# Anisotropic Fast-Marching on cartesian grids using Lattice Basis Reduction

Jean-Marie Mirebeau

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**Abstract**

We introduce a modification of the Fast Marching Algorithm, which solves the generalized eikonal equation associated to an arbitrary continuous riemannian metric $\mathcal{M}$, on a two or three dimensional box domain. The algorithm has a logarithmic complexity in the maximum anisotropy ratio $\kappa(\mathcal{M})$ of the riemannian metric $\mathcal{M}$, which allows to handle extreme anisotropies for a reduced numerical cost. We establish that the output of the algorithm converges towards the viscosity solution of continuous problem, as the discretization step tends to zero. The algorithm is based on the computation at each grid point $z$ of a reduced basis of the lattice $\mathbb{Z}^d$, with respect to the symmetric positive definite matrix $\mathcal{M}(z)$ encoding the desired anisotropy at this point.

## Introduction

The eikonal equation, and its generalization the Hamilton-Jacobi equation, is a partial differential differential equation (PDE) which describes an elementary front propagation model : the speed of the front depends only on the front position and orientation. This PDE is encountered in numerous applications, such as motion planning control problems [12], modeling of bio-medical phenomena [11], and image analysis [9]. It was also recently used in the context of medical image analysis [2] for extracting vessels in two dimensional projections or three dimensional scans of the human body, and for performing virtual endoscopies. This application requires to solve a highly anisotropic generalized eikonal equation with a high resolution on a cartesian grid, at a computational cost compatible with user interaction. It is one of the key motivations of this paper.

We introduce in this paper a modification of the Fast Marching Algorithm [12, 13], designed to solve the eikonal generalized equation on a two or three dimensional cartesian grid, and which can handle large or even extreme anisotropies. The proposed algorithm relies on algebraic tools, called $M$-reduced meshes and bases of $\mathbb{Z}^d$, where $M$ is a symmetric positive definite matrix, which are introduced and studied in the first section. We describe the algorithm itself in the second section, where we also analyze its computational cost, and establish its consistency : the discrete approximations produced by this algorithm converge towards the viscosity solution of the continuous eikonal equation as the discretization step tends to zero. We present some numerical experiments in the third section, which confirm the small numerical cost of the algorithm and show a remarkable accuracy in test cases related to our envisioned medical application.

We consider a positive integer $d$, the dimension, which is fixed throughout this paper. Let us mention that our algorithm only applies in dimension $d \in \{2, 3\}$, unless the problem of interest

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has a special structure, see Point iii at the end of this introduction. We denote by $\Omega$ the periodic unit box

$$\Omega := (\mathbb{R}/\mathbb{Z})^d. \tag{1}$$

The domain $\Omega$ coincides with the unit box $[-1/2,\, 1/2]^d$ equipped with periodic boundary conditions. The periodicity assumption is not essential, as shown in the numerical experiments and discussed in Point ii at the end of this introduction, but it simplifies the description and the proof of convergence of our algorithm. We consider a fixed Riemannian metric

$$\mathcal{M} \in C^0(\Omega, S_d^+)$$

where $S_d^+$ denotes the set of $d \times d$ symmetric positive definite matrices. For each $u \in \mathbb{R}^d$ and each $M \in S_d^+$ we denote $\|u\|_M := \sqrt{u^{\mathrm{T}} M u}$. Our objective is to compute (an approximation of) the *viscosity solution* $\mathrm{d} : \Omega \to \mathbb{R}_+$, see [7] and Definition 2.9, of the generalized eikonal equation

$$\left\{ \begin{array}{l} \|\nabla \mathrm{d}(z)\|_{\mathcal{M}(z)^{-1}} = 1 \text{ for almost every } z \in \Omega \setminus \{0\} \\ \mathrm{d}(0) = 0. \end{array} \right. \tag{2}$$

The only element of general theory used in this paper is the *uniqueness* of the viscosity solution to the above problem. Another point of view on this problem is provided by the following characterization [7]: for each $z \in \Omega$, the quantity $\mathrm{d}(z)$ is the riemannian length of the shortest path $\gamma$ joining $z$ to the origin.

$$\mathrm{d}(z) \quad = \quad \inf \{\mathrm{length}(\gamma);\, \gamma \in C^1([0,1], \Omega),\, \gamma(0) = 0,\, \gamma(1) = z\}, \tag{3}$$

$$\mathrm{length}(\gamma) \quad := \quad \int_0^1 \|\gamma'(t)\|_{\mathcal{M}(\gamma(t))} dt. \tag{4}$$

We define the measure of anisotropy $\kappa(M)$ of a matrix $M \in S_d^+$, and the maximum anisotropy $\kappa(\mathcal{M})$ of the riemannian metric $\mathcal{M}$, as follows

$$\kappa(M) := \sqrt{\|M\| \|M^{-1}\|}, \quad \kappa(\mathcal{M}) := \max_{z \in \Omega} \kappa(\mathcal{M}(z)). \tag{5}$$

Typically $\kappa(\mathcal{M}) \approx 100$ in the test cases presented int §3, which are relevant benchmark for the envisioned medical applications [2]. The periodic box $\Omega$ is discretized on a cartesian grid of step size $1/n$

$$\Omega_n := \{0,\, 1/n,\, 2/n, \cdots,\, (n-1)/n\}^d \subset \Omega.$$

Denoting by $N := n^d = \#(\Omega_n)$ the number of grid points, our algorithm has complexity

$$\mathcal{O}(N \ln N + N \ln \kappa(\mathcal{M})), \tag{6}$$

where the constant behind the $\mathcal{O}(\cdot)$ notation only depends on the dimension $d$. The Orthogonal Upwind Method (OUM) and the Adaptive Gauss-Siedel Iteration (AGI) are two alternative consistent methods for solving the eikonal equation, introduced respectively by Sethian and Vladimirsky in [12], and by Bornemann and Rasch in [3]. The first method has complexity

$$\mathcal{O}(\Upsilon(\mathcal{M})^d N \ln N), \qquad \Upsilon(\mathcal{M}) := \sqrt{\max_{z \in \Omega} \|\mathcal{M}(z)\| \max_{z' \in \Omega} \|\mathcal{M}(z')^{-1}\|}.$$

Note that $\Upsilon(\mathcal{M}) \geq \kappa(\mathcal{M})$. The *asymptotic* complexity of this method drops to $\mathcal{O}(\Upsilon(\mathcal{M})^{d-1} N \ln N)$, for a fixed $\mathcal{M}$ and as $N \to \infty$, but this is still substantially larger than (6) for pronounced anisotropies. The second method AGI has complexity $\mathcal{O}(\lambda(\mathcal{M}) N^{1+1/d})$, where $\lambda(\mathcal{M})$ is a non

explicit constant which depends on global geometrical features of the metric. Based on numerical experiments in [3] (Figure 4), on a Hamiton-Jacobi equation, this method is competitive with the OUM when the anisotropy coefficient $\Upsilon$ is pronounced and the resolution is not too large (namely $\Upsilon \approx 19$ and $N = n \times n$, $n \leq 725$), the heuristic reason being that the anisotropy penalty $\Upsilon^{d-1}$ in the OUM compensates for the larger power $N^{1+1/d}$ in the IGM. This is no longer the case with our algorithm, as our numerical experiments in §3 illustrate.

Let us finally compare the complexity of our method with that of classical isotropic fast marching algorithms, which are consistent only if $\mathcal{M}(z) = m(z)\,\mathrm{Id}$ for each $z \in \Omega$, but can be executed nonetheless (with variable results, see §3) for a general anisotropic metric $\mathcal{M}$. Depending on the implementation [13, 8], involving either a sorted list or a bucket sort, these methods have complexity $\mathcal{O}(N \ln N)$ or $\mathcal{O}(\Upsilon(\mathcal{M})N)$, see [8]. In the applications for which our method is intended one typically has $\ln(N) \lesssim \kappa(\mathcal{M}) \leq \Upsilon(\mathcal{M}) \ll N$, in such way that the complexity (6) of the proposed method is comparable to $\mathcal{O}(N \ln N)$ and smaller than $\mathcal{O}(\Upsilon(\mathcal{M})N)$ : the anisotropy of $\mathcal{M}$ has essentially no performance impact on our method.

Let us emphasize that the efficiency of the proposed algorithm comes at the price of its specialization. We review below its limitations :

i The algorithm only applies to the generalized eikonal equation. The underlying riemannian structure plays an important role in our approach, and our algorithm therefore cannot handle more general Hamilton-Jacobi equations, contrary to the OUM [12] and the AGI [3] mentioned above, in which the local euclidean norm $\|\cdot\|_{\mathcal{M}(z)}$ is replaced for each $z \in \Omega$ with a more general norm or pseudo-norm $|\cdot|_z \in C^0(\mathbb{R}^d, \mathbb{R}+)$, usually non euclidean and non symmetric but homogeneous and satisfying the triangle inequality.

ii The domain needs to be discretized on a cartesian grid. Our algorithm therefore does not applies to domains provided under the form of general unstructured meshes, a more difficult setting which has attracted an important research effort [1, 3, 5, 6, 12]. At the price of a higher technicality of the proof, the specific periodic domain (1) could however be replaced with a more general smooth bounded domain $\Omega \subset \mathbb{R}^d$, still discretized on a grid. In that case the PDE (2) is replaced with

$$\begin{cases} \|\nabla \mathrm{d}(z)\|_{\mathcal{M}(z)^{-1}} = 1 \text{ for almost every } z \in \Omega. \\ \mathrm{d}(z) = f(z) \text{ for all } z \in \partial\Omega \end{cases}$$

where the boundary data $f$ needs to satisfy [7] the compatibility condition $|f(z) - f(z')| \leq \mathrm{length}(\gamma)$ for any path $\gamma \in C^1([0,1], \overline{\Omega})$ joining two points $z, z' \in \partial\Omega$.

iii The algorithm applies to arbitrary continuous riemannian metrics on a two or three dimensional domain. The algorithm can be extended to dimension $d \geq 4$ if the riemannian metric $\mathcal{M}$ has a specific diagonal block structure, with blocks of size 1, 2 or 3, see Proposition 2.6. Such block diagonal structures are not uncommon in the context of medical imaging, see [2]. They are inherited from the cartesian product structure of the fast marching domain : $\Omega = \Omega_0 \times \Omega_1$, where $\Omega_0$ is a physical domain of dimension $\leq 3$, and $\Omega_1$ is an abstract parameter domain of dimension $\leq 2$.

We introduce and study in §1 the notion of $M$-reduced mesh, where $M \in S_d^+$ is a symmetric positive definite matrix. This notion is the main originality of this paper and the key of our modified fast marching algorithm. Following a more classical approach, we describe the proposed variant of the fast marching algorithm in §2, and we establish the related convergence result. We finally we present some numerical experiments in §3.

**Notations and conventions** Slightly abusing notations, we denote by $z + v \in \Omega$, the offset of a point $z \in \Omega$ by a vector $v \in \mathbb{R}^d$. We denote by $\mathbb{R}_+ := \{z \in \mathbb{R};\, z \geq 0\}$ the collection of non-negative reals, and by $S_d^+$ the collection of symmetric *positive definite* matrices. We denote by $\|u\|$ the euclidean norm of a vector $u \in \mathbb{R}^d$, and by $\|u\|_M := \sqrt{u^{\mathrm{T}} M u}$ the norm associated to a symmetric positive definite matrix $M \in S_d^+$.

# 1 Reduced meshes and bases

We introduce in this section the notion of $M$-reduced meshes, where $M \in S_d^+$ is a given symmetric positive definite matrix. These meshes are a key ingredient in our construction §2 of a consistent anisotropic fast marching algorithm.

A simplex $T \subset \mathbb{R}^d$ is the convex envelope of $d+1$ points $v_0, \cdots, v_d \in \mathbb{R}^d$, not lying on a common hyperplane, which are called the vertices of $T$. A mesh is a finite collection $\mathcal{T}$ of simplices which satisfy the following conformity condition : for all $S, T \in \mathcal{T}$ the intersection $S \cap T$ is the convex envelope of the common vertices of $S$ and $T$.

**Definition 1.1.** *A $M$-reduced mesh is a mesh $\mathcal{T}$ which satisfies the following properties.*

*(a) The union of the elements of $\mathcal{T}$ is a neighborhood of the origin*

*(b) The vertices of each simplex $T \in \mathcal{T}$ lie on the lattice $\mathbb{Z}^d$, and $T$ has volume $1/d!$.*

*(c) For each $T \in \mathcal{T}$, one of the vertices of $T$ is the origin $0$, and the others denoted by $v_1, \cdots, v_d$ satisfy for all $1 \leq i, j \leq d$*

$$v_i^{\mathrm{T}} M v_j \geq 0. \tag{7}$$

Note that the condition (c) alone involves the matrix $M$. The algorithm proposed in this paper is based on the construction, for each point $z$ in the discrete set $\Omega_n$, of a $\mathcal{M}(z)$-reduced mesh which dictates the local definition of the numerical scheme. Heuristically point (a) of Definition 1.1 ensures that the numerical information is propagated in all directions. Point (b) implies that the non-zero vertices of any simplex $T \in \mathcal{T}$ form a basis of $\mathbb{Z}^d$, see Definition 1.4. Point (c) is an acuteness condition related to the celebrated Causality property, see [12] and Corollary 2.2, which is a prerequisite for the Fast Marching Algorithm and other one-pass solvers of eikonal or Hamilton-Jacobi equations.

The next proposition, illustrated by Figure 1, gives some simple examples of $M$-reduced meshes.

**Proposition 1.2.** *Let $\mathcal{T}$ be a $d$-dimensional mesh which satisfies the requirements (a) and (b) of Definition 1.1. Let*

$$\kappa(\mathcal{T}) := \sqrt{\frac{1+\gamma(\mathcal{T})}{1-\gamma(\mathcal{T})}}, \quad \textit{where} \quad \gamma(\mathcal{T}) := \min_{T,(u,v)} \frac{u^{\mathrm{T}} v}{\|u\|\|v\|}$$

*in which the infimum is taken among all simplices $T \in \mathcal{T}$ and all common non-zero vertices $u, v$ of $T$. Then $\mathcal{T}$ is a $M$ reduced mesh for any $M \in S_d^+$ such that $\kappa(M) \leq \kappa(\mathcal{T})$.*

*Proof.* Let $u, v$ be two non-zero vertices of a common simplex $T \in \mathcal{T}$, and let $M \in S_d^+$. Let $u' := u/\|u\|$ and let $v' := v/\|v\|$. By construction we have

$$\|u' + v'\|^2 = 2(1 + u'^{\mathrm{T}} v') \geq 2(1+\gamma(\mathcal{T})), \qquad \|u' - v'\|^2 = 2(1 - u'^{\mathrm{T}} v') \leq 2(1-\gamma(\mathcal{T})).$$

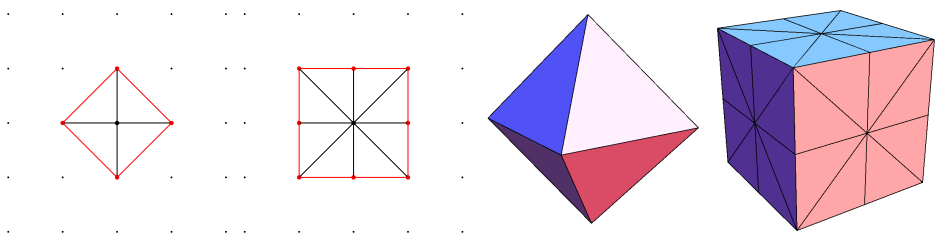

Figure 1: The above meshes are $M$-reduced meshes for any *diagonal* matrix $M \in S_d^+$, (left : $d = 2$, right : $d = 3$). Furthermore, according to Proposition 1.2, the two-dimensional 8-neighborhood (center left) is a $M$-reduced mesh for any $M \in S_2^+$ such that $\kappa(M) \leq 1 + \sqrt{2} \approx 2.41\ldots$ The three-dimensional 26-neighborhood (right, originally introduced in [4]) is a $M$-reduced mesh for any $M \in S_3^+$ such that $\kappa(M) \leq (\sqrt{3}+1)/\sqrt{2} \approx 1.93\ldots$

Let us assume for contradiction that $uMv < 0$, which implies that $\|u'+v'\|_M < \|u'-v'\|_M$. Observing that

$$\kappa(M)^2 = \|M\|\|M^{-1}\| \geq \frac{\|u'-v'\|_M^2}{\|u'-v'\|^2}\frac{\|u'+v'\|^2}{\|u'+v'\|_M^2} > \frac{1+\gamma(\mathcal{T})}{1-\gamma(\mathcal{T})},$$

we obtain that $\kappa(M) > \kappa(\mathcal{T})$, which concludes the proof of this proposition. □

**Remark 1.3** (Block diagonal matrices)**.** *Let $d_1, d_2$ be positive integers, let $M_1 \in S_{d_1}^+$, and let $M_2 \in S_{d_2}^+$. Let $\mathcal{T}_1$ be a $M_1$-reduced mesh, and let $\mathcal{T}_2$ a $M_2$-reduced mesh. Let $d := d_1 + d_2$ and let $\mathcal{T}$ be the $d$-dimensional mesh defined as follows : for any $T_1 \in \mathcal{T}_1$ of vertices $0 = u_0^1, \cdots u_{d_1}^1$, and any $T_2 \in \mathcal{T}_2$ of vertices $0 = u_0^2, \cdots, u_{d_2}^2$, the $d$-dimensional simplex $T$ of vertices*

$$(0,0), (u_1^1, 0), \cdots, (u_{d_1}^1, 0), (0, u_1^2), \cdots, (0, u_{d_2}^2)$$

*belongs to $\mathcal{T}$. Then one easily checks that $\mathcal{T}$ is a $M$-reduced mesh, where $M \in S_d^+$ denotes the matrix of diagonal blocks $M_1$ and $M_2$.*

We introduce in subsection §1.1 the algebraic notion of $M$-reduced basis of $\mathbb{Z}^d$, where $M \in S_d^+$ and $1 \leq d \leq 4$. We show that the collection of vertices of a $M$-reduced mesh contains a $M$-reduced basis, a property later used in the analysis of our algorithm. We next give in §1.2 an explicit construction of a $M$-reduced mesh of bounded cardinality, for each $M \in S_d^+$, $d \in \{2, 3\}$.

## 1.1 Bases of the lattice $\mathbb{Z}^d$

The results of this section fall in the framework of low-dimensional lattice basis reduction. We refer to [8] and reference therein for an introduction to this rich theory, from which we use only one result : Theorem 1.5 stated below.

For all $u_1, \cdots u_k \in \mathbb{Z}^d$ we denote

$$u_1\mathbb{Z} + \cdots + u_k\mathbb{Z} := \{u_1 z_1 + \cdots + u_k z_k;\ (z_1, \cdots, z_k) \in \mathbb{Z}^k\}.$$

If $k = 0$ then the above sum equals $\{0\}$ by convention.

**Definition 1.4.** *A basis of $\mathbb{Z}^d$ is a $d$-plet $(u_1, \cdots, u_d)$ of elements of $\mathbb{Z}^d$ such that*

$$|\det(u_1, \cdots, u_d)| = 1. \tag{8}$$

*Assume that $1 \leq d \leq 4$. A $M$-reduced basis of $\mathbb{Z}^d$, where $M \in S_d^+$, is a basis $(u_1, \cdots, u_d)$ of $\mathbb{Z}^d$ which satisfies for all $1 \leq k \leq d$*

$$u_k \in \operatorname{argmin}\{\|z\|_M;\, z \in \mathbb{Z}^d \setminus (u_1\mathbb{Z} + \cdots + u_{k-1}\mathbb{Z})\}. \tag{9}$$

For instance the canonical basis $(e_1, \cdots, e_d)$ of $\mathbb{R}^d$ is also a basis of $\mathbb{Z}^d$. If $M \in S_d^+$ is a diagonal matrix of coefficients $(\lambda_1, \cdots, \lambda_d)$, and if $\lambda_{\sigma(1)} \leq \cdots \leq \lambda_{\sigma(d)}$ for some permutation $\sigma$, then $(e_{\sigma(1)}, \cdots e_{\sigma(d)})$ is a $M$-reduced basis of $\mathbb{Z}^d$.

The above definition of a $M$-reduced basis of $\mathbb{Z}^d$ is adequate only up to dimension 4, since in dimension $d \geq 5$ there exists matrices $M \in S_d^+$ such that no basis of $\mathbb{Z}^d$ satisfies the relations (9), see [8] (these relations state that $\|u_i\|_M$ equals the $i$-th Minkowski's minimum $\lambda_i(M)$). The proper generalization of Definition 1.4 in dimension $d \geq 5$, is Minkowski's reduction [8].

**Theorem 1.5** (Nguyen, Stelhé, 2009)**.** *There exists an algorithm which, given a matrix $M \in S_d^+$ as input, $1 \leq d \leq 4$, produces a $M$-reduced basis of $\mathbb{Z}^d$ and has the numerical cost $\mathcal{O}(1+\ln \kappa(M))$.*

*Proof.* The proof is contained in [8], and we only point out here the precise reference within the paper and the slight differences in notations. The algorithm described in [8] takes as input a basis $(b_1, \cdots, b_d)$ (here : the canonical matrix of $\mathbb{R}^d$) of a lattice $L$ (here : $\mathbb{Z}^d$), and its Gram matrix with respect to some scalar product (here : the Gram matrix is $M$). The algorithm outputs a greedy reduced basis of the lattice $L$, a notion which coincides with Minkowski's reduction if $d \leq 4$ (Lemma 4.3.2 in [8]), which itself coincides with Definition 1.4 if $d \leq 4$.

The main loop of the iterative algorithm is executed at most the following number of times (Theorem 6.0.5 in [8]):

$$\mathcal{O}\left(1 + \ln \max_{1 \leq i \leq d} \|b_i\|_M - \ln \min_{u \in L} \|u\|_M\right),$$

hence $\mathcal{O}(1 + \ln \|M\|^{\frac{1}{2}} - \ln \|M^{-1}\|^{-\frac{1}{2}}) = \mathcal{O}(1 + \ln \kappa(M))$ times in our setting. The complexity of each of these iterations is dominated by a closest vector search, described in Theorem 5.0.4 in [8], which consists of a the inversion of a $k \times k$ Gram matrix, where $1 \leq k \leq d-1$, and a $\mathcal{O}(1)$ exhaustive search. In terms of elementary operations $(+, -, \times, /)$ among reals, each iteration of this algorithm thus has cost $\mathcal{O}(1)$, and the overall cost is the number of iterations $\mathcal{O}(1+\ln \kappa(M))$.

Note that an important part of the discussion in [8] is devoted to the special case where the vectors $(b_1, \cdots, b_d)$ have *large integer* coefficients, the Gram matrix is computed with respect to the standard euclidean scalar product, and the complexity of an elementary operation $(+, -, \times, /)$ among integers is not unitary but depends on the size of these integers. This more subtle notion of complexity, named bit complexity, is not relevant in our setting. □

**Proposition 1.6.** *Assume that $1 \leq d \leq 4$. Let $M \in S_d^+$ and let $(u_1, \cdots, u_d)$ be a $M$-reduced basis of $\mathbb{Z}^d$. Then for all $1 \leq i \leq d$*

$$\|u_i\| \leq \kappa(M), \tag{10}$$

$$\|u_i\|_M \leq \kappa(M)\|u_1\|_M. \tag{11}$$

*For any integer combination $z$ of the elements of the basis distinct from $u_i$, in other words $z = \alpha_1 u_1 + \cdots \alpha_{i-1}u_{i-1} + \alpha_{i+1}u_{i+1} + \alpha_d u_d$, where $\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_d \in \mathbb{Z}$, one has*

$$2|u_i^{\mathrm{T}} M z| \leq \|z\|_M^2 \tag{12}$$

*Proof.* We consider a fixed $1 \leq i \leq d$, and we claim that

$$\|M^{-1}\|^{-\frac{1}{2}} \leq \|u_1\|_M \leq \|u_i\|_M \leq \|M\|^{\frac{1}{2}}. \tag{13}$$

The left inequality follows from $1 \leq \|u_1\| \leq \|M^{-\frac{1}{2}}\|\|u_1\|_M$. The central inequality follows from the fact that $u_1$ minimizes the norm $\|\cdot\|_M$ among all elements of $\mathbb{Z}^d \setminus \{0\}$, see Definition 1.4. Denoting by $(e_1, \cdots, e_d)$ the canonical basis of $\mathbb{R}^d$, we observe comparing dimensions that there exists $1 \leq j \leq d$ such that $e_j \notin u_1\mathbb{Z} + \cdots + \cdots u_{i-1}\mathbb{Z}$. Therefore $\|u_i\|_M \leq \|e_j\|_M \leq \|M\|^{\frac{1}{2}}$ using Definition 1.4, which establishes (13), hence also (11). We obtain (10) combining (13) with the observation $\|u_i\| \leq \|u_i\|_M \|M^{-1}\|^{\frac{1}{2}}$.

We next turn to the proof of (12), and for that purpose we remark that $u_i + z \notin u_1\mathbb{Z} + \cdots + u_{i-1}\mathbb{Z}$. Indeed otherwise $u_i$ would be a linear combination of $u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_d$, which contradicts (8). Definition 1.4 thus implies that

$$\|u_i\|_M^2 \leq \|u_i + z\|_M^2 = \|u_i\|_M^2 + 2u_i^{\mathrm{T}} M z + \|z\|_M^2,$$

which implies that $-2u_i^{\mathrm{T}} M z \leq \|z\|_M^2$. We obtain likewise $2u_i^{\mathrm{T}} M z \leq \|z\|_M^2$, which concludes the proof of (12). □

We say that point $z$ is a vertex of a mesh $\mathcal{T}$ if it is a vertex of one of the simplices $T \in \mathcal{T}$. We denote by $d_\times$ the distance on the collection $S_d^+$ of symmetric positive definite matrices, which is defined as follows : for all $M, N \in S_d^+$

$$d_\times(M, N) := \sup_{u \neq 0} \left| \ln \|u\|_M - \ln \|u\|_N \right|.$$

**Lemma 1.7.** *Assume that $1 \leq d \leq 4$. Let $M, N \in S_d^+$. Let $u_1, \cdots, u_d$ be an arbitrary $M$-reduced basis of $\mathbb{Z}^d$, and let $\mathcal{T}$ be a $N$-reduced mesh. Consider a point $z \in \mathbb{Z}^d$ which is not a vertex of $\mathcal{T}$. Then there exists $1 \leq l \leq d$ such that*

$$z \in u_1\mathbb{Z} + \cdots + u_l\mathbb{Z} \quad \text{and} \quad \|z\|_M^2 \, e^{4d_\times(M,N)} \geq \|u_l\|_M^2 + \|u_1\|_M^2.$$

*Proof.* Since the union of the elements of $\mathcal{T}$ is a neighborhood of the origin, there exists a simplex $T \in \mathcal{T}$ and a real $\lambda > 0$ such that $\lambda z \in T$. Denoting by $0 = v_0, v_1, \cdots, v_d$ the vertices of $T$, there exists therefore non-negative reals $\alpha_1, \cdots, \alpha_d \in \mathbb{R}_+$ such that $z = \alpha_1 v_1 + \cdots + \alpha_d v_d$.

Since $|\det(v_1, \cdots, v_d)| = d!|T| = 1$, the coefficient $\alpha_i$ is an integer for all $1 \leq i \leq d$. We denote by $l$ the smallest integer such that : $\big(v_i \in u_1\mathbb{Z} + \cdots + u_l\mathbb{Z}$ or $\alpha_i = 0\big)$ for all $1 \leq i \leq d$. Observe that $z \in u_1\mathbb{Z} + \cdots + u_l\mathbb{Z}$, and, using definition 1.4, that there exists $1 \leq i \leq d$ such that : $(\alpha_i > 0$ and $\|v_i\|_M^2 \geq \|u_l\|_M^2)$. We thus obtain

$$\begin{aligned}
e^{4d_\times(M,N)}\|z\|_M^2 &\geq e^{2d_\times(M,N)}\|z\|_N^2 \\
&= e^{2d_\times(M,N)} \left( \sum_{i=1}^d \alpha_i^2 \|v_i\|_N^2 + 2 \sum_{1 \leq i < j \leq d} \alpha_i \alpha_j v_i^{\mathrm{T}} N v_j \right) \\
&\geq e^{2d_\times(M,N)} \sum_{i=1}^d \alpha_i^2 \|v_i\|_N^2 \\
&\geq \sum_{i=1}^d \alpha_i^2 \|v_i\|_M^2 \\
&\geq \|u_l\|_M^2 + \left( \sum_{i=1}^d \alpha_i^2 - 1 \right) \|u_1\|_M^2.
\end{aligned}$$

Since $z$ is not a vertex of $T$ we have $\alpha_1^2 + \cdots + \alpha_d^2 \geq 2$, which concludes the proof. □

**Corollary 1.8.** *Assume that* $1 \leq d \leq 4$. *Let* $M, N \in S_d^+$ *be such that*

$$d_\times(M,N) < \ln(1+\kappa(M)^{-2})/4. \tag{14}$$

*Let* $(u_1, \cdots, u_d)$ *be a* $M$*-reduced basis of* $\mathbb{Z}^d$, *and let* $\mathcal{T}$ *be a* $N$*-reduced mesh. Then* $u_1, \cdots, u_d$ *and* $-u_1, \cdots, -u_d$ *are vertices of* $\mathcal{T}$.

*Proof.* We consider $1 \leq l \leq d$ and we assume for contradiction that $u_l$ (or $-u_l$) is not a vertex of $\mathcal{T}$. It follows from (14) and (11) that

$$\|u_l\|_M^2 e^{4d_\times(M,N)} < \|u_l\|_M^2 + \kappa(M)^{-2}\|u_l\|_M^2 \leq \|u_l\|_M^2 + \|u_1\|_M^2.$$

It thus follows from the previous lemma that $u_l \in u_1\mathbb{Z} + \cdots + u_k\mathbb{Z}$ for some $1 \leq k < l$. This contradicts Definition 1.4, and concludes the proof. □

## 1.2 Explicit construction of $M$-reduced meshes

This subsection is devoted to the explicit construction of a $M$-reduced mesh of bounded cardinality for any $M \in S_d^+$, where $d = 2$ in Proposition 1.9, and that $d = 3$ in Proposition 1.10. This construction uses as a starting point a $M$-reduced basis of $\mathbb{Z}^d$.

Let $A$ be a $d \times d$ invertible matrix. For any simplex $T$ we denote $A(T) := \{Az;\, z \in T\}$, and for any mesh $\mathcal{T}$ we denote $A(\mathcal{T}) := \{A(T);\, T \in \mathcal{T}\}$.

**Proposition 1.9.** *Let* $M \in S_2^+$ *and let* $(u_1, u_2)$ *be a* $M$*-reduced basis of* $\mathbb{Z}^2$. *Let* $u := u_1$ *and let* $v := \varepsilon u_2$, *where* $\varepsilon \in \{-1, 1\}$ *is chosen in such way that* $u^{\mathrm{T}} M v \geq 0$. *Consider the collection* $\mathcal{T}$ *of triangles of vertices* $0$ *and one of the following pairs*

$$\{u, v\}, \{v, v-u\}, \{v-u, -u\}$$

*or their opposites (the opposite of the pair* $\{a, b\}$ *is* $\{-a, -b\}$*). Then* $\mathcal{T}$ *is a* $M$*-reduced mesh.*

*Proof.* We denote by (a),(b),(c) the corresponding points of the Definition 1.1 of $M$-reduced meshes.

Consider a matrix $A$ such that $Au = (1, 0)$ and $Av = (0, 1)$. Then the mesh $A(\mathcal{T})$ does not depend on $u$ and $v$, and the reunion of its elements is easily checked to be a neighborhood of the origin. This immediately implies (a).

The equality of the determinants

$$\det(u, v) = \det(v, v-u) = \det(v-u, -u)$$

shows that all the elements of $\mathcal{T}$ have volume $1/2$. Since their vertices clearly lie on $\mathbb{Z}^2$, this establishes (b).

We have by construction $u^{\mathrm{T}} M v \geq 0$. Furthermore, applying (12) to the vectors $u_1$ and $z = u_2$ (resp. $u_2$ and $z = u_1$) we obtain $2|u^{\mathrm{T}} M v| \leq \|v\|_M^2$ and $2|u^{\mathrm{T}} M v| \leq \|u\|_M^2$, hence

$$v^{\mathrm{T}} M (v-u) = \|v\|_M^2 - u^{\mathrm{T}} M v \geq 0, \quad (v-u)^{\mathrm{T}} M (-u) = \|u\|_M^2 - u^{\mathrm{T}} M v \geq 0,$$

which establishes point (c) and concludes the proof.

For a visual illustration of point (c) we suggest the reader to look at Figure 2 (center, left), which displays the *image* of the mesh $\mathcal{T}$ by a linear transformation $P$ such that $Pu = (1, 0)$ and $P^{\mathrm{T}} P = M$. For any two vectors $a, b \in \mathbb{R}^2$, we have $aMb \geq 0$ if and only if the images of $a$ and $b$ by $P$ form an acute angle. The blue region corresponds to all possible values of $v$ which satisfy the constraints $2|v^{\mathrm{T}} M u| \leq \|u\|_M^2 \leq \|v\|_M^2$. □

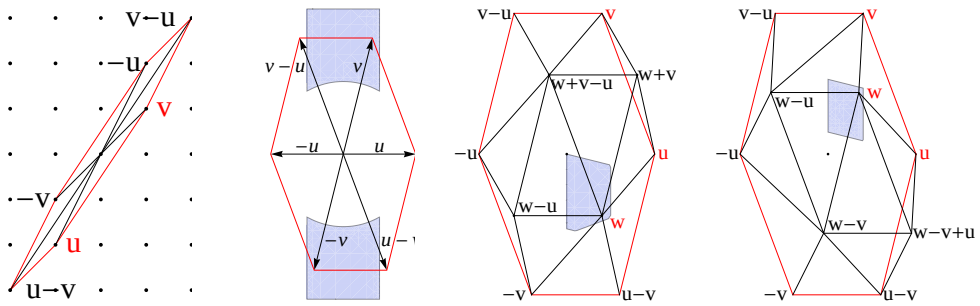


Figure 2: A $M$-reduced mesh for particular a matrix $M \in S_2^+$, such that $\kappa(M) = 10$ (left). Illustrations for the proof of Proposition 1.9. Connectivity defined by (16) (center, right) and by (18) (right).

**Proposition 1.10.** *Let $M \in S_3^+$ and let $(u_1, u_2, u_3)$ be a $M$-reduced basis of $\mathbb{Z}^3$. We distinguish two cases depending on the parity of the number of non-negative scalar products among $u_1^{\mathrm{T}} M u_2$, $u_2^{\mathrm{T}} M u_3$, $u_3^{\mathrm{T}} M u_1$.*

- *Odd parity. We denote $u := \varepsilon_1 u_1$, $v := \varepsilon_2 u_2$, $w := \varepsilon_3 u_3$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$ are chosen in such way that*

$$u^{\mathrm{T}} M v \geq 0, \quad u^{\mathrm{T}} M w \geq 0, \quad v^{\mathrm{T}} M w \leq 0. \tag{15}$$

*Consider the collection $\mathcal{T}$ of tetrahedra of vertices $0$ and one of the following triplets*

$$\begin{array}{ccc} \{w, u, w+v\} & \{w, w+v, w+v-u\} & \{w, w+v-u, w-u\} \\ \{w, w-u, -v\} & \{w, -v, u-v\} & \{w, u-v, u\} \\ \{v, w+v, u\} & \{v, w+v-u, w+v\} & \{v, v-u, w+v-u\} \\ \{-u, -v, w-u\} & \{-u, w-u, w+v-u\} & \{-u, w+v-u, v-u\} \end{array} \tag{16}$$

*or their opposites (the opposite of the triplet $\{a, b, c\}$ is $\{-a, -b, -c\}$). Then $\mathcal{T}$ is a $M$-reduced mesh.*

- *Even parity. We denote $u := \varepsilon_1 u_{\sigma_1}$, $u := \varepsilon_2 u_{\sigma_2}$, $w := \varepsilon_3 u_{\sigma_3}$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$ and the permutation $\sigma$ of $\{1, 2, 3\}$ are chosen in such way that*

$$v^{\mathrm{T}} M w \geq u^{\mathrm{T}} M v \geq u^{\mathrm{T}} M w \geq 0. \tag{17}$$

*Consider the collection $\mathcal{T}$ of tetrahedra of vertices $0$ and one of the following triplets*

$$\begin{array}{ccc} \{u, v, w\} & \{w, v, w-u\} & \{w-u, v, v-u\} \\ \{w-u, v-u, -u\} & \{w-u, -u, w-v\} & \{w-v, -u, -v\} \\ \{w-v, -v, u-v\} & \{w-v, u-v, w-v+u\} & \{w-v+u, u-v, u\} \\ \{u, w, w-v+u\} & \{w, w-v, w-v+u\} & \{w, w-u, w-v\} \end{array} \tag{18}$$

*or their opposites (the opposite of the triplet $\{a, b, c\}$ is $\{-a, -b, -c\}$. Then $\mathcal{T}$ is a $M$-reduced mesh.*

*Proof.* We denote by (a),(b),(c) the corresponding points of the Definition 1.1 of $M$-reduced meshes.

In each case of the parity, if $A$ is a matrix such that $Au = (1,0,0)$, $Av = (0,1,0)$ and $Aw = (0,0,1)$, then the mesh $A(\mathcal{T})$ is independent of $u, v, w$. The union of the elements of this constant mesh $A(\mathcal{T})$ is easily seen to be a neighborhood of the origin, which implies (a).

The determinant of each triplets of vectors appearing in (16) and (18), is equal to $\det(u,v,w)$. Since $|\det(u,v,w)| = 1$, see (8), the volume of all the constructed simplices is $1/6$. Furthermore their vertices obviously lie on $\mathbb{Z}^3$, which establishes (b).

We next need to establish (c) : the non-negativeness of the scalar products between two common vertices of any tetrahedron $T \in \mathcal{T}$. To avoid notational clutter we denote in this proof

$$\langle u, v\rangle := u^{\mathrm{T}} M v.$$

We first remark that the positivity of the scalar products $\langle u, v\rangle$, $\langle v, v-u\rangle$,$\langle v-u, -u\rangle$ (associated to edges which lie in the plane generated by $(u, v)$) follows as in Proposition 1.9 from the inequalities $\langle u, v\rangle \geq 0$, $2|\langle u, v\rangle| \leq \|u\|_M^2$ and $2|\langle u, v\rangle| \leq \|v\|_M^2$. For the other scalar products we need to distinguish between the two cases of the parity.

**Odd parity.** From the inequality (15) we obtain

$$\langle u, v\rangle \geq 0, \langle u, w\rangle \geq 0, \langle -v, w\rangle \geq 0, \langle u, w+v\rangle \geq 0, \langle w, u-v\rangle \geq 0, \langle w-u, -v\rangle \geq 0.$$

Applying (12) to $z = u$ we obtain $2|\langle u, w\rangle| \leq \|u\|_M^2$, hence $\langle -u, w-u\rangle = \|u\|_M^2 - \langle u, w\rangle \geq 0$. Likewise

$$\langle v, w+v\rangle \geq 0, \langle w, w+v\rangle \geq 0, \langle w, w-u\rangle \geq 0$$

Applying (12) to $z = u$ we obtain $2|\langle u, v\rangle| \leq \|u\|_M^2$ and $2|\langle u, w\rangle| \leq \|u\|_M^2$, which implies $\langle -u, w+v-u\rangle = \|u\|_M^2 - \langle u, v\rangle - \langle u, w\rangle \geq 0$. Likewise

$$\langle v, w+v-u\rangle \geq 0,\ \langle w, w+v-u\rangle \geq 0.$$

Applying (12) to $z = v - u$ we obtain $2|\langle w, v-u\rangle| \leq \|v-u\|_M^2$, hence $\langle v-u, w+v-u\rangle = \|v-u\|_M^2 + \langle w, v-u\rangle \geq 0$. Likewise

$$\langle w-u, w+v-u\rangle \geq 0,\ \langle w+v, w+v-u\rangle \geq 0.$$

**Even parity.** From the inequality (17) we obtain

$$\langle v, w\rangle \geq \langle u, v\rangle \geq \langle u, w\rangle \geq 0,\ \langle v, w-u\rangle \geq 0,\ \langle -u, w-v\rangle \geq 0.$$

Applying (12) to $z = u$ we obtain $2|\langle u, v\rangle| \leq \|u\|_M^2$, and therefore $\langle -u, w-u\rangle \geq 0$. Likewise

$$\langle -v, w-v\rangle \geq 0,\ \langle w, w-u\rangle \geq 0,\ \langle w, w-v\rangle \geq 0.$$
$$\langle u, w-v+u\rangle \geq 0,\ \langle w, w-v+u\rangle \geq 0.$$

Applying (12) to $z = u$ we obtain $2|\langle u, v\rangle| \leq \|u\|_M^2$ and $2|\langle u, w\rangle| \leq \|u\|_M^2$, which implies $\langle v-u, w-u\rangle = \langle v, w\rangle + (\|u_M\|^2 - \langle u, v\rangle - \langle u, w\rangle) \geq 0$. Likewise

$$\langle w-u, w-v\rangle \geq 0,\ \langle w-v, u-v\rangle \geq 0.$$

Applying (12) to $z = u - v$ we obtain $|\langle w, u-v\rangle| \leq \|u-v\|_M^2$, hence $\langle u-v, w-v+u\rangle \geq 0$. Likewise $\langle w-v, w-v+u\rangle \geq 0$. □

## 2 The algorithm

We present in this section our modified fast marching algorithm. We estimate its complexity, and we establish its consistence : the convergence of the discrete approximations towards the solution of the continuous problem. Our presentation of this section is fairly classical, and we invoke (variants of) arguments previously seen in the literature on numerical schemes for the eikonal equation [13, 12, 3]. A detailed description and proof is nevertheless necessary, since the original constructions of §1 do not exactly fit in earlier framework.

We consider a fixed integer $n > 0$, and for each $z \in \Omega_n$ we assume that a $\mathcal{M}(z)$-reduced mesh $\mathcal{T}(z)$ has been constructed. A few additional properties of the meshes $\mathcal{T}(z)$, namely Assumption 2.4 and Assumptions 2.5, will be required for the complexity and the convergence analyses of this algorithm. These additional properties are established in Proposition 2.6 for the mesh constructions proposed in §1.

We introduce the Hopf-Lax update function [3]. For each map $\mathrm{d} : \Omega_n \to \mathbb{R}_+ \cup \{\infty\}$, each boolean table $b : \Omega_n \to \{\mathit{trial}, \mathit{computed}\}$, and each $z \in \Omega_n$ we denote

$$\Lambda^b_{\mathrm{d}}(z)_n := \min_{k,(\alpha_i),(v_i)} \left\{ \left\| \sum_{1\le i\le k} \alpha_i \frac{v_i}{n} \right\|_{\mathcal{M}(z)} + \sum_{1\le i\le k} \alpha_i \, \mathrm{d}\left(z + \frac{v_i}{n}\right) \right\}, \tag{19}$$

where the minimum is taken among all $1 \le k \le d$, all $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$ and all $v_1, \cdots, v_k \in \mathbb{Z}^d$, such that

- $\alpha_i + \cdots + \alpha_k = 1$, and $\alpha_i \ge 0$ for all $1 \le i \le k$.
- $v_1, \cdots, v_k$ are non-zero vertices of a common simplex $T \in \mathcal{T}(z)$.
- $b(z + v_1/n) = \cdots = b(z + v_k/n) = \mathit{computed}$.

We denote by $\Lambda_{\mathrm{d}}(z)_n$ the operator $\Lambda^b_{\mathrm{d}}(z)_n$ associated to the constant boolean table defined by $b(z) = \mathit{computed}$ for all $z \in \Omega_n$. We describe in the first part of this section an algorithm which computes a map $\mathrm{d}_n : \Omega_n \to \mathbb{R}_+ \cup \{\infty\}$, satisfying

$$\begin{cases} \mathrm{d}_n(z) = \Lambda_{\mathrm{d}_n}(z)_n, & \forall z \in \Omega_n \setminus \{0\}, \\ \mathrm{d}_n(0) = 0. \end{cases} \tag{20}$$

In the second part of this section we establish, in Theorem 2.10, the convergence of $\mathrm{d}_n$ to the viscosity solution d of (2) as $n \to \infty$.

The next lemma, and the subsequent corollary, constitute the causality property which is a key ingredient in the analysis of the fast marching algorithm and its variants.

**Lemma 2.1** (Sethian and Vladimirsky 2000, [12])**.** *Let $M \in S_d^+$ and let $w_1, \cdots, w_k \in \mathbb{R}^d$ be linearly independent vectors such that $w_i^{\mathrm{T}} M w_j \ge 0$ for all $1 \le i \le j \le k$. Consider the compact and convex set*

$$\Xi := \{\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k_+;\ \alpha_1 + \cdots + \alpha_k = 1\}$$

*Choose $\delta = (\delta_1, \cdots, \delta_k) \in \mathbb{R}$, denote*

$$D := \min_{\alpha\in\Xi} \left\| \sum_{1\le i\le k} \alpha_i w_i \right\|_M + \sum_{1\le i\le k} \alpha_i \delta_i, \tag{21}$$

*and consider a minimizer* $\alpha = (\alpha_1, \cdots, \alpha_k) \in \Xi$.

*If* $\alpha_i > 0$ *for all* $1 \leq i \leq d$, *then denoting by* $\hat{M}$ *the matrix of entries* $\hat{M}_{ij} = w_i^{\mathrm{T}} M w_j$ *one has the relations*

$$1 = \|D\mathbf{1} - \delta\|_{\hat{M}^{-1}}, \tag{22}$$

$$\hat{M}\alpha = \|\alpha\|_{\hat{M}}(D\mathbf{1} - \delta), \tag{23}$$

*and furthermore* $D > \delta_i$ *for each* $1 \leq i \leq k$.

*Proof.* For completeness, and due to notation changes, we give the proof of this lemma, which follows the steps of Property A.1 in [12].

The gradients of the maps $\alpha \mapsto \|\alpha\|_{\hat{M}} + \alpha^{\mathrm{T}}\delta$ and $\alpha \mapsto \alpha^{\mathrm{T}}1$, respectively the minimized function and the constraint in (21), are proportional at the minimizer $\alpha$ according to Lagrange's Theorem. Hence there exists $\lambda \in \mathbb{R}$, the Lagrange multiplier, such that

$$\frac{\hat{M}\alpha}{\|\alpha\|_{\hat{M}}} + \delta = \lambda\mathbf{1}. \tag{24}$$

Therefore

$$\|\lambda\mathbf{1} - \delta\|_{\hat{M}^{-1}} = \left\| \frac{\hat{M}\alpha}{\|\alpha\|_{\hat{M}}} \right\|_{\hat{M}^{-1}} = 1. \tag{25}$$

Using again (24) we obtain

$$D = \|\alpha\|_{\hat{M}} + \alpha^{\mathrm{T}}\delta = \alpha^{\mathrm{T}}\left( \frac{\hat{M}\alpha}{\|\alpha\|_{\hat{M}}} + \delta \right) = \alpha^{\mathrm{T}}(\lambda\mathbf{1}) = \lambda.$$

Combining the last equation with (25) (resp. (24)) we obtain (22) (resp. (23)).

Since $\hat{M}$ has non negative coefficients, positive diagonal coefficients, and since we have assumed that $\alpha$ has positive coefficients, we obtain that the product $\hat{M}\alpha$ has positive coefficients. Therefore $D > d_i$ for all $1 \leq i \leq k$ in view of (23), which concludes the proof. □

The following corollary, the causality property, heuristically states that if the value $\Lambda_{\mathrm{d}}^{b'}(z)$ of the Hopf-Lax update is smaller that $\mathrm{d}(z_*)$, for some $z, z_* \in \Omega_n$, then $\Lambda_{\mathrm{d}}^{b'}(z)$ does not depend upon $\mathrm{d}(z_*)$.

In order to avoid notational clutter, we denote in the rest of this subsection $\Lambda_{\mathrm{d}}^{b}(z) := \Lambda_{\mathrm{d}}^{b}(z)_n$. We also set by convention, $\min \emptyset = +\infty$.

**Corollary 2.2** (Causality property)**.** *Let* $z, z' \in \Omega_n$, *let* $\mathrm{d} : \Omega_n \to \mathbb{R}_+ \cup \{\infty\}$ *and* $b' : \Omega_n \to \{trial, computed\}$. *One has the following implication*

$$\Lambda_{\mathrm{d}}^{b'}(z) \leq \mathrm{d}(z_*) \;\Rightarrow\; \Lambda_{\mathrm{d}}^{b'}(z) = \Lambda_{\mathrm{d}}^{b}(z). \tag{26}$$

*where* $b : \Omega_n \to \{trial, computed\}$ *coincides with* $b'$ *on* $\Omega_n \setminus \{z_*\}$ *and satisfies* $b(z_*) = trial$.

*Proof.* We consider the integer $k$, the vertices $v_1, \cdots, v_k \in \mathbb{Z}^d \setminus \{0\}$ of $T \in \mathcal{T}(z)$ and the non-negative coefficients $\alpha_1, \cdots, \alpha_k \in \mathbb{R}_+$ for which the minimum defining the Hopf-Lax update $\Lambda_{\mathrm{d}}^{b'}(z)$, see (19), is attained. We assume without loss of generality (up to reducing the integer $k$) that the coefficients $\alpha_1, \cdots, \alpha_k$ are positive.

If $z_* \neq z + v_i/n$ for all $1 \leq i \leq k$, then $\left\| \sum_{1 \leq i \leq k} \alpha_i \frac{v_i}{n} \right\|_{\mathcal{M}(z)} + \sum_{1 \leq i \leq k} \alpha_i \, \mathrm{d}\left(z + \frac{v_i}{n}\right)$ is also an upper bound for $\Lambda_{\mathrm{d}}^{b}(z)$, and therefore $\Lambda_{\mathrm{d}}^{b}(z) \leq \Lambda_{\mathrm{d}}^{b'}(z)$. Since by construction $\Lambda_{\mathrm{d}}^{b}(z) \geq \Lambda_{\mathrm{d}}^{b'}(z)$ we obtain $\Lambda_{\mathrm{d}}^{b}(z) = \Lambda_{\mathrm{d}}^{b'}(z)$.

On the other hand if $z_* = z + v_i/n$ for some $1 \leq i \leq k$, then applying Lemma 2.1 we obtain $\Lambda^{b'}_{\mathrm{d}}(z) > \mathrm{d}(z_*)$. We have established that either $\Lambda^{b}_{\mathrm{d}}(z) = \Lambda^{b'}_{\mathrm{d}}(z)$ or $\Lambda^{b'}_{\mathrm{d}}(z) > \mathrm{d}(z_*)$, which concludes the proof. □

For each $z_* \in \Omega_n$ we denote by $\Omega_n(z_*)$ the collection of "reversed" neighbors of $z_*$ in $\Omega_n$ :

$$\Omega_n(z_*) := \{z \in \Omega_n;\ z_* = z + v/n \text{ for some non-zero vertex } v \text{ of } \mathcal{T}(z)\}.$$

Our variant of the fast marching algorithm, described below, uses two main variables $\delta$, $\beta$ and a "while" loop.

- $\delta$ is a (mutable) map, $\delta : \Omega_n \to \mathbb{R}_+ \cup \{+\infty\}$, initialized to $+\infty$, except for $\delta(0) = 0$.
- $\beta$ is a (mutable) boolean map $\beta : \Omega_n \to \{\mathit{trial}, \mathit{computed}\}$, initialized to *trial*.

**While** $\beta^{-1}(\mathit{trial}) \neq \emptyset$**.**

1. Denote by $z_*$ the minimizer of $\delta$ in $\beta^{-1}(\mathit{trial})$. Set $\beta(z_*) \leftarrow \mathit{computed}$.
2. For each $z \in \Omega_n(z_*)$ such that $\beta(z) = \mathit{trial}$, set $\delta(z) \leftarrow \Lambda^{\beta}_{\delta}(x)$.

**Proposition 2.3.** *Before and after each iteration of the "while" loop, the variables $\delta$ and $\beta$ satisfy*

*(i)* $\begin{cases} \delta(z) = \Lambda^{\beta}_{\delta}(z), & \forall z \in \Omega_n \setminus \{0\} \\ \delta(0) = 0, \end{cases}$

*(ii)* $\delta(x) \leq \delta(y)$ *for all* $x, y \in \Omega_n$ *such that* $\beta(x) = \mathit{computed}$ *and* $\beta(y) = \mathit{trial}$.

*As a result the map* $\mathrm{d}_n := \delta$ *obtained at the end of the execution of this algorithm satisfies (20).*

*Proof.* We first observe that properties (i) and (ii) hold at the initialization of the algorithm. We next assume these two properties hold for the maps $\mathrm{d} = \delta$ and $b = \beta$ obtained before the execution of the contents of a which statement, and our objective is to show that they hold for the maps $\mathrm{d}' = \delta$, $b' = \beta$ obtained after the execution of that which statement. We denote by $z_*$ the point selected in step 1. of this statement.

Using (i) and (ii) for d and $b$ we obtain that $\Lambda^{b}_{\mathrm{d}}(x) = \mathrm{d}(x) \leq \mathrm{d}(z_*)$ for all $x \in b^{-1}(\mathit{computed}) \setminus \{0\}$. Since $\Lambda^{b'}_{\mathrm{d}}(x) \leq \Lambda^{b}_{\mathrm{d}}(x)$, (26) implies that $\Lambda^{b'}_{\mathrm{d}}(x) = \Lambda^{b}_{\mathrm{d}}(x) = \mathrm{d}(x) = \mathrm{d}'(x)$. On the other hand for all $x \in \Omega_n \setminus (\Omega_n(y) \cup \{0\})$ we have $\Lambda^{b'}_{\mathrm{d}}(x) = \Lambda^{b}_{\mathrm{d}}(x)$ since $\Lambda^{b}_{\mathrm{d}}(x)$ does not depend upon $b(z)$ by definition. We also have $\Lambda^{b}_{\mathrm{d}}(x) = \mathrm{d}(x)$ by hypothesis, and $\mathrm{d}(x) = \mathrm{d}'(x)$ since $\delta(x)$ was left unchanged, therefore $\Lambda^{b'}_{\mathrm{d}}(x) = \mathrm{d}'(x)$. Finally for all $x \in \Omega_n(y)$ such that $b(x) = \mathit{trial}$, the equality $\Lambda^{b'}_{\mathrm{d}}(z) = \mathrm{d}'(z)$ is enforced in step 2. of the algorithm.

We thus have established that $\Lambda^{b'}_{\mathrm{d}}(z) = \mathrm{d}'(z)$ for all $z \in \Omega_n \setminus \{0\}$. Since $\mathrm{d}(z) = \mathrm{d}'(z)$ for all $z \in \Omega_n$ such that $b'(z) = \mathit{computed}$, we have furthermore $\Lambda^{b'}_{\mathrm{d}'}(z) = \Lambda^{b'}_{\mathrm{d}}(z)$ for all $z \in \Omega_n$. This establishes (i) for $\mathrm{d}'$ and $b'$.

Using (ii) for d and $b$ and recalling that $b(z_*) = \mathit{trial}$, we obtain $\mathrm{d}'(x) = \mathrm{d}(x) \leq \mathrm{d}(z_*) = \mathrm{d}'(z_*)$ for all $x \in b^{-1}(\mathit{computed})$. We next consider $y \in b^{-1}(\mathit{trial})$, and we observe that $\mathrm{d}(y) \geq \mathrm{d}(z_*) = \mathrm{d}'(z_*)$ by choice of $z_*$. Furthermore if $\mathrm{d}'(y) \neq \mathrm{d}(y)$ then $\Lambda^{b'}_{\mathrm{d}}(y) = \mathrm{d}'(y) \neq \mathrm{d}(y) = \Lambda^{b}_{\mathrm{d}}(y)$. The implication (26) thus implies that $\mathrm{d}(z_*) < \Lambda^{b'}_{\mathrm{d}}(y) = \mathrm{d}'(y)$. We thus have shown that for all $x \in b^{-1}(\mathit{computed})$ and all $y \in b^{-1}(\mathit{trial})$ one has $\mathrm{d}'(x) \leq \mathrm{d}'(z_*) \leq \mathrm{d}'(y)$. Since the boolean tables $b$ and $b'$ only differ by their value at $z_*$, this implies (ii) for $\mathrm{d}'$ and $b'$, which concludes the proof. □

In order to analyze the complexity of the proposed algorithm, we make the following assumption. As shown in Proposition 2.6 below, this assumption holds for the mesh construction proposed in this paper.

**Assumption 2.4** (Complexity)**.** *We assume that each mesh $\mathcal{T}(z)$ has $\mathcal{O}(1)$ vertices. We also assume that the construction of each mesh $\mathcal{T}(z)$, $z \in \Omega_n$, costs at most $\mathcal{O}(1 + \ln \kappa(\mathcal{M}))$.*

We denote by $\#(E)$ the cardinality of a finite set $E$, and we set $N := n^d = \#(\Omega_n)$. It follows from the above assumption that the overall numerical cost of the construction of the meshes $\mathcal{T}(z)$, $z \in \Omega_n$, is $\mathcal{O}(N + N \ln \kappa(\mathcal{M}))$. Each mesh $\mathcal{T}(z)$, $z \in \Omega_n$, has $\mathcal{O}(1)$ vertices by assumption. Therefore evaluating the Hopf-Lax formula (19) amounts to solve $\mathcal{O}(1)$ optimization problems of the form (21), which solution is obtained as the root $D$ of a univariate quadratic equation (22). Assuming that computing the square root of a positive real has unit cost, computing the Hopf-Lax update $\Lambda^b_{\mathrm{d}}(z)_n$ therefore has numerical cost $\mathcal{O}(1)$.

The algorithm requires to maintain a list of the elements of $\beta^{-1}(\mathit{trial})$, sorted by increasing value of $\delta$. Elementary insertions and deletions in this list have cost $\mathcal{O}(\ln N)$ (if this sorted lists is implemented numerically using an appropriate tree structure), and occur each time the Hopf-Lax formula is evaluated. This happens at most the following number of times :

$$\sum_{z \in \Omega_n} \#(\Omega_n(z)) = \sum_{z \in \Omega_n} n(z) = \mathcal{O}(N),$$

where $n(z)$ denotes the number of non-zero vertices of $\mathcal{T}(z)$ ($n(z) = \mathcal{O}(1)$ by assumption). As announced in the introduction, the overall cost of the execution of the proposed algorithm is $\mathcal{O}(N \ln N + N \ln \kappa(\mathcal{M}))$, under assumption (2.4).

## 2.1 Convergence analysis

In this section the riemannian metric $\mathcal{M} \in C^0(\Omega, S^+_d)$ is fixed, and we consider for each $z \in \Omega$ a $\mathcal{M}(z)$-reduced mesh $\mathcal{T}(z)$. For each $n > 0$ we denote by $\mathrm{d}_n : \Omega_n \to \mathbb{R}_+ \cup \{\infty\}$, a map which satisfies (20). The main result of this section is Theorem 2.10 which establishes that the maps $\mathrm{d}_n$ converge in a certain sense to a viscosity solution of the continuous eikonal equation (2). This theorem relies on two additional assumptions, below, which hold for the mesh construction proposed in this paper, see Proposition 2.6.

**Assumption 2.5.**

- *(Limited extension) There exists a constant $V_0 = V_0(\mathcal{M})$ such that each vertex $v$ of each mesh $\mathcal{T}(z)$, $z \in \Omega$, satisfies $\|v\| \leq V_0$.*
- *(Consistency) There exists a constant $r_0 = r_0(\mathcal{M}) > 0$ such that the following holds. For each $z \in \Omega$ there exists a basis $(u_1, \cdots, u_d)$ of $\mathbb{Z}^d$ such that $u_1, \cdots, u_d, -u_1, \cdots, -u_d$ are vertices of the mesh $\mathcal{T}(z + w)$, for all $w \in \mathbb{R}^d$ such that $\|w\| \leq r_0$.*

**Proposition 2.6.** *Assumption 2.4 and Assumptions 2.5 hold in the following cases.*

- *If the dimension is $d = 2$ (resp. $d = 3$) and the $\mathcal{M}(z)$-reduced mesh $\mathcal{T}(z)$ is built using Proposition 1.9 (resp. Proposition 1.10), for each $z \in \Omega$.*
- *If the following "block diagonal" structure holds : there exists $d_1, \cdots, d_k \in \{1, 2, 3\}$ such that $d_1 + \cdots + d_k = d$ and for each $z \in \Omega$, the matrix $\mathcal{M}(z)$ has a block diagonal structure, where the dimensions of the blocks are the consecutive integers $d_1, \cdots, d_k$. For each $1 \leq i \leq k$ we denote the $i$-th block by $\mathcal{M}_i(z) \in S^+_{d_i}$, and we consider a $\mathcal{M}_i(z)$-reduced mesh $\mathcal{T}_i(z)$ built using proposition 1.9 or 1.10. The $\mathcal{M}(z)$-reduced mesh $\mathcal{T}(z)$ is built by combining the meshes $\mathcal{T}_i(z)$, as shown in Remark 1.3.*

*Proof.* We only prove the first case, since the second "block diagonal" case immediately follows by considering each block separately.

As observed in §1, computing a $M$-reduced basis of $\mathbb{Z}^d$ has the numerical cost $\mathcal{O}(1+\ln\kappa(M))$. Once this basis is computed the construction of a two or three dimensional $M$-reduced mesh using Proposition 1.9 or 1.10 only costs $\mathcal{O}(1)$. Furthermore this mesh has only 6 vertices if $d=2$ (resp. 14 vertices if $d=3$) which establishes Assumption 2.4 in this case.

The elements of a $M$-reduced basis of $\mathbb{Z}^d$ have a norm bounded by $\kappa(M)$ according to Proposition 1.6. Hence any vertex of a $M$-reduced mesh $\mathcal{T}$ built as described in Proposition 1.9 satisfies $\|v\| \leq 2\kappa(M)$ (resp. Proposition 1.10 and $\|v\| \leq 3\kappa(M)$). This establishes point (Consistency) of Assumption 2.5.

Last we denote by $r_0 = r_0(\mathcal{M})$ the largest positive constant such that for all $z \in \Omega$ and all $w \in \mathbb{R}^d$

$$|w| \leq r_0 \;\Rightarrow\; d(\mathcal{M}(z), \mathcal{M}(z+w)) \leq \ln(1+\kappa(\mathcal{M})^{-2})/4.$$

If $z \in \Omega$ and $\|w\| \leq r_0$ then it follows from Corollary 1.8 that the vertices $\mathcal{T}(z+w)$, which is $\mathcal{M}(z+w)$-reduced mesh, contain as a subset any $\mathcal{M}(z)$-reduced basis $(u_1, \cdots, u_d)$ of $\mathbb{Z}^d$. This establishes point (Limited extension) of assumption 2.5, and concludes the proof. □

Our first lemma shows that the discrete maps $\mathrm{d}_n : \Omega_n \to \mathbb{R}_+ \cup \{\infty\}$ obey a regularity property if $n$ is sufficiently large.

**Lemma 2.7.** *For any $n > 0$, any $z \in \Omega_n$ and any vertex $v$ of $\mathcal{T}(z)$ one has*

$$\mathrm{d}_n(z) - \mathrm{d}_n(z + v/n) \leq \|v\|_{\mathcal{M}(z)}/n. \tag{27}$$

*Under Assumptions 2.5, for any $n \geq n_0$, any $z \in \Omega_n$, and any $v \in \{-1,0,1\}^d$ one has*

$$|\,\mathrm{d}_n(z) - \mathrm{d}_n(z+v/n)| \leq \Delta_0/n, \tag{28}$$

*where $n_0 := d^2V_0^d/r_0$, $\Delta_0 := d^2V_0^dM_0$, and $M_0 := \max\{\sqrt{\|\mathcal{M}(z)\|};\, z \in \Omega\}$.*

*Proof.* If $z \in \Omega_n \setminus \{0\}$, then (27) follows from the equality $\mathrm{d}_n(z) = \Lambda_{\mathrm{d}_n}(z)_n$, and from the definition (19) of $\Lambda_{\mathrm{d}_n}$. If $z = 0$, then the left hand side of (27) is negative, while the right hand side is non-negative. This concludes the proof of the first part of this lemma.

We next turn to the proof of (28), and for that purpose we consider a fixed $n \geq n_0$. Consider a point $z \in \Omega_n$ and a basis $(u_1, \cdots, u_d)$ of $\mathbb{Z}^d$ obeying Assumption 2.5 (Consistency), and observe that $\|u_i\| \leq V_0$ for all $1 \leq i \leq d$. Let $A$ be the $d \times d$ matrix which columns are $u_1, \cdots, u_d$. We denote by $\mathrm{com}(A)$ the comatrix of $A$, and we observe that the coefficients of this matrix are bounded in absolute value by $V_0^{d-1}$ (indeed they are determinants of $(d-1)\times(d-1)$ sub-matrices of $A$ the norm of which columns is bounded by $V_0$). Since $|\det A| = 1$, the absolute value of the coefficients of $A^{-1} = \mathrm{com}(A)^{\mathrm{T}}/\det A$ is also bounded by $V_0^{d-1}$.

Let $(\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}^d$ be such that $v = \alpha_1 u_1 + \cdots + \alpha_d u_1$, in other words $(\alpha_1, \cdots, \alpha_d) = A^{-1}v$. We denote $s := |\alpha_1| + \cdots + |\alpha_d|$, and we observe that $s \leq d^2V_0^{d-1}$, since the absolute value of the coefficients of $v$ is bounded by 1. There exists $v_0, v_1, \cdots, v_s \in \mathbb{Z}^d$, such that $v_0 = 0$, $v_s = v$ and $v_{i+1} - v_i \in \{u_1, \cdots, u_d, -u_1, \cdots, -u_d\}$ for all $1 \leq i \leq s$. Furthermore $\|v_i\| \leq sV_0 \leq d^2V_0^d$, hence $u_1, \cdots, u_d, -u_1, \cdots, -u_d$ are vertices of $\mathcal{T}(z + v_i/n)$ according to Assumption 2.5 (Consistency). It thus follows from (27) that

$$n(\mathrm{d}_n(z+v_i/n) - \mathrm{d}_n(z+v_{i+1}/n)) \leq \|v_{i+1}-v_i\|_{\mathcal{M}(z+v_i)} \leq \|v_{i+1}-v_i\|\sqrt{\|\mathcal{M}(z+v_i)\|} \leq M_0V_0,$$

which implies that $\mathrm{d}_n(z) - \mathrm{d}_n(z+v/n) \leq sM_0V_0/n \leq d^2M_0V_0^d/n = \Delta_0/n$. Exchanging the roles of $z$ and $z+v/n$ we obtain likewise $\mathrm{d}_n(z+v/n) - \mathrm{d}_n(z) \leq \Delta_0/n$, which concludes the proof. □

Let $\varphi : \mathbb{R}^d \to \Omega$ be the canonical surjection. We denote by $\mathrm{d}_{\mathrm{per}}$ the distance on $\Omega$ defined for all $z, z' \in \Omega$ by

$$\mathrm{d}_{\mathrm{per}}(z, z') := \min\{\|Z - Z'\|;\, Z \in \varphi^{-1}(z),\, Z' \in \varphi^{-1}(z')\}$$

**Corollary 2.8.** *Under Assumptions 2.5, and using the constants $n_0$, $\Delta_0$ defined in Lemma 2.7. For each $n \geq n_0$ and for all $z \in \Omega$ we define*

$$\hat{\mathrm{d}}_n(x) := \min_{z \in \Omega_n} \mathrm{d}_n(z) + \Delta_0\, \mathrm{d}_{\mathrm{per}}(x, z), \tag{29}$$

*The map $\hat{\mathrm{d}}_n : (\Omega, \mathrm{d}_{\mathrm{per}}) \to I\!R_+$ is $\Delta_0$-Lipschitz, and $\mathrm{d}_n(x) = \hat{\mathrm{d}}_n(x)$ for all $x \in \Omega_n$.*

*Proof.* Consider $x, y \in \Omega$. It follows from (29) that

$$\hat{\mathrm{d}}_n(x) \leq \min_{z \in \Omega_n} \mathrm{d}_n(z) + \Delta_0(\mathrm{d}_{\mathrm{per}}(x, y) + \mathrm{d}_{\mathrm{per}}(y, z)) = \hat{\mathrm{d}}_n(y) + \Delta_0\, \mathrm{d}_{\mathrm{per}}(x, y).$$

Likewise $\hat{\mathrm{d}}_n(y) \leq \hat{\mathrm{d}}_n(x) + \Delta_0\, \mathrm{d}_{\mathrm{per}}(x, y)$, which establishes that $\hat{\mathrm{d}}_n$ is $\Delta_0$-Lipschitz.

Let $x \in \Omega_n$, let $v = (v_1, \cdots, v_d) \in \mathbb{Z}^d$ and let $v_{\max} := \max\{|v_1|, \cdots, |v_d|\}$. Applying $v_{\max}$ times (28) we obtain

$$|\,\mathrm{d}_n(x) - \mathrm{d}_n(x + v/n)| \leq \Delta_0 v_{\max}/n \leq \Delta_0 \|v/n\|.$$

It follows that $\mathrm{d}_n(x) \leq \mathrm{d}_n(z) + \Delta_0\, \mathrm{d}_{\mathrm{per}}(x, z)$ for all $x, z \in \Omega_n$. This immediately implies as announced that $\mathrm{d}_n(x) = \hat{\mathrm{d}}_n(x)$ for all $x \in \Omega_n$, which concludes the proof. □

Before stating the main result of this section, we recall the definition of the viscosity solution of an eikonal equation [7].

**Definition 2.9.** *The viscosity solution of the eikonal equation (2), is the unique continuous function $\mathrm{d} : \Omega \to I\!R_+$ such that for any $\varphi \in C^1(\Omega, I\!R)$ and any $z \in \Omega \setminus \{0\}$ the following holds :*

- *If $\mathrm{d} - \varphi$ attains its unique global maximum at $z$, then $\|\nabla\varphi(z)\|_{\mathcal{M}(z)^{-1}} \leq 1$.*
- *If $\mathrm{d} - \varphi$ attains its unique global minimum at $z$, then $\|\nabla\varphi(z)\|_{\mathcal{M}(z)^{-1}} \geq 1$.*

In alternative definitions of the notion of viscosity solution, the assumption that "$\mathrm{d} - \varphi$ attains its unique global maximum at $z$" is often replaced with "$\mathrm{d} - \varphi$ attains a local maximum at $z$" (resp. minimum). These two definitions are equivalent, since one may subtract (resp. add) to $\varphi$ a suitable smooth function $\psi \in C^1(\Omega, \mathbb{R}_+)$, large far from $z$, and with a parabolic behavior close to $z$ : $\psi(z + h) \approx \lambda\|h\|^2$ for $h$ sufficiently small.

We finally establish the convergence of the discrete maps $\mathrm{d}_n$ produced by our modified algorithm towards the viscosity solution of the generalized equation. The proof is similar in essence to the proof provided in [3], yet with a number of minor modifications due to our specific context.

**Theorem 2.10.** *Under Assumptions 2.5. There exists a sequence $(\varepsilon_n)_{n>0}$, converging to 0, such that for all $n > 0$ and all $z \in \Omega_n$ one has*

$$|\,\mathrm{d}(z) - \mathrm{d}_n(z)| \leq \varepsilon_n,$$

*where $\mathrm{d} : \Omega \to I\!R_+$ denotes the viscosity solution of (2).*

*Proof.* We denote by $\hat{\mathrm{d}}_n : \Omega \to \mathbb{R}_+$ the function introduced in Lemma 2.8, and we show in the following that $\hat{\mathrm{d}}_n$ converges uniformly toward the viscosity solution of (2) as $n \to \infty$. This immediately implies the announced result.

Since $(\Omega, \mathrm{d}_{\mathrm{per}})$ is a compact metric space, since $\hat{d}_n(0) = 0$ and since $\hat{\mathrm{d}}_n$ is $\Delta_0$-Lipschitz for all $n \geq n_0$, we the theorem of Arzelà-Ascoli implies that the sequence $\hat{d}_n$ is pre-compact. In the rest of the proof we consider an arbitrary subsequence $(\hat{\mathrm{d}}_{\sigma(n)})_{n\geq 0}$ which converges uniformly on $\Omega$ towards a limit $\mathrm{d} : \Omega \to \mathbb{R}_+$. Our objective is to establish that d is a viscosity solution of the eikonal equation (2). Once this point is established, we conclude as announced that $\hat{\mathrm{d}}_n$ converges uniformly toward the viscosity solution of (2), using the uniqueness of the viscosity solution and the pre-compactness of the sequence $\hat{d}_n$.

We consider for each $n \geq 1$ a point $z_n \in \Omega_n$, some vertices $v_1^n, \cdots, v_d^n \in \mathbb{Z}^d$ of a common simplex $T \in \mathcal{T}(z_n)$, and some coefficients $\alpha_1^n, \cdots, \alpha_d^n \in \mathbb{R}_+$, $\alpha_1^n + \cdots + \alpha_d^n = 1$, which will all be specified later. We consider $\lambda_n > 0$, and $v_n \in \mathbb{R}^d$, $\|v_n\| = 1$, such that

$$\lambda_n v_n = \alpha_1^n v_1^n + \cdots + \alpha_d^n v_d^n.$$

Note that $\lambda_n \leq V_0$. On the other hand using the acuteness property (7) and denoting $M := \mathcal{M}(z_n)$ we obtain

$$\begin{aligned}
\lambda_n^2 \|M\| &\geq \|\lambda_n v_n\|_M^2 \\
&= \sum_{1\leq i\leq n} (\alpha_i^n)^2 \|v_i^n\|_M^2 + 2 \sum_{1\leq i<j\leq n} \alpha_i^n \alpha_j^n (v_i^n)^{\mathrm{T}} M v_j^n \\
&\geq \left( \sum_{1\leq i\leq n} (\alpha_i^n)^2 \right) \min_{1\leq i\leq n} \|v_i^n\|_M^2 \\
&\geq d^{-1} \|M^{-\frac{1}{2}}\|^{-1},
\end{aligned}$$

hence $\lambda_n \geq (d\kappa(\mathcal{M}))^{-\frac{1}{2}}$. Denoting

$$D_n := \mathrm{d}_n(z_n) - \sum_{1\leq i\leq d} \alpha_i^n \, \mathrm{d}_n(z_n + v_i^n/n),$$

it follows from the definition (20) of the Hopf-Lax update that

$$D_n \leq \lambda_n \|v_n\|_{\mathcal{M}(z_n)}/n \tag{30}$$

but also that, given $n > 0$ and $z_n \in \Omega_n$, we may choose $v_1^n, \cdots, v_d^n$ and $\alpha_1^n, \cdots, \alpha_d^n$ in such way that the above inequality is an equality.

Consider an arbitrary map $\varphi \in C^1(\Omega, \mathbb{R})$ and a point $z \in \Omega \setminus \{0\}$. Denoting $L := \nabla\varphi(z)^{\mathrm{T}}$ we have for any $x \in \Omega$ and any $h \in \mathbb{R}^d$ the Taylor development

$$|\varphi(x+h) - \varphi(x) - Lh| \leq \int_0^1 \|\nabla\varphi(x+th) - \nabla\varphi(z)\| \|h\| dt \leq \omega(\mathrm{d}_{\mathrm{per}}(x,z) + \|h\|)\|h\|, \tag{31}$$

where $\omega$ denotes the modulus of continuity of the continuous function $\nabla\varphi : \Omega \to \mathbb{R}^d$. We denote

$$S_n := \varphi(z_n) - \sum_{1\leq i\leq d} \alpha_i^n \varphi(z_n + v_i^n/n),$$

and we observe that, using (31) and denoting $r_n := \mathrm{d}_{\mathrm{per}}(z, z_n) + V_0/n$,

$$|S_n + \lambda_n L v_n/n| \leq \sum_{1\leq i\leq n} \alpha_i^n \, |\varphi(z_n + v_i^n/n) - \varphi(z_n) - L(v_i^n)/n| \leq \omega(r_n) V_0/n. \tag{32}$$

We now explicit the choice of $z_n$, $(v_i^n)$ and $(\alpha_i^n)$, and we use the above inequalities to conclude the proof. Let us assume that $\mathrm{d}-\varphi$ has a strict global maximum (resp. minimum) at $z$. For each $n>0$ we denote by $z_n \in \Omega_n$ a maximizer (resp. minimizer) of $\mathrm{d}-\mathrm{d}_n$ on $\Omega_n$. It follows from the uniform convergence $\hat{\mathrm{d}}_{\sigma(n)} \to \mathrm{d}$, as $n \to \infty$, that $z_{\sigma(n)} \to z$.

We consider a fixed $v \in \mathbb{R}^d$ such that $\|v\|=1$, and we choose $\alpha_1^n, \cdots, \alpha_d^n$ and $v_1^n, \cdots, v_d^n$ as above in such way that $v_n = v$ for all $n>0$. (resp. We choose $\alpha_1^n, \cdots, \alpha_d^n$ and $v_1^n, \cdots, v_d^n$ in such way that (30) is an equality for all $n>0$, and we denote by $v$ an arbitrary cluster value of the sequence $(v_{\sigma(n)})_{n>0}$.)

Using successively (32), the definition of $z_n$, and (30) we obtain

$$-\lambda_n L v_n/n - \omega(r_n)V_0/n \leq S_n \leq D_n \leq \lambda_n \|v_n\|_{\mathcal{M}(z_n)}/n$$

(resp. $-\lambda_n L v_n/n + \omega(r_n)V_0/n \geq S_n \geq D_n = \lambda_n \|v_n\|_{\mathcal{M}(z_n)}/n$). It follows that

$$-Lv_n \leq \|v_n\|_{\mathcal{M}(z_n)} + \omega(r_n)V_0/\lambda_n$$

(resp. $-Lv_n \geq \|v_n\|_{\mathcal{M}(z_n)} - \omega(r_n)V_0/\lambda_n$). Since $r_{\sigma(n)} \to 0$ as $n \to \infty$ (resp. $r_{\sigma(n)} \to 0$ and $v$ is a cluster value of the sequence $(v_{\sigma(n)})_{n>0}$), and since $\lambda_n$ is bounded below independently of $n$, we obtain

$$-Lv \leq \|v\|_{\mathcal{M}(z)}$$

(resp. $-Lv \geq \|v\|_{\mathcal{M}(z)}$). Observing that

$$\|\nabla\varphi(z)\|_{\mathcal{M}(z)^{-1}} = \sup\left\{\frac{-Lv}{\|v\|_{\mathcal{M}(z)}};\, v \in \mathbb{R}^d,\, \|v\|=1\right\}$$

we obtain as announced that $\|\nabla\varphi(z)\|_{\mathcal{M}(z)} \leq 1$ (resp. $\|\nabla\varphi(z)\|_{\mathcal{M}(z)} \geq 1$). It follows that d is the viscosity solution of (2), which concludes the proof. □

# 3 Numerical experiments

We conducted some numerical experiments on a 2-d and a 3-d test case, which are heuristically defined as follows. In both cases the riemannian metric $\mathcal{M}$, defined on a box domain $\Omega \subset \mathbb{R}^d$ containing 0, is euclidean (equal to Id) except in the neighborhood of a given smooth curve $\Gamma \subset \Omega$. The matrix $\mathcal{M}(z)$, when $z \in \Omega$ is close two $\Gamma$, has two eigenspaces : one of dimension 1, directed "tangentially" to the curve $\Gamma$, and associated to a small eigenvalue, and one of co-dimension 1, associated to the eigenvalue 1. Note that, in contrast with the theoretical part of this paper, but consistently with the envisioned application, the riemannian metric is discontinuous and we do not apply periodic boundary conditions on the domain. In each case we compute an approximation of $\mathrm{d}(z)$, the viscosity solution of (2) which is also the riemannian distance to the origin $0 \in \Omega$ see (3), using different algorithms. We use this approximated distance to compute an approximate geodesic $\gamma : [0, L] \to \Omega$, which joins a given fixed point $P \in \Omega$ to the origin, and which is obtained by gradient descent : for each $t \in [0, L]$

$$\gamma'(t) = -\mathcal{M}(\gamma(t))^{-1} \nabla\, \mathrm{d}(\gamma(t)).$$

Qualitatively, as shown in Figures 3 (left) and 5 (center), this geodesic goes in straight line from $P$ to the curve $\Gamma$, then follows the curve $\Gamma$ up to the origin 0 in the 2-d case (resp. follows the curve $\Gamma$ and later goes in straight line from $\Gamma$ to the origin 0 in the 3-d case).

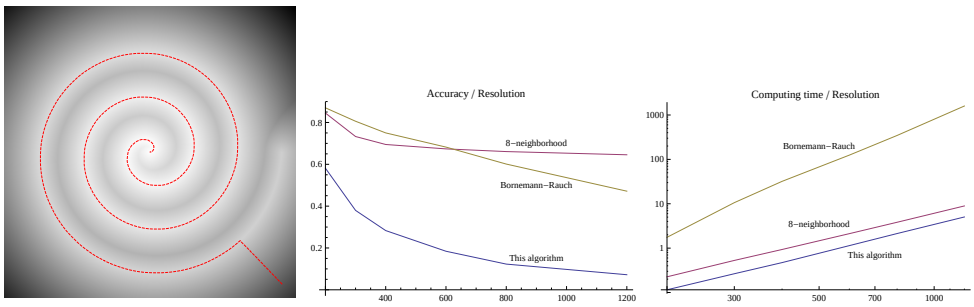


Figure 3: Reference solution for the 2-d test case (left). Accuracy of these algorithms (center). Computing time of the different algorithms, at several resolutions (right, log-log scale).

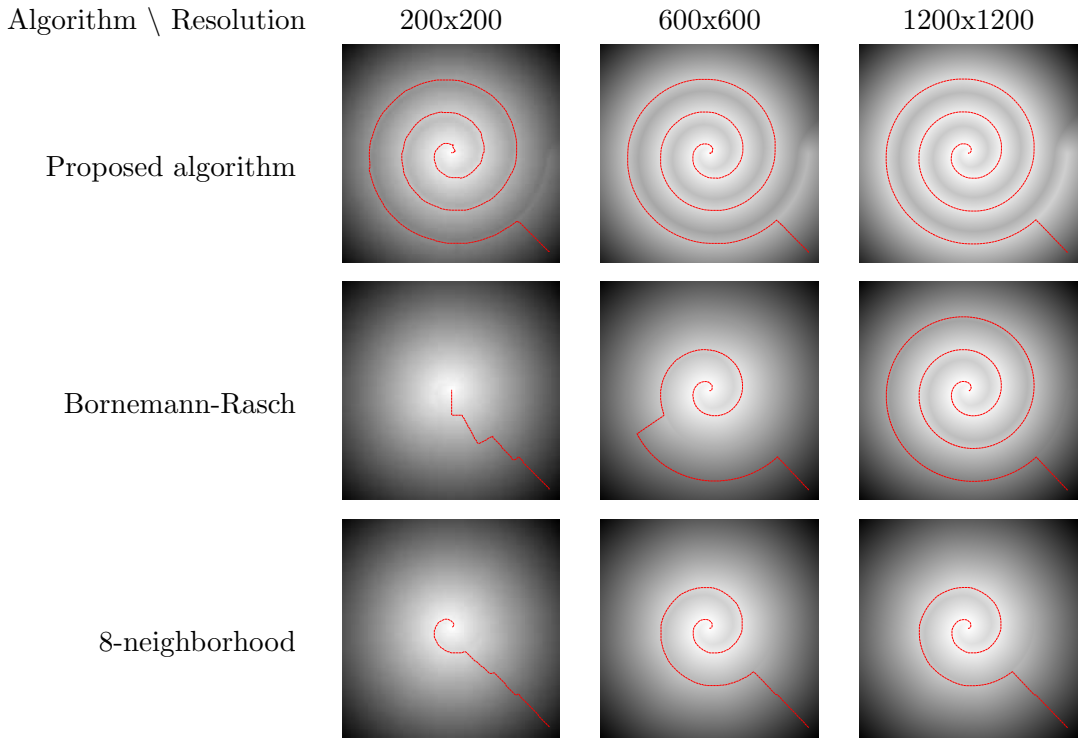


Figure 4: Visual comparison of the accuracy of three algorithms, at three resolutions, in the 2-d test case. Qualitatively the approximate geodesic has the right behavior for a resolution as low as $200 \times 200$ with the proposed algorithm, and $1200 \times 1200$ with Bornemann and Rasch's algorithm. This is presumably never the case for the (non-consistent) classical Fast Marching algorithm with non-adaptive 8-neighborhoods.

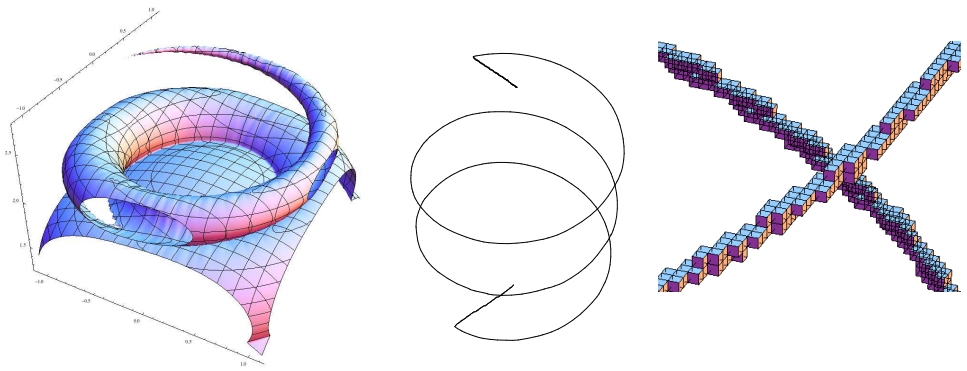

Figure 5: Iso-surface $\{\mathrm{d}(z) = 2\}$ for the 3-d test case (left). Geodesic joining the points $(0,0,0)$ and $(3,0,0)$ (center). Detail of the discrete points (represented by small cubes), in the neighborhood of the curve $\Gamma(t) = (\cos\omega_0 t, \sin\omega_0 t, t)$, for which the riemannian metric is not euclidean (right).

The 2-d test case was originally proposed in [2]. These two test cases are relevant benchmarks if one's objective is to use fast marching methods in the context medical image processing, for the segmentation of tubular structures [2]. The maximum anisotropy, defined by (5), is $\kappa(\mathcal{M}) = 100$ in the 2-d experiment, and $\kappa(\mathcal{M}) = 50$ in the 3-d experiment. The neighborhood of the curve $\Gamma$, on which the metric is not euclidean (i.e. not equal to Id), is only 1 pixel large on average in the 2-d experiment at the resolution $200 \times 200$, and is consequently 6 pixels large at the resolution $1200 \times 1200$. In the 3-d experiment this neighborhood is 2 pixels large on average and the resolution is $200 \times 200 \times 272$.

In our implementation of the algorithm proposed in this paper, we use for each discrete point $z$ the $\mathcal{M}(z)$-reduced neighborhood described by Proposition 1.9 in 2-d (resp. or 1.10 in 3-d), except if the matrix $\mathcal{M}(z)$ is detected to be exactly diagonal. In that case we use the standard 4 vertices neighborhood in 2-d (resp. 6 vertices in 3-d), which is a $\mathcal{M}(z)$-reduced mesh, see Figure 1 (left and center right) (this modification has little impact on accuracy or computing time, but avoids to pointlessly break the symmetry of the numerical scheme). A C++ source code, provided as an ancillary file to the Arxiv version of this paper, allows to reproduce the above experiments.

In the 2-d case, we also implemented for comparison Bornemann and Rasch's "Adaptive Gauss-Seidel" algorithm [3], with tolerance $10^{-10}$, and the classical fast marching algorithm with non-adaptive 8-neighborhoods, see Figure 1 (center left). Figure 3 (center) shows the accuracy of the methods at different resolutions $n \times n$, $n \in \{200, 300, 400, 600, 800, 1200\}$ : the $L^\infty$ norm of the difference between the approximated distance and a reference solution computed on a $3600 \times 3600$ grid (see Figure 4 for a visual comparison). This figure illustrates the fact that the approximations produced by the proposed algorithm and Bornemann and Rasch's algorithm converge towards the solution of the continuous problem as the resolution increases, which is not the case for the classical fast marching algorithm in this setting. Figure 3 (right) shows the computing times, obtained on a 2.4 GHz Core 2 Duo laptop. This figure illustrates the fact that the computing cost of the proposed algorithm and of the classical fast marching algorithm behave similarly, namely $\mathcal{O}(N \ln N)$, whereas Bornemann and Rasch's algorithm has an intrinsically larger computing cost, namely $\mathcal{O}(N^{1+\frac{1}{d}})$ according to [3].

## Conclusion

In this paper, we have introduced a modified version of the Fast Marching algorithm, based on the algebraic concept of reduced lattice bases, which strongpoints are the following. (I) The algorithm is consistent for any generalized eikonal equation, where the underlying continuous riemannian metric may have arbitrary anisotropy. (II) It has a numerical cost comparable to that of the classical fast marching. (III) Numerically, the efficiency of the algorithm is striking in test cases, related to tubular segmentation in medical images, where the riemannian metric has a pronounced anisotropy close to and tangentially to a curve. These strongpoints come at the price of the specialization of the algorithm : (i) it only applies to the generalized eikonal equation, (ii) discretized on a cartesian grid, and (iii) the domain needs to be of dimension 2, 3, or in higher dimension the underlying riemannian metric needs to have a block diagonal structure. Hopefully these requirements are met in many applications, and future work will be devoted to the application of the proposed algorithm in the context of medical image processing.

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