# Some two-dimensional extensions of Bougerol's identity in law for the exponential functional of linear Brownian motion

Jean Bertoin, Daniel Dufresne, Marc Yor

# Some two-dimensional extensions of Bougerol's identity in law for the exponential functional of linear Brownian motion

J. Bertoin* D. Dufresne† M. Yor‡§

January 6, 2012

**Abstract**

We present a two-dimensional extension of an identity in distribution due to Bougerol [Bou] that involves the exponential functional of a linear Brownian motion. Even though this identity does not extend at the level of processes, we point at further striking relations in this direction.

**Key words:** Brownian motion, exponential functional, Bougerol's identity, local time, Bessel processes.

## 1 Introduction

**(1.1)** To a linear Brownian motion $(B_s,\, s \geq 0)$ starting from 0, we associate the exponential functional

$$A_t = \int_0^t ds \exp(2B_s)\,, \qquad t \geq 0.$$

The distribution of $A_t$ is made accessible thanks to Bougerol's identity in law

$$\text{for fixed} \quad t\,, \quad \sinh(B_t) \overset{(\text{law})}{=} \beta(A_t), \tag{1}$$

*Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland. E-mail: jean.bertoin@math.uzh.ch

†Centre of Actuarial Studies, University of Melbourne, 3010 Victoria, Australia. E-mail:dufresne@unimelb.edu.au

‡Laboratoire de Probabilités et Modèles Aléatoires, UPMC, 4 place Jussieu, F-75252 Paris cedex 05, France. E-mail: deaproba@proba.jussieu.fr

§Institut Universitaire de France

where $(\beta(u),\, u \geq 0)$ denotes a Brownian motion which is independent of $(B_s\,, s \geq 0)$, hence of $A_t$. Assuming (1), elementary computations yield the following characterization of the law of $A_t$:

$$\mathbb{E}\left[\frac{1}{\sqrt{A_t}}\exp\left(-\frac{x^2}{2A_t}\right)\right] = \frac{a'(x)}{\sqrt{t}}\exp\left(-\frac{a^2(x)}{2t}\right)\,, \qquad x \in \mathbb{R}\,, \tag{2}$$

where

$$a(x) = \arg\sinh(x) \equiv \log(x + \sqrt{1+x^2}) \quad \text{and} \quad a'(x) = \frac{1}{\sqrt{1+x^2}}\,.$$

For further reference, we note some simple, but useful, consequences of (2), *i.e.*:

$$\mathbb{E}\left(\frac{1}{\sqrt{A_t}}\right) = \frac{1}{\sqrt{t}}\,, \tag{3}$$

and, differentiating both sides with respect to $t$:

$$\mathbb{E}\left(\frac{\exp(B_t)}{A_t^{3/2}}\right) \overset{(*)}{=} \mathbb{E}\left(\frac{\exp(2B_t)}{A_t^{3/2}}\right) = \frac{1}{t^{3/2}}\,, \tag{4}$$

where $(*)$ is obtained by time reversal of $(B_s, s \leq t)$ from time $t$.

**(1.2)** It took some time, despite the original proof in [Bou], to understand simply and deeply why (1) holds. In [ADY], one finds the following arguments, among which the (essential) time reversal one:

$$\text{for fixed } t\,,\ \beta(A_t) \ \text{ is distributed as: } \int_0^t \exp(B_s)\, d\beta(s)\,,$$

which, by time reversal (at time $t$) is also distributed as

$$\exp(B_t)\int_0^t \exp(-B_s)\, d\beta(s)\,. \tag{5}$$

Now, it is easily shown, using Itô's formula, that the process in (5) is distributed as the process $(\sinh(B_t)\,,\, t \geq 0)$.

**(1.3)** In the present paper, we obtain an extension of (1), by considering the two-dimensional vector $(\sinh(B_t), \sinh(L_t))$, where $(L_t\,,\, t \geq 0)$ denotes the local time at 0 of $B$. Our main result is:

**Theorem 1.** *For fixed $t$, the three following two-dimensional random variables are identically distributed:*

$$(\sinh(B_t)\,,\, \sinh(L_t)) \overset{(law)}{=} (\beta(A_t)\,,\, \exp(-B_t)\lambda(A_t)) \overset{(law)}{=} (\exp(-B_t)\beta(A_t)\,,\, \lambda(A_t))\,, \tag{6}$$

*where $(\beta(u), u \geq 0)$ is a one-dimensional Brownian motion, with local time at 0, $(\lambda(u), u \geq 0)$,*

*and $\beta$ is independent from $B$.*

It may be interesting to observe right now that Tanaka's formula shows that the local time at level 0 and time $t$ of the process $(\sinh(B_s), s \geq 0)$ is simply $L_t$, whereas that of the process $(\exp(-B_s)\beta(A_s), s \geq 0)$ can be expressed as $\int_0^t \exp(-B_s)d\lambda(A_s)$. Hence we have also the identity in distribution between two-dimensional processes

$$(\sinh(B_t), L_t)_{t\geq 0} \overset{(\text{law})}{=} \left(\exp(-B_t)\beta(A_t), \int_0^t \exp(-B_s)d\lambda(A_s)\right)_{t\geq 0}. \tag{7}$$

We stress that (6) cannot be extended to the level of processes; see the forthcoming Section 2.2. Hence the two identities in distribution (6) and (7) differ profoundly.

Theorem 1 is proved in Section 3. In Section 2, we discuss a number of consequences and equivalent statements to that of Theorem 1. For instance, the well-known equivalence in law, due to Paul Lévy, between the processes $((\overline{B}_t - B_t, \overline{B}_t), t \geq 0)$ and $((|B_t|, L_t), t \geq 0)$ allows to present a version involving the supremum $\overline{B}_t = \sup_{s\leq t} B_s$ instead of the local time version of Theorem 1.

Apart from this, Section 2 consists in the statements and discussions of four other theorems. Roughly speaking, these theorems were motivated by our desire to understand whether in (6), the two extreme identities hold for processes. This question has now been solved in the negative (see [BY]), but nonetheless there are some rather remarkable identities between jump intensity measures, which are described in Theorems 2-5, and which made us believe for some time in a 2-dimensional process identity extending (6). We let the reader discover the precise statements of these theorems in Section 2; their proofs are found in Section 4.

**(1.4)** Before we get into the precise arguments of the proofs of our various theorems, we should like to present an overall appreciation of the present work, by making a parallel with the way our understanding of Bougerol's identity (1) has improved: in [$Y_b$], a Mellin transform proof was given, based on the identity in law (23) below. Later, in [ADY], a time-reversal argument and stochastic calculus proof of (1) were found. We estimate that, at the moment, our understanding of Theorem 1 lies at the level of [$Y_b$], and that we should be able to develop some kind of understanding similar to [ADY]. However, such a proof eludes us for now; it is not clear that a time-reversal argument may be the missing stone... Nevertheless, we present some further extensions of Bougerol's identity different from the ones found in the volume [$Y_d$], which, hopefully should lead us in the future to a better understanding of Theorem 1.

## 2 Discussion of and some theorems closely related to Theorem 1

**(2.1)** Firstly we note that we may rewrite the identity in law (6) in a seemingly slightly weaker form

$$(\sinh(|B_t|)\,,\,\sinh(L_t)) \overset{(\text{law})}{=} (|\beta|(A_t)\,,\,\exp(-B_t)\lambda(A_t)) \overset{(\text{law})}{=} (\exp(-B_t)|\beta|(A_t)\,,\,\lambda(A_t))\,. \quad (8)$$

This induces no loss of generality, since the left-hand sides (without absolute values) of the expressions in (6) only differ from the ones with absolute values in (8) by multiplication by a symmetric Bernoulli variable, independent of the remaining quantities.

Secondly, it is well-known that the law of the two-dimensional vector $(|B_t|, L_t)$ is symmetric. More precisely, it is given by:

$$\mathbb{P}(|B_t| \in dx\,,\, L_t \in d\ell) = \frac{2(x+\ell)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(x+\ell)^2}{2t}\right) dx\, d\ell\,, \qquad x \geq 0, \ell \geq 0\,, \quad (9)$$

as it can be checked from Lévy's identity (Theorem VI.2.3 in [RY], p.240) and the reflexion principle (Exercise 3.14 in [RY], p.110, or Proposition 2.8.1 in [KS], p. 95). Hence, the common law of (8) is also symmetric.

**(2.2)** We now consider the right-hand sides of the first and third vectors in (6) (or (8)), and we deduce therefrom

$$\sinh(L_t) \overset{(\text{law})}{=} \lambda(A_t)\,. \quad (10)$$

Now, unlike for Bougerol's identity (1), for which the possibility of an identity in law between processes is immediatley ruled out, since the left-hand side of (1) is not a martingale, whereas the right-hand side is, when one considers (10) it seems reasonable to wonder whether this identity might be valid at the level of the two increasing processes involved. However, the recent results in [BY] also rule out this possibility. In fact, it is this uncertainty which prevented us from publishing an earlier version of this paper, as in [DY].

We point out also that for each fixed $t \geq 0$, we deduce from (7) the rather puzzling identity in law

$$\lambda(A_t) \overset{(\text{law})}{=} \sinh\left(\int_0^t \exp(-B_s) d\lambda(A_s)\right)\,,$$

which complements (10).

**(2.3)** Reformulation in terms of Brownian suprema. A celebrated identity in distribution due to Paul Lévy states that

$$((\overline{B}_t - B_t, \overline{B}_t)\,, t \geq 0) \overset{(\text{law})}{=} ((|B_t|\,, L_t)\,, t \geq 0)$$

where $\overline{B}_t = \sup_{s \leq t} B_s$ denotes the supremum of the Brownian trajectory up to time $t$. This enables us to reformulate (8) in the form

$$\begin{aligned}(\sinh(\overline{B}_t - B_t)\,,\, \sinh(\overline{B}_t)) &\overset{(\text{law})}{=} ((\overline{\beta} - \beta)(A_t)\,,\, \exp(-B_t)\overline{\beta}(A_t)) \\ &\overset{(\text{law})}{=} (\exp(-B_t)(\overline{\beta} - \beta)(A_t)\,,\, \overline{\beta}(A_t))\,,\end{aligned}$$

with $\overline{\beta}(t) = \sup_{s \leq t} \beta(s)$. We leave to the interested reader further alternative reformulations of this identity in the same vein.

**(2.4)** A partial interpretation in terms of the Bessel clock.

We now discuss Bougerol's identity (1) in terms of a two-dimensional Bessel process. Specifically, let $(R_h, h \geq 0)$ denote 2-dimensional Bessel process starting from 1, and let

$$H_u = \int_0^u \frac{dh}{R_h^2}\,, \qquad u \geq 0\,,$$

the clock associated with $R$. The well-known skew-product decomposition of planar Brownian motion ([IM], p.270; [M1];[M2]) shows that the clock $H$ can be viewed as the inverse of the exponential Brownian functional $A$; consequently, considering the inverses of the increasing processes involved in (10), we obtain the following:

**Corollary 1.** *Let $(\sigma_t, t \geq 0)$ denote a stable $(1/2)$ subordinator, precisely:*

$$\sigma_t := \inf\{u : \lambda_u \geq t\}\,,\ t \geq 0\,,$$

*independent from $(R_h, h \geq 0)$. Then, for fixed $s$, one has:*

$$H_{\sigma_s} \overset{(law)}{=} \sigma_{a(s)} \tag{11}$$

*where $a(s) \equiv \arg\sinh(s)$.*

It is interesting to compare Corollary 1 with the following consequence of (7):

**Corollary 2.** *There is the identity in law between processes*

$$(\sigma_t)_{t \geq 0} \overset{(law)}{=} \left(H_{\sigma_{\eta(t)}}\right)_{t \geq 0}$$

*where $\sigma$ is as in Corollary 1 and $\eta : [0, \infty) \to [0, \infty)$ is the inverse bijection of the continuous strictly increasing process*

$$s \mapsto \int_0^s \frac{du}{R_{\sigma_u}}\,.$$

**Proof:** Indeed, (7) yields in terms of the Bessel clock

$$(L_t)_{t\geq 0} \overset{(\mathrm{law})}{=} \left(\int_0^{A_t} \frac{d\lambda(u)}{R_u}\right)_{t\geq 0}.$$

Our statement now follows from the easy fact that the process $\big(\sigma_{\eta(t)}, t \geq 0\big)$ is the right-inverse of the continuous increasing process $s \mapsto \int_0^s \frac{d\lambda(u)}{R_u}$. □

Again, one may wonder whether identity (11) holds at the level of increasing processes, however the results in [BY] rule out this possibility. It is then natural to ask for which functionals $\Phi : \mathcal{C}^{\uparrow} \to \mathbb{R}$ the identity

$$\mathbb{E}(\Phi(H_{\sigma.})) = \mathbb{E}(\Phi(\sigma_{a(.)})) \tag{12}$$

may hold, where $\mathcal{C}^{\uparrow}$ stands for the space of càdlàg increasing paths $\omega : \mathbb{R}_+ \to \mathbb{R}_+$. An element of positive response is provided by the following result.

**Theorem 2.** *Consider a measurable function $\Gamma : \mathbb{R}_+^3 \to \mathbb{R}_+$ with $\Gamma(\cdot, 0, \cdot) = 0$ and define*

$$\Phi(\omega) = \sum_{s\geq 0} \Gamma(\omega_{s-}, \Delta\omega_s, s)\,, \qquad \omega \in \mathcal{C}^{\uparrow}\,,$$

*where $\Delta\omega_s = \omega_s - \omega_{s-}$. Then (12) holds.*

*More precisely, if $\Gamma(x, y, s) = f(x, s)g(y)$ for some measurable nonnegative functions $f$ and $g$ with $g(0) = 0$, then*

$$\mathbb{E}(\Phi(H_{\sigma.})) = \mathbb{E}(\Phi(\sigma_{a(.)})) = C(f)D(g)\,,$$

*with*

$$C(f) = \int_0^{\infty} \frac{d\lambda}{\sqrt{1+\lambda^2}} \mathbb{E}[f(H_{\sigma_\lambda}, \lambda)] \;=\; \int_0^{\infty} \frac{d\lambda}{\sqrt{1+\lambda^2}} \mathbb{E}[f(\sigma_{a(\lambda)}, \lambda)]$$

*and*

$$D(g) = \int_0^{\infty} \frac{dt}{\sqrt{2\pi t^3}} g(t)\,.$$

We observe that Corollary 2 implies that the range of subordinated clock $H_\sigma$ has the same distribution as the range of subordinator $\sigma$ (and hence is a regenerative set). In particular we see that (12) holds whenever for a generic increasing path $\omega$, $\Phi(\omega)$ only depends on the range of $\omega$. This provides a quick check of the identity $\mathbb{E}(\Phi(H_{\sigma.})) = \mathbb{E}(\Phi(\sigma_{a(.)}))$ in the special case when the function $\Gamma$ does not depend on the time parameter, i.e. $\Gamma(x, y, s) = \Gamma(x, y)$.

**(2.5)** An amplification and a variant of Theorem 2.

**Theorem 3.** *Let $a, b \geq 0$ and $\Gamma : \mathbb{R}_+^2 \to \mathbb{R}_+$ a measurable function with $\Gamma(\cdot, 0) = 0$. Introduce*

$$\mathcal{H}^{a,b}(\Gamma)(\ell) = \mathbb{E}\left[\sum_{\lambda\leq\ell} (R_{\sigma_{\lambda-}})^a \frac{\Gamma(H_{\sigma_{\lambda-}}, H_{\sigma_\lambda} - H_{\sigma_{\lambda-}})}{R_{\sigma_\lambda}^b}\right].$$

*The following formula holds for* $\Gamma = f \otimes g$*:*

$$\mathcal{H}^{a,b}(f \otimes g)(\ell) = h^-_{a-b}(f,\ell) h^+_b(g) \ ,$$

*where:*

$$\begin{aligned} h^-_c(f,\ell) &= \int_0^\infty dt\, f(t) \mathbb{E}\left[ \frac{e^{(c+1)B_t}}{\sqrt{2\pi A_t}} (1 - e^{-\ell^2/2A_t}) \right] \\ h^+_b(g) &= \int_0^\infty \frac{dt\, g(t)}{\sqrt{2\pi}} \mathbb{E}\left[ \frac{e^{(2-b)B_t}}{A_t^{3/2}} \right] . \end{aligned}$$

It may be interesting to point out that these formulas become simpler in the special case when $a = 0$ and $b = 1$. Indeed, one gets using (4) that

$$h^+_1(g) = \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\, g(t)$$

and then, using (3) and (2) that

$$h^-_{-1}(f,\ell) = \int_0^\infty \frac{dt\, f(t)}{\sqrt{2\pi t}} \left( 1 - a'(\ell) \exp\left( -\frac{a^2(\ell)}{2t} \right) \right)$$

where $a$ and $a'$ are defined below (2).

We also stress that the quantities

$$m_{p,q}(t) = \mathbb{E}\left[ \frac{\exp(pB_t)}{A_t^q} \right] \tag{13}$$

arising in Theorem 3, have been studied in [CMY] and [Duf$_b$, Duf$_c$].

**(2.6)** A variant of Theorem 2 which involves the windings of planar Brownian motion.

The following variant of Theorem 2 bears upon a relationship between the continuous winding process of planar Brownian motion, subordinated with $(\sigma_\lambda, \lambda \geq 0)$, and the standard Cauchy process.

**Theorem 4.** *Let* $Z_u = |Z_u| \exp(i\theta_u)$*,* $u \geq 0$*, denote complex valued Brownian motion, starting from:* $1 + i0$*, with* $(\theta_u, u \geq 0)$ *its continuous winding process. Let* $(\sigma_\lambda, \lambda \geq 0)$ *denote the inverse local time process of a linear Brownian motion, so* $\sigma$ *is a stable(1/2) subordinator, which is assumed to be independent from* $(Z_u, u \geq 0)$*. Finally, let* $(C_\alpha, \alpha \geq 0)$ *be a standard Cauchy process.*

*For any measurable* $\Gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ *with* $\Gamma(x, 0) = 0$*, we have*

$$\mathbb{E}\left[\sum_{\lambda \leq \ell} \Gamma(\theta_{\sigma_\lambda -}, \theta_{\sigma_\lambda} - \theta_{\sigma_\lambda -})\right] = \mathbb{E}\left[\sum_{\lambda \leq \ell} \Gamma(C_{a(\lambda)-}, C_{a(\lambda)} - C_{a(\lambda)-})\right]$$

*for all* $\ell$*'s. In particular, for fixed* $\ell \geq 0$*, there is the equality in law:*

$$\theta_{\sigma_\ell} \overset{(law)}{=} C_{a(\ell)}\,. \tag{14}$$

The reader interested in some applications of these identities in law to functionals of the winding process $(\theta_u, u \geq 0)$ may refer to Vakeroudis [V]. In particular, the identity (14) allows to apply D. Williams' pinching method to yield yet another proof of Spitzer's celebrated theorem:

$$\frac{2}{\log t}\,\theta_t \xrightarrow[t\to\infty]{\text{(law)}} C_1\,. \tag{15}$$

**(2.7)** The joint Laplace-Mellin transform of $(H_{\sigma_\lambda}, R_{\sigma_\lambda})$.

We now come back to Theorem 2, or rather we discuss part of its proof, as given in paragraph (3.2) below. A by product of Lemma 1 therein is:

$$\mathbb{E}\left[\frac{1}{R_{\sigma_\lambda}} \,\middle|\, H_{\sigma_\lambda} = h\right] = \frac{1}{\sqrt{1+\lambda^2}}\,, \tag{16}$$

an intriguing identity, which made us suspect for a moment that $R_{\sigma_\lambda}$ and $H_{\sigma_\lambda}$ might be independent. This is not the case, as we discovered by computing the joint Laplace-Mellin transform of $(H_{\sigma_\lambda}, R_{\sigma_\lambda})$:

**Theorem 5.** *The following formulae hold:*

$$\begin{aligned}\mathbb{E}\left[\frac{1}{(R_{\sigma_\lambda})^{2b}}\exp(-\frac{\mu^2}{2}H_{\sigma_\lambda})\right] &= \mathbb{E}^{(\mu)}\left[\frac{1}{(R_{\sigma_\lambda})^{2b+\mu}}\right] \\ &= C_{b,\mu} \times \frac{F\left(\frac{\mu+1}{2} - b, \frac{\mu}{2} + 1 - b, \mu + 1; -1/\lambda^2\right)}{(1+\lambda^2)^{2b-1/2}(\lambda^2)^{\frac{\mu+1}{2}-b}}\,,\end{aligned} \tag{17}$$

*where* $F \equiv {}_2F_1$ *denotes the classical hypergeometric family of functions with three parameters, and* $\mathbb{E}^{(\mu)}$ *refers to the expectation with respect to the probability measure* $\mathbb{P}^{(\mu)}$ *under which* $(R_t, t \geq 0)$ *is a Bessel process with index* $\mu$ *(i.e. of dimension* $2 + 2\mu$*) and started from* $R_0 = 1$*, and*

$$C_{b,\mu} = \frac{\Gamma(b + \frac{\mu}{2} + \frac{1}{2})\Gamma(1 + \frac{\mu}{2} - b)}{\Gamma(\frac{1}{2})\Gamma(1+\mu)}.$$

As a partial check for formula (17), we have made verifications with $b = 1/2$, $b = -\mu/2$ (the

result should be 1), and $b = 0$ (the result is $\exp(-\mu a(\lambda)) = (\lambda + \sqrt{1+\lambda^2})^{-\mu}$). Let us give some details for $b = 1/2$: We note that, for $b = 1/2$, (17) simplifies, as in the numerator

$$F\left(\frac{\mu}{2}, \frac{\mu}{2} + \frac{1}{2}, \mu + 1; -y\right) = \left(\frac{2}{1 + \sqrt{1+y}}\right)^{\mu},$$

and in the denominator

$$(1+\lambda^2)^{1/2}(\lambda^2)^{\mu/2} \equiv \lambda^{\mu}(1+\lambda^2)^{1/2}.$$

Hence, using the fact that $C_{1/2,\mu} = 2^{-\mu}$, formula (17) simplifies to:

$$2^{-\mu} \cdot \frac{2^{\mu}}{\left(1 + \sqrt{1 + \frac{1}{\lambda^2}}\right)^{\mu}} \cdot \frac{1}{\lambda^{\mu}\sqrt{1+\lambda^2}} \equiv \frac{1}{(\sqrt{1+\lambda^2} + \lambda)^{\mu}\sqrt{1+\lambda^2}},$$

which confirms identity (16), since the previous RHS expression equals

$$\frac{1}{\sqrt{1+\lambda^2}}\mathbb{E}\left(\exp\left(-\frac{\mu^2}{2}H_{\sigma_\lambda}\right)\right).$$

# 3 Proof of Theorem 1

**(3.1)** We start by recalling some well-known facts about Brownian motion running up to an independent exponential time, which will be useful for the proof. In this subsection, $S_p$ denotes an exponential random variable with parameter $p > 0$, independent from the Brownian motion $B$. For $t \geq 0$, denote $g_t = \sup\{u < t : B_u = 0\}$ the last zero of $B$ before $t$. It is known that the processes $(B_u, u \leq g_{S_p})$ and $(B_{g_{S_p}+u}, u \leq S_p - g_{S_p})$ are independent. As a consequence, the variables $L_{S_p} (\equiv L_{g_{S_p}})$ and $B_{S_p}$ are independent. Moreover, since $L_t$ and $|B_t|$ have the same law (see (9)), the same applies to $L_{S_p}$ and $|B_{S_p}|$. Their common density is

$$\sqrt{2p}\exp(-\sqrt{2p}u), \qquad u \geq 0$$

(this is because $\mathbb{P}(L_{S_p} \geq \ell) = \mathbb{P}(S_p \geq \tau_\ell) = \mathbb{E}[\exp(-p\tau_\ell)] = \exp(-\ell\sqrt{2p})$, if $\tau_\ell$ is the time $L$. reaches $\ell$). An equivalent way to express this property is

$$\sqrt{2\mathbf{e}}(|\beta(1)|, \lambda(1)) \overset{(\text{law})}{=} (\mathbf{e}, \mathbf{e}'),$$

where on the left $\mathbf{e} \overset{(\text{law})}{=} S_1$ is independent of $\beta$, and on the right the two variables are independent copies of $S_1$.

**(3.2)** Recall the discussion in paragraph (2.1). Our main goal is to show that

$$(\sinh(|B_t|), \sinh(L_t)) \overset{(\text{law})}{=} (\exp(-B_t)\sqrt{A_t}|\beta(1)|, \sqrt{A_t}\lambda(1)). \tag{18}$$

This will be done by computing the joint Mellin transforms on either side, but before doing so we replace $t$ with an exponential time $S_p$ and multiply both sides by $\sqrt{2\mathbf{e}}$, assuming implicitly that $S_p$, $\mathbf{e}$, $B$ and $\beta$ are independent. What will be proved is:

$$\sqrt{2\mathbf{e}}(\sinh(|B_{S_p}|), \sinh(L_{S_p})) \overset{(\text{law})}{=} \sqrt{2\mathbf{e}}(\exp(-B_{S_p})\sqrt{A_{S_p}}|\beta(1)|, \sqrt{A_{S_p}}\lambda(1))\,. \tag{19}$$

From the one-dimensional Bougerol identity (1), the left-hand side of (19) has the same distribution as

$$\sqrt{2\mathbf{e}}(|N|\sqrt{A_{S_p}}, |N'|\sqrt{A'_{S'_p}})\,, \tag{20}$$

where $N$, $N'$ are independent standard normals and $A'$ a copy of $A$ which is also independent of the other quantities. On the right-hand side of (19), use the facts in paragraph (3.1) to obtain

$$\begin{aligned}\sqrt{2\mathbf{e}}(\exp(-B_{S_p})\sqrt{A_{S_p}}|\beta(1)|, \sqrt{A_{S_p}}\lambda(1)) &\overset{(\text{law})}{=} (\exp(-B_{S_p})\sqrt{A_{S_p}}\mathbf{e}\,, \sqrt{A_{S_p}}\mathbf{e}')\\ &\overset{(\text{law})}{=} (\exp(-B_{S_p})\sqrt{A_{S_p}}|N|\sqrt{2\mathbf{e}}\,, \sqrt{A_{S_p}}|N'|\sqrt{2\mathbf{e}'})\,.\end{aligned} \tag{21}$$

Squaring, we are left with calculating the joint Mellin transforms of

$$\mathbf{e}(A_{S_p}, A'_{S'_p}) \quad \text{and} \quad (\exp(-2B_{S_p})A_{S_p}\mathbf{e}\,, A_{S_p}\mathbf{e}')\,, \tag{22}$$

and verifying that they are equal (the $N, N'$ on both sides of (20) and (21) can be cancelled).

The essential ingredient for these Mellin transforms is (see [Y$_a$], paper ♯6, p. 94)

$$A^{(\nu)}_{S_p} \overset{(\text{law})}{=} \frac{\beta_{1,a}}{2\gamma_b}\,, \tag{23}$$

where $A^{(\nu)}$ denotes the exponential functional

$$A^{(\nu)}_t = \int_0^t ds \exp(2B^{(\nu)}_s)\,, \qquad t \geq 0,$$

of a Brownian motion with drift $\nu$, $B^{(\nu)}_s = B_s + \nu s$, $\beta_{u,v}$ is a beta variable with parameters $(u, v)$, $\gamma_b$ is a gamma variable with parameter $b$, and

$$a = a(\nu, p) = \frac{1}{2}(\nu + \sqrt{2p + \nu^2})\,, \qquad b = b(\nu, p) = \frac{1}{2}(-\nu + \sqrt{2p + \nu^2})\,.$$

The Mellin transform of $A^{(\nu)}_{S_p}$ is then

$$\mathbb{E}(A^{(\nu)}_{S_p})^r = 2^{-r}\frac{\Gamma(1+a)\Gamma(1+r)\Gamma(b-r)}{\Gamma(1+a+r)\Gamma(b)}\,.$$

On the one hand, since $a(0,p) = b(0,p) = \sqrt{\frac{p}{2}}$,

$$\begin{aligned}\mathbb{E}[(\mathbf{e}A_{S_p})^c(\mathbf{e}A'_{S'_p})^d] &= 2^{-c-d}\frac{\Gamma(1+c+d)\Gamma(1+a)\Gamma(1+c)\Gamma(b-c)\Gamma(1+a)\Gamma(1+d)\Gamma(b-d)}{\Gamma(1+a+c)\Gamma(1+a+d)\Gamma(b)^2} \\ &= 2^{-c-d}\,\frac{p}{2}\,\frac{\Gamma(1+c+d)\Gamma(1+c)\Gamma(\sqrt{\frac{p}{2}}-c)\Gamma(1+d)\Gamma(\sqrt{\frac{p}{2}}-d)}{\Gamma(1+c+\sqrt{\frac{p}{2}})\Gamma(1+d+\sqrt{\frac{p}{2}})}\,. \end{aligned} \tag{24}$$

On the other hand, recalling the definition of $A_t^{(\nu)}$ and using the Girsanov-Cameron-Martin theorem, we obtain

$$\begin{aligned}\mathbb{E}[(\exp(-2B_{S_p})A_{S_p}\mathbf{e})^c(A_{S_p}\mathbf{e}')^d] &= \Gamma(1+c)\Gamma(1+d)\mathbb{E}[\exp(-2cB_{S_p})(A_{S_p})^{c+d}] \\ &= \Gamma(1+c)\Gamma(1+d)\mathbb{E}[\exp(2c^2S_p)(A_{S_p}^{(-2c)})^{c+d}].\end{aligned}$$

From the elementary identity

$$\mathbb{P}(S_p \in ds\,;\,\exp(\eta S_p)) = \frac{p}{p-\eta}\mathbb{P}(S_{p-\eta}\in ds)\,, \qquad \eta < p\,,$$

we find, letting $q = p - 2c^2$,

$$\begin{aligned}&\mathbb{E}[(\exp(-2B_{S_p})A_{S_p}\mathbf{e})^c(A_{S_p}\mathbf{e}')^d] \\ &= \frac{p}{q}\Gamma(1+c)\Gamma(1+d)\mathbb{E}[(A_{S_q}^{(-2c)})^{c+d}] \\ &= 2^{-c-d}\frac{p}{q}\Gamma(1+c)\Gamma(1+d)\frac{\Gamma(1+a(-2c,q))\Gamma(1+c+d)\Gamma(b(-2c,q)-c-d)}{\Gamma(1+a(-2c,q)+c+d)\Gamma(b(-2c,q))}\,.\end{aligned} \tag{25}$$

Now, $a(-2c,q) = -c+\sqrt{\frac{p}{2}}$, $b(-2c,q) = c+\sqrt{\frac{p}{2}}$, and thus, comparing (24)-(25),

$$\begin{aligned}&\frac{p}{q}\frac{\Gamma(1+a(-2c,q))\Gamma(b(-2c,q)-c-d)}{\Gamma(1+a(-2c,q)+c+d)\Gamma(b(-2c,q))} \\ &= \frac{p}{p-2c^2}\frac{\Gamma(1-c+\sqrt{\frac{p}{2}})\Gamma(-d+\sqrt{\frac{p}{2}})}{\Gamma(1+d+\sqrt{\frac{p}{2}})\Gamma(c+\sqrt{\frac{p}{2}})} \\ &= \frac{p}{p-2c^2}(\sqrt{\tfrac{p}{2}}-c)(\sqrt{\tfrac{p}{2}}+c)\frac{\Gamma(\sqrt{\frac{p}{2}}-c)\Gamma(\sqrt{\frac{p}{2}}-d)}{\Gamma(1+c+\sqrt{\frac{p}{2}})\Gamma(1+d+\sqrt{\frac{p}{2}})} \\ &= \frac{p}{2}\frac{\Gamma(\sqrt{\frac{p}{2}}-c)\Gamma(\sqrt{\frac{p}{2}}-d)}{\Gamma(1+c+\sqrt{\frac{p}{2}})\Gamma(1+d+\sqrt{\frac{p}{2}})}\,.\end{aligned}$$

Note that, along the way, it was necessary to assume $q = p - 2c^2 > 0$, so that $c$ needed to be taken small enough, and likewise for $d$, precisely: $c, d < \sqrt{\frac{p}{2}}$. But, even with these restrictions, we can conclude the proof of the identity in law of the two vectors in (22), thus ending the proof of (18).

**(3.3)** The second identity in Theorem 1 may be proved rather simply, by first noting that

$$(\beta(A_t), \exp(-B_t)\lambda(A_t)) \overset{(\text{law})}{=} (\sqrt{A_t}\beta(1), \exp(-B_t)\sqrt{A_t}\lambda(1)) \tag{26}$$

and then recalling (from the proof of Bougerol's identity in [ADY]) that time reversal

$$(B_t - B_{(t-u)}\,,\, 0 \le u \le t) \overset{(\text{law})}{=} (B_u\,,\, 0 \le u \le t)$$

implies

$$(A_t\,,\, \exp(-2B_t)A_t) \overset{(\text{law})}{=} (\exp(-2B_t)A_t\,,\, A_t)\,.$$

This completes the proof of Theorem 1.

# 4 Proofs of Theorems 2 to 5

**(4.1)** Proof of Theorem 2. (a) A key for the proof of Theorem 2 is the following interesting, and puzzling, identity, as discussed in Subsection 2.7.

**Lemma 1.** *For any measurable function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$, and any $s \ge 0$, we have:*

$$\mathbb{E}\left[\frac{1}{R_{\sigma_s}} f(H_{\sigma_s})\right] = \frac{1}{\sqrt{1+s^2}}\mathbb{E}[f(\sigma_{a(s)})].$$

**Proof:** For all $q, t, \varepsilon > 0$, we have from (11) that

$$\mathbb{E}(\exp(-q\sigma_{a(t)}) - \exp(-q\sigma_{a(t+\varepsilon)})) = \mathbb{E}(\exp(-qH_{\sigma_t}) - \exp(-qH_{\sigma_{t+\varepsilon}}))\,.$$

The LHS can be computed explicitly and we obtain

$$\exp(-a(t)\sqrt{2q}) - \exp(-a(t+\varepsilon)\sqrt{2q})) \sim \frac{\varepsilon}{\sqrt{1+t^2}}\sqrt{2q}\exp(-a(t)\sqrt{2q})\,, \quad \varepsilon \to 0\,.$$

We next turn our attention to the RHS and apply the Markov property. In this direction, it is convenient to introduce a two-dimensional Bessel process $R'$ which is independent of $R$ and write $H'$ for its clock. Likewise, $\sigma'$ refers to an independent subordinator which has the same distribution as $\sigma$. For every $r > 0$, the notation $\mathbb{P}'_r$ refers to the law under which $R'_0 = r$ and $\mathbb{E}'_r$ to the mathematical expectation under $\mathbb{P}'_r$. We point out that the scaling property implies the identities

$$\mathbb{E}'_r\left(1 - \exp(-qH'_{\sigma'_\varepsilon})\right) = \mathbb{E}'_1\left(1 - \exp(-qH'_{r^{-2}\sigma'_\varepsilon})\right) = \mathbb{E}'_1\left(1 - \exp(-qH'_{\sigma'_{\varepsilon/r}})\right).$$

Of course we can also express the RHS as $\mathbb{E}\left(1-\exp(-qH_{\sigma_{\varepsilon/r}})\right)$. Writing $\sigma_{t+\varepsilon}=\sigma_t+\sigma'_\varepsilon$, with $\sigma'_\varepsilon$ independent from $\sigma_t$ and $R$, we get from an application of the Markov property

$$\begin{aligned}\mathbb{E}(\exp(-qH_{\sigma_t})-\exp(-qH_{\sigma_{t+\varepsilon}})) &= \mathbb{E}\left(\exp(-qH_{\sigma_t})\mathbb{E}'_{R_{\sigma_t}}\left(1-\exp(-qH'_{\sigma'_\varepsilon})\right)\right)\\ &= \mathbb{E}\left(\exp(-qH_{\sigma_t})\mathbb{E}'_1\left(1-\exp(-qH'_{\sigma'_{\varepsilon/R_{\sigma_t}}})\right)\right)\\ &= \mathbb{E}\left(\exp(-qH_{\sigma_t})\left(1-\exp\{-a(\varepsilon/R_{\sigma_t})\sqrt{2q}\}\right)\right)\end{aligned}$$

where the third equality stems from (11). Note that when $\varepsilon\to 0+$, the preceding quantity is equivalent to

$$\varepsilon\sqrt{2q}\mathbb{E}\left(\exp(-qH_{\sigma_t})\frac{1}{R_{\sigma_t}}\right).$$

Putting the pieces together, we arrive at

$$\mathbb{E}\left(\exp(-qH_{\sigma_t})\frac{1}{R_{\sigma_t}}\right)=\frac{1}{\sqrt{1+t^2}}\exp(-a(t)\sqrt{2q})=\frac{1}{\sqrt{1+t^2}}\mathbb{E}(\exp(-q\sigma_{a(t)}))$$

for all $t,q>0$, which establishes Lemma 1. □

(b) To end the proof of Theorem 2, we introduce the following notation concerning jump intensity measures:

$$\begin{aligned}\mathcal{H}(\Gamma)(s)&=\mathbb{E}\left[\sum_{\lambda\le s}\Gamma(H_{\sigma_{\lambda^-}},H_{\sigma_\lambda}-H_{\sigma_{\lambda^-}})\right]\\ \text{and}\qquad&\qquad(27)\\ \mathcal{K}(\Gamma)(s)&=\mathbb{E}\left[\sum_{\alpha\le a(s)}\Gamma(\sigma_{\alpha^-},\sigma_\alpha-\sigma_{\alpha^-})\right]\end{aligned}$$

for given $s$, and $\Gamma:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, Borel, such that $\Gamma(x,0)=0$. Then, in order to finish the proof of Theorem 2, it suffices to take: $\Gamma=f\otimes g$ and to show the following:

$$\begin{aligned}\mathcal{H}(f\otimes g)(s) &= h(f)(s)\int_0^\infty\frac{dt}{\sqrt{2\pi t^3}}g(t)\\ \mathcal{K}(f\otimes g)(s) &= k(f)(s)\int_0^\infty\frac{dt}{\sqrt{2\pi t^3}}g(t)\end{aligned}$$

with, furthermore, the quantities $h(f)(s)$ and $k(f)(s)$ being equal, and equal to:

$$\begin{aligned}h(f)(s) &= \int_0^s d\lambda\mathbb{E}\left[\frac{1}{R_{\sigma_\lambda}}f(H_{\sigma_\lambda})\right]=\int_0^s\frac{d\lambda}{\sqrt{1+\lambda^2}}\mathbb{E}[f(H_{\sigma_\lambda})]\\ \| &\\ k(f)(s) &= \int_0^{a(s)}d\alpha\mathbb{E}[f(\sigma_\alpha)]=\int_0^s\frac{d\lambda}{\sqrt{1+\lambda^2}}\mathbb{E}[f(\sigma_{a(\lambda)})]\,.\end{aligned}$$

Now, concerning $\mathcal{K}(\Gamma)(s)$, since the Lévy measure of the subordinator $(\sigma_\alpha, \alpha \geq 0)$ is $\frac{dt}{\sqrt{2\pi t^3}}$, we have:

$$\begin{aligned}\mathcal{K}(\Gamma)(s) &= \mathbb{E}\left[\int_0^{a(s)} d\alpha f(\sigma_{\alpha-})\right] \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} g(t) \\ &= \int_0^s \frac{du}{\sqrt{1+u^2}} \mathbb{E}[f(\sigma_{a(u)})] \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} g(t)\,.\end{aligned}$$

Concerning $\mathcal{H}(\Gamma)(s)$, starting again with the same argument (i.e: the knowledge of the Lévy measure of $(\sigma_u, u \geq 0)$), we obtain:

$$\begin{aligned}\mathcal{H}(\Gamma)(s) &= \mathbb{E}\left[\int_0^s d\lambda \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} f(H_{\sigma_\lambda}) g(H_{\sigma_\lambda + t} - H_{\sigma_\lambda})\right] \\ &= \mathbb{E}\left[\int_0^s d\lambda \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} f(H_{\sigma_\lambda}) \mathbb{E}'_{R_{\sigma_\lambda}}(g(H'_t))\right] \qquad (28)\end{aligned}$$

(from the Markov property for $R$). However, by scaling, we have:

$$\mathbb{E}'_\rho[g(H'_t)] = \mathbb{E}[g(H_{t/\rho^2})] \qquad (29)$$

so that, plugging (29) in (28), we obtain:

$$\begin{aligned}\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \mathbb{E}'_\rho[g(H'_t)] &= \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \mathbb{E}[g(H_{t/\rho^2})] \\ &= \frac{1}{\rho} \int_0^\infty \frac{du}{\sqrt{2\pi u^3}} \mathbb{E}[g(H_u)]\,. \qquad (30)\end{aligned}$$

Note that, since the inverse of $\{u \to H_u\}$ is: $t \to A_t = \int_0^t dv e^{2B_v}$, we have:

$$\int_0^\infty \frac{du}{\sqrt{2\pi u^3}} \mathbb{E}[g(H_u)] = \frac{1}{\sqrt{2\pi}} \mathbb{E}\left[\int_0^\infty dt g(t) \frac{e^{2B_t}}{\sqrt{A_t^3}}\right] = \int_0^\infty dt g(t) \frac{1}{\sqrt{2\pi t^3}}$$

by (4).

Going back to (28), we have obtained:

$$\mathcal{H}(\Gamma)(s) = \mathbb{E}\left[\int_0^s d\lambda \frac{1}{R_{\sigma_\lambda}} f(H_{\sigma_\lambda})\right] \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} g(t) \qquad (31)$$

b) Finally, to obtain the equality between $\mathcal{H}(\Gamma)(s)$ and $\mathcal{K}(\Gamma)(s)$, it remains to show, with the notation in the statement of Theorem 3, that:

$$h(f)(s) = k(f)(s)\,, \text{ for every } f \geq 0, \text{ Borel.}$$

Again, it suffices to prove this for $f_\lambda(a) = e^{-\lambda a}$, for any $\lambda \geq 0$. Now we have:

$$\begin{aligned}
\mathbb{E}[\exp(-\nu H_{\sigma_s})] &= 1 + \mathbb{E}\left[\sum_{\theta \leq s} \left(e^{-\nu H_{\sigma_\lambda}} - e^{-\nu H_{\sigma_{\lambda^-}}}\right)\right] \\
&= 1 + \mathbb{E}\left[\sum_{\theta \leq s} e^{-\nu H_{\sigma_{\lambda^-}}} \left(e^{-\nu(H_{\sigma_\lambda} - H_{\sigma_{\lambda^-}})} - 1\right)\right] \\
&= 1 + \mathcal{H}(f_\nu \otimes g_\nu)(s) \\
(\text{where} \quad &: \quad f_\nu(a) = \exp(-\nu a); g_\nu(b) = (e^{-\nu b} - 1)) \\
&= 1 + h(f_\nu)(s) \left(\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} g_\nu(t)\right).
\end{aligned}$$

On the other hand:

$$\mathbb{E}[\exp(-\nu \sigma_{a(s)})] = 1 + \int_0^{a(s)} d\alpha f_\nu(\alpha) \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} g_\nu(t)\,.$$

Thus, explicitly:

$$\mathbb{E}[\exp(-\nu H_{\sigma_s})] = 1 - (h(f_\nu)(s))\sqrt{2\nu} \tag{32}$$

whereas:

$$\mathbb{E}[\exp(-\nu \sigma_{a(s)})] = 1 - k(f_\nu)(s)\sqrt{2\nu}\,. \tag{33}$$

Since the left hand sides of (32) and (33) are equal, so are the right hand sides, therefore:

$$\forall f \geq 0\,,\ \text{Borel}\,,\ h(f)(s) = k(f)(s)\,. \tag{34}$$

Hence, in complete generality:

$$\mathcal{H}(\Gamma)(s) = \mathcal{K}(\Gamma)(s)\,, \tag{35}$$

which finishes the proof of Theorem 2.

**(4.2)** Proof of Theorem 3. Here are the main steps of this proof, which is quite similar to that of Theorem 2:

1) We first transform

$$\begin{aligned}
\mathcal{H}^{a,b}(\Gamma)(\ell) &= \mathbb{E}\left[\sum_{\lambda \leq \ell} (R_{\sigma_{\lambda_-}})^a f(H_{\sigma_{\lambda_-}}) \frac{g(H_{\sigma_\lambda} - H_{\sigma_{\lambda_-}}) 1_{(\sigma_\lambda > \sigma_{\lambda-})}}{(R_{\sigma_\lambda})^b}\right] \\
&= \mathbb{E}\left[\int_0^\ell d\lambda (R_{\sigma_\lambda})^a f(H_{\sigma_\lambda}) \int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \frac{g(H_{\sigma_{\lambda+t}} - H_{\sigma_\lambda})}{(R_{\sigma_{\lambda+t}})^b}\right] \\
&= \mathbb{E}\left[\int_0^\ell d\lambda (R_{\sigma_\lambda})^a f(H_{\sigma_\lambda}) E_{R_{\sigma_\lambda}}\left[\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}} \frac{g(H_t)}{(R_t)^b}\right]\right].
\end{aligned}$$

We begin by studying

$$\begin{aligned} h^{(+)}(r,g) &= \mathbb{E}_r\left[\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\frac{g(H_t)}{R_t^b}\right] \\ &= \mathbb{E}\left[\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\frac{g(H_{t/r^2})}{(rR_{t/r^2})^b}\right] \text{ (by scaling)} \\ &= \frac{1}{r^b}\mathbb{E}\left[\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\frac{g(H_{t/r^2})}{(R_{t/r^2})^b}\right] \\ &= \frac{1}{r^b}\left(\frac{1}{r}\right)\mathbb{E}\left[\int_0^\infty \frac{du}{\sqrt{2\pi u^3}}\frac{g(H_u)}{(R_u)^b}\right] \\ &\equiv \frac{1}{r^{b+1}}h_b^{(+)}(g)\,. \end{aligned}$$

We then study:

$$\begin{aligned} h_b^{(+)}(g) &= \mathbb{E}\left[\int_0^\infty \frac{du}{\sqrt{2\pi u^3}}\frac{g(H_u)}{(R_u)^b}\right] \\ &= \mathbb{E}\left[\int_0^\infty \frac{dte^{2B_t}}{\sqrt{2\pi A_t^3}}\frac{g(t)}{(\exp(bB_t))}\right] \\ &= \mathbb{E}\left[\int_0^\infty \frac{dte^{(2-b)B_t}}{\sqrt{2\pi A_t^3}}g(t)\right] \\ &= \int_0^\infty \frac{dtg(t)}{\sqrt{2\pi}}\mathbb{E}\left[\frac{e^{(2-b)B_t}}{\sqrt{A_t^3}}\right] \\ &= \int_0^\infty \frac{dtg(t)}{\sqrt{2\pi}}m_{2-b,3/2}(t)\,, \end{aligned}$$

where the quantity $m_{p,q}(t)$ has been defined in (13).

3) Let us come back to

$$\begin{aligned} \mathcal{H}^{a,b}(\Gamma)(\ell) &= \mathbb{E}\left[\int_0^\ell d\lambda (R_{\sigma_\lambda})^{a-b-1}f(H_{\sigma_\lambda})\right]h_b^{(+)}(g) \\ &= h_{a-b}^{(-)}(f,\ell)h_b^{(+)}(g), \text{ for } F = f\otimes g\,. \end{aligned}$$

Thus, our next aim is to study:

$$h_c^{(-)}(f,\ell) := \mathbb{E}\left[\int_0^\ell d\lambda (R_{\sigma_\lambda})^{c-1}f(H_{\sigma_\lambda})\right].$$

We can re-express this quantity as

$$\begin{aligned}
h_c^{(-)}(f,\ell) &= \mathbb{E}\left[\int_0^{\tau_\ell} dL_u (R_u)^{c-1} f(H_u)\right] \\
&= \int_0^\infty \frac{du}{\sqrt{2\pi u}} \mathbb{P}(L_u < \ell | B_u = 0)\mathbb{E}[(R_u)^{c-1} f(H_u)] \\
&= \int_0^\infty \frac{du}{\sqrt{2\pi u}} \mathbb{P}(\sqrt{u}\sqrt{2\mathbf{e}} < \ell)\mathbb{E}[(R_u)^{c-1} f(H_u)] \\
&= \int_0^\infty \frac{du}{\sqrt{2\pi u}} \mathbb{P}(\mathbf{e} \leq \ell^2/2u)\mathbb{E}[(R_u)^{c-1} f(H_u)] \\
&= \int_0^\infty \frac{du}{\sqrt{2\pi u}} (1 - \exp(-\ell^2/2u))\mathbb{E}[(R_u)^{c-1} f(H_u)] \\
&= \mathbb{E}\left[\int_0^\infty \frac{dA_t}{\sqrt{2\pi A_t}} (1 - e^{-\ell^2/2A_t}) \exp((c-1)B_t) f(t)\right] \\
&= \mathbb{E}\left[\int_0^\infty dt \frac{e^{(c+1)B_t}}{\sqrt{2\pi A_t}} (1 - e^{-\ell^2/2A_t}) f(t)\right] \\
&= \int_0^\infty dt f(t) \mathbb{E}\left[\frac{e^{(c+1)B_t}}{\sqrt{2\pi A_t}} (1 - e^{-\ell^2/2A_t})\right] .
\end{aligned}$$

**(4.3)** Proof of Theorem 4. It is a simple consequence of Theorem 2, once one uses the well-known skew product reresentation of $\theta_t = \gamma_{H_t}$, where $(\gamma_u, u \geq 0)$ is a real-valued Brownian motion independent from $(H_t, t \geq 0)$ (we already gave some references before Corollary 1). Then all one needs to do is to "freeze" $\gamma$ first, then apply Theorem 2, and finally use Spitzer's representation of the Cauchy process as

$$(C_\alpha, \alpha \geq 0) \overset{(\text{law})}{=} (\gamma_{\sigma_\alpha}, \alpha \geq 0).$$

**(4.4)** Proof of Theorem 5. The first equality follows from the (local) absolute continuity relationship between the laws of different Bessel processes, see, *e.g.*, [Y$_a$].

Thus, it remains to prove the second equality. For this purpose, we use the same arguments as in the proof of (16) in [DY]; here are some details.

Let

$$J \overset{\text{def}}{=} \mathbb{E}^{(\mu)}\left[\frac{1}{(R_{\sigma_\lambda})^{2b+\mu}}\right] = \frac{1}{\Gamma(b+\frac{\mu}{2})} \int_0^\infty du\ u^{b+\frac{\mu}{2}-1} \mathbb{E}^{(\mu)}\left[\exp(-uR_{\sigma_\lambda}^2)\right] .$$

There is a classical expression for $\mathbb{E}^{(\mu)}\left[\exp(-uR_t^2)\right]$ (see, e.g. [RY] on page 441), which yields

$$\mathbb{E}^{(\mu)}\left[\exp(-uR_{\sigma_\lambda}^2)\right] = \mathbb{E}\left((1+2u\sigma_\lambda)^{-1-\mu} \exp\left(-\frac{u}{1+2u\sigma_\lambda}\right)\right) .$$

Using Fubini and making the change of variables $v = 2u\sigma_\lambda/(1+2u\sigma_\lambda)$, we obtain the following

expression for the integral

$$I \overset{\text{def}}{=} \int_0^\infty du\ u^{b+\frac{\mu}{2}-1}\,(1+2u\sigma_\lambda)^{-1-\mu}\exp\left(-\frac{u}{1+2u\sigma_\lambda}\right)$$
$$= \frac{1}{(2\sigma_\lambda)^{b+\mu/2}}\int_0^1 dv\ v^{b+\frac{\mu}{2}-1}(1-v)^{\frac{\mu}{2}-b}\exp\left(-\frac{v}{2\sigma_\lambda}\right).$$

Hence, using Fubini again, we obtain:

$$J = \frac{1}{\Gamma(b+\mu/2)}\int_0^1 dv\ v^{b+\frac{\mu}{2}-1}(1-v)^{\frac{\mu}{2}-b}\mathbb{E}\left[\frac{1}{(2\sigma_\lambda)^{b+\mu/2}}\exp\left(-\frac{v}{2\sigma_\lambda}\right)\right]. \tag{36}$$

To compute this last expectation, which we denote by $K$, we use

$$\frac{1}{2\sigma_\lambda} \overset{\text{(law)}}{=} \frac{N^2}{2\lambda^2} \overset{\text{(law)}}{=} \frac{\gamma_{1/2}}{\lambda^2}$$

where $\gamma_{1/2}$ is a standard gamma(1/2)-variable. Then we obtain

$$K = \frac{\Gamma(b+\frac{\mu}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\lambda^{2b+\mu}(1+v\lambda^{-2})^{b+\frac{\mu}{2}+\frac{1}{2}}}.$$

Plugging this in (36), we obtain:

$$J = \frac{\Gamma(b+\frac{\mu}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(b+\mu/2)}\int_0^1 dv\ v^{b+\frac{\mu}{2}-1}\frac{(1-v)^{\frac{\mu}{2}-b}}{\lambda^{2b+\mu}(1+v\lambda^{-2})^{b+\frac{\mu}{2}+\frac{1}{2}}}.$$

To derive the desired formula, we finally use a classical integral representation of ${}_2F_1$, together with

$${}_2F_1\left(\alpha,\beta,\gamma;-z\right) = (1+z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma;-z)\,;$$

see formula (9.5.3) in Lebedev [Leb]. We leave the details to the reader.

# References


[ADY] L. Alili, D. Dufresne, M. Yor. Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement brownien avec drift, p.3–14. In: M. Yor (ed.): Exponential functionals and principal values related to Brownian motion. A collection of research papers, Biblioteca de la Revista Matematica Iberoamericana, Madrid, 1997.

[BDY] J. Bertoin, D. Dufresne and M. Yor. In preparation, December 2011.

[BY] J. Bertoin and M. Yor. Retrieving information from subordination. To appear in Festschrift volume for Professor Prokhorov, Springer, 2011.

[Bou] P. Bougerol. Exemples de théorèmes locaux sur les groupes résolubles. Ann. Inst. H. Poincaré 19, pages 369–391, 1983.

[CY] L. Chaumont, M. Yor. Exercises in Probability. A guided tour from Measure Theory to Random Processes, via Conditioning. Cambridge University Press, 2003.

[CMY] C. Donati-Martin, H. Matsumoto, M. Yor. On positive and negative moments of the integral of geometric Brownian motions. Stat. and Prob. Letters 49, pages 45–52, 2000.

[Duf$_a$] D. Dufresne. Bessel processes and Asian options. In: Numerical Methods in Finance, pages 35–57, eds: H. Ben Ameur, M. Breton, Kluwer, 2005.

[Duf$_b$] D. Dufresne. Laguerre series for Asian and other options. Math. Finance 10, No.4, pages 407–428, 2000.

[Duf$_c$] D. Dufresne. The integral of geometric Brownian motion. Adv. in Appl. Probab. 33, No. 1, pages 223–241, 2001.

[DY] D. Dufresne, M. Yor. A two dimensional extension of Bougerol's identity in law for the exponential of Brownian motion. Working paper no.222, Centre for Actuarial Studies, University of Melbourne, 2011.

[GY] H. Geman, M. Yor. Bessel processes, Asian options and perpetuities. Mathematical Finance, vol. 3, No. 4, pages 349–375, 1993.

[KS] I. Karatzas, and S. E. Shreve. Brownian Motion and Stochastic Calculus, 2nd Ed., Springer-Verlag, New York, 1991.

[Leb] N.N. Lebedev. Special Functions and their Applications. Dover, New York (1972).

[IM] K. Itô, H.P. McKean. Diffusions and their Sample Paths, Springer, 1965.

[M1] H.P. McKean. Brownian motion and the general diffusion: Scale & clock. In: Bachelier Congress 2000, eds H. Geman, D. Madan, S. R. Pliska and T. Vorst. Springer Finance, Springer, Berlin, 2002

[M2] H.P. McKean. Stochastic Integrals, Academic Press, 1969.

[RY] D. Revuz, M. Yor. Continuous Martingales and Brownian Motion, 3rd. Ed., Springer-Verlag, Berlin, 1999.

[V] S. Vakeroudis. On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity. Ph.D. thesis of the University Pierre et Marie Curie (April 2011).

[$Y_a$] M. Yor. Exponential Functionals of Brownian Motion and Related Processes. Springer Finance (2001).

[$Y_b$] M. Yor. On some exponential functionals of Brownian motion. Adv. App. Probab. 24, pages 509–531, 1992 (a). Paper ♯2 in [$Y_a$].

[$Y_c$] M. Yor. The laws of exponential functionals of Brownian motion, taken at various random times. C. R. Acad. Sci., Paris, Sér. I 314, pages 951–956, 1992 (b).

[$Y_d$] M. Yor (Ed.) Exponential functionals and principal values related to Brownian motion. A collection of research papers, Biblioteca de la Revista Matematica Iberoamericana, Madrid, 1997.