique et Analyse* 18 (1975), no 71–72, 408–428.
- [5] Franklin Haimo, ‘The FC -chain of a group’, *Canadian Journal of Mathematics* 5 (1953) 498–511.
- [6] Cédric Milliet, ‘On properties of (weakly) small groups’, to appear in *The Journal of Symbolic Logic*.
- [7] Bernhard H. Neumann, ‘Groups covered by permutable subsets’, *Journal of the London Mathematical Society* 29 (1954) 236–248.
- [8] Anand Pillay, ‘Definability and Definable Groups in Simple Theories’, *The Journal of Symbolic Logic* 63 (1998) 788–796.
- [9] Jacob M. Plotkin, ‘ZF and locally finite groups’, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 27 (1981) 375–379.
- [10] Bruno Poizat, Cours de théorie des modèles, Nur Al-Mantiq Wal-Ma’rifah, 1985.
- [11] Bruno Poizat, Groupes Stables, Nur Al-Mantiq Wal-Ma’rifah, 1987.
- [12] Saharon Shelah, ‘Simple unstable theories’, *Annals of Mathematical Logic* 19 (1980) 177–203.
- [13] Saharon Shelah, ‘Dependent first order theories, continued’, (Shelah 783), to be published in the *Israel Journal of Mathematics*.
- [14] Günter Schlichting, ‘Operationen mit periodischen Stabilisatoren’, *Archiv der Mathematik* 34 (1980) 97–99.
- [15] Frank O. Wagner, ‘Groups in simple theories’, in M. Baaz, S.-D. Friedman, J. Krajicek (eds.) *Logic Colloquium 2001, Lecture Notes in Logic* 20 (2005) 440–467.
- [16] Frank O. Wagner, Simple Theories, Kluwer Academic Publishers, Dordrecht, NL, 2000.

UNIVERSITÉ GALATASARAY
FACULTÉ DE SCIENCES ET DE LETTRES
DÉPARTEMENT DE MATHÉMATIQUES
ÇIRAĞAN CADDESİ N°36
34357 ORTAKÖY, İSTANBUL, TURQUIE

E-mail address: `milliet@math.univ-lyon1.fr`