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| Michel Mollard. Maximal hypercubes in Fibonacci and Lucas cubes. 2012. hal-00657456

HAL Id: hal-00657456

<https://hal.science/hal-00657456v1>

Preprint submitted on 6 Jan 2012

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Maximal hypercubes in Fibonacci and Lucas cubes

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Abstract

5 The Fibonacci cube Γ_n is the subgraph of the hypercube induced by the binary strings that contain no two consecutive 1's. The Lucas cube Λ_n is obtained from Γ_n by removing vertices that start and end with 1. We characterize maximal induced hypercubes in Γ_n and Λ_n and deduce for any $p \leq n$ the number of maximal p -dimensional hypercubes in these graphs.

Key words: hypercubes; cube polynomials; Fibonacci cubes; Lucas cubes;

10 **AMS subject classifications:** 05C31, 05A15, 26C10

1 Introduction

An interconnection topology can be represented by a graph $G = (V, E)$, where V denotes the processors and E the communication links. The *distance* $d_G(u, v)$ between two vertices u, v of a graph G is the length of a shortest path connecting u and v . An *isometric* subgraph H of a graph G is an induced subgraph such that for
15 any vertices u, v of H we have $d_H(u, v) = d_G(u, v)$.

The *hypercube* of dimension n is the graph Q_n whose vertices are the binary strings of length n where two vertices are adjacent if they differ in exactly one coordinate. The *weight* of a vertex, $w(u)$, is the number of 1 in the string u . Notice that
20 the graph distance between two vertices of Q_n is equal to the *Hamming distance* of the strings, the number of coordinates they differ. The hypercube is a popular interconnection network because of its structural properties.

Fibonacci cubes and Lucas cubes were introduced in [4] and [11] as new interconnection networks. They are isometric subgraphs of Q_n and have also recurrent
25 structure.

A *Fibonacci string* of length n is a binary string $b_1 b_2 \dots b_n$ with $b_i b_{i+1} = 0$ for $1 \leq i < n$. The *Fibonacci cube* Γ_n ($n \geq 1$) is the subgraph of Q_n induced by the

*Partially supported by the ANR Project GraTel (Graphs for Telecommunications), ANR-blan-09-blan-0373-01

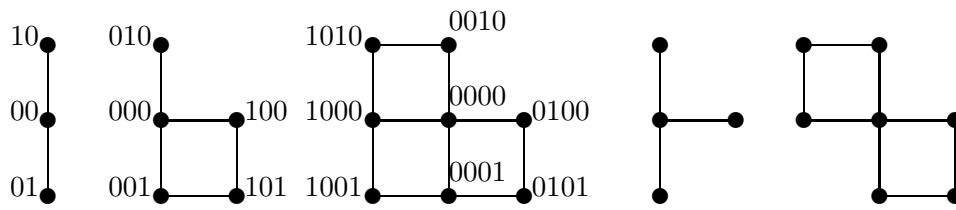


Figure 1: $\Gamma_2 = \Lambda_2$, Γ_3 , Γ_4 and Λ_3 , Λ_4

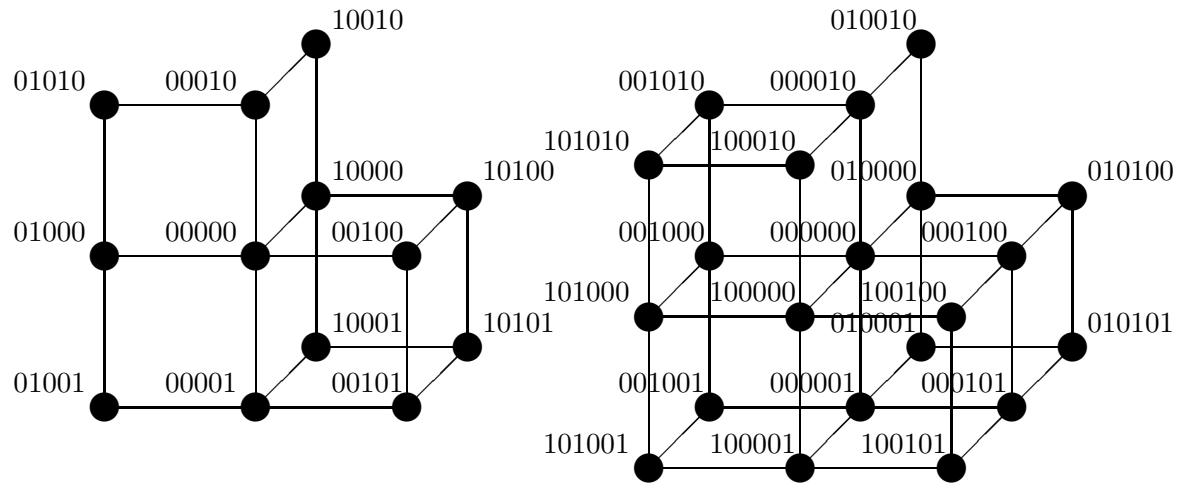


Figure 2: Γ_5 and Γ_6

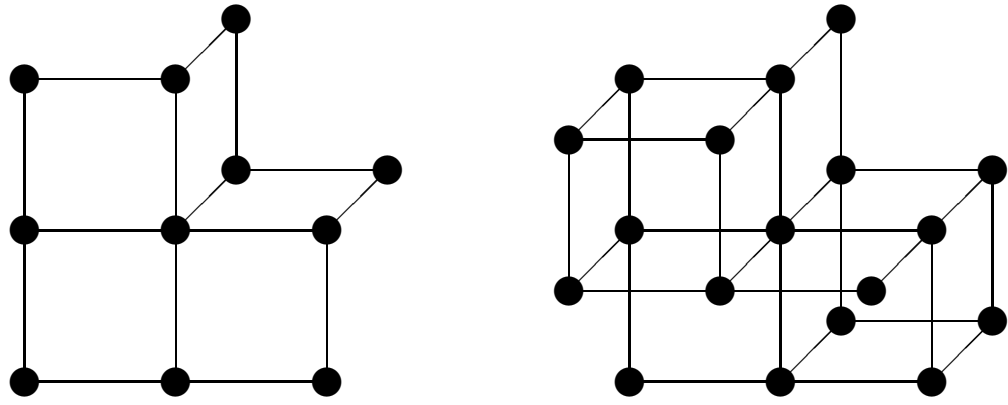


Figure 3: Λ_5 and Λ_6

Fibonacci strings of length n . For convenience we also consider the empty string and set $\Gamma_0 = K_1$. Call a Fibonacci string $b_1b_2 \dots b_n$ a *Lucas string* if $b_1b_n \neq 1$. Then the *Lucas cube* Λ_n ($n \geq 1$) is the subgraph of Q_n induced by the Lucas strings of length n . We also set $\Lambda_0 = K_1$.

Since their introduction Γ_n and Λ_n have been also studied for their graph theory properties and found other applications, for example in chemistry (see the survey [7]). Recently different enumerative sequences of these graphs have been determined. Among them: number of vertices of a given degree[10], number of vertices of a given eccentricity[3], number of pair of vertices at a given distance[8] or number of isometric subgraphs isomorphic so some Q_k [9]. The counting polynomial of this last sequence is known as cubic polynomial and has very nice properties[1].

We propose to study an other enumeration and characterization problem. For a given interconnection topology it is important to characterize maximal hypercubes, for example from the point of view of embeddings. So let us consider *maximal hypercubes of dimension p* , i.e. induced subgraphs H of Γ_n (respectively Λ_n) that are isomorphic to Q_p , and such that there exists no induced subgraph H' of Γ_n (respectively Λ_n), $H \subset H'$, isomorphic to Q_{p+1} .

Let $f_{n,p}$ and $g_{n,p}$ be the numbers of maximal hypercubes of dimension p of Γ_n , respectively Λ_n , and $C'(\Gamma_n, x) = \sum_{p=0}^{\infty} f_{n,p}x^p$, respectively $C'(\Lambda_n, x) = \sum_{p=0}^{\infty} g_{n,p}x^p$, their counting polynomials.

By direct inspection, see figures 1 to 3, we obtain the first of them:

$$\begin{aligned}
C'(\Gamma_0, x) &= 1 & C'(\Lambda_0, x) &= 1 \\
C'(\Gamma_1, x) &= x & C'(\Lambda_1, x) &= 1 \\
C'(\Gamma_2, x) &= 2x & C'(\Lambda_2, x) &= 2x \\
C'(\Gamma_3, x) &= x^2 + x & C'(\Lambda_3, x) &= 3x \\
C'(\Gamma_4, x) &= 3x^2 & C'(\Lambda_4, x) &= 2x^2 \\
C'(\Gamma_5, x) &= x^3 + 3x^2 & C'(\Lambda_5, x) &= 5x^2 \\
C'(\Gamma_6, x) &= 4x^3 + x^2 & C'(\Lambda_6, x) &= 2x^3 + 3x^2
\end{aligned}$$

The intersection graph of maximal hypercubes (also called cube graph) in a graph have been studied by various authors, for example in the context of median graphs[2]. Hypercubes playing a role similar to cliques in clique graph. Nice result have been obtained on cube graph of median graphs, and it is thus of interest, from the graph theory point of view, to characterize maximal hypercubes in families of graphs and thus obtain non trivial examples of such graphs. We will first characterize maximal induced hypercubes in Γ_n and Λ_n and then deduce the number of maximal p -dimensional hypercubes in these graphs.

2 Main results

For any vertex $x = x_1 \dots x_n$ of Q_n and any $i \in \{1, \dots, n\}$ let $x + \epsilon_i$ be the vertex of Q_n defined by $(x + \epsilon_i)_i = 1 - x_i$ and $(x + \epsilon_i)_j = x_j$ for $j \neq i$.

60 Let H be an induced subgraph of Q_n isomorphic to some Q_k . The *support* of H is the subset set of $\{1 \dots n\}$ defined by $Sup(H) = \{i / \exists x, y \in V(H) \text{ with } x_i \neq y_i\}$. Let $i \notin Sup(H)$, we will denote by $H \widetilde{+} \epsilon_i$ the subgraph induced by $V(H) \cup \{x + \epsilon_i / x \in V(H)\}$. Note that $H \widetilde{+} \epsilon_i$ is isomorphic to Q_{k+1} .

The following result is well known[6].

65 **Proposition 2.1** *In every induced subgraph H of Q_n isomorphic to Q_k there exists a unique vertex of minimal weight, the bottom vertex $b(H)$. There exists also a unique vertex of maximal weight, the top vertex $t(H)$. Furthermore $b(H)$ and $t(H)$ are at distance k and characterize H among the subgraphs of Q_n isomorphic to Q_k .*

We can precise this result. A basic property of hypercubes is that if $x, x + \epsilon_i, x + \epsilon_j$ are vertices of H then $x + \epsilon_i + \epsilon_j$ must be a vertex of H . By connectivity we deduce that if $x, x + \epsilon_i$ and y are vertices of H then $y + \epsilon_i$ must be also a vertex of H . We have thus by induction on k :

Proposition 2.2 *if H is an induced subgraph of Q_n isomorphic to Q_k then*

- 75 (i) $|Sup(H)| = k$
(ii) If $i \notin Sup(H)$ then $\forall x \in V(H) \ x_i = b(H)_i = t(H)_i$
(iii) If $i \in Sup(H)$ then $b(H)_i = 0$ and $t(H)_i = 1$
(iv) $V(H) = \{x = x_1 \dots x_n / \forall i \notin Sup(H) \ x_i = b(H)_i\}$.

If H is an induced subgraph of Γ_n , or Λ_n , then, as a set of strings of length n , it defines also an induced subgraph of Q_n ; thus Propositions 2.1 and 2.2 are still true for induced subgraphs of Fibonacci or Lucas cubes.

A Fibonacci string can be view as blocks of 0's separated by isolated 1's, or as isolated 0's possibly separated by isolated 1's. These two points of view give the two following decompositions of the vertices of Γ_n .

85 **Proposition 2.3** *Any vertex of weight w from Γ_n can be uniquely decomposed as $0^{l_0} 10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $p = w$; $\sum_{i=0}^p l_i = n - w$; $l_0, l_p \geq 0$ and $l_1, \dots, l_{p-1} \geq 1$.*

Proposition 2.4 *Any vertex of weight w from Γ_n can be uniquely decomposed as $1^{k_0} 01^{k_1} \dots 01^{k_i} \dots 01^{k_q}$ where $q = n - w$; $\sum_{i=0}^q k_i = w$ and $k_0, \dots, k_q \leq 1$.*

Proof. A vertex from Γ_n , $n \geq 2$ being the concatenation of a string of $V(\Gamma_{n-1})$ with 0 or a string of $V(\Gamma_{n-2})$ with 01, both properties are easily proved by induction on n .

□

Using the the second decomposition, the vertices of weight w from Γ_n are thus obtained by choosing, in $\{0, 1, \dots, q\}$, the w values of i such that $k_i = 1$ in . We have then the classical result:

Proposition 2.5 For any $w \leq n$ the number of vertices of weight w in Γ_n is $\binom{n-w+1}{w}$.

Considering the constraint on the extremities of a Lucas string we obtain the two following decompositions of the vertices of Λ_n .

100 **Proposition 2.6** Any vertex of weight w in Λ_n can be uniquely decomposed as $0^{l_0}10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $p = w$, $\sum_{i=0}^p l_i = n - w$, $l_0, l_p \geq 0$, $l_0 + l_p \geq 1$ and $l_1, \dots, l_{p-1} \geq 1$.

Proposition 2.7 Any vertex of weight w in Λ_n can be uniquely decomposed as $1^{k_0}01^{k_1} \dots 01^{k_i} \dots 01^{k_q}$ where $q = n - w$; $\sum_{i=0}^q k_i = w$; $k_0 + k_q \leq 1$ and $k_0, \dots, k_q \leq 1$.

105 From Propositions 2.2 and 2.3 it is possible to characterize the bottom and top vertices of maximal hypercubes in Γ_n .

Lemma 2.8 If H is a maximal hypercube of dimension p in Γ_n then $b(H) = 0^n$ and $t(H) = 0^{l_0}10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $\sum_{i=0}^p l_i = n - p$; $0 \leq l_0 \leq 1$; $0 \leq l_p \leq 1$ and $1 \leq l_i \leq 2$ for $i = 1, \dots, p - 1$. Furthermore any such vertex is the top vertex of a
110 unique maximal hypercube.

Proof. Let H be a maximal hypercube in Γ_n . Assume there exists an integer i such that $b(H)_i = 1$. Then $i \notin \text{sup}(H)$ by Proposition 2.2. Therefore, for any $x \in V(H)$, $x_i = b(H)_i = 1$ thus $x + \epsilon_i \in V(\Gamma_n)$. Then $H \tilde{+} \epsilon_i$ must be an induced subgraph of Γ_n , a contradiction with H maximal.

115 Consider now $t(H) = 0^{l_0}10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$.

If $l_0 \geq 2$ then for any vertex x of H we have $x_0 = x_1 = 0$, thus $x + \epsilon_0 \in V(\Gamma_n)$. Therefore $H \tilde{+} \epsilon_0$ is an induced subgraph of Γ_n , a contradiction with H maximal. The case $l_p \geq 2$ is similar by symmetry.

120 Assume now $l_i \geq 3$, for some $i \in \{1, \dots, p - 1\}$. Let $j = i + \sum_{k=0}^{i-1} l_k$. We have thus $t(H)_j = 1$ and $t(H)_{j+1} = t(H)_{j+2} = t(H)_{j+3} = 0$. Then for any vertex x of H we have $x_{j+1} = x_{j+2} = x_{j+3} = 0$, thus $x + \epsilon_{j+2} \in V(\Gamma_n)$ and H is not maximal, a contradiction.

125 Conversely consider a vertex $z = 0^{l_0}10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $\sum_{i=0}^p l_i = n - p$; $0 \leq l_0 \leq 1$; $0 \leq l_p \leq 1$ and $1 \leq l_i \leq 2$ for $i = 1, \dots, p - 1$. Then, by Propositions 2.1 and 2.2, $t(H) = z$ and $b(H) = 0^n$ define a unique hypercube H in Q_n isomorphic to Q_p and clearly all vertices of H are Fibonacci strings. Notice that for any $i \notin \text{Sup}(H)$ $z + \epsilon_i$ is not a Fibonacci string thus H is maximal. \square

With the same arguments we obtain for Lucas cube:

130 **Proposition 2.9** If H is a maximal hypercube of dimension $p \geq 1$ in Λ_n then $b(H) = 0^n$ and $t(H) = 0^{l_0}10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $\sum_{i=0}^p l_i = n - p$; $0 \leq l_0 \leq 2$; $0 \leq l_p \leq 2$; $1 \leq l_0 + l_p \leq 2$ and $1 \leq l_i \leq 2$ for $i = 1, \dots, p - 1$. Furthermore any such vertex is the top vertex of a maximal hypercube.

Theorem 2.10 Let $0 \leq p \leq n$ and $f_{n,p}$ be the number of maximal hypercubes of dimension p in Γ_n then:

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$$f_{n,p} = \binom{p+1}{n-2p+1}$$

Proof. This is clearly true for $p = 0$ so assume $p \geq 1$. Since maximal hypercubes of Γ_n are characterized by their top vertex, let us consider the set T of strings which can be write $0^{l_0}10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $\sum_{i=0}^p l_i = n - p$; $0 \leq l_0 \leq 1$; $0 \leq l_p \leq 1$ and $1 \leq l_i \leq 2$ for $i = 1, \dots, p-1$. Let $l'_i = l_i - 1$ for $i = 1, \dots, p-1$; $l'_0 = l_0$; $l'_p = l_p$. We have thus a 1 to 1 mapping between T and the set of strings $D = \{0^{l'_0}10^{l'_1} \dots 10^{l'_i} \dots 10^{l'_p}\}$ where $\sum_{i=0}^p l'_i = n - 2p + 1$ any $l'_i \leq 1$ for $i = 0, \dots, p$. This set is in bijection with the set $E = \{1^{l'_0}01^{l'_1} \dots 01^{l'_i} \dots 01^{l'_p}\}$. By Proposition 2.4, E is the set of Fibonacci strings of length $n - p + 1$ and weight $n - 2p + 1$ and we obtain the expression of $f_{n,p}$ by Proposition 2.5. \square

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Corollary 2.11 The counting polynomial $C'(\Gamma_n, x) = \sum_{p=0}^{\infty} f_{n,p}x^p$ of the number of maximal hypercubes of dimension p in Γ_n satisfies:

$$\begin{aligned} C'(\Gamma_n, x) &= x(C'(\Gamma_{n-2}, x) + C'(\Gamma_{n-3}, x)) \quad (n \geq 3) \\ C'(\Gamma_0, x) &= 1, \quad C'(\Gamma_1, x) = x, \quad C'(\Gamma_2, x) = 2x \end{aligned}$$

The generating function of the sequence $\{C'(\Gamma_n, x)\}$ is:

$$\sum_{n \geq 0} C'(\Gamma_n, x)y^n = \frac{1 + xy(1+y)}{1 - xy^2(1+y)}$$

Proof. By theorem 2.10 and Pascal identity we obtain $f_{n,p} = f_{n-2,p-1} + f_{n-3,p-1}$ for $n \geq 3$ and $p \geq 1$. Notice that $f_{n,0} = 0$ for $n \neq 0$. The recurrence relation for $C'(\Gamma_n, x)$ follows. Setting $f(x, y) = \sum_{n \geq 0} C'(\Gamma_n, x)y^n$ we deduce from the recurrence relation $f(x, y) - 1 - xy - 2xy^2 = x(y^2(f(x, y) - 1) + y^3 f(x, y))$ thus the value of $f(x, y)$. \square

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Theorem 2.12 Let $1 \leq p \leq n$ and $g_{n,p}$ be the number of maximal hypercubes of dimension p in Λ_n then:

$$g_{n,p} = \frac{n}{p} \binom{p}{n-2p}$$

Proof. The proof is similar to the previous result with three cases according to the value of l_0 :

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By Proposition 2.9 the set T of top vertices that begin with 1 is the set of strings which can be write $10^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $\sum_{i=1}^p l_i = n - p$ and $1 \leq l_i \leq 2$ for $i = 1, \dots, p$. Let $l'_i = l_i - 1$ for $i = 1, \dots, p$. We have thus a 1 to 1 mapping between T and the set of strings $D = \{10^{l'_1} \dots 10^{l'_i} \dots 10^{l'_p}\}$ where $\sum_{i=1}^p l'_i = n - 2p$, $0 \leq l'_i \leq 1$ for $i = 1, \dots, p$. Removing the first 1, and by complement, this set is in bijection with the set $E = \{1^{l'_1} \dots 01^{l'_i} \dots 01^{l'_p}\}$. By Proposition 2.4, E is the set of Fibonacci strings of length $n - p - 1$ and weight $n - 2p$. Thus $|T| = \binom{p}{n-2p}$.

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The set U of top vertices that begin with 01 is the set of strings which can be write $010^{l_1} \dots 10^{l_i} \dots 10^{l_p}$ where $\sum_{i=1}^p l_i = n - p - 1$; $1 \leq l_i \leq 2$ for $i = 1, \dots, p-1$ and $l_p \leq 1$. Let $l'_i = l_i - 1$ for $i = 1, \dots, p-1$ and $l'_p = l_p$. We have thus a 1 to 1 mapping between U and the set of strings $F = \{010^{l'_1} \dots 10^{l'_i} \dots 10^{l'_p}\}$ where $\sum_{i=1}^p l'_i = n - 2p$ and $l'_i \leq 1$ for $i = 1, \dots, p$. Removing the first 01, and by complement, this set is in bijection with the set $G = \{1^{l'_1} \dots 01^{l'_i} \dots 01^{l'_p}\}$. By Proposition 2.4, G is the set of Fibonacci strings of length $n - p - 1$ and weight $n - 2p$. Thus $|U| = \binom{p}{n-2p}$.

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The last set, V , of top vertices that begin with 001, is the set of strings which can be write $0010^{l_1} \dots 10^{l_i} \dots 0^{l_{p-1}}1$ where $\sum_{i=1}^{p-1} l_i = n - p - 2$ and $1 \leq l_i \leq 2$ for $i = 1, \dots, p-1$. Let $l'_i = l_i - 1$ for $i = 1, \dots, p-1$. We have thus a 1 to 1 mapping between V and the set of strings $H = \{0010^{l'_1} \dots 10^{l'_i} \dots 0^{l'_{p-1}}1\}$ where $\sum_{i=1}^{p-1} l'_i = n - 2p - 1$ and $l'_i \leq 1$ for $i = 1, \dots, p-1$. Removing the first 001 and the last 1, this set, again by complement, is in bijection with the set $K = \{1^{l'_1} \dots 01^{l'_i} \dots 01^{l'_{p-1}}\}$. The set K is the set of Fibonacci strings of length $n - p - 3$ and weight $n - 2p - 1$. Thus $|V| = \binom{p-1}{n-2p-1}$ and $g_{n,p} = 2\binom{p}{n-2p} + \binom{p-1}{n-2p-1} = \frac{n}{p}\binom{p}{n-2p}$. \square

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Corollary 2.13 *The counting polynomial $C'(\Lambda_n, x) = \sum_{p=0}^{\infty} g_{n,p}x^p$ of the number of maximal hypercubes of dimension p in Λ_n satisfies:*

$$C'(\Lambda_n, x) = x(C'(\Lambda_{n-2}, x) + C'(\Lambda_{n-3}, x)) \quad (n \geq 5)$$

$$C'(\Lambda_0, x) = 1, \quad C'(\Lambda_1, x) = 1, \quad C'(\Lambda_2, x) = 2x, \quad C'(\Lambda_3, x) = 3x, \quad C'(\Lambda_4, x) = 2x^2$$

The generating function of the sequence $\{C'(\Lambda_n, x)\}$ is:

$$\sum_{n \geq 0} C'(\Lambda_n, x)y^n = \frac{1 + y + xy^2 + xy^3 - xy^4}{1 - xy^2(1 + y)}$$

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Proof. Assume $n \geq 5$. Here also by theorem 2.12 and Pascal identity we get $g_{n,p} = g_{n-2,p-1} + g_{n-3,p-1}$ for $n \geq 5$ and $p \geq 2$. Notice that when $n \geq 5$ this equality occurs also for $p = 1$ and $g_{n,0} = 0$. The recurrence relation for $C'(\Lambda_n, x)$ follows and $g(x, y) = \sum_{n \geq 0} C'(\Lambda_n, x)y^n$ satisfies $g(x, y) - 1 - y - 2xy^2 - 3xy^3 - 2x^2y^4 = x(y^2(g(x, y) - 1 - y - 2xy^2) + y^3(g(x, y) - 1 - y))$. \square

190 Notice that $f_{n,p} \neq 0$ if and only if $\lceil \frac{n}{3} \rceil \leq p \leq \lfloor \frac{n+1}{2} \rfloor$ and $g_{n,p} \neq 0$ if and only if $\lceil \frac{n}{3} \rceil \leq p \leq \lfloor \frac{n}{2} \rfloor$ (for $n \neq 1$). Maximal induced hypercubes of maximum dimension are maximum induced hypercubes and we obtain again that cube polynomials of Γ_n , respectively Λ_n , are of degree $\lfloor \frac{n+1}{2} \rfloor$, respectively $\leq \lfloor \frac{n}{2} \rfloor$ [9].

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