Linearized perturbations of the Kerr spacetime and outer boundary conditions in numerical relativity

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Abstract

This paper discusses linearised vacuum gravitational perturbations of the Kerr space-time in a neighbourhood of future null infinity $\mathscr{I}^+$. Unlike earlier discussion of perturbations of the Kerr space-time we avoid the use of spheroidal harmonics and harmonic time dependence. Instead we develop the theory in terms of Hertz potentials and spherical harmonics with coupling between modes. The "master equation" is a single complex scalar wave equation which, in the Minkowski limit, reduces to the Euler-Poisson-Darboux equation. We solve this by Picard iteration making extensive use of the flat space-time Riemann-Green function. As an application we consider the problem of outer boundary conditions for numerical relativity and generalise earlier results of Buchman & Sarbach (2006), (2007) for the Schwarzschild case.

1 Introduction, motivation and conclusions

This paper is about linearised vacuum gravitational perturbations of the Kerr space-time in a neighbourhood of future null infinity $\mathscr{I}^+$. This study is motivated by the increased interest, both theoretical and observational, in the gravitational radiation emitted by collapsing rotating objects. (In particular an understanding of how zero rest mass fields propagate far from the source in a curved space-time background is essential for the prescription of outer boundary conditions in numerical relativity.) One might have thought that this had already been achieved. Linearised perturbations of the Schwarzschild space-time were discussed originally by Regge & Wheeler (1957) and Zerilli (1970), and their work has inspired many successors, including Buchman & Sarbach (2006),(2007), whose contribution was the inspiration for the current work. Linearised perturbations of the Kerr

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space-time were first treated systematically by Teukolsky (1973) and the state of the art is described in the classic monograph, Chandrasekhar (1983). Later developments have been reviewed recently by Sasaki & Tagoshi (2003). So what more is there to say? Well the computational techniques employed in this paper start from a very different viewpoint to this earlier work, and lead to different insights. We review these differences below.

1.1 Separability

The Einstein field equations involve four independent variables and many dependent ones, leading to a complicated set of partial differential equations. However the Schwarzschild space-time is spherically symmetric and if two of the independent variables are chosen to be standard spherical polar angles θ and ϕ one can factor out their dependence in the equations governing linear problems by decomposing the dependent variables as a sum of terms involving spherical harmonics $Y_{l\,m}(\theta,\phi)$ and their vector and tensor generalisations, or equivalently in terms of spin-weighted spherical harmonics ${}_sY_{l\,m}(\theta,\phi)$. (In all cases one factors out an $e^{im\phi}$ dependence.) The resulting equations now involve only two independent variables, usually called t and r. Most studies carry the separability further by noting that the Schwarzschild space-time is static. They make an additional assumption of "normal modes" with an $e^{i\omega t}$ dependence, thereby reducing the problem to the study of the properties of ordinary differential equations. This is an excellent way to study problems such as linearised stability, where one formulates boundary conditions as $r \to 2M$ and as $r \to \infty$. However for the applications we have in mind, e.g., a collapsing star, only part of the Schwarzschild space-time is present and $r \gg 2M$.

The Kerr space-time also possesses symmetries—it is axisymmetric and stationary. Traditionally spherical harmonics are replaced by spheroidal harmonics $Z(\theta,\phi,a\omega)$, where the $e^{i\omega t}$ "normal mode" dependence is mandatory. However we believe that normal modes can detract from physical understanding, and so we do not make this assumption. We shall however use a spherical harmonic decomposition because those functions are complete on the unit sphere. Because we have axisymmetry rather than spherical symmetry there will be a cross-coupling of our (l,m)-modes (which involve functions of two variables, usually t and r), and we show how to deal with this in section 5. In particular we publish, apparently for the first time, expansions for $\cos\theta\,{}_sY_{l\,m}(\theta,\phi)$ and $\sin\theta(\partial/\partial\theta)\,{}_sY_{l\,m}(\theta,\phi)$ as sums of the ${}_sY_{l\,m}(\theta,\phi)$ for contiguous l.

1.2 Charts and null tetrads

Almost all treatments of perturbations of the Kerr space-time use the well-known Boyer-Lindquist chart, referred to here as $(\tilde{t},\tilde{r},\tilde{\theta},\tilde{\phi})$. Those based on the approach of Teukolsky (1973) use also the Newman & Penrose (1962) (NP) null tetrad associated with this chart, in which the only nonzero Weyl curvature scalar is Ψ_2. This chart (and associated NP tetrad) has many convenient properties but we believe that it is suboptimal for the description of zero rest mass fields near future null infinity $\mathscr{I}^+$. We prefer to use a Newman & Unti (1962) chart (u,r,θ,ϕ) where u is a retarded time variable, and take great care about its choice in section 2.

1.3 Formalisms for linearised perturbation theory

The Einstein field equations are an exceedingly complicated set of coupled nonlinear partial differential equations. However in the Regge-Wheeler-Zerilli theory it was soon recognised that perturbations could be described completely in terms of a single complex scalar function satisfying a "master equation", a sort of time-independent one dimensional Schrödinger equation. Teukolsky (1973) realised that the equations governing linearised perturbations could also be reduced to a single "master equation" for a complex scalar field, essentially the linearised Ψ_0 or Ψ_4. Chandrasekhar (1983) then showed how to transform the master equation into Schwarzschild form. As was pointed out above, we wish here to avoid normal modes, but we would like a master equation formalism.

In fact the master equation property is rather fundamental and was already known at the end of the nineteenth century. Hertz (1889) considered classical electrodynamics and introduced a Hertz bivector potential $H^{ab} = H^{[ab]}$ whose divergence generated automatically a vector potential A^a in Lorentz gauge. H^{ab} has sufficient gauge freedom that it can be determined in terms of a single complex scalar field which satisfies a linear wave equation. Sachs & Bergmann (1958) extended the idea to linearised perturbations of Minkowski space-time, where the Hertz potential H^{abcd} has the usual Riemann tensor symmetries and $H^{abcd}{}_{,bd}$ generates a metric perturbation in de Donder gauge. Again H^{abcd} has sufficient gauge freedom to be described by a single complex scalar field satisfying a linear wave equation. Penrose (1965) generalised these ideas to the case of spin s zero rest mass scalar fields propagating in Minkowski space-time. They could be described in terms of a totally symmetric 2-component spinor of valence $2s$ and, by a suitable choice of gauge, the Hertz spinor could be specified in terms of a single complex scalar field satisfying a linear wave equation. Stewart (1979) generalised Penrose's work, restricted to integral s, to arbitrary vacuum background space-times in the electromagnetic case and vacuum algebraically special space-times in the gravitational case. Stewart's work, restricted to vacuum space-times of Petrov type D (which includes Schwarzschild and Kerr) is reviewed in section 4. (Nearly similar results can be found in Chrzanowski (1975), Kegeles & Cohen (1979) and references therein.) Because we will be considering Newman & Penrose (1962) theory in a Petrov type D space-time we utilise the formalism of Geroch, Held & Penrose (1973) which greatly simplifies the calculations. We give a brief introduction to this formalism in section 3.

1.4 Evolution equations

Whichever formalism we use for linearised theory we have to solve a second order linear partial differential equation in four independent variables. As explained above we can factor out the angular dependence leaving us with a countable set of linear differential equations with two independent variables, say t and r. We are not making a Fourier decomposition with respect to t and so we have to solve a partial differential equation for say $\chi(t, r)$. In section 6 we write this formally as $E[\chi] = C[\chi]$ where all of the terms on the right hand side vanish when $M = a = 0$, the "curvature terms". The flat space equation $E[\chi] = 0$ turns out to be the Euler-Poisson-Darboux (EPD) equation which, in this context, can be solved exactly, Darboux (1899), Stewart (2009). We next argue that in a neighbourhood of $\mathscr{I}^+$ where $a/r < M/r \ll 1$ the coefficients of the derivatives in $C[\chi]$ are at least one order of magnitude smaller than the corresponding coefficients in $E[\chi]$.

This means that we can solve $E[\chi] = C[\chi]$ by Picard iteration, i.e., by solving a sequence of problems $E[\chi_{(0)}] = 0$, $E[\chi_{(n)}] = C[\chi_{(n-1)}]$ for $n = 1, 2, \ldots$. Then the solution of the original problem is $\chi = \lim_{n\to\infty} \chi_{(n)}$. We carry out the $n = 1$ step, and show how, for reasonably simple initial data, one can obtain $\chi_{(1)}$ analytically.

Solutions of the homogeneous flat space-time equation $E[\chi] = 0$ can be split in an unambiguous way into "outgoing" and "incoming" modes which, except in the simplest cases, do not satisfy a Huygens's principle. For prescribed initial data we show how $\chi_{(1)}$ can be written as the sum of an inhomogeneous "mixed" mode, and two outgoing and one incoming homogeneous modes. As far as we know this result has not appeared before in the relativity literature, and so offers new insight into the propagation of waves on a curved space-time background.

1.5 Outer boundary conditions

Numerical relativists face a difficult problem modelling radiation far from a compact source, because a space-time grid cannot be arbitrarily large. There are various ways to circumvent this problem e.g., conformal compactification or hyperboloidal slicing. However a simple approach requires the specification of boundary conditions. Suppose we use a spherical polar chart (T, R, Θ, Φ), with $R \leqslant R_0$. We would need to specify boundary conditions on the timelike hypersurface $R = R_0$. We would like to specify conditions at this artificial boundary which are transparent to "outgoing" radiation (including possible backscattering), and do not produce spurious radiation flowing back into the computational domain. There are very many approaches to this problem on a flat space-time background, and perhaps the simplest is that of Bayliss & Turkel (1980). This work has been generalised to propagation on a Schwarzschild space-time background by Buchman & Sarbach (2006), (2007), and this work is the inspiration for the current study. In section 7 we use our techniques to extend their work to propagation on a Kerr space-time background. The reader may be disappointed because we do not include a detailed comparison with their results. They made clever use of the rich properties of the Schwarzschild solution and offered a different representation of solutions of the perturbed theory. We have used only properties which generalise to the Kerr case, and so our representation is different. One cannot recover their results by simply taking the limit $a \to 0$ of ours. Bearing this in mind our results, in this limit, appear to be fully consistent with theirs. We find that the Bayliss-Turkel boundary conditions are indeed transparent to homogeneous (flat space-time) outgoing modes and block (or at least reduce the magnitude of) the incoming ones. The "reflection coefficient" for the incoming modes is of the same order of magnitude as that in Buchman & Sarbach (2007). Miraculously the Bayliss-Turkel conditions are transparent also to the inhomogeneous (curved space-time) mixed modes! These results appear to be novel.

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2 Coordinate chart and tetrad for the Kerr space-time

We shall be considering perturbations on a background Kerr space-time. This is often described using the Boyer & Lindquist (1967) chart $x^{\tilde{a}} = (\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$. A space-time containing a mass M with angular momentum Ma placed at the origin has line element

$$\begin{aligned} ds^2 = &\left(1 - \frac{2M\tilde{r}}{\tilde{\rho}^2}\right) d\tilde{t}^2 - \frac{\tilde{\rho}^2}{\Delta} d\tilde{r}^2 + \frac{4Ma\tilde{r}\sin^2\tilde{\theta}}{\tilde{\rho}^2} d\tilde{t}\, d\tilde{\phi} \\ &- \tilde{\rho}^2 d\tilde{\theta}^2 - \left(\tilde{r}^2 + a^2 + \frac{2Ma^2\tilde{r}}{\tilde{\rho}^2}\sin^2\tilde{\theta}\right)\sin^2\tilde{\theta}\, d\tilde{\phi}^2, \end{aligned} \tag{1}$$

where

$$\Delta = \tilde{r}^2 - 2M\tilde{r} + a^2, \qquad \tilde{\rho}^2 = \tilde{r}^2 + a^2\cos^2\tilde{\theta}. \tag{2}$$

It turns out that this chart is not well suited for our purposes. Following earlier work of Bai et al (2007), we introduce a Newman & Unti (1962) (NU) coordinate chart (u, r, θ, ϕ) together with an adapted Newman & Penrose (1962) NP tetrad. The u-coordinate will be a *retarded time*, i.e., a solution of the relativistic eikonal equation

$$g^{\tilde{a}\tilde{b}} u_{,\tilde{a}} u_{,\tilde{b}} = 0, \tag{3}$$

which has, in the Boyer-Lindquist chart, the asymptotic form $u \sim \tilde{t} - \tilde{r}$ as $\tilde{r} \to \infty$. From the work of Bai et al (2007) it has an asymptotic expansion in the Boyer-Lindquist chart as $\tilde{r} \to \infty$ given by $u = \tilde{t} - \tilde{r}^*$ where $\tilde{r}^*(r, \theta)$ is defined by

$$\begin{aligned} \tilde{r}^* = &\left(\tilde{r} + 2M\log\frac{\tilde{r}}{2M} - \frac{4M^2 - \frac{1}{2}a^2\sin^2\tilde{\theta}}{\tilde{r}} - \frac{4M^3 - Ma^2}{\tilde{r}^2} - \right. \\ &\frac{16M^4 - 8M^2a^2 + \frac{3}{8}a^4\sin^4\tilde{\theta}}{3\tilde{r}^3} - \\ &\left. \frac{32M^5 - 24M^3a^2 + Ma^4(2 + \frac{1}{4}\sin^4\tilde{\theta})}{4\tilde{r}^4} + O\left(\frac{1}{\tilde{r}^5}\right)\right). \end{aligned} \tag{4}$$

(We could also define an *advanced time* $v = \tilde{t} + \tilde{r}^*$.) Note that there is no natural place to truncate this and subsequent asymptotic expansions. Truncating too early may lose useful information and truncating late leads to unwieldy formulae. The underlying calculations could be done to arbitrary accuracy, and indeed they included about twice as many terms as those presented here.

Bai et al (2007) chose their NP tetrad as follows. First $l = du$, i.e., $l_{\tilde{a}}$ is normal to the hypersurfaces $u = const.$, viz.,

$$l_{\tilde{a}} = u_{,\tilde{a}} = (1, -h_1, -h_2, 0), \tag{5}$$

where $h_1 = \partial\tilde{r}^*/\partial\tilde{r}$ and $h_2 = \partial\tilde{r}^*/\partial\tilde{\theta}$. Next they chose $n \propto dv$ and the requirement $g^{\tilde{a}\tilde{b}} l_{\tilde{a}} n_{\tilde{b}} = 1$ gives

$$n_{\tilde{a}} = \frac{1}{g^{\tilde{0}\tilde{0}} - g^{\tilde{1}\tilde{1}}{h_1}^2 - g^{\tilde{2}\tilde{2}}{h_2}^2}(1, h_1, h_2, 0). \tag{6}$$

The requirements that m and $\overline{m}$ be null, orthogonal to l and n, and $g^{\tilde{a}\tilde{b}}m_{\tilde{a}}\overline{m}_{\tilde{b}} = -1$ fix m and $\overline{m}$ up to (a) complex conjugation, $m \to \overline{m}$, $\overline{m} \to m$, and (b) a *spin* $m \to e^{i\psi}m$ where ψ is real. We choose

$$\begin{aligned} m_{\tilde{a}} = \Bigg(& -g_{\tilde{0}\tilde{3}}\frac{i}{\sin\tilde{\theta}}\sqrt{\frac{\tilde{\rho}^2}{2\Sigma^2}},\ -g_{\tilde{1}\tilde{1}}\sqrt{\frac{-{h_2}^2}{2g_{\tilde{1}\tilde{1}}{h_2}^2 + 2g_{\tilde{2}\tilde{2}}{h_1}^2}}, \\ & g_{\tilde{2}\tilde{2}}\sqrt{\frac{-{h_1}^2}{2g_{\tilde{1}\tilde{1}}{h_2}^2 + 2g_{\tilde{2}\tilde{2}}{h_1}^2}},\ -g_{\tilde{3}\tilde{3}}\frac{i}{\sin\tilde{\theta}}\sqrt{\frac{\tilde{\rho}^2}{2\Sigma^2}}\Bigg), \end{aligned} \tag{7}$$

(where $\Sigma = (\tilde{r}^2 + a^2)^2 - \Delta a^2 \sin^2\tilde{\theta}$), which is the complex conjugate of Bai's choice. We can now compute $l^{\tilde{a}} = g^{\tilde{a}\tilde{b}}l_{\tilde{b}}$ etc.

So far we have worked in the Boyer-Lindquist chart $x^{\tilde{a}} = (\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$. Following Newman & Unti (1962) we introduce a new chart $x^a = (u, r, \theta, \phi)$ adapted to the NP tetrad as follows. The first coordinate is $u = \tilde{t} - \tilde{r}^*$ defined above. The surfaces $u = const.$ are null hypersurfaces with geodesic null generators $l^{\tilde{a}}$. We choose the second coordinate r to be an affine parameter along the geodesics, normalised so that $r \sim \tilde{r}$ as $\tilde{r} \to \infty$. (Note Bai et al (2007) wrote λ instead of r.)

The angular coordinates θ and ϕ are required to be constant along the null geodesic generators with $\theta = \lim_{\tilde{r}\to\infty}\tilde{\theta}$ etc.

Now consider the definition of $l^{\tilde{a}}$,

$$\frac{\partial x^{\tilde{a}}}{\partial r} = l^{\tilde{a}}(x^{\tilde{c}}), \tag{8}$$

where the right hand sides can be expressed as power series in $1/\tilde{r}$. By reversion of series we can obtain asymptotic expansions for $x^{\tilde{a}}(x^c)$,

$$\begin{aligned} \tilde{t} = {} & u + r + 2M\log\left(\frac{r}{2M}\right) + \frac{4M^2}{r} - \frac{8M^3 - Ma^2(2 - 3\sin^2\theta)}{2r^2} - \\ & \frac{16M^4 - M^2a^2(8 - 9\sin^2\theta)}{3r^3} - \\ & \frac{8M^5 - 6M^3a^2\cos^2\theta + \frac{1}{16}Ma^4(8 - 40\sin^2\theta + 35\sin^4\theta)}{r^4} + O\left(\frac{1}{r^5}\right), \\ \tilde{r} = {} & r - \frac{a^2\sin^2\theta}{2r} - \frac{Ma^2\sin^2\theta}{2r^2} + \frac{a^4\sin^2\theta(4 - 5\sin^2\theta)}{8r^3} + \\ & \frac{Ma^4\sin^2\theta(4 - 7\sin^2\theta)}{8r^4} + O\left(\frac{1}{r^5}\right), \\ \tilde{\theta} = {} & \theta - \frac{a^2\cos\theta\sin\theta}{2r^2} + \frac{3a^4\cos\theta\sin\theta(1 - 2\sin^2\theta)}{8r^4} + O\left(\frac{1}{r^5}\right), \\ \tilde{\phi} = {} & \phi - \frac{Ma}{r^2} - \frac{4M^2a}{3r^3} - \frac{8M^3a - Ma^3(4 - 5\sin^2\theta)}{4r^4} + O\left(\frac{1}{r^5}\right). \end{aligned} \tag{9}$$

These expressions were generated using the computer algebra package *REDUCE 3.8* (script available on request), and are consistent with (and extend) the results given in Bai et al (2007).

From (9) we can readily obtain an asymptotic expansion of the Jacobian for the transformation $x^{\tilde{a}} \to x^a$, and its inverse. Thus we can re-express the components of the NP tetrad (5)–(7) in the basis defined by the x^a-chart. Before we do so we make two changes. In the original definition (Newman & Unti, 1962), m and n were required to be parallely transported along l, i.e. the NP scalars $\epsilon - \overline{\epsilon}$ and π have to vanish. For the Bai tetrad this is not the case. One can compute that $\epsilon - \overline{\epsilon} = O(r^{-5})$. We may enforce its vanishing by means of a suitable spin. Here we are neglecting terms of order $O(r^{-5})$ and so we have not done this. We also find $\pi = O(r^{-3})$. Following Bai et al (2007) we make a Lorentz transformation, a *null rotation about* l,

$$l \to l, \qquad n \to n + \overline{c}m + c\overline{m} + c\overline{c}l, \qquad m \to m + cl,$$

given by

$$c = -\frac{3iMa\sin\theta}{2^{3/2}r^2} - \frac{Ma^2\cos\theta\sin\theta}{2^{1/2}r^3} + \frac{iMa^3\sin\theta(20-25\sin^2\theta)}{2^{7/2}r^4} + O\left(\frac{1}{r^5}\right),$$

which ensures $\pi = O(r^{-6})$.

Carrying out this transformation we find

$$\begin{aligned}
l_a = & \left(1, 0, O\left(\frac{1}{r^7}\right), 0\right), \\
n_a = & \left(\frac{1}{2} - \frac{M}{r} + \frac{Ma^2(2-3\sin^2\theta)}{2r^3} + O\left(\frac{1}{r^5}\right), 1,\right. \\
& \left. - \frac{3Ma^2\cos\theta\sin\theta}{r^2} + \frac{5Ma^4\cos\theta\sin\theta(4-7\sin^2\theta)}{4r^4} + O\left(\frac{1}{r^5}\right), 0\right) \\
m_a = & \left(- \frac{iMa\sin\theta}{2^{1/2}r^2} - \frac{2^{1/2}Ma^2\cos\theta\sin\theta}{r^3} - \frac{3iMa^3\sin\theta(4-5\sin^2\theta)}{2^{5/2}r^4} + O\left(\frac{1}{r^5}\right),\right. \\
& 0, -\frac{r}{2^{1/2}} + \frac{Ma^2\sin^2\theta}{2^{3/2}r^2} - \frac{5iMa^3\cos\theta\sin^2\theta}{2^{3/2}r^3} - \\
& \frac{3Ma^4\sin^2\theta(6-7\sin^2\theta)}{2^{7/2}r^4} + O\left(\frac{1}{r^5}\right), \\
& \left.\frac{ir\sin\theta}{2^{1/2}} + \frac{iMa^2\sin^3\theta}{2^{3/2}r^2} - \frac{3Ma^4\sin^2\theta(6-7\sin^2\theta)}{2^{7/2}r^4} + O\left(\frac{1}{r^5}\right)\right),
\end{aligned} \tag{10}$$

and

$$
\begin{aligned}
l^a = &\ \big(0,1,0,0\big),\\
n^a = &\ \left(1+O\left(\frac{1}{r^8}\right), -\frac{1}{2}+\frac{M}{r}-\frac{Ma^2(2-3\sin^2\theta)}{2r^3}+\frac{3M^2a^2\sin^2\theta}{r^4}+O\left(\frac{1}{r^5}\right),\right.\\
&\ \left. -\frac{2Ma^2\cos\theta\sin\theta}{r^4}+O\left(\frac{1}{r^7}\right), \frac{Ma}{r^3}+O\left(\frac{1}{r^5}\right)\right),\\
m^a = &\ \left(O\left(\frac{1}{r^8}\right), \frac{3iMa\sin\theta}{2^{1/2}r^2}+\frac{Ma^2\cos\theta\sin\theta}{2^{1/2}r^3}-\frac{5iMa^3\sin\theta(4-5\sin^2\theta)}{2^{5/2}r^4}+O\left(\frac{1}{r^5}\right),\right.\\
&\ \left. \frac{1}{2^{1/2}r}+\frac{Ma^2\sin^2\theta}{2^{3/2}r^4}+O\left(\frac{1}{r^6}\right), -\frac{i}{2^{1/2}r\sin\theta}+\frac{iMa^2\sin\theta}{2^{3/2}r^4}+O\left(\frac{1}{r^5}\right)\right).
\end{aligned}
\tag{11}
$$

The NP scalars for this tetrad will be given in the next section.

3 The GHP formalism

The Kerr space time is of course of Petrov type D, and so two preferred null directions are singled out, and we have aligned the l and n tetrad vectors along them. There is an extension of the NP formalism due to Geroch, Held & Penrose (1973) (GHP) which adds additional structure in such situations, and we review, very briefly, its main features here.

The GHP formalism makes use of of three transformations, of which only two are relevant here. The first, complex conjugation, is of course familiar, e.g., $m^a \to \overline{m}^a$. The second, dash transform, can be defined by

$$(l^a)' = n^a, \quad (n^a)' = l^a, \quad (m^a)' = \overline{m}^a, \quad (\overline{m}^a)' = m^a.$$

This can be used to relabel the NP scalars (connection coefficients) as ρ, σ, κ, τ, ϵ, β and their dash transforms, since

$$\mu = -\rho', \quad \lambda = -\sigma', \quad \nu = -\sigma', \quad \pi = -\tau', \quad \gamma = -\epsilon', \quad \alpha = -\beta'.$$

In addition the Weyl scalars satisfy $\Psi_n' = \Psi_{4-n}$ for $n \in [0,4]$.

Since the directions of l and n are determined as the repeated principal null directions the only remaining gauge freedoms are boosts of l and n and spins of m and $\overline{m}$ defined by

$$l \to \widehat{l} = A^2 l, \quad n \to \widehat{n} = A^{-2} n, \quad m \to \widehat{m} = e^{2i\psi} m, \quad \overline{m} \to \widehat{\overline{m}} = e^{-2i\psi}\overline{m}, \tag{12}$$

where the parameters A and ψ are real. The GHP formalism copes with the tetrad gauge (boost and spin) freedom described by (12) as follows. If η transforms under (12) according to $\eta \to A^{p+q}e^{i(p-q)\psi}\eta$ then η is said to be of *type* (p,q). Alternatively η has *boost weight* $\frac{1}{2}(p+q)$ and *spin weight* $\frac{1}{2}(p-q)$. Clearly the types of l, n, m and $\overline{m}$ are $(1,1)$, $(-1,-1)$, $(1,-1)$ and $(-1,1)$ respectively. All of the Weyl scalars and most of the NP scalars transform homogeneously, and their types are shown in figure 1.

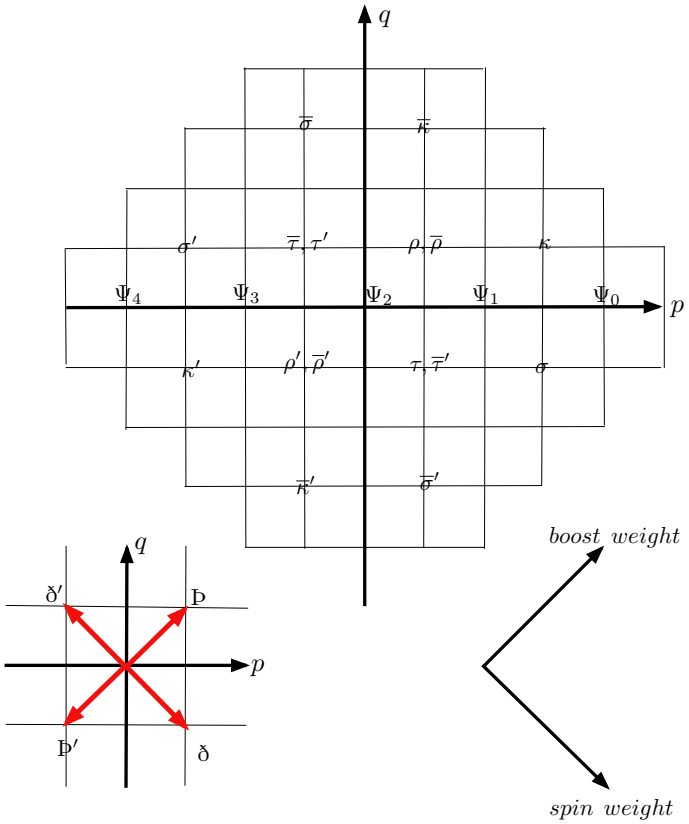

Figure 1: The main part of this figure shows the (p, q) types of those NP scalars which transform homogeneously under boosts and spins . (The squares have side 1.) At bottom left the effects of the GHP derivations $\text{Þ}, \text{Þ}', \text{ð}, \text{ð}'$ are shown. At bottom right the conversion algorithm to boost and spin weight is illustrated.

The NP scalars ϵ, ϵ', β, β' and the NP directional derivatives D, Δ, δ, $\overline{\delta}$ do not transform homogeneously under (12). However they can be composed into *derivations* which do transform homogeneously. Acting on a quantity η of type (p, q) define

$$\begin{aligned} \text{Þ}\eta &= (D - p\epsilon - q\overline{\epsilon})\eta, \qquad & \text{Þ}'\eta &= (\Delta + p\epsilon' + q\overline{\epsilon}')\eta, \\ \text{ð}\eta &= (\delta - p\beta + q\overline{\beta}')\eta, \qquad & \text{ð}'\eta &= (\overline{\delta} + p\beta' - q\overline{\beta})\eta. \end{aligned} \tag{13}$$

The types of these derivations are shown in figure 1. Clearly Þ and Þ′ are boost weight raising and lowering operators, and ð and ð′ are spin weight raising and lowering operators. (In all four cases the increment is 1.)

A routine calculation now gives the non-zero NP scalars for the null tetrad (10), (11)

as

$$
\begin{aligned}
\Phi_{ik} &= 0, \qquad \forall i,k, \\
\Psi_0 &= O\left(\frac{1}{r^5}\right), \\
\Psi_1 &= -\frac{3iMa\sin\theta}{2^{1/2}r^4} + O\left(\frac{1}{r^5}\right), \\
\Psi_2 &= -\frac{M}{r^3} + \frac{3iMa\cos\theta}{r^4} + O\left(\frac{1}{r^5}\right), \\
\Psi_3 &= \frac{3iMa\sin\theta}{2^{3/2}r^4} + O\left(\frac{1}{r^5}\right), \\
\Psi_4 &= O\left(\frac{1}{r^5}\right),
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
\beta &= \frac{\cos\theta}{2^{3/2}r\sin\theta} + \frac{3iMa\sin\theta}{2^{3/2}r^3} + \frac{15Ma^2\cos\theta\sin\theta}{2^{5/2}r^4} + O\left(\frac{1}{r^5}\right), \\
\beta' &= \frac{\cos\theta}{2^{3/2}r\sin\theta} - \frac{Ma^2\cos\theta\sin\theta}{2^{5/2}r^4} + O\left(\frac{1}{r^5}\right), \\
\epsilon &= O\left(\frac{1}{r^5}\right), \\
\epsilon' &= -\frac{M}{2r^2} + \frac{3iMa\cos\theta}{4r^3} + \frac{3Ma^2(2-3\sin^2\theta)}{4r^4} + O\left(\frac{1}{r^5}\right), \\
\rho &= -\frac{1}{r} + O\left(\frac{1}{r^7}\right), \\
\rho' &= \frac{1}{2r} - \frac{M}{r^2} + \frac{3iMa\cos\theta}{2r^3} + \frac{Ma^2(2-3\sin^2\theta)}{r^4} + O\left(\frac{1}{r^5}\right), \\
\sigma &= -\frac{3Ma^2\sin^2\theta}{2r^4} + O\left(\frac{1}{r^5}\right), \\
\sigma' &= \frac{Ma^2\sin^2\theta}{4r^4} + O\left(\frac{1}{r^5}\right), \\
\kappa &= O\left(\frac{1}{r^9}\right), \\
\kappa' &= \frac{3iMa\sin\theta}{2^{5/2}r^3} + \frac{Ma^2\cos\theta\sin\theta}{2^{1/2}r^4} + O\left(\frac{1}{r^5}\right), \\
\tau &= \frac{3iMa\sin\theta}{2^{3/2}r^3} + \frac{4Ma^2\cos\theta\sin\theta}{2^{1/2}r^4} + O\left(\frac{1}{r^5}\right), \\
\tau' &= O\left(\frac{1}{r^6}\right).
\end{aligned}
\tag{15}
$$

4 Gravitational Perturbations via Hertz Potentials

The study of vacuum gravitational perturbations of the Schwarzschild space-time is reasonably mature with many alternative formalisms, usually based on the pioneering work of Regge & Wheeler (1957) and Zerilli (1970). Because they make essential use of the spherical symmetry, these approaches have not yet been extended to perturbations of Kerr space-times.

However an alternative approach, Teukolsky (1973) overcomes this difficulty, and allows some discussion of perturbations of any vacuum Petrov type D space-time, of which the most familiar examples are Minkowski, Schwarzschild and Kerr. For such space-times a NP null tetrad can be chosen so that all of the Weyl scalars except Ψ_2 are zero. Teukolsky examined the Bianchi identities for vacuum perturbations of such space-times. He was able to construct decoupled wave equations for the perturbation of both Ψ_0 and Ψ_4. This is significant because these quantities are scalar fields which vanish in the background. The Stewart-Walker lemma, Stewart & Walker (1974), guarantees that they are invariant under both infinitesimal coordinate transformations and infinitesimal Lorentz transformations of the NP tetrad, and so should be physically meaningful.

Subsequently Chrzanowski (1975) realised that from either of the Teukolsky scalars one could construct a (highly gauge dependent) perturbed metric tensor. Later Wald (1978) explained why this was possible using a very general argument.

Because the work cited above relies heavily on normal mode decomposition which we eschew, we shall reintroduce a very different approach. Consider first the familiar problem of vacuum perturbations of flat Minkowski space-time. One often sets

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} + O(\epsilon^2).$$

Then if one imposes the de Donder gauge condition $\nabla^b h_{ab} = 0$, the perturbed metric satisfies

$$\Box h_{ab} \equiv \eta^{cd}\nabla_c\nabla_d h_{ab} = 0.$$

There are three technical problems with this approach:

- one is looking for h_{ab} which satisfies not only the wave equation but also the de Donder gauge condition,
- the tensor wave equation decomposes into uncoupled wave equations only in a Cartesian chart,
- one needs to impose supplementary gauge conditions in order to identify the two "gravitational degrees of freedom".

Analogous problems arise in the solution of Maxwell's equations for the vacuum electromagnetic field, and of course in this context they have been resolved. The fundamental insight was due to Hertz (1889). For a brief survey of further developments see Stewart (1979).

The first implementation of Hertz potentials for gravitational perturbations of Minkowski space-time appears to be that of Sachs & Bergmann (1958). They considered a "superpotential" H_{abcd} with the Riemann symmetries

$$H_{abcd} = H_{[ab][cd]} = H_{cdab},$$

and set

$$h_{ab} = \nabla^c \nabla^d H_{acbd}.$$

Then h_{ab} is symmetric and satisfies the de Donder gauge condition. Thus if $\Box H_{abcd} = 0$ we have a solution of the linearised Einstein equations on a flat space background. Of course as a potential H_{abcd} has gauge freedom. Suppose that $Q_{abcdef} = Q_{[abc][def]} = Q_{[def][abc]}$ is arbitrary. Then

$$H_{abcd} \to H_{abcd} + \nabla^e \nabla^f Q_{abecdf},$$

generates the same metric perturbation. Using this gauge freedom there are many ways to reduce the linearised Einstein equations to the solution of two decoupled scalar wave equations, thus solving the technical problems enumerated above.

Manipulating valence 4 tensors with Weyl symmetry is always awkward, and Penrose (1965) realised that these calculations could be simplified greatly using the language of 2-component spinors, and he constructed the Hertz potential theory for zero rest mass fields of half-integer spin propagating on a flat space-time background. Stewart (1979) extended this theory to propagation of linearised zero rest mass fields with integer spin on certain curved space-time backgrounds. Here we are concerned with spin 2 fields, and the Penrose theory generalises to vacuum algebraically special space-time backgrounds. Note that the Schwarzschild and Kerr space-times belong to the Petrov type D subclass. These have two repeated principal spinors o^A and ι^A, and the Hertz spinor χ_{ABCD} itself has to be of Petrov type N. We state here the results assuming $\chi_{ABCD} = \chi \iota_A \iota_B \iota_C \iota_D$, where χ is a scalar field of GHP type $(4, 0)$. In Stewart (1979) the results are given in greater detail for the other case $\chi_{ABCD} = \chi o_A o_B o_C o_D$, which is the GHP $'$-transform of this one. The results stated here are more concise than the earlier version because of our judicious use of the commutation relations between the GHP operators.

Suppose χ is a scalar field of GHP type $(4, 0)$ which satisfies the wave equation

$$\mathcal{H}[\chi] \equiv (\text{Þ}\text{Þ}' - \eth\eth' + 3\rho'\text{Þ} - \overline{\rho}\text{Þ}' - 3\tau'\eth + \overline{\tau}'\eth' - 6\Psi_2)\,\chi = 0, \tag{16}$$

where the quantities inside the round brackets are to be evaluated in the NP tetrad adapted to the principal null directions, i.e., $l_a = o_A \overline{o}_{A'}$, $n_a = \iota_A \overline{\iota}_{A'}$ and $m_a = o_A \overline{\iota}_{A'}$ in the background Petrov type D space-time with line element $g^{(B)}_{ab}$. Then

$$g_{ab} = g^{(B)}_{ab} + \widehat{g}_{ab} \tag{17}$$

is a solution of the linearised vacuum Einstein field equations, where

$$\widehat{g}_{ab} = (X + \overline{X}) n_a n_b + Y m_a m_b + \overline{Y} \overline{m}_a \overline{m}_b - 2Z n_{(a} m_{b)} - 2\overline{Z} n_{(a} \overline{m}_{b)}, \tag{18}$$

and

$$X = \eth'\eth'\chi + 2\tau'\eth'\chi, \qquad Y = \text{Þ}'\text{Þ}'\overline{\chi} + 2\overline{\rho}'\text{Þ}'\overline{\chi}, \qquad Z = \text{Þ}'\eth\overline{\chi} + (\tau + \overline{\tau}')\text{Þ}'\overline{\chi} + \overline{\rho}'\eth\overline{\chi}. \tag{19}$$

(Here the gauge choice is such that $\widehat{g}_{ab} n^b = \widehat{g}_a{}^a = 0$.) Of course we can regard a perturbation to the metric tensor as being equivalent to a perturbation to the NP tetrad. The relevant formulae are

$$\widehat{l}_a = \tfrac{1}{2}(X + \overline{X}) n_a - Z m_a - \overline{Z} \overline{m}_a, \qquad \widehat{n}_a = 0, \qquad \widehat{m}_a = -\tfrac{1}{2} \overline{Y} \overline{m}_a, \tag{20}$$

and

$$\widehat{l}^a = -\tfrac{1}{2}(X+\overline{X})n^a, \qquad \widehat{n}^a = 0, \qquad \widehat{m}^a = -\overline{Z}n^a + \tfrac{1}{2}\overline{Y}\,\overline{m}^a. \tag{21}$$

The NP connection coefficients can be expressed in terms of antisymmetrised partial derivatives of the tetrad vectors, Cocke (1989), e.g.,

$$\kappa = 2m^a l^b\, m_{[a,b]},$$

and these equations can be perturbed in a straightforward manner. The resulting perturbed NP connection coefficients are given by

$$\begin{aligned}
\widehat{\kappa} &= \Big[\tfrac{1}{2}\text{Þ}\text{Þ}'\text{ð}' + \tfrac{1}{2}(\overline{\tau}+2\tau')\text{Þ}\text{Þ}' - \tfrac{1}{2}\tau\text{ð}'^2 + \big(\text{Þ}\tau' + \tfrac{1}{2}\overline{\rho}(\overline{\tau}-\tau')\big)\,\text{Þ}' - \\
&\quad (\tau\tau' - \tfrac{1}{2}\Psi_2)\text{ð}'\Big]\chi + \Big[-\tfrac{1}{2}\text{ð}^3 - \tfrac{1}{2}(\tau+\overline{\tau}')\text{ð}^2 - \tau\tau'\text{ð}\Big]\overline{\chi}, && (22)\\
\widehat{\kappa}' &= 0, && (23)\\
\widehat{\sigma} &= \big[\tfrac{1}{2}\text{Þ}\text{Þ}'^2 + \tfrac{1}{2}(\rho-\overline{\rho})\text{Þ}'^2 + \rho'\text{Þ}\text{Þ}' + (\overline{\tau}'-\tau)\text{Þ}'\text{ð}' + \\
&\quad (\text{ð}\tau' + \overline{\tau}\overline{\tau}' - \tau\overline{\tau} - \tau\tau' + \rho\rho' - \Psi_2)\text{Þ}' + \rho'(\overline{\tau}'-\tau)\text{ð}'\big]\chi, && (24)\\
\widehat{\sigma}' &= \big[\tfrac{1}{2}\text{Þ}'^3 + \tfrac{1}{2}(\rho'+\overline{\rho}')\text{Þ}'^2 + \rho'\overline{\rho}'\text{Þ}'\big]\overline{\chi}, && (25)\\
\widehat{\rho} &= \Big[\tfrac{1}{2}\text{Þ}'\text{ð}'^2 + (\overline{\tau}+\tau')\text{Þ}'\text{ð}' - \tfrac{1}{2}\overline{\rho}'\text{ð}'^2 + (\overline{\tau}^2 + \overline{\tau}\tau' + \tau'^2)\text{Þ}' - \overline{\rho}'\tau'\text{ð}'\Big]\chi + \\
&\quad \Big[-\tfrac{1}{2}\text{Þ}'\text{ð}^2 - 2\tau\text{Þ}'\text{ð} - \tfrac{1}{2}\overline{\rho}'\text{ð}^2 - \tau(2\tau+\overline{\tau}')\text{Þ}' - \overline{\rho}'\tau\text{ð}\Big]\overline{\chi}, && (26)\\
\widehat{\rho}' &= 0, && (27)\\
\widehat{\tau} &= \big[\tfrac{1}{2}\text{Þ}'^2\text{ð}' + (\tfrac{1}{2}\overline{\tau}+\tau')\text{Þ}'^2 + (\tfrac{1}{2}\text{Þ}'\overline{\tau} + \rho'(\tau'-\overline{\tau}))\text{Þ}'\big]\chi, && (28)\\
\widehat{\tau}' &= \big[\tfrac{1}{2}\text{Þ}'^2\text{ð} + (\tau+\tfrac{1}{2}\overline{\tau}')\text{Þ}'^2 + \tfrac{1}{2}(\text{Þ}'\tau)\text{Þ}'\big]\overline{\chi}, && (29)\\
\widehat{\beta} &= \big[-\tfrac{1}{2}\beta'\text{Þ}'^2 + (\epsilon' + \tfrac{1}{2}\rho')\text{Þ}'\text{ð}' + \\
&\quad \big((\overline{\tau}+\tau')\epsilon' - \rho'\beta' + \rho'(\tfrac{1}{2}\overline{\tau}+\tau')\big)\,\text{Þ}' + \rho'\epsilon'\text{ð}'\big]\chi, && (30)\\
\widehat{\beta}' &= \big[\tfrac{1}{2}\text{Þ}'^2\text{ð} + (\tau - \tfrac{1}{2}\beta)\text{Þ}^2 + (\tfrac{1}{2}\rho' - \epsilon')\text{Þ}'\text{ð} \\
&\quad + \big(\tfrac{1}{2}\text{Þ}'\tau + \overline{\rho}(\tau - \tfrac{1}{2}\beta) - \epsilon'(\tau+\overline{\tau}')\big)\,\text{Þ}' - \overline{\rho}'\epsilon'\text{ð}\big]\overline{\chi}, && (31)\\
\widehat{\epsilon} &= \big[\tfrac{1}{2}\tau'\text{Þ}'\text{ð}' + \tfrac{1}{2}\epsilon'\text{ð}'^2 + \tau'(\tau' + \tfrac{1}{2}\overline{\tau})\text{Þ}' + \epsilon'\tau'\text{ð}'\big]\chi \\
&\quad + \big[-\tfrac{1}{2}\text{Þ}'\text{ð}^2 - (\tau + \tfrac{1}{2}\overline{\tau}')\text{Þ}'\text{ð} + \tfrac{1}{2}\epsilon'\text{ð}^2 - \tau(\tau+\tfrac{1}{2}\overline{\tau}')\text{Þ}' + \epsilon'\overline{\tau}'\text{ð}\big]\overline{\chi}, && (32)\\
\widehat{\epsilon}' &= 0. && (33)
\end{aligned}$$

The perturbed Weyl tensor components are then given by

$$\begin{aligned}
\widehat{\Psi}_4 &= \tfrac{1}{2}\text{Þ}'^4\overline{\chi},\\
\widehat{\Psi}_3 &= \big[\tfrac{1}{2}\text{Þ}'^3\text{ð} + \tfrac{3}{2}\tau\text{Þ}'^3 + \tfrac{3}{2}(\text{Þ}'\tau)\text{Þ}'^2 + \tfrac{1}{2}(\text{Þ}'^2\tau)\text{Þ}'\big]\overline{\chi},\\
\widehat{\Psi}_2 &= \big[\tfrac{1}{2}\text{Þ}'^2\text{ð}^2 + 2\tau\text{Þ}'^2\text{ð}\overline{\chi} + 3\tau^2\text{Þ}'^2 + (\text{Þ}'\tau)\text{Þ}'\text{ð} + 3\tau(\text{Þ}'\tau)\text{Þ}'\big]\overline{\chi},\\
\widehat{\Psi}_1 &= -\tfrac{3}{2}\Psi_2\big[\text{Þ}'\text{ð}' + (\overline{\tau}+\tau')\text{Þ}' + \rho'\text{ð}'\big]\chi \\
&\qquad \big[\tfrac{1}{2}\text{Þ}'\text{ð}^3 + \tfrac{3}{2}\text{Þ}'\text{ð}^2 + 3\tau^2\text{Þ}'\text{ð} + 3\tau^3\text{Þ}'\big]\overline{\chi},\\
\widehat{\Psi}_0 &= \tfrac{3}{2}\Psi_2\big[-\rho'\text{Þ} + \rho\text{Þ}' + \tau'\text{ð} - \tau\text{ð}' + 2\Psi_2\big]\chi + \tfrac{1}{2}\text{ð}^4\overline{\chi}.
\end{aligned} \tag{34}$$

Although all of these equations are chart independent they do require the NP tetrad to be aligned along the repeated principal null directions of the background.

We now consider solving (16) with the tetrad given by (10), (11). This tetrad has the alignment property only asymptotically, i.e., as $r \to \infty$. However the relative errors we make by ignoring this effect can be estimated as $O(r^{-5})$, and so will be ignored. Note that the individual tetrad components are homogeneous polynomials in r, M and a. (Physically this is obvious since each of r, M and a has the dimensions of length.) Thus the same is true for the NP scalars, as can be verified by inspection of (14) and (15). It follows that the coefficients in the partial differential equation $\mathcal{H}[\chi] = 0$, (16), are homogeneous polynomials in r, M and a.

Consider first the simpler equation $\mathcal{H}_M[\chi] = 0$, obtained by setting $M = a = 0$. This equation governs Hertz potentials on a Minkowski background. We write it as

$$\mathcal{H}_M[\chi] \equiv \mathcal{H}_{MD}[\chi] - \mathcal{H}_{MS}[\chi] = 0, \tag{35}$$

where

$$\mathcal{H}_{MD}[\chi] \equiv \chi_{,ru} - \tfrac{1}{2}\chi_{,rr} + \frac{\chi_{,u} + \chi_{,r}}{r}, \tag{36}$$

and

$$\mathcal{H}_{MS}[\chi] \equiv \frac{1}{2r^2}\left\{\chi_{,\theta\theta} + \frac{1}{\sin^2\theta}\chi_{,\phi\phi} + \frac{\cos\theta}{\sin\theta}\chi_{,\theta} - 4i\frac{\cos\theta}{\sin^2\theta}\chi_{,\phi} - 2(2\cot^2\theta + 1)\chi\right\}. \tag{37}$$

In section 5 we examine the properties of the operator $\mathcal{H}_{MS}[]$. This information is used in the following section 6 to construct the general solution of $\mathcal{H}_M[\chi] = 0$ and the Riemann-Green function for $\mathcal{H}_M[]$.

The equation we actually want to solve, $\mathcal{H}[\chi] = 0$, can be written as

$$\mathcal{H}_M[\chi] = (\mathcal{H}_M - \mathcal{H})[\chi]. \tag{38}$$

Because u, r, M and a all have the dimension of length L we see that all terms on the left hand side have dimension $[\chi]/L^2$, and this must also be true for the terms on the right hand side. These terms are homogeneous polynomials in M, a and r which vanish when $M = a = 0$. It follows that they can be written as asymptotic expansions in M/r and a/r which will converge for $r \gg M > a$, i.e., in a neighbourhood of future null infinity $\mathscr{I}^+$. In such a neighbourhood we should be able to solve (38) by a Picard iteration. Setting $\mathcal{S} = \mathcal{H}_M - \mathcal{H}$ and $\chi = \sum_{k=0}^{\infty} \chi_{(k)}$ we try to solve the sequence

$$\mathcal{H}_M[\chi_{(0)}] = 0, \qquad \mathcal{H}_M[\chi_{(k+1)}] = \mathcal{S}[\chi_{(k)}], \quad k = 0, 1, 2, \ldots, \tag{39}$$

with suitable initial data. Recall that we will know the Riemann-Green function for $\mathcal{H}_M$, so that both equations above can be solved. Clearly we can solve this system for a finite number of terms only, but $\sum_0^N \chi_k$ should furnish a useful asymptotic expansion for χ in the neighbourhood of interest.

We shall implement this procedure after a minor change of coordinates, $(u, r, \theta, \phi) \to (u, v, \theta, \phi)$ where $v = u + 2r$. In Minkowski space-time v is an advanced time coordinate. Using this chart the right hand side of (38), which we shall henceforth refer to as the "source terms", can be written as

$$\mathcal{S}[\chi] = \mathcal{C}_{vv}\chi_{,vv} + \mathcal{C}_{\theta\theta}\chi_{,\theta\theta} + \mathcal{C}_{\phi\phi}\chi_{,\phi\phi} + \mathcal{C}_u\chi_{,u} + \mathcal{C}_v\chi_{,v} + \mathcal{C}_\theta\chi_{,\theta} + \mathcal{C}_\phi\chi_{,\phi} + \mathcal{C}\chi, \tag{40}$$

where

$$\begin{aligned}
\mathcal{C}_{vv} &= -\frac{4M}{r} + \frac{2Ma^2(3\cos^2\theta - 1)}{r^3} + \frac{6M^2a^2(1-\cos^2\theta)}{r^4} + O\left(\frac{1}{r^5}\right), \\
\mathcal{C}_{\theta\theta} &= O\left(\frac{1}{r^5}\right), \quad \mathcal{C}_{\phi\phi} = O\left(\frac{1}{r^5}\right), \quad \mathcal{C}_u = O\left(\frac{1}{r^6}\right), \\
\mathcal{C}_v &= \frac{2M}{r^2} + O\left(\frac{1}{r^5}\right), \quad \mathcal{C}_\theta = -\frac{3iMa\sin\theta}{r^4} + O\left(\frac{1}{r^5}\right), \quad \mathcal{C}_\phi = \frac{4Ma}{r^4} + O\left(\frac{1}{r^5}\right), \\
\mathcal{C} &= -\frac{4M}{r^3} + \frac{6iMa\cos\theta}{r^4} + O\left(\frac{1}{r^5}\right).
\end{aligned} \tag{41}$$

For the moment we shall ignore the "source terms". We reconsider them in section 6.

We turn next to an analysis of the operator $\mathcal{H}_{MS}[\chi]$.

5 Spin weighted spherical harmonics

In this section we are examining the operator $\mathcal{H}_{MS}$ defined by (37) when acting on the quantity χ of spin weight 2. This equation originated from (16) restricted to Minkowski space-time where $\tau = \tau' = 0$ (see (15)), i.e., from the term $-\eth\eth'\chi$. Thus we need to consider the restrictions of the derivations $\eth$ and $\eth'$ to Minkowski space-time. Using the definitions (13) and the scalars (15) we may write these restrictions acting on a quantity $\eta(u, v, \theta, \phi)$ of spin weight s as

$$\eth_M\eta = \frac{1}{\sqrt{2}r}\left(\eta_{,\theta} - s\frac{\cos\theta}{\sin\theta}\eta - i\frac{1}{\sin\theta}\eta_{,\phi}\right), \quad \eth'_M\eta = \frac{1}{\sqrt{2}r}\left(\eta_{,\theta} + s\frac{\cos\theta}{\sin\theta}\eta + i\frac{1}{\sin\theta}\eta_{,\phi}\right), \tag{42}$$

which imply

$$\left(\eth'_M\eth_M - \eth_M\eth'_M\right)\eta = \frac{s}{r^2}\eta. \tag{43}$$

In this section we regard η as a function defined on the unit sphere (with coordinates θ and ϕ) depending on additional parameters u and v. The reader is warned that there is little consistency about these definitions in the literature and some authors have frequently changed their conventions between papers! Including or omitting the r factor is common as is complex conjugation while holding s fixed. Our conventions follow Penrose & Rindler (1984) and Stewart (1990).

Next we define the standard scalar spherical harmonic functions via

$${}_0Y_{lm}(\theta,\phi) = Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\phi}, \tag{44}$$

where $P_l^m(x)$ is an associated Legendre function (Abramowitz & Stegun, 1964), $l = 0, 1, 2, \ldots$, and m is an integer with $|m| \leqslant l$. These functions are orthonormal on the unit sphere

$$\iint {}_0Y_{lm}(\theta,\phi)\,{}_0Y_{LM}(\theta,\phi)\,\sin\theta d\theta d\phi = \delta_{lL}\delta_{mM}, \tag{45}$$

and any smooth function of spin weight 0 defined on the unit sphere can be expressed as a linear combination of the ${}_0Y_{lm}$, the completeness property. Further

$$\eth_M \eth'_M {}_0Y_{lm} = -\frac{l(l+1)}{2r^2}\,{}_0Y_{lm}. \tag{46}$$

If we write this equation as an ordinary differential equation for $P_l^m(x)$ (where $x = \cos\theta$) we see that we have a variant of the hypergeometric equation. It follows (Abramowitz & Stegun, 1964) that the $P_l^m(x)$ satisfy various recurrence relations including

$$\begin{aligned}(l-m+1)P_{l+1}^m(x) =& (2l+1)xP_l^m(x) - (l-m)P_{l-1}^m(x), \\ (1-x^2)(d/dx)P_l^m(x) =& -lxP_l^m(x) + (l+m)P_{l-1}^m(x),\end{aligned} \tag{47}$$

which imply similar relations for the ${}_0Y_{lm}$.

We now define the spin-weighted spherical harmonics ${}_sY_{lm}(\theta,\phi)$ for integer s with $0 \leqslant |s| \leqslant l$ by

$${}_sY_{lm}(\theta,\phi) = \begin{cases} (-1)^s r^s \sqrt{\frac{2^s(l-s)!}{(l+s)!}} (\eth_M)^s\, {}_0Y_{lm}(\theta,\phi), & 0 \leqslant s \leqslant l, \\ r^{-s}\sqrt{\frac{2^{-s}(l+s)!}{(l-s)!}} (\eth'_M)^{-s}\, {}_0Y_{lm}(\theta,\phi), & 0 \geqslant s \geqslant -l. \end{cases} \tag{48}$$

Another way of writing this is

$$\eth_M\, {}_sY_{lm} = \frac{1}{\sqrt{2}r}\sqrt{\frac{l-s}{l+s+1}}\,{}_{s+1}Y_{lm}, \qquad \eth'_M\, {}_sY_{lm} = -\frac{1}{\sqrt{2}r}\sqrt{\frac{l+s}{l-s+1}}\,{}_{s-1}Y_{lm}, \tag{49}$$

which implies the eigenvalue equation

$$\eth_M \eth'_M\, {}_sY_{lm} = -\frac{(l+s)(l-s+1)}{2r^2}\,{}_sY_{lm}. \tag{50}$$

Another form for the spin weighted spherical harmonics is

$${}_sY_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-s)!(l-m)!}{2^{-2s}4\pi(l+s)!(l+m)!}} P_l^{m+s,m-s}(\cos\theta)e^{im\phi}, \tag{51}$$

where $P_l^{m+s,m-s}(x)$ is a generalised associated Legendre function, Virchenko & Fedotova (2001). Again these functions form a complete set of orthonormal functions on the unit sphere.

Unfortunately a full set of the recurrence relations satisfied by the ${}_sY_{lm}$ has not been written down yet. From the work of Kuipers (1959) we have deduced

$$\cos\theta\, {}_sY_{lm}(\theta,\phi) = {}_sA_{lm}\, {}_sY_{l+1\,m}(\theta,\phi) + {}_sB_{lm}\, {}_sY_{lm}(\theta,\phi) + {}_sC_{lm}\, {}_sY_{l-1\,m}(\theta,\phi), \tag{52}$$

where

$${}_sA_{lm} = \sqrt{\frac{(l+s+1)(l-s+1)(l+m+1)(l-m+1)}{(2l+1)(2l+3)(l+1)^2}}, \tag{53}$$

$${}_sB_{lm} = \frac{sm}{l(l+1)}, \qquad {}_sC_{lm} = {}_sA_{l-1m}. \tag{54}$$

We have also found

$$\sin\theta\frac{\partial}{\partial\theta}{}_sY_{lm}(\theta,\phi) = {}_sD_{lm}\,{}_sY_{l+1\,m}(\theta,\phi) + {}_sE_{lm}\,{}_sY_{l\,m}(\theta,\phi) + {}_sF_{lm}\,{}_sY_{l-1\,m}(\theta,\phi), \quad (55)$$

where

$${}_sD_{lm} = l\,{}_sA_{lm}, \qquad {}_sE_{lm} = -{}_sB_{lm}, \qquad {}_sF_{lm} = -(l+1)\,{}_sC_{lm}\,. \quad (56)$$

Note that (52)–(55) are the generalisations of (47) to the general spin case. As far as we know they have not been published before. (For another approach to cross-coupling see Núñez et al (2010).) The derivation of these results will be given elsewhere.

We now return to the study of the operator $H_{MS}[\chi]$ defined by (37). It should be obvious that $H_{MS}[\chi] = \eth_M\eth'_M\chi$ where χ has spin weight 2. We use the completeness property to expand

$$\chi(u,v,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\chi^{lm}(u,v)\,{}_2Y_{lm}(\theta,\phi), \quad (57)$$

so that relation (50) implies

$$\mathcal{H}_{MS}[\chi] = -\frac{1}{2r^2}\sum_{l,m}(l-1)(l+2)\chi^{lm}(u,v)\,{}_2Y_{lm}(\theta,\phi). \quad (58)$$

We turn next to the operator $\mathcal{H}_M[\chi]$.

6 The Euler-Poisson-Darboux equation

6.1 Minkowski space-time

We re-examine the operator $\mathcal{H}_M$ defined by (35), (36) and (37). Making the change of coordinates $(u,r,\theta,\phi) \to (u,v,\theta,\phi)$ where $v = u + 2r$, and the spherical harmonic expansion (57) we have

$$\mathcal{H}_M[\chi] = \sum_{l,m}\left(\chi^{lm}(u,v)_{,uv} + \frac{\chi^{lm}(u,v)_{,u} + 3\chi^{lm}(u,v)_{,v}}{v-u} + \frac{(l-1)(l+2)\chi^{lm}(u,v)}{(v-u)^2}\right){}_2Y_{lm}(\theta,\phi). \quad (59)$$

Suppose we set

$$\chi^{lm}(u,v) = (v-u)^{l+2}\varpi^{l\,m}(u,v). \quad (60)$$

Then (59) becomes

$$\mathcal{H}_M[\chi] = \sum_{l,m}(v-u)^{l+2}\Big(\mathcal{E}^{l\,m}[\varpi^{lm}(u,v)]\Big){}_2Y_{lm}(\theta,\phi), \quad (61)$$

where

$$\mathcal{E}^{l\,m}[\varpi^{l\,m}] = \varpi^{l\,m}_{,uv} + \frac{l+3}{v-u}\varpi^{l\,m}_{,u} - \frac{l-1}{v-u}\varpi^{l\,m}_{,v}. \quad (62)$$

Concentrate first on the solution of $\mathcal{H}_M[\chi] = 0$. Then (61) implies that for all l and m,

$$\mathcal{E}^{l\,m}[\varpi^{l\,m}(u,v)] = 0. \tag{63}$$

These are instances of the *Euler-Poisson-Darboux equation*, Darboux (1899), Stewart (2009). Because the original χ had spin $s = 2$ we must have $l = 2, 3, \ldots$. If l is an integer we can write down the general solution of (63). Let $U(u)$ and $V(v)$ be arbitrary smooth functions of one variable. Then the general solution of (63) is

$$\varpi^{l\,m}(u,v) = \left(\frac{\partial}{\partial u}\right)^{l-2}\left(\frac{\partial}{\partial v}\right)^{l+2}\left(\frac{U(u)+V(v)}{v-u}\right). \tag{64}$$

Next we need to pose an initial value problem, e.g., a *Cauchy problem*. However in Minkowski space-time both u and v are null coordinates. Thus it is marginally simpler to set a *characteristic initial value problem* as follows. Choose initial null hypersurfaces $u = u_0$ and $v = v_0$ where u_0 and v_0 are constants. Specify

$$\varpi^{l\,m}(u,v_0) = F(u), \qquad \varpi^{l\,m}(u_0,v) = G(v), \tag{65}$$

where $F(u_0) = G(v_0)$, and determine the solution in the region $u \geqslant u_0$, $v \geqslant v_0$. Note that because $\varpi^{l\,m} = const.$ is obviously a solution of (63) we may, without loss of generality assume $F(u_0) = G(v_0) = 0$.

We next relate $F(u)$ to $U(u)$ and $G(v)$ to $V(v)$. First assume that $G(v) = 0$. Then

$$F(u) = \varpi^{l\,m}(u,v_0) = (-1)^l\left(\frac{\partial}{\partial u}\right)^{l-2}\frac{U(u)}{((l+2)!(v_0-u)^{l+3}}, \tag{66}$$

and we can obtain $U(u)$ from $F(u)$ by quadratures. By next setting $F(u) = 0$ we can in a similar fashion find

$$G(v) = \varpi^{l\,m}(u_0,v) = \left(\frac{\partial}{\partial v}\right)^{l+2}\frac{V(v)}{((l-2)!(v-u_0)^{l-1}}, \tag{67}$$

and we then obtain $V(v)$ from $G(v)$ by quadratures. It should be clear that F specifies the "outgoing wave" while G specifies the "incoming wave", and the general case is the linear superposition of the two..

Consider first the case of purely outgoing waves $G(v) = 0$, so that after some harmless rescaling

$$\varpi^{l\,m}(u,v) = \left(\frac{\partial}{\partial u}\right)^{l-2}\frac{U(u)}{(v-u)^{l+3}}. \tag{68}$$

The case $l = 2$ is special, for the solution to the initial value problem (63), (65) requires

$$U(u) = (v_0-u)^5 F(u).$$

Then if the data function F has compact support, so does U, and we have "sharp propagation". This is not true if $l > 2$. Consider e.g., the case $l = 3$, $F(u) = \delta(u-u_1)$ where $u_1 > u_0$, for which

$$\varpi^{2\,m}(u,v) = \left(\frac{u-v_0}{u-v}\right)^6\delta(u-u_1) - 6(v-v_0)\frac{(u-v_0)^5}{(u-v)^7}H(u-u_1),$$

where $H(x)$ is the Heaviside step function. For $v > v_0$ the support of $\varpi^{l\,m}(u,v)$ involves values of u with $u > u_1$. This effect in Minkowski space-time will be called the "geometric tail".

Thus the propagation of solutions of a characteristic initial value problem for the Hertz equation on a Minkowski background is extremely simple. We use the initial data to determine the functions $U(u)$ and $V(v)$ in the general solution (64), and the solution is fixed. Although geometric tails can occur there is no mixing of incoming and outgoing modes.

6.2 Kerr space-time

We saw earlier that, at least in the neighbourhood of future null infinity $\mathscr{I}$, we can reduce the problem of solving the Hertz equation on a Kerr background to solving a countable sequence of problems of the form (39). Some care is needed in specifying initial data. As in the Minkowski case u is a retarded time coordinate, but this is no longer the case for v, although it does hold asymptotically as $r \to \infty$ holding u fixed. This means that the limit $v \to \infty$ holding u fixed is at future null infinity $\mathscr{I}$, but the limit $u \to -\infty$ holding v fixed is *not* past null infinity. Nevertheless the local operator $\mathcal{H}_{\mathcal{M}}$ regards u and v as characteristic coordinates. It is therefore appropriate to posit an *asymptotic characteristic initial value problem*, where we specify χ on the null hypersurface $u = u_0$ and the asymptotically null hypersurface $v = v_0$, see figure 2. We have discussed the solution of the first problem (39) in section 6.1, and here we show how to solve the second problem

$$\mathcal{H}[\chi_{(1)}] = \mathcal{S}[\chi_{(0)}], \tag{69}$$

with zero data on the initial surfaces $u = u_0$ and $v = v_0$, for known $\chi_{(0)}(u,v)$.

We have already discussed the operator on the left hand side of (69) in section 6.1. There we wrote $\chi_{(0)}$ as an expansion in terms of spin weighted spherical harmonics with coefficients $\chi^{l\,m}_{(0)}$ which we denote by $\chi_{(0)}(u,v,\theta,\phi) \leftrightarrow \{\chi^{l\,m}_{(0)}(u,v)\}$. See e.g., (57). We then chose to use new coefficients $\varpi^{l\,m}_{(0)}$ via $\chi^{l\,m}_{(0)}(u,v) = (v-u)^{l+2}\varpi^{l\,m}_{(0)}(u,v)$ so that $\chi_{(0)} \leftrightarrow \{\varpi^{l\,m}_{(0)}\}$. Now $\mathcal{S}[\chi_{(0)}]$ was defined by (40). Suppose we construct its spin-weighted spherical harmonic decomposition

$$\mathcal{S}[\chi_{(0)}] = \sum_{l,m} \mathcal{S}^{l\,m}[\{\chi^{l\,m}_{(0)}\}]_2 Y_{l\,m}. \tag{70}$$

With these decompositions (69) becomes

$$\mathcal{E}^{l\,m}[\varpi^{l\,m}_{(1)}(u,v)] = \text{source}^{l\,m}(u,v), \tag{71}$$

where

$$\text{source}^{l\,m}(u,v) = (v-u)^{-(l+2)}\mathcal{S}^{l\,m}[\{(v-u)^{l+2}\varpi^{l\,m}_{(0)}(u,v)\}], \tag{72}$$

a sequence of inhomogeneous Euler-Poisson-Darboux equations with known sources.

Without further restrictions solutions of (71) are not unique. For if $\varpi^{l\,m}_{(1)}(u,v)$ is a solution then so is $\varpi^{l\,m}_{(1)}(u,v) + \varpi^{l\,m}_{(H)}(u,v)$, where $\varpi^{l\,m}_{(H)}(u,v)$ is any solution of the homogeneous equation (63). We can make the solution unique by imposing data on the initial surfaces $u = u_0$ and $v = v_0$. Taking into account the underlying iteration scheme the

most obvious choice is to require $\varpi^{l\,m}_{(1)}(u,v)$ to vanish on these initial surfaces, but the choice is arbitrary. This imposition of data makes the solution of (71) unique, and it is most elegantly expressed in terms of the Riemann-Green function, as we now explain.

The left hand side of (71) is an Euler-Poisson-Darboux operator, Darboux (1899), Stewart (2009). The Riemann-Green function is

$$R(u',v';u,v) = \frac{(v'-u')^{2l+2}}{(v'-u)^{l-1}(v-u')^{l+3}}\,{}_2F_1(l-1,l+3;1;z), \tag{73}$$

where

$$z = \frac{(v'-v)(u'-u)}{(v'-u)(u'-v)}, \tag{74}$$

and ${}_2F_1$ is the hypergeometric function. For integer l this is a rational function, e.g.,

$${}_2F_1(l-1,l+3;1;z) = \begin{cases} (1-z)^{-5} & l=2, \\ (1-z)^{-7}(1+5z) & l=3, \\ (1-z)^{-9}(1+12z+15z^2) & l=4. \end{cases} \tag{75}$$

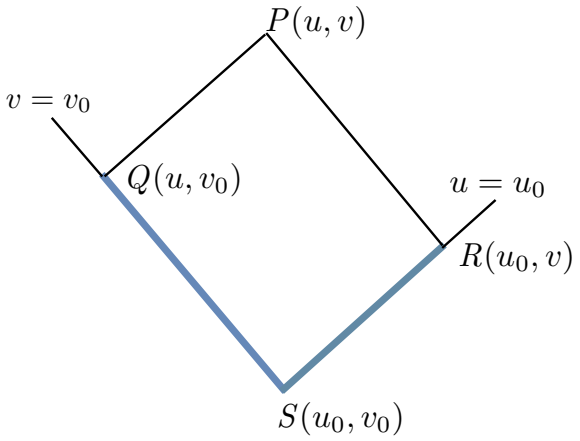

Figure 2: The characteristic initial value problem. The initial surfaces $u=u_0$ and $v=v_0$ are shown. P with coordinates (u,v) is a typical field point to the future of the initial surfaces. The line of constant u through P meets $v=v_0$ at Q. The line of constant v through P meets $u=u_0$ at R. In order to determine the solution at P of the homogeneous problem we need to specify initial data along SQ and SR. In order to determine the solution at P of the inhomogeneous problem we need to specify initial data along SQ and SR and perform an integral involving the source terms over the rectangle $SRPQ$. For simplicity, our inhomogeneous problems have zero data on SQ and SR.

The solution of (71) with zero (asymptotically) characteristic initial data is, see fig. 2,

$$\begin{aligned} \varpi^{l\,m}_{(1)}(u,v) &= \iint\limits_{PQRS} R(u',v';u,v)\text{source}^{l\,m}(u',v')\,du'dv' \\ &= \int_{u_0}^{u} du' \int_{v_0}^{v} dv'\, R(u',v';u,v)\text{source}^{l\,m}(u',v'), \end{aligned} \tag{76}$$

where source$^{l\,m}$ was defined in (72). (For non-trivial initial data we need to add on the right hand side line integrals along SQ and SR, see Stewart (2009).) It will be convenient to denote the indefinite integral implied by the right hand side of (76) by rhs$(u', v'; u, v)$, a function of (u', v') and (u, v), so that

$$\begin{aligned}\varpi^{l\,m}_{(1)}(u, v) &= \text{rhs}(u, v; u, v) - \text{rhs}(u, v_0; u, v) - \text{rhs}(u_0, v; u, v) + \text{rhs}(u_0, v_0; u, v)\\ &= \text{rhs}(P) - \text{rhs}(Q) - \text{rhs}(R) + \text{rhs}(S),\end{aligned} \tag{77}$$

for short.

We now show how, at least for the simplest cases, we can compute $\varpi^{l\,m}_{(1)}(u, v)$ in closed form.

6.3 Outgoing solutions

For simplicity of presentation we shall consider in detail the $l = 2$, $m = 0$ flat space-time outgoing solution

$$\varpi^{2\,0}_{(0)}(u, v) = U^{2\,0}(u)/(v - u)^5, \tag{78}$$

but the techniques we present can handle any choice of l and m. We start by substituting (78) in the right hand side of (71) obtaining, at the point (u', v'),

$$\text{source}(u', v') = \left[c_{21}(v' - u')^{-8} + c_{22}(v' - u')^{-9}\right] U^{20}(u'), \tag{79}$$

where

$$c_{21} = -56M, \qquad c_{22} = Ma(96i\cos\theta - 48i\sin\theta d_\theta + 64d_\phi). \tag{80}$$

Here the c_{21} term measures effects common to both Schwarzschild and Kerr backgrounds while the c_{22} one includes angular momentum effects. The symbols d_θ and d_ϕ in c_{22} mean that ${}_2Y_{lm}(\theta, \phi)$ is to be replaced by the partial derivatives $({}_2Y_{lm})_{,\theta}$ and $({}_2Y_{lm})_{,\phi}$. Because of (52) and (55), the first two c_{22} terms generate terms for neighbouring values of l, which we shall investigate later. We note also that for $l = 2$,

$$R(u', v'; u, v) = \frac{(v' - u)^4(v' - u')}{(v - u)^5}. \tag{81}$$

Now the indefinite integral in (76), (77) becomes

$$\text{rhs}(u', v'; u, v) = (v - u)^{-5}\left[c_{21}I_{47000} + c_{22}I_{48000}\right], \tag{82}$$

where

$$I_{mnpqr} = \int du'\, U^{20}(u')(u - u')^p(v - u')^q \int dv'\, (v' - u)^m(v - v')^r(v' - u')^{-n}. \tag{83}$$

(The terms with powers p, q and r do not occur for $l = 2$ but are needed for other values of l.)

It turns out that we can carry out the v'-integration explicitly. Let us define

$$J_{mnpqr} = (v' - u)^m(v - v')^r \int du'\, U^{20}(u')(u - u')^p(v - u')^q(v' - u')^{-n}. \tag{84}$$

Note that integration by parts of (83) gives

$$I_{mnpqr} = \left(-J_{m\,n-1\,pqr} + mI_{m-1\,n-1\,pqr} - rI_{m\,m-1\,pq\,r-1}\right)/(n-1). \tag{85}$$

In all cases of interest $n > m + r$ and so repeated application of (85) converts I-integrals to J-integrals. In particular

$$\begin{aligned}\text{rhs} = -(v-u)^{-5}\Big[&c_{21}\left(\tfrac{1}{30}J_{02000} + \tfrac{1}{15}J_{13000} + \tfrac{1}{10}J_{24000} + \tfrac{2}{15}J_{35000} + \tfrac{1}{6}J_{46000}\right) + \\ &c_{22}\left(\tfrac{1}{105}J_{03000} + \tfrac{1}{35}J_{14000} + \tfrac{2}{35}J_{25000} + \tfrac{2}{21}J_{36000} + \tfrac{1}{7}J_{47000}\right)\Big]\end{aligned} \tag{86}$$

If $l > 2$ the expression above will include instances of J_{mnpqr} with $p > 0$ or $q > 0$ or both. By repeated application of the identities

$$J_{mnpqr} = -J_{m+1\,n\,p-1\,qr} + J_{m\,n-1\,p-1\,qr} \qquad \text{for } p > 0,$$

$$J_{mnpqr} = J_{mnp\,q-1\,r+1} + J_{m\,n-1\,p\,q-1\,r} \qquad \text{for } q > 0,$$

we may assume that all terms in the expression analogous to (86) have $p = q = 0$.

In order to make further progress we need to specify $U^{20}(u')$. Various choices are possible, but we have chosen to examine a Fourier mode of the initial data with wavelength λ,

$$U^{20}(u') = e^{iku'}, \tag{87}$$

where $k = 1/\lambda$, which implies

$$J_{mn00r} = (v'-u)^m(v-v')^r(ik)^{n-1}e^{ikv'}\Gamma(1-n, ik(v'-u')), \tag{88}$$

where $\Gamma(a,z)$ is the *Incomplete Gamma function*, Abramowitz & Stegun (1964). Eventually we shall need to impose $v_0 \leqslant v' \leqslant v$, where $v_0 \gg M, a$. Also we are interested in the wave-zone where $v_0 \gg \lambda$. Setting $z = ik(v'-u')$ we see that we are interested in the asymptotic limit $|z| \to \infty$ with $\arg(z) = \frac{1}{2}\pi$. In this limit (Abramowitz & Stegun, 1964),

$$\Gamma(a,z) \sim z^{a-1}e^{-z}\left(1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \cdots\right). \tag{89}$$

These formulae allow us to estimate asymptotic expansions for rhs, after which $\varpi^{l\,m}_{(1)}(u,v)$ is found from (77).

We examine the first order correction to $\chi^{2\,0}_{(0)}(u,v) = (v-u)^4\varpi^{2\,0}_{(0)}(u,v) = e^{iku}/(v-u)$. The contribution from the point P is

$$\chi^{2\,0}_{(1)} = \frac{e^{iku}\left(\frac{1}{2}ic_{21}\lambda(v-u)^{-2} + (\frac{1}{3}ic_{22}\lambda - \frac{7}{3}c_{21}\lambda^2)(v-u)^{-3} + O\left((v-u)^{-4}\right)\right)}{(v-u)}, \tag{90}$$

where c_{21} and c_{22} were defined in (80). This is a solution of the inhomogeneous equation, a mixed mode, neither pure incoming nor pure outgoing. The leading term is of order $O(M/r)O(\lambda/r)$ smaller than $\chi^{2\,0}_{(0)}(u,v)$. The "Kerr contribution" appears first in the second term which is of order $O(Ma/r^2)O(\lambda/r)$ smaller than $\chi^{2\,0}_{(0)}(u,v)$. The contribution from the point Q is also given by (90) provided we set $v \to v_0$ in the numerator. Since we can write it in the form $\tilde{U}(u)/(v-u)$ it is a solution of the homogeneous EPD equation

(63), and represents a pure outgoing mode, again two orders of magnitude smaller than $\chi^{2\,0}_{(0)}(u,v)$. The leading term in the contribution from the point R is

$$\chi^{2\,0}_{(1)} = \frac{e^{iku_0}\left(\frac{1}{2}i\lambda c_{21}v^{-2} - \frac{1}{3}[((4u-7u_0)i\lambda + 7\lambda^2)c_{21} + i\lambda c_{22}]v^{-3} + O(v^{-4})\right)}{(v-u)}\,. \quad (91)$$

This again is a solution of the homogeneous equation (63). If we temporarily ignore the $O(v^{-4})$ term we see that it can be written in the form (95) where

$$V(v) = e^{iku_0}\left[\frac{c_{21}}{30}\left(\frac{i\lambda}{(v-u_0)^2} - \frac{2\lambda^2}{(v-u_0)^2} - \frac{6i\lambda^3}{(v-u_0)^4} + \frac{24\lambda^4}{(v-u_0)^5}\right) + \frac{c_{22}}{105}\left(\frac{i\lambda}{(v-u_0)^3} - \frac{3\lambda^2}{(v-u_0)^4} - \frac{12i\lambda^3}{(v-u_0)^5} + \frac{60\lambda^4}{(v-u_0)^6}\right)\right]. \quad (92)$$

Thus it is a pure incoming mode, again two orders of magnitude smaller than $\chi^{2\,0}_{(0)}(u,v)$. The exponential dependence suggests that it is propagating information about the initial data. Finally the leading term in the contribution from the point S is given by setting $v \to v_0$ in the numerator of (91). This is of the form $U(u)/(v-u)$ and so is a pure outgoing wave. It is clear that the contribution (90) from P is the only inhomogeneous one. The other three terms are a solution of the homogeneous equation required so that the total $\chi^{2\,0}_{(1)}$ satisfies the initial conditions on $u = u_0$ and $v = v_0$.

Next we briefly summarise the results for purely outgoing $l = 3$ data. Assuming for simplicity a Fourier mode we set

$$\chi^{3\,0}_{(0)}(u,v) = A_3 e^{iku}\left(\frac{1}{v-u} + \frac{6}{ik(v-u)^2}\right). \quad (93)$$

In the wave zone the second term is $O(\lambda/v)$ smaller than the leading one which differs from $\chi^{2\,0}_{(0)}(u,v)$ by the amplitude A_3. The contribution to $\chi^{3\,0}_{(1)}(u,v)$ from the point P is

$$\chi^{3\,0}_{(1)} = A_3 e^{iku}\left(\frac{\frac{1}{2}ic_{21}\lambda}{(v-u)^3} + \frac{\frac{1}{3}ic_{22}\lambda + \frac{44}{7}c_{21}\lambda^2}{(v-u)^4} + O\left((v-u)^{-5}\right)\right), \quad (94)$$

which bears a striking similarity to (90). The contributions to $\chi^{3\,0}_{(1)}$ from the points Q, R and S behave similarly.

Next we describe the interaction between different "Kerr modes", and for the sake of generality we assume $m = 1$, and small values of l. It is straightforward to show that $\chi^{2\,1}_{(1)}$ contains a d_θ term $-16i\lambda e^{iku}(v-u)^{-4}\sin\theta(\partial/\partial\theta)\ {}_2Y_{2\,1}(\theta,\phi)$, which originates from the c_{22} term in (90). But from (55) we have

$$\sin\theta\,\frac{\partial}{\partial\theta}\ {}_2Y_{2\,1}(\theta,\phi) = \tfrac{4\sqrt{2}}{3\sqrt{7}}\ {}_2Y_{3\,1}(\theta,\phi) - \tfrac{1}{3}\ {}_2Y_{2\,1}(\theta,\phi).$$

This means that the d_θ term in $\chi^{2\,1}_{(1)}$ has to be replaced by a term in $\chi^{2\,1}_{(1)}$ and a term in $\chi^{3\,1}_{(1)}$. The d_ϕ term in $\chi^{2\,1}_{(1)}$ is easier; one replaces d_ϕ by the factor $im = i$. Consider finally the c_{22} term in $\chi^{3\,1}_{(1)}$ which is proportional to $A_3\cos\theta\ {}_2Y_{3\,1}$. From (52) we know

$$\cos\theta\ {}_2Y_{3\,1}(\theta,\phi) = \tfrac{\sqrt{5}}{2\sqrt{7}}\ {}_2Y_{4\,1}(\theta,\phi) + \tfrac{1}{6}\ {}_2Y_{3\,1}(\theta,\phi) + \tfrac{2\sqrt{2}}{3\sqrt{7}}\ {}_2Y_{2\,1}(\theta,\phi),$$

and so this term has to be reapportioned between three different modes. In principle this process continues indefinitely involving arbitrarily high values of l. In concrete problems the $l = 2$ mode is likely to dominate with $|A_4| \ll |A_3| \ll 1$, and so truncation can be a good approximation.

6.4 Incoming solutions

We need also to examine pure flat space-time incoming modes with

$$\begin{aligned}\chi^{2\,0}_{(0)} &= (v-u)^4 \left(\frac{\partial}{\partial v}\right)^4 \frac{V(v)}{(v-u)} \\ &= (v-u)^3 V^{(4)} - 4(v-u)^2 V^{(3)} + 12(v-u)V^{(2)} - 24V' + \frac{24V}{(v-u)}.\end{aligned} \tag{95}$$

(If we were to require the usual peeling conditions to hold for the perturbed Weyl curvature, in particular the requirement $\Psi_0 = O(v^{-5})$ as $v \to \infty$, then we would need to require $V^{(n)} = O(v^{-n})$ for $0 \leqslant n \leqslant 4$. We could satisfy these conditions by requiring,

$$V(v) = V_{20} + V_{21}/v + V_{22}/v^2 + V_{23}/v^3 + V_{24}/v^4 + \cdots, \tag{96}$$

for example.) Then just as for the outgoing modes studied in the previous subsection we can compute the first order correction. All of the integrands are rational functions of u' and v' and so the quadratures can be performed analytically. The leading order terms in the contribution from the point P are

$$\chi^{2\,0}_{(1)} \sim -\frac{4c_{21}V_{20}(v-u)^{-1} + (\frac{12}{7}c_{22}V_{20} - \frac{480}{7}c_{21}V_{21}u(1-u/v)^2)(v-u)^{-2}}{(v-u)}, \tag{97}$$

where c_{21} and c_{22} were defined by (80). This is a solution of the inhomogeneous equation and hence a mixed mode. The contribution from the point Q is obtained by letting $v \to v_0$ in the numerator of (97). It is a solution of the homogeneous equation and represents a pure outgoing mode. The leading order terms in the contribution from the point R are

$$\chi^{2\,0}_{(1)} \sim -\frac{4c_{21}V_{20}(v^{-1} - (2u-3u_0)v^{-2}) + \frac{12}{7}c_{22}V_{20}v^{-2} - \frac{480}{7}c_{21}V_{21}u_0v^{-2}}{(v-u)}. \tag{98}$$

This too is a solution of the homogeneous equation (63), and represents a pure ingoing mode. Finally we obtain the contribution from the point S by letting $v \to v_0$ in the numerator of (98), which gives a solution of the homogeneous equation, a pure outgoing mode.

The treatment of incoming modes is thus seen to be very similar to that of outgoing modes, and the interaction between modes with different l and m parallels that for the outgoing modes.

7 Boundary conditions for numerical relativity

Buchman & Sarbach (2006) have given an excellent extensive review of the many varied approaches to this problem and so we shall not present this material again. We shall concentrate on simple local boundary conditions.

7.1 Bayliss-Turkel boundary conditions

Notice that the principal part of our equation (16) is, in a neighbourhood of $\mathscr{I}^+$, asymptotically Minkowskian in form. This means that we can use results developed for the study of wave-like equations in Minkowski space-time. Perhaps the simplest local outer boundary conditions for wave-like equations are based on an idea of Bayliss & Turkel (1980) for the scalar wave equation. If we use characteristic coordinates (u, v, θ, ϕ) consider an expansion of the form

$$\psi_{out} = \sum_{m=k}^{\infty} \frac{f_m(u, \theta, \phi)}{(v-u)^m}. \tag{99}$$

We assume this is convergent, or at least asymptotic, in the limit $|v-u| \to \infty$. With a slight extension of the idea of Bayliss and Turkel, we introduce the operator $L_{k,n}[]$ defined by

$$L_{k,n}[\psi] = \left(\frac{\partial}{\partial v} + \frac{2n+k-2}{v-u}\right)\psi, \tag{100}$$

as well as the operator $B_{k,n}[]$ defined recursively by

$$B_{k,1}[\psi] = L_{k,1}[\psi], \qquad B_{k,m}[\psi] = L_{k,m}\left[B_{k,m-1}[\psi]\right], \qquad m = 2, 3, \ldots. \tag{101}$$

Notice that $B_{k,1}[\psi_{out}]$ annihilates the leading term in ψ_{out} for any choice of $f_k(u, \theta, \phi)$, while $B_{k,n}[\psi_{out}]$ removes the first n leading terms, indeed $B_{k,n}[\psi_{out}] = O((v-u)^{-(2n+k)})$.

To see the significance of these operators let us consider outer boundary conditions for the Hertz potential equation (38). We start by applying them to the zero order iterate $\mathcal{H}_M[\chi] = 0$. First consider pure outgoing modes in Minkowski space-time

$$\chi^{l\,m}_{(0)}(u, v) = (v-u)^{2l}\left(\frac{\partial}{\partial u}\right)^{l-2}\frac{U(u)}{(v-u)^{l+3}}. \tag{102}$$

These form a finite sum of the form (99) and so are annihilated by $B_{k,n}[]$ for suitable k and sufficiently large n. Next consider the first order correction to the Hertz scalar $\chi^{l\,m}_{(1)}$. The contributions from the point P for the cases $l = 2$ and $l = 3$ are given by (90) and (94). They too are annihilated by $B_{k,n}[]$ for suitable k and n. Since the contributions from the points Q and S are pure outgoing modes they share the annihilation property. This is not true for the contributions from the point R which are pure incoming modes. So how does $B_{k,n}[]$ affect incoming modes?

The analogue of (99) for incoming modes is

$$\psi_{in} = \sum_{m=k}^{\infty} \frac{f_m(v, \theta, \phi)}{(v-u)^m}, \tag{103}$$

but it is more profitable to study special cases, in particular the pure incoming mode for the EPD equation (95),

$$\begin{aligned}\chi^{2\,0}_{(0)} &= (v-u)^4\left(\frac{\partial}{\partial v}\right)^4\frac{V(v)}{(v-u)} \\ &= (v-u)^3V^{(4)} - 4(v-u)^2V^{(3)} + 12(v-u)V^{(2)} - 24V' + \frac{24V}{(v-u)}.\end{aligned} \tag{104}$$

It is easy to see that $B_{1,1}\chi^{2\,0}_{(0)} = (v-u)^3 V^{(5)}(v)$, so requiring the left hand side to vanish is equivalent to enforcing $V^{(5)}(v) = 0$. For the particular choice of $V(v)$ given by (96), the only solution is a constant $V(v)$.

We see that in Minkowski space-time the boundary condition $B_{1,n}[]$ is transparent to outgoing modes of the EPD equation and absorbs (at least partially) incoming modes of $\mathcal{H}_M[\chi] = 0$.

Next consider the first order iterate $\chi^{2\,0}_{(1)}$ generated from $\chi^{2\,0}_{(0)}$ which takes into account the leading order corrections for a Kerr space-time background. This has to be the solution of an inhomogeneous EPD equation given by (71) and (72). In general a solution will not be unique, for we may always add on an arbitrary solution of the homogeneous equation, and we need to impose initial conditions to fix it. Recall that we chose boundary conditions for $\chi^{2\,0}_{(0)}$ appropriate for an pure outgoing mode in Minkowski space-time, and imposed trivial data for the correction $\chi^{2\,0}_{(1)}$. This is required by the Riemann-Green approach, which delivers a unique solution. Notice the way we have decomposed $\chi^{2\,0}_{(1)}$ into four terms corresponding to the vertices of the characteristic rectangle $PQRS$, see figure 2 and (77). The contribution from the point P is a genuine solution of the inhomogeneous equation, while the contributions from Q, R and S are the homogeneous corrections needed to fit the solution to the initial data.

The contribution from the point P given by (90) is clearly of the form (99) with $k = 1$. Somewhat miraculously it is totally transparent to the boundary condition $B_{1,n}[]$. As pointed out in the last section the contributions from Q and S are also pure outgoing modes of the homogeneous equation, and by the reasoning above they are transparent for the boundary condition. The contribution from the point R is a solution of the homogeneous equation and can be written as a pure incoming mode with $V(v)$ given by (92). This means that it can be very substantially damped by $B_{1,n}[]$.

There are several ways to interpret this result. We could interpret the boundary condition as providing a "transmission coefficient" of $1 - |\chi^{2\,0}_{(1)}(R)|/|\chi^{2\,0}_{(0)} + \chi^{2\,0}_{(1)}| = 1 - O(\lambda M/(v-u)^2)$, or alternatively a "reflection coefficient" of magnitude $O(\lambda M/(v-u)^2)$, consistent with the results of Buchman & Sarbach (2006), (2007).

From another point of view the transmitted solution, i.e., the corrected solution, omitting the contribution to $\chi^{2\,0}_{(1)}$ from the point R, satisfies the inhomogeneous equation, but for different initial conditions. To see this note that the solution of the homogeneous equation $\varpi^{2\,0}_{(0)}$ satisfied (65) with $F(u) = e^{iku}/(v_0-u)^5$ on SQ and $G(v) = 0$ on SR. Equivalently we imposed $\chi^{2\,0}_{(0)} = e^{iku}/(v_0-u)$ on SQ and $\chi^{2\,0}_{(0)} = 0$ on SR. The correction satisfied $\chi^{2\,0}_{(1)} = 0$ on SQ and $\chi^{2\,0}_{(1)} = 0$ on SR. The contribution from the point R satisfies the homogeneous equation for initial data appropriate to the form (95) with $V(v)$ given by (92). The corresponding $G(v)$ can be obtained from (67). Thus we can mimic the effect of deleting the contribution from R by imposing not $G(v) = 0$ on SR but $G(v)$ given by the negative of the value specified in the previous sentence. By linearity we will obtain the transmitted solution of the inhomogeneous equation.

This raises a third point—what are the appropriate initial conditions for the correction terms? In section 6 we chose, for simplicity, pure outgoing data for the Minkowski space-time contribution and trivial data for the correction. We see that this arbitrary choice led to a pure ingoing contribution term in the curved space-time correction. Had we chosen initial data for the correction as in the previous paragraph, there would be no pure

incoming mode. So what are the appropriate initial conditions?

In many problems in relativistic astrophysics we do not know what the initial conditions were. We hope that solutions settle down to a form independent of the actual initial conditions. We can model such situations by retaining only the contribution from the point P to the correction term. For such a procedure the Bayliss-Turkel boundary conditions are transparent to modes which would be pure outgoing im Minkowski space-time.

In practice numerical relativists do not evolve the Hertz potential χ but would prefer to apply outer boundary conditions to one of the Weyl curvature coefficients Ψ_n, because they are Lorenz scalars and, apart from Ψ_2 they are coordinate gauge invariant quantities. These are obtained from χ by the formulae (34). For each part of $\chi^{2,0} = \chi^{2\,0}_{(0)} + \chi^{2\,0}_{(1)}$ the expansions (99) will still be valid, but the parameter k will change away from 1. That is why k is included in the definition of $B_{k,n}[]$. The same applies when we consider modes with $l \geq 2$.

7.2 Bayliss-Turkel boundary conditions for numerical relativity

We have introduced the Bayliss-Turkel boundary conditions (100) and (101) in the (u, v, θ, ϕ) chart. They appear to be independent of M and a, but this is a feature of our chosen chart. We need to transfer the definitions to a numerical relativity chart (T, R, Θ, Φ) via the sequence

$$(u, v, \theta, \phi) \to (u, r, \theta, \phi) \to (\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi}) \to (T, R, \Theta, \Phi),$$

where $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$ is the standard Boyer-Lindquist chart.

The first step is easy. Since $v = u + 2r$

$$L_{k,n}[] = \frac{1}{2}\left[\left(\frac{\partial}{\partial r}\right)_{u,\theta,\phi} + \frac{2n+k-2}{r}\right], \tag{105}$$

and for convenience we drop the factor $\frac{1}{2}$.

For the next step we start from (9). Clearly

$$\begin{aligned}\left(\frac{\partial}{\partial r}\right)_{u,\theta,\phi} &= \left(\frac{\partial \tilde{t}}{\partial r}\right)\frac{\partial}{\partial \tilde{t}} + \left(\frac{\partial \tilde{r}}{\partial r}\right)\frac{\partial}{\partial \tilde{r}} + \left(\frac{\partial \tilde{\theta}}{\partial r}\right)\frac{\partial}{\partial \tilde{\theta}} + \left(\frac{\partial \tilde{\phi}}{\partial r}\right)\frac{\partial}{\partial \tilde{\phi}} \\ &= \left[1 + \frac{2M}{r} - \frac{4M^2}{r^2} + \frac{M(8M^2 - a^2(2-3\sin^2\theta))}{r^3} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \tilde{t}} + \\ &\quad \left[1 + \frac{a^2\sin^2\theta}{2r^2} + \frac{Ma^2\sin^2\theta}{r^3} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \tilde{r}} + \\ &\quad \left[\frac{a^2\cos\theta\sin\theta}{r^3} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \tilde{\theta}} + \left[\frac{2Ma}{r^3} + O\left(\frac{1}{r^4}\right)\right]\frac{\partial}{\partial \tilde{\phi}}\,.\end{aligned} \tag{106}$$

Next we see by inverting (9) that

$$r = \tilde{r} + \frac{a^2\sin^2\tilde{\theta}}{2\tilde{r}} + \frac{Ma^2\sin^2\tilde{\theta}}{2\tilde{r}^2} - \frac{a^4\sin^2\tilde{\theta}(8-7\sin^2\tilde{\theta})}{8\tilde{r}^3} + O\left(\frac{1}{\tilde{r}^4}\right),$$

and

$$\theta = \tilde{\theta} + \frac{a^2 \cos\tilde{\theta}\sin\tilde{\theta}}{2\tilde{r}^2} + O\left(\frac{1}{\tilde{r}^4}\right).$$

Thus

$$\begin{aligned} L_{k,n}[] = & \left[1 + \frac{2M}{\tilde{r}} + \frac{4M^2}{\tilde{r}^2} + \frac{M(8M^2 - 2a^2\cos^2\tilde{\theta})}{\tilde{r}^3} O\left(\frac{1}{\tilde{r}^4}\right)\right]\frac{\partial}{\partial\tilde{t}} + \\ & \left[1 + \frac{a^2\sin^2\tilde{\theta}}{2\tilde{r}^2} + \frac{Ma^2\sin^2\tilde{\theta}}{\tilde{r}^3} + O\left(\frac{1}{\tilde{r}^4}\right)\right]\frac{\partial}{\partial\tilde{r}} + \\ & \left[\frac{a^2\cos\tilde{\theta}\sin\tilde{\theta}}{\tilde{r}^3} + O\left(\frac{1}{\tilde{r}^4}\right)\right]\frac{\partial}{\partial\tilde{\theta}} + \left[\frac{2Ma}{\tilde{r}^3} + O\left(\frac{1}{\tilde{r}^4}\right)\right]\frac{\partial}{\partial\tilde{\phi}} + \\ & (2n + k - 2)\left[\frac{1}{\tilde{r}} - \frac{a^2\sin^2\tilde{\theta}}{2\tilde{r}^3} + O\left(\frac{1}{\tilde{r}^4}\right)\right]. \end{aligned} \tag{107}$$

It is of course impossible to write down algorithms covering all of the possible coordinate systems used by numerical relativists. We have chosen to use a standard chart (T, R, Θ, Φ) in which the metric is asymptotically Minkowskian

$$ds^2 \sim dT^2 - dR^2 - R^2(d\Theta^2 + \sin^2\Theta d\Phi^2), \tag{108}$$

see e.g., the extensive discussion in Deadman & Stewart (2009). A numerical relativist would need to consider the transformation between her/his chart and this one. The line element (108) matches asymptotically the Boyer-Lindquist one if

$$T = \tilde{t}, \quad R = \sqrt{\tilde{r}^2 + a^2\sin^2\tilde{\theta}}, \quad \sin\Theta = \sqrt{\frac{\tilde{r}^2 + a^2}{\tilde{r}^2 + a^2\sin^2\tilde{\theta}}}\sin\tilde{\theta}, \quad \Phi = \tilde{\phi}. \tag{109}$$

Carrying out a similar calculation to that used to derive (107) we obtain

$$\begin{aligned} L_{k,n}[] = & \left[1 + \frac{2M}{R} + \frac{4M^2}{R^2} + \frac{M(8M^2 - a^2(2 - 3\sin^2\Theta))}{R^3} + O\left(\frac{1}{R^4}\right)\right]\frac{\partial}{\partial T} + \\ & \left[1 + \frac{Ma^2\sin^2\Theta}{R^3} + O\left(\frac{1}{R^4}\right)\right]\frac{\partial}{\partial R} + \\ & \left[\frac{0}{R} + \frac{a^2\cos\Theta\sin\Theta}{R^3} + O\left(\frac{1}{R^4}\right)\right]\frac{\partial}{\partial\Theta} + \left[\frac{0}{R} + \frac{2Ma}{R^3} + O\left(\frac{1}{R^4}\right)\right]\frac{\partial}{\partial\Phi} + \\ & (2n + k - 2)\left[\frac{1}{R} + O\left(\frac{1}{R^4}\right)\right]. \end{aligned} \tag{110}$$

Examining (110) we see that to leading order $L_{k,n}[]$ takes on its Minkowskian form, as expected. The next order makes a Schwarzschild correction to the coefficient of $\partial/\partial T$. To second order we have another similar correction to this coefficient. (Indeed if $a = 0$ the coefficient of $\partial/\partial T$ would appear to be $(1 - 2M/R)^{-1}$.) These results are fully consistent with those of Buchman & Sarbach (2006),(2007). However angular momentum effects

creep into the coefficients of the angular derivatives. The second order correction to the coefficient of $\partial/\partial\Theta$ persists even if $M = 0$. It occurs because the asymptotic form of Boyer-Lindquist chart is not the usual Minkowski one in spherical polar coordinates, see (109). The second order correction to the coefficient of $\partial/\partial\Phi$ is the first genuine Kerr term. The third order correction introduces Kerr terms in all coefficients.

We have of course to insert this operator into the recursively defined, see (101),

$$B_{k,1}[\psi] = L_{k,1}[\psi], \qquad B_{k,m}[\psi] = L_{k,m}\left[B_{k,m-1}[\psi]\right], \tag{111}$$

and the Bayliss-Turkel outer boundary condition is to impose

$$B_{k,m}[\psi] = 0, \tag{112}$$

at the outer numerical boundary for suitable choices of k, m and ψ. The choice of of k is determined by the choice of which Lorentz scalar ψ is to be subject to the boundary condition. (Typically ψ is one of the perturbed Weyl scalars Ψ_i.) The choice of n is more problematic, and there are two issues to be considered.

If the numerical relativist wishes to consider pure linearised theory then from a theoretical viewpoint the choice of n is arbitrary. However most numerical relativists will hope to solve the full nonlinear Einstein field equations. Deadman & Stewart (2009) looked at the asymptotic expansion near $\mathscr{I}^+$ of vacuum asymptotically flat solutions of the full field equations and, inter alia, compared them with the results deduced from linearised theory. This suggests that only the first few terms in the asymptotic expansion of a given ψ are given by linearised theory. There is a surprisingly low upper bound, depending on the choice of ψ, for values of n which can be justified theoretically.

The second issue is the practical one of the actual implementation of the condition (112). This is discussed in general terms in e.g., Givoli & Neta (2003), Givoli (2004) and Hagstrom & Warburton (2004). A general relativistic implementation is given in Rinne et al (2009). Choosing a large value for n leads to a surprisingly complicated numerical algorithm.

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