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## Plus ultra

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# PLUS ULTRA

FRANK O. WAGNER

ABSTRACT. We define a reasonably well-behaved class of ultraimaginar-ies, i.e. classes modulo invariant equivalence relations, namely the *tame* ones, and establish some basic simplicity-theoretic facts. We also show feeble elimination of supersimple ultraimaginar-ies: If  $e$  is an ultraimaginary definable over a tuple  $a$  with  $SU(a) < \omega^{\alpha+1}$ , then  $e$  is eliminable up to rank  $< \omega^\alpha$ . Finally, we prove some uniform versions of the weak canonical base property.

## 1. INTRODUCTION

This paper arose out of an attempt to understand and generalize Chatzidakis' results on the weak canonical base property [5, Proposition 1.16 and Lemma 1.17]. In doing so, we realized that certain stability-theoretic phenomena were best explained using ultraimaginar-ies. It should be noted that ultraimaginar-ies occur naturally in simplic-ity theory and were in fact briefly considered in [3] before specializing to the more restricted class of almost hyperimaginar-ies. However, they have faded into oblivion since Ben Yaacov [1] has shown that no satisfac-tory independence theory can exist for them, as there are problems both with the finite character and with the extension axiom for inde-pendence. Nevertheless, at least finite character can be salvaged if one restricts to quasi-finitary ultraimaginar-ies in a supersimple theory, or more generally to what we called *tame* ultraimaginar-ies.

We shall define ultraimaginar-ies in Section 2 and give various exam-ples. We also give a first example of a natural general result involving them, Proposition 2.9, which for a supersimple theory of finite rank specializes to a theorem of Lascar. In Section 3 we define tame ul-traimaginar-ies and recover certain tools from simplicity theory, even though, due to the lack of extension, canonical bases are not available

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in our context. One may thus hope to extend the techniques of this section for instance to the superrosy context, where the lack of canonical bases has been one of the main technical problems.

In Section 4 we prove feeble elimination of ultrimaginaries. In particular ultrimaginaries of finite rank are interbounded with hyperimaginaries. This is used in Section 5 to generalize some of Chatzidakis' results [5] on the weak canonical base property from sets of finite  $SU$ -rank to arbitrary ordinal  $SU$ -rank. It is interesting to compare this generalization to the coarser [8, Theorem 5.4] which uses  $\alpha$ -closure. We expect that this is a general phenomenon: The use of ultrimaginaries allows for a more direct and more refined proof without explicit use of  $SU$ -rank; rank considerations principally intervene via the feeble elimination result and the technical results of Section 3.

All elements, tuples and parameter sets are hyperimaginary, unless stated otherwise. For an introduction to simplicity and hyperimaginaries, the reader is invited to consult [4] or [11].

## 2. ULTRAIMAGINARIES

**Definition 2.1.** An *ultraimaginary* is the class  $a_E$  of a tuple  $a$  under an  $\emptyset$ -invariant equivalence relation  $E$ .

Note that tuples of ultrimaginaires are again ultraimaginary. Alternatively, any tuple of ultrimaginaries is interdefinable with a tuple of countable ultrimaginaries.

**Definition 2.2.** An ultraimaginary  $a_E$  is *definable* over a set  $A$  if any automorphism of the monster model fixing  $A$  stabilises the  $E$ -class of  $a$ . It is *bounded* over  $A$  if the orbit of  $a$  under the group of automorphisms of the monster model which fix  $A$  is contained in boundedly many  $E$ -classes. A *representative* of an ultraimaginary  $e$  is any tuple  $a$  such that  $e$  is definable over  $a$ .

**Remark 2.3.** As usual, if  $E_A(x, y)$  is an  $A$ -invariant equivalence relation, one considers the  $\emptyset$ -invariant relation  $E(xX, yY)$  given by

$$(X = Y \wedge X \equiv A \wedge E_X(x, y)) \vee (X = Y \wedge x = y).$$

This is an equivalence relation, and  $(aA)_E$  is interdefinable over  $A$  with  $a_{E_A}$ .

**Remark 2.4.** As any  $\emptyset$ -invariant relation,  $E$  is given by a union of types over  $\emptyset$ .

We shall say that two ultrimaginaries have the same (Lascar strong) type over some set  $A$  if they have representatives which do. Clearly, two ultrimaginaries are conjugate by a (Lascar strong) automorphism if and only if they have the same (Lascar strong) type over  $A$ .

Ultrimaginaries arise quite naturally in stability and simplicity theory.

**Example 2.5.** Let  $p_A \in S(A)$  be a regular type in a stable theory. For  $A', A'' \models \text{tp}(A)$  put  $E(A', A'')$  if  $p_{A'} \not\perp p_{A''}$ . Then  $E$  is an  $\emptyset$ -invariant equivalence relation, and  $A_E$  codes the non-orthogonality class of  $p_A$ .

The work with ultrimaginaries requires caution, as some basic properties become problematic.

**Example 2.6.** [1] Let  $E$  be the  $\emptyset$ -invariant equivalence relation on infinite sequences which holds if they differ only on finitely many elements. Consider a sequence  $I = (a_i : i < \omega)$  of elements such that no finite subtuple is bounded over the remaining elements. Then every finite tuple  $\bar{a} \in I$  can be moved to a disjoint conjugate over  $I_E$ , but  $I$  cannot. Similarly, if  $I$  is a Morley sequence in a simple theory, then  $\bar{a} \perp\!\!\!\perp I_E$  for any finite  $\bar{a} \in I$ , but  $I \not\perp\!\!\!\perp I_E$ . (We call two ultrimaginaries independent if they have representatives which are.)

Even in the  $\omega$ -stable context, for classes of finite tuples, the theory is not smooth.

**Example 2.7.** Let  $T$  be the theory of a cycle-free graph (forest) of infinite valency, with predicates  $P_n(x, y)$  for couples of points of distance  $n$  for all  $n < \omega$ . It is easy to see by back-and-forth that  $T$  eliminates quantifiers and is  $\omega$ -stable of rank  $\omega$ ; the formula  $P_n(x, a)$  has rank  $n$  over  $a$ . Let  $E$  be the  $\emptyset$ -invariant equivalence relation of being in the same connected component. Then existence of non-forking extensions fails over  $a_E$ , as any two points in the connected component of  $a$  have some finite distance  $n$ , and hence rank  $n$  over one another, but rank  $\geq k$  over  $a_E$  for all  $k < \omega$ , since  $a_E$  is definable over any point of distance at least  $k$ .

The same phenomenon can be observed for any type  $p$  of rank  $SU(p) = \omega$  in a simple theory, with the relation  $E(x, y)$  on  $p$  which holds if  $SU(x/y) < \omega$  and  $SU(y/x) < \omega$  (actually, one follows from the other by Lascar's inequalities).

The behaviour of Example 2.7 is inconvenient and signifies that we shall avoid working *over* an ultrimaginary. The behaviour of Example

2.6 is outright vexatious; we shall restrict the class of ultrimaginaries under consideration in order to preserve the finite character of independence.

**Definition 2.8.** An ultrimaginary  $e$  is *quasi-finitary* if there is a finite tuple  $a$  such that  $e$  is bounded over  $a$ .

For hyperimaginary tuples contained in the bounded closure of a finite set, we shall use *quasi-finite* rather than *quasi-finitary*, in order to emphasize the distinction between usual hyperimaginaries and ultrimaginaries. The set of all / all quasi-finitary ultrimaginaries definable over  $A$  will be denoted by  $\text{dcl}^u(A)$  /  $\text{dcl}^{qfu}(A)$ , respectively. Similarly,  $\text{bdd}^u(A)$  /  $\text{bdd}^{qfu}(A)$  will denote the corresponding bounded closures.

Recall that two tuples  $a$  and  $b$  have the same *Lascar strong type* over  $A$ , denoted  $a \equiv_A^{\text{lstp}} b$  or  $b \models \text{lstp}(a/A)$ , if they lie in the same class modulo all  $A$ -invariant equivalence relations with only boundedly many classes. This is the finest bounded  $A$ -invariant equivalence relation, so  $\text{bdd}^u(A)$  is bounded by the number of Lascar strong types over  $A$ .

**Proposition 2.9.** *The following are equivalent:*

- (1)  $\text{bdd}^u(a) \cap \text{bdd}^u(b) = \text{bdd}^u(\emptyset)$ .
- (2) *For any  $a' \models \text{lstp}(a)$  there is  $n < \omega$  and a sequence  $(a_i b_i : i \leq n)$  such that*

$$a_0 = a, \quad b_0 = b, \quad a_n = a'$$

and for each  $i < n$

$$b_{i+1} \models \text{lstp}(b_i/a_i) \quad \text{and} \quad a_{i+1} \models \text{lstp}(a_i/b_{i+1}).$$

*If  $a$  and  $b$  are quasi-finite, this is also equivalent to  $\text{bdd}^{qfu}(a) \cap \text{bdd}^{qfu}(b) = \text{bdd}^{qfu}(\emptyset)$ .*

*Proof:* (1)  $\Rightarrow$  (2) Suppose  $\text{bdd}^u(a) \cap \text{bdd}^u(b) = \text{bdd}^u(\emptyset)$ , and define an  $\emptyset$ -invariant relation on  $\text{lstp}(ab)$  by  $E(xy, x'y')$  if there is a sequence  $(x_i y_i : i \leq n)$  such that

$$ab \equiv^{\text{lstp}} x_0 y_0, \quad x_0 y_0 = xy, \quad x_n y_n = x' y'$$

and for each  $i < n$

$$y_{i+1} \models \text{lstp}(y_i/x_i) \quad \text{and} \quad x_{i+1} \models \text{lstp}(x_i/y_{i+1}).$$

Note that this implies  $x_i y_i \equiv^{\text{lstp}} ab$  for all  $i \leq n$ , so  $E$  is an equivalence relation. Now if  $b' \models \text{lstp}(b/a)$ , then  $\models E(ab, ab')$ . Hence  $(ab)_E \in \text{bdd}^u(a)$ . Similarly  $(ab)_E \in \text{bdd}^u(b)$ , whence  $(ab)_E \in \text{bdd}^u(\emptyset)$ . But for any  $a' \models \text{lstp}(a)$  there is  $b'$  with  $ab \equiv^{\text{lstp}} a'b'$ . Then  $\models E(ab, a'b')$ , in particular (2) holds.

(2)  $\Rightarrow$  (1) Suppose not, and consider  $e \in (\text{bdd}^u(a) \cap \text{bdd}^u(b)) \setminus \text{bdd}^u(\emptyset)$ . As  $e \notin \text{bdd}^u(\emptyset)$  there is  $a' \models \text{lstp}(a)$  with  $e \notin \text{bdd}^u(a')$ . Consider a sequence  $(a_i, b_i : i \leq n)$  as in (2). Since  $b_{i+1} \models \text{lstp}(b_i/a_i)$  and  $a_{i+1} \models \text{lstp}(a_i/b_{i+1})$  we have

$$\begin{aligned} \text{bdd}^u(a_i) \cap \text{bdd}^u(b_i) &= \text{bdd}^u(a_i) \cap \text{bdd}^u(b_{i+1}) \\ &= \text{bdd}^u(a_{i+1}) \cap \text{bdd}^u(b_{i+1}). \end{aligned}$$

In particular,

$$\begin{aligned} e \in \text{bdd}^u(a) \cap \text{bdd}^u(b) &= \text{bdd}^u(a_0) \cap \text{bdd}^u(b_0) \\ &= \text{bdd}^u(a_n) \cap \text{bdd}^u(b_n) \subseteq \text{bdd}^u(a'), \end{aligned}$$

a contradiction.

The last assertion follows from the fact that for quasi-finite  $ab$  the ultramaginary  $(ab)_E$  in the proof of (1)  $\Rightarrow$  (2) is quasi-finitary.  $\square$

Using weak elimination of ultramaginarities proven in Section 4, we recover a Lemma of Lascar [6] (see also [7, Lemma 2.2]), proved originally for stable theories of finite Lascar rank.

**Corollary 2.10.** *Let  $T$  be a simple theory of finite SU-rank,  $A$  a parameter set and  $a, b$  quasi-finite hyperimaginary tuples. The following are equivalent:*

- (1)  $\text{bdd}(Aa) \cap \text{bdd}(Ab) = \text{bdd}(A)$ .
- (2) *For any  $a' \models \text{lstp}(a/A)$  independent of  $a$  over  $A$  there are sequences  $a = a_0, \dots, a_n = a'$  and  $b = b_0, \dots, b_n$ , such that  $b_{i+1} \models \text{lstp}(b_i/a_i)$  and  $a_{i+1} \models \text{lstp}(a_i/b_{i+1})$  for each  $i < n$ .*

*Proof:* We add  $A$  to the language. By Theorem 4.6 supersimple theories of finite rank have weak elimination of quasi-finitary ultramaginarities. Hence condition (1) is equivalent to  $\text{bdd}^u(a) \cap \text{bdd}^u(b) = \text{bdd}^u(\emptyset)$ . So (1)  $\Rightarrow$  (2) follows from Proposition 2.9; for the converse given arbitrary  $a' \models \text{lstp}(a/A)$  we consider  $a'' \models \text{lstp}(a/A)$  with  $a'' \downarrow_A aa'$  and compose the sequence  $(a_i b_i : i \leq n)$  from  $ab = a_0 b_0$  to  $a_n = a''$  with the sequence  $(a_i b_i : n \leq i \leq \ell)$  from  $a_n b_n$  to  $a_\ell = a'$ . Hence (2) holds for arbitrary  $a' \models \text{lstp}(a/A)$ , so we can again apply Proposition 2.9.  $\square$

### 3. ULTRAIMAGINARIES IN SIMPLE THEORIES

From now on the ambient theory will be simple. Our notation is standard and follows [11]. We shall be working in a sufficiently saturated model of the ambient theory. Tuples are tuples of hyperimaginaries,

and closures (definable, algebraic and bounded closures) will include hyperimaginaries.

**Remark 3.1.** Since in a simple theory Lascar strong type equals Kim-Pillay strong type, we have  $\text{bdd}^u(A) = \text{dcl}^u(\text{bdd}(A))$ . But of course, as with real and imaginary algebraic closures,  $\text{bdd}(A) \cap \text{bdd}(B) = \text{bdd}(\emptyset)$  does not imply  $\text{bdd}^u(A) \cap \text{bdd}^u(B) = \text{bdd}^u(\emptyset)$  unless the theory weakly eliminates ultrimaginaries.

**Definition 3.2.** We shall say that two ultrimaginaries  $e$  and  $e'$  are *independent* over  $A$ , denoted  $e \perp_A e'$ , if they have representatives which are.

**Remark 3.3.** If  $e$  or  $e'$  is a sequence of ultrimaginaries, we require sequences of representatives which are independent. In particular, it is not clear even for real  $e'$  that an infinite sequence  $e$  of ultrimaginaries is independent of  $e'$  if every finite subsequence is independent of  $e'$ . One should thus avoid to work with infinite tuples of ultrimaginaries.

On the other hand, as we have seen in Example 2.6, if  $e$  is a hyperimaginary set and  $e'$  a single ultrimaginary, finite character can also fail. This will give rise to Definition 3.7.

In a simple theory, ultrimaginary independence is clearly symmetric, and satisfies local character and extension (but recall that we only consider hyperimaginary base sets), since this is inherited from suitable representatives. As for transitivity, we have the following.

**Fact 3.4.** [3, Lemma 1.10] *Let  $A, a$  be hyperimaginary, and  $e, e'$  ultrimaginary.*

- *If  $e \perp_A e'e''$  and  $e' \perp_A e''$ , then  $ee' \perp_A e''$  and  $e \perp_A e'$ .*
- *$e \perp_A ae'$  if and only if  $e \perp_A a$  and  $e \perp_{Aa} e'$ .*

The Independence Theorem and Boundedness axiom also hold.

**Fact 3.5.** [3, page 189] *Let  $A$  be hyperimaginary and  $e, e'$  ultrimaginary with  $e \perp_A e'$ .*

- *If  $f, f'$  are ultrimaginary with  $f \perp_A e$ ,  $f' \perp_A e'$  and  $f \equiv_A^{Lstp} f'$ , then there is  $f'' \perp_A ee'$  with  $ef'' \equiv_A^{Lstp} ef$  and  $e'f'' \equiv_A^{Lstp} e'f'$ .*
- *If  $e'' \in \text{bdd}^u(Ae)$  then  $e'' \perp_A e'$ . Moreover, if  $e \perp_a e$  for every representative  $a$  of an ultrimaginary  $e''$ , then  $e \in \text{bdd}^u(e'')$ .*

Next, ultrimaginary bounded closures of independent sets intersect trivially.

**Lemma 3.6.** *If  $A$  is hyperimaginary and  $e, e'$  ultraimaginary with  $e \perp_A e'$ , then  $\text{bdd}^u(Ae) \cap \text{bdd}^u(Ae') = \text{bdd}^u(A)$ .*

*Proof:* Replacing  $e$  and  $e'$  by  $A$ -independent representatives, we may assume that  $e$  and  $e'$  are hyperimaginary. Consider  $a_E \in \text{bdd}^u(Ae) \cap \text{bdd}^u(Ae')$ . We may assume  $a \perp_{Ae} e'$ , whence  $ae \perp_A e'$ . Let  $(a_i : i < \omega)$  be a Morley sequence in  $\text{lstp}(a/Ae')$ . Then  $E(a_i, a_j)$  for all  $i, j < \omega$ . But  $a_i \perp_A a_j$  for  $i \neq j$ , so  $\pi(x, a_j) = \text{tp}(a_i/a_j)$  does not fork over  $A$ , and neither does  $\pi(x, a)$ . Note that  $\pi(x, y)$  implies  $E(x, y)$ .

Now suppose  $a_E \notin \text{bdd}^u(A)$ . We can then find a long sequence  $(a'_i : i < \alpha)$  of  $A$ -conjugates of  $a$  such that  $\neg E(a'_i, a'_j)$  for  $i \neq j$ . By the Erdős-Rado theorem there is an infinite  $A$ -indiscernible sequence  $(a''_i : i < \omega)$  whose 2-type over  $A$  is among the 2-types of  $(a'_i : i < \alpha)$ . In particular  $\neg E(a''_i, a''_j)$  for  $i \neq j$ , and  $(\pi(x, a''_i) : i < \omega)$  is 2-inconsistent. Since  $a''_0 \models \text{tp}(a/A)$ , we see that  $\pi(x, a)$  divides over  $A$ , a contradiction.  $\square$

As we have seen in Remark 3.3, finite character may fail for ultraimaginaries. The next definition singles out the subclass of ultraimaginaries where this does not happen, at least for hyperimaginary sets.

**Definition 3.7.** Let  $T$  be simple. An ultraimaginary  $e$  is *tame* if for all sets  $A, B$  of hyperimaginaries we have  $e \perp_A B$  if and only if  $e \perp_{A} B_0$  for all finite subsets  $B_0 \subseteq B$ . It is *supersimple* if it has a representative of ordinal  $SU$ -rank.

**Remark 3.8.** A supersimple ultraimaginary in a simple theory is quasi-finitary; in a supersimple theory the converse holds as well.

*Proof:* Suppose  $A$  is a representative for an ultraimaginary  $e$  with  $SU(A) < \infty$ , and let  $B$  be a real tuple with  $A \in \text{bdd}(B)$ . Let  $b \in B$  be a finite subtuple with  $SU(A/b)$  minimal; it follows that  $A \perp_b B$ . Hence  $A \subseteq \text{bdd}(b)$  and  $e$  is bounded over  $b$ , so  $e$  is quasi-finitary. In a supersimple theory the converse is obvious.  $\square$

We are really interested in the set of tame ultraimaginaries. However, we do not have a good criterion when an ultraimaginary is tame; moreover, an ultraimaginary definable over a tame ultraimaginary need not be tame itself. For instance, the sequence  $I$  in Example 2.6 is tame (since it is real), but  $I_E$  is not. Clearly, an ultraimaginary definable (or even bounded) over a quasi-finitary / supersimple ultraimaginary is itself quasi-finitary / supersimple.

**Lemma 3.9.** *A supersimple ultraimaginary is tame. In particular, quasi-finitary ultraimaginaries in a supersimple theory are tame.*



*Proof:* Let  $e$  be a supersimple ultrimaginary, and  $a$  a representative with  $SU(a) < \infty$ . Consider sets  $A$  and  $B$ . There is a finite  $b \in B$  with  $a \downarrow_{Ab} B$ . So  $e \downarrow_A B$  if and only if  $e \downarrow_A b$  by Fact 3.4. Thus  $e$  is tame.  $\square$

In a supersimple theory quasi-finitary ultrimaginaries are the correct ones to consider: Due to elimination of hyperimaginaries all parameters consist of imaginaries of ordinal  $SU$ -rank; as canonical bases of such imaginaries are finite, we can always reduce to a quasi-finitary situation.

Another kind of tame ultrimaginaries arose in the generalization of the group configuration theorem to simple theories [2, 3].

**Definition 3.10.** An invariant equivalence relation  $E$  is *almost type-definable* if there is a type-definable symmetric and reflexive relation  $R$  finer than  $E$  such that any  $E$ -class can be covered by boundedly many  $R$ -classes (i.e. sets of the form  $\{x : xRa\}$  for varying  $a$ ). A class modulo an almost type-definable equivalence relation is called an *almost hyperimaginary*.

**Fact 3.11.** [3, page 188] *Almost hyperimaginaries are tame. In fact, they satisfy finite character.*

The following two Propositions tells us how to obtain invariant equivalence relations, and hence ultrimaginaries.

**Proposition 3.12.** *Let  $T$  be stable. For algebraically closed  $A$  and an  $\emptyset$ -invariant equivalence relation  $E$  on  $\text{tp}(b)$ , consider the relation  $R(X, Y)$  given by*

$$\exists xy [Xx \equiv Yy \equiv Ab \wedge x \underset{X}{\downarrow} Y \wedge y \underset{Y}{\downarrow} X \wedge E(x, y)].$$

*Then  $R$  is an  $\emptyset$ -invariant equivalence relation on  $\text{tp}(A)$ .*

*Proof:* Clearly,  $R$  is  $\emptyset$ -invariant, reflexive and symmetric. So suppose that  $R(A, A')$  and  $R(A', A'')$  both hold, and let this be witnessed by  $b, b'$  and  $b^*, b''$ . Let  $b_1 \models \text{tp}(b'/A') = \text{tp}(b^*/A')$  with  $b_1 \downarrow_{A'} AA''$ . Since  $A'$  is algebraically closed,  $b' \downarrow_{A'} A$  and  $b^* \downarrow_{A'} A''$  we have  $b_1 \equiv_{AA'} b'$  and  $b_1 \equiv_{A'A''} b^*$  by stationarity. Hence there are  $b_0, b_2$  with  $bb' \equiv_{AA'} b_0 b_1$  and  $b^* b'' \equiv_{A'A''} b_1 b_2$ . In particular  $E(b_0, b_1)$  and  $E(b_1, b_2)$  hold, and so does  $E(b_0, b_2)$ . Moreover, we may assume  $b_0 \downarrow_{AA'b_1} A''$  and  $b_2 \downarrow_{A'A''b_1} A'$ . Now  $b_1 \downarrow_{A'} AA''$  implies  $b_0 \downarrow_{AA'} A''$  and  $b_2 \downarrow_{A'A''} A$ . Then  $b_0 \downarrow_A A'$  and  $b_2 \downarrow_{A''} A'$  imply  $b_0 \downarrow_A A''$  and  $b_2 \downarrow_{A''} A$ , whence  $R(A, A'')$  holds. So  $R$  is transitive.  $\square$

Recall that a reflexive and symmetric binary relation  $R(x, y)$  on a partial type  $\pi(x)$  is *generically transitive* if whenever  $x, y, z \models \pi$  and  $x \downarrow_y z$ , then  $R(x, y)$  and  $R(y, z)$  together imply  $R(x, z)$ .

For a (regular) type  $p$  let  $SU_p$  denote *SU-rank relativized to  $p$*  (see [11, Remark 5.1.19]), and for a set  $A$  put

$$\text{cl}_p(A) = \{a : SU_p(a/A) = 0\},$$

the  $p$ -closure of  $A$ .

**Proposition 3.13.** *Let  $T$  be simple. Suppose  $R$  is an  $\emptyset$ -invariant, reflexive, symmetric and generically transitive relation on  $\text{lstp}(a)$ , and  $p$  is a regular type such that  $SU_p(a)$  is finite. Let  $E$  be the transitive closure of  $R$ , and suppose  $a_E \in \text{bdd}^u(\text{cl}_p(\emptyset))$ . Then there is  $a' \downarrow_{\text{cl}_p(\emptyset)} a$  with  $R(a, a')$ .*

*Proof:* Put  $c = \text{bdd}(a) \cap \text{cl}_p(\emptyset)$ . Then  $a \downarrow_c \text{cl}_p(\emptyset)$ , whence  $a_E \in \text{bdd}^u(c)$  by Lemma 3.6. Let  $a' \equiv_c^{lstp} a$  with  $a' \downarrow_c a$ . Then  $a_E = a'_E$ , so there is  $n < \omega$  and a chain  $a = a_0, a_1, \dots, a_n = a'$  such that  $R(a_i, a_{i+1})$  holds for all  $i < n$ . Put  $a'_0 = a_0$ , and for  $0 < i < n$  let

$$a'_i \equiv_{a_i}^{lstp} a'_{i-1} \quad \text{with} \quad a'_i \downarrow_{a_i} a_{i+1}.$$

**Claim.**  $\text{bdd}^u(a'_i) \cap \text{bdd}^u(a_{i+1}) \subseteq \text{bdd}^u(a_0)$ .

*Proof of Claim:* For  $i = 0$  this is trivial. For  $i > 0$ , as  $a'_i \equiv_{a_i}^{lstp} a'_{i-1}$  and  $\text{bdd}^u(a_i) = \text{dcl}^u(\text{bdd}(a_i))$ , we get

$$\text{bdd}^u(a'_i) \cap \text{bdd}^u(a_i) = \text{bdd}^u(a'_{i-1}) \cap \text{bdd}^u(a_i).$$

Next,  $a'_i \downarrow_{a_i} a_{i+1}$  implies

$$\text{bdd}^u(a'_i a_i) \cap \text{bdd}^u(a_i a_{i+1}) = \text{bdd}^u(a_i)$$

by Lemma 3.6. Hence inductively

$$\begin{aligned} \text{bdd}^u(a'_i) \cap \text{bdd}^u(a_{i+1}) &\subseteq \text{bdd}^u(a'_i) \cap \text{bdd}^u(a_i) \\ &= \text{bdd}^u(a'_{i-1}) \cap \text{bdd}^u(a_i) \\ &\subseteq \text{bdd}^u(a_0). \quad \square \end{aligned}$$

Now by generic transitivity and induction,  $R(a'_i, a_{i+1})$  holds for all  $i < n$ . In particular  $R(a'_{n-1}, a_n)$  holds, and by Lemma 3.6

$$\text{bdd}^u(a'_{n-1}) \cap \text{bdd}^u(a_n) \subseteq \text{bdd}^u(a_0) \cap \text{bdd}^u(a_n) = \text{bdd}^u(c).$$

Choose  $a''$  with  $R(a'', a'_{n-1})$  such that  $SU_p(a''/a'_{n-1})$  is maximal possible. We may choose it such that  $a'' \downarrow_{a'_{n-1}} a_n$ . Then

$$\text{bdd}^u(a'') \cap \text{bdd}^u(a_n) \subseteq \text{bdd}^u(a_n) \cap \text{bdd}^u(a'_{n-1}) \subseteq \text{bdd}^u(c)$$

and

$$SU_p(a''/a_n) \geq SU_p(a''/a'_{n-1}a_n) = SU_p(a''/a'_{n-1}).$$

Rename  $a''a_n$  as  $a_1a_2$ , and note that  $\text{bdd}^u(a_1) \cap \text{bdd}^u(a_2) \subseteq \text{bdd}^u(c)$ ,  $c \subseteq \text{bdd}(a_2)$ , and  $SU_p(a_1/a_2)$  is maximal possible among tuples  $(x, y)$  with  $R(x, y)$ . Moreover,

$$SU_p(a_2/a_1) = SU_p(a_1a_2) - SU_p(a_1) = SU_p(a_1a_2) - SU_p(a_2) = SU_p(a_1/a_2),$$

so this is also maximal.

Choose  $a_3 \downarrow_{a_2} a_1$  with  $a_3 \equiv_{a_2}^{lstp} a_1$ . By generic transitivity  $R(a_1, a_3)$  holds. Moreover,

$$SU_p(a_3/a_1) \geq SU_p(a_3/a_1a_2) = SU_p(a_3/a_2),$$

so equality holds. Similarly,

$$SU_p(a_1/a_3) = SU_p(a_1/a_2a_3) = SU_p(a_1/a_2).$$

Now  $SU_p(a_i/a_j) = SU_p(a_i/a_ja_k)$  for  $\{i, j, k\} = \{1, 2, 3\}$  means that

$$\text{cl}_p(a_i) \downarrow_{\text{cl}_p(a_j)} \text{cl}_p(a_k).$$

In particular,

$$\text{cl}_p(a_i) \cap \text{cl}_p(a_k) = \text{cl}_p(a_1) \cap \text{cl}_p(a_2) \cap \text{cl}_p(a_3).$$

Let  $b = \text{cl}_p(a_1) \cap \text{cl}_p(a_2) \cap \text{bdd}(a_1a_2)$ . Then  $\text{cl}_p(a_1) \cap \text{cl}_p(a_2) = \text{cl}_p(b)$  by [8, Lemma 3.18]. Let  $F(x, y)$  be the  $\emptyset$ -invariant equivalence relation on  $\text{lstp}(b)$  given by  $\text{cl}_p(x) = \text{cl}_p(y)$ . As  $b_F$  is fixed by the  $\text{bdd}(a_2)$ -automorphism moving  $a_1$  to  $a_3$  and  $a_1 \downarrow_{a_2} a_3$ , we get  $b_F \in \text{bdd}^u(a_2)$  by Lemma 3.6. Similarly, considering an  $a'_3 \downarrow_{a_1} a_2$  with  $a'_3 \equiv_{a_1}^{lstp} a_2$  we obtain  $b_F \in \text{bdd}^u(a_1)$ , whence

$$b_F \in \text{bdd}^u(a_1) \cap \text{bdd}^u(a_2) \subseteq \text{bdd}^u(c).$$

So if  $b' \downarrow_c b$  satisfies  $\text{lstp}(b/c)$ , then  $b'_F = b_F$  and

$$\text{cl}_p(b') = \text{cl}_p(b) = \text{cl}_p(c) = \text{cl}_p(\emptyset).$$

But now

$$\text{Cb}(a_3/\text{cl}_p(a_1)\text{cl}_p(a_2)) \subseteq \text{cl}_p(a_1) \cap \text{cl}_p(a_2) = \text{cl}_p(b) = \text{cl}_p(\emptyset),$$

so  $a_3 \downarrow_{\text{cl}_p(\emptyset)} a_2$ , as required.  $\square$

**Remark 3.14.** We cannot generalize [11, Lemma 3.3.1] and strengthen Proposition 3.13 to say that if  $R$  is  $\emptyset$ -invariant, reflexive, symmetric and generically transitive on a Lascar strong type, then the transitive closure  $E$  of  $R$  equals the 2-step iteration of  $R$ . Consider on the forest of Example 2.7 the relation  $R(a, b)$  which holds if 3 divides the distance between  $a$  and  $b$ . This is generically transitive, as for  $a' \downarrow_a a''$  the distance between  $a'$  and  $a''$  is the sum of the distances between  $a'$  and  $a$  and between  $a$  and  $a''$ . However, two points of distance 2 are easily seen to be  $R^2$ -related, so the transitive closure  $E$  of  $R$  is just the relation of being in the same connected component. But no two points of distance 1 are  $R^2$ -related.

From now on, let  $\Sigma$  be an  $\emptyset$ -invariant family of partial types.

**Definition 3.15.** We shall say that an ultrimaginary  $e$  is (*almost*)  $\Sigma$ -*internal*, or is  $\Sigma$ -*analysable*, if it has a representative which is. Similarly,  $e$  is *orthogonal* over  $A$  to some type  $p$  if for all  $B \downarrow_A e$  such that  $p$  is over  $B$  and for any realization  $b \models p|B$  we have  $e \downarrow_A Bb$ .

**Remark 3.16.** This definition does not imply that we define the notion of an analysis of an ultrimaginary. Moreover,  $e$  orthogonal to  $p$  over  $A$  does not imply that  $e$  has a representative which is orthogonal to  $p$ . Moreover, orthogonality of  $e$  over  $A$  to  $p$  does not imply orthogonality to  $p^{(\omega)}$ , unless  $e$  is tame.

**Definition 3.17.** For an ordinal  $\alpha$  the  $\alpha$ -*th*  $\Sigma$ -*level* of  $a$  over  $A$  is defined inductively by  $\ell_0^\Sigma(a/A) = \text{bdd}(A)$ , and for  $\alpha > 0$

$$\ell_\alpha^\Sigma(a/A) = \{b \in \text{bdd}(aA) : \text{tp}(b/\bigcup_{\beta < \alpha} \ell_\beta(a/A)) \text{ is almost } \Sigma\text{-internal}\}.$$

We shall write  $\ell_\infty^\Sigma(a/A)$  for  $\bigcup_\alpha \ell_\alpha^\Sigma(a/A)$ , i.e. the set of all hyperimaginaries  $b \in \text{bdd}(aA)$  such that  $\text{tp}(b/A)$  is  $\Sigma$ -analysable.

**Remark 3.18.** So  $a \in \ell_\alpha^\Sigma(a/A)$  is and only if  $\text{tp}(a/A)$  is  $\Sigma$ -analysable in  $\alpha$  steps.

**Lemma 3.19.** *If  $\text{tp}(a/A)$  is  $\Sigma$ -analysable in  $\alpha$  steps for some ordinal  $\alpha$  or  $\alpha = \infty$  and  $A \downarrow b$ , put  $c = \ell_\alpha^\Sigma(b)$ . Then  $Aa \downarrow_c b$ .*

*Proof:*  $\text{Cb}(Aa/b)$  is definable over a Morley sequence  $(A_i a_i : i < \omega)$  in  $\text{lstp}(a/b)$ . Then  $(A_i : i < \omega) \downarrow b$  and  $\text{tp}(a_i/A_i)$  is  $\Sigma$ -analysable in  $\alpha$  steps for all  $i < \omega$ . Hence  $\text{Cb}(Aa/b)$  is also  $\Sigma$ -analysable in  $\alpha$  steps. Thus  $\text{Cb}(Aa/b) \subseteq c$ , and  $Aa \downarrow_c b$ .  $\square$

**Proposition 3.20.** *Let  $T$  be simple. Suppose  $b_E$  is an ultrimaginary non-orthogonal to some regular type  $p$ , and  $SU_p(\ell_1^p(b)) < \omega$ . Then there is an almost  $p$ -internal ultrimaginary  $e \in \text{bdd}^u(b_E) \setminus \text{bdd}^u(\text{cl}_p(\emptyset))$ . Moreover,  $e \in \text{bdd}^u(\ell_1^p(b))$ .*

*Proof:* Let  $c = \ell_1^p(b)$ . Define an  $\emptyset$ -invariant relation  $R$  on  $\text{tp}(c)$  by

$$R(c', c'') \iff \exists b' b'' [b' c' \equiv b'' c'' \equiv bc \wedge E(b', b'')].$$

This is reflexive and symmetric; moreover for  $c' \downarrow_{c''} c'''$  with  $R(c', c'')$  and  $R(c'', c''')$  we can find  $b', b'', b^*, b'''$  with

$$b' c' \equiv b'' c'' \equiv b^* c'' \equiv b''' c''' \equiv bc,$$

such that  $E(b', b'')$  and  $E(b^*, b''')$  hold. Since  $c''$  is boundedly closed,  $b'' \equiv_{c''}^{\text{lstp}} b^*$ ; moreover  $b'' \downarrow_{c''} c'$  and  $b^* \downarrow_{c''} c'''$  by Lemma 3.19. By the Independence Theorem we can assume  $b'' = b^*$ , so  $E(b', b''')$  and  $R(c', c''')$  hold. Hence  $R$  is generically transitive; let  $F$  be its transitive closure. The class  $c_F$  is clearly almost  $p$ -internal. Moreover, if  $E(b', b)$  holds there is  $c'$  with  $b' c' \equiv bc$ . Thus  $F(c', c)$  holds, so  $c_F$  is bounded over  $b_E$ .

Finally, suppose  $c_F \in \text{bdd}^u(\text{cl}_p(\emptyset))$ . By Proposition 3.13 there is  $c' \downarrow_{\text{cl}_p(\emptyset)} c$  with  $R(c', c)$ . Hence there are  $b', b^*$  with  $b' c' \equiv b^* c \equiv bc$  and  $\models E(b', b^*)$ . Applying a  $c$ -automorphism (and moving  $c'$ ), we may assume  $b = b^*$ . Let  $A \downarrow b$  be some parameters and  $a$  some realization of  $p$  over  $A$  with  $a \not\downarrow_A b_E$ ; we may assume  $Aa \downarrow_b b'$ , whence  $A \downarrow bb'$ . Moreover  $b \downarrow_c Aa$  by Lemma 3.19, whence  $b' \downarrow_c Aa$ . Thus  $b' \downarrow_{\text{cl}_p(c)} Aa$ . Now  $c' \downarrow_{\text{cl}_p(\emptyset)} c$  yields  $c' \downarrow_{\text{cl}_p(\emptyset)} \text{cl}_p(c)$ , and hence  $c' \downarrow_{\text{cl}_p(\emptyset)} Aa$ . Then  $a \downarrow_A \text{cl}_p(\emptyset)$  implies  $a \downarrow_A c'$ . Now  $b' \downarrow_{c'} Aa$  by Lemma 3.19, whence  $b' \downarrow_A a$ . As  $b_E = b'_E$  we obtain  $a \downarrow_A b_E$ , a contradiction.  $\square$

**Corollary 3.21.** *Let  $e$  be a supersimple ultrimaginary. Suppose  $e$  is non-orthogonal to some regular type  $p$  over some set  $B$ . Then there is an almost  $p$ -internal supersimple  $e' \in \text{bdd}^{\text{qfu}}(Be) \setminus \text{bdd}^{\text{qfu}}(\text{cl}_p(B))$ .*

*Proof:* Let  $a$  be a representative of  $e$  with  $SU(a) < \infty$  and put  $b = \text{Cb}(a/B)$ . Then  $SU(b) < \infty$ , as  $b$  is bounded over a finite initial segment of a Morley sequence in  $\text{lstp}(a/B)$ . Now  $e \downarrow_b B$ , so  $\text{tp}(e/b)$  is non-orthogonal to  $p$ . Note that  $SU_p(\ell_1^p(a/b)/b)$  is finite by supersimplicity. By Proposition 3.20 applied over  $b$  there is an almost  $p$ -internal ultrimaginary  $e' \in \text{bdd}^u(be) \setminus \text{bdd}^u(\text{cl}_p(b))$ ; moreover  $e' \in \text{bdd}^u(\ell_1^p(a/b)) \subseteq \text{bdd}(ab)$ . Thus  $e'$  is supersimple, almost  $p$ -internal over  $b$  and thus over  $B$ ; it is quasi-finitary by Remark 3.8.  $\square$

**Remark 3.22.** For hyperimaginary  $e$  in a simple theory, the proof of Corollary 3.21 uses the canonical base of some type over  $e$ . As we cannot consider types over ultraimaginarys, this does not make sense in our context.

**Proposition 3.23.** *Let  $T$  be supersimple. If  $AB \downarrow D$  and  $\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset)$ , then  $\text{bdd}^{qfu}(AD) \cap \text{bdd}^{qfu}(BD) = \text{bdd}^{qfu}(D)$ .*

*Proof:* We may assume that  $A$ ,  $B$  and  $D$  are boundedly closed. Consider

$$e \in (\text{bdd}^{qfu}(AD) \cap \text{bdd}^{qfu}(BD)) \setminus \text{bdd}^{qfu}(D).$$

Let  $p$  be a regular type of least  $SU$ -rank non-orthogonal to  $e$  over  $D$ . This exists by transitivity since  $e$  is tame. By Corollary 3.21 we may assume that  $e$  is almost  $p$ -internal of finite  $SU_p$ -rank over  $D$ ; let  $a'$  be a representative which is almost  $p$ -internal over  $D$ . Put  $a = \text{Cb}(a'D/A)$ . As  $a \downarrow D$  we obtain that  $\text{tp}(a)$  is almost  $p$ -internal; note that  $SU(a) < \infty$ . Since  $e \downarrow_{aD} A$ , Lemma 3.6 implies  $e \in \text{bdd}^{qfu}(aD)$ . So we may assume that  $A = \text{bdd}(a)$  and  $SU_p(A) < \omega$ . Moreover, we may assume that  $D = \text{bdd}(\text{Cb}(aa'/D))$  is the bounded closure of a finite set.

Let  $(A_i : i < \omega)$  be a Morley sequence in  $\text{lstp}(A/BD)$  with  $A_0 = A$ , and put  $B' = \text{bdd}(A_1A_2)$ . Then  $B'$  is almost  $p$ -internal of finite  $SU_p$ -rank. Since  $e \in \text{bdd}^{qfu}(AD) \cap \text{bdd}^{qfu}(BD)$  we have  $e \in \text{bdd}^{qfu}(A_iD)$  for all  $i < \omega$ . Let  $e'$  be the set of  $B'D$ -conjugates of  $e$ , again a quasi-finitary ultraimaginary. Since any  $B'D$ -conjugate of  $e$  is again in

$$\begin{aligned} \text{bdd}^{qfu}(A_1D) \cap \text{bdd}^{qfu}(A_2D) &= \text{bdd}^{qfu}(BD) \cap \text{bdd}^{qfu}(A_1D) \\ &= \text{bdd}^{qfu}(BD) \cap \text{bdd}^{qfu}(AD), \end{aligned}$$

we have  $e' \in \text{dcl}^{qfu}(B'D) \cap \text{bdd}^{qfu}(AD)$ . Moreover,  $B' \downarrow_{BD} A$ , whence  $B' \downarrow_B A$  and

$$\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B') \subseteq \text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset).$$

Choose  $A' \equiv_{AD}^{\text{lstp}} B'$  with  $A' \downarrow_{AD} B'$ . Then  $e' \in \text{dcl}^{qfu}(A'D) \cap \text{dcl}^{qfu}(B'D)$ . Furthermore,  $D \downarrow_B A$  implies  $D \downarrow_B AB'$ ; as  $D \downarrow B$  we get  $D \downarrow ABB'$ . Therefore  $D \downarrow_A B'$ , whence  $A' \downarrow_A B'$  and

$$\text{bdd}^{qfu}(A') \cap \text{bdd}^{qfu}(B') \subseteq \text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B') = \text{bdd}^{qfu}(\emptyset).$$

We may assume  $e' = (A'D)_E$  for some  $\emptyset$ -invariant equivalence relation  $E$ . Define a  $\emptyset$ -invariant reflexive and symmetric relation  $R$  on  $\text{lstp}(A')$  by

$$R(X, Y) \Leftrightarrow \exists Z [XZ \equiv YZ \equiv A'D \wedge Z \downarrow XY \wedge E(XZ, YZ)].$$

By the independence theorem, if  $A_1 \downarrow_{A_2} A_3$  such that  $R(A_1, A_2)$  and  $R(A_2, A_3)$  hold, we have  $R(A_1, A_3)$ . Hence  $R$  is generically transitive; let  $E'$  be the transitive closure of  $R$ . Clearly  $A'_{E'}$  is quasi-finitary.

Next, consider  $A'' \equiv_{B'} A'$  with  $A'' \downarrow_{B'} A'$ . By the independence theorem there is  $D'$  with  $A'D \equiv_{B'} A'D' \equiv_{B'} A''D'$  and  $D' \downarrow_{B'} A'A''$ . Then  $D' \downarrow B'$ , whence  $D' \downarrow A'A''$  and  $(A'D')_E = (A''D')_E \in \text{dcl}^{qfu}(B'D')$ . Therefore  $E'(A', A'')$  holds and  $A'_{E'} \in \text{dcl}^{qfu}(B')$ . Thus

$$A'_{E'} \in \text{dcl}^{qfu}(A') \cap \text{dcl}^{qfu}(B') \subseteq \text{bdd}^{qfu}(\emptyset).$$

By Proposition 3.13 there is  $A'' \downarrow_{\text{cl}_p(\emptyset)} A'$  with  $R(A', A'')$ . Let  $D'$  witness  $R(A', A'')$ . Then  $D' \equiv_{A'} D$ , so we may assume  $D' = D$ . Since  $\text{cl}_p(D) \downarrow_{\text{cl}_p(\emptyset)} \text{cl}_p(A'A'')$  and  $\text{cl}_p(A') \downarrow_{\text{cl}_p(\emptyset)} \text{cl}_p(A'')$  we obtain

$$\text{cl}_p(A') \downarrow_{\text{cl}_p(\emptyset)} \text{cl}_p(A'')\text{cl}_p(D)$$

and hence  $A' \downarrow_{\text{cl}_p(D)} A''$ . But now

$$e' = (A'D)_E = (A''D)_E \in \text{dcl}^{qfu}(A'D) \cap \text{dcl}^{qfu}(A''D) \subseteq \text{bdd}^{qfu}(\text{cl}_p(D))$$

by Lemma 3.6. Since  $e \in \text{bdd}^{qfu}(e')$ , this contradicts non-orthogonality of  $e$  to  $p$  over  $D$ .  $\square$

**Remark 3.24.** Again, the proof of the hyperimaginary analogue of Proposition 3.23 for simple theories uses canonical bases and does not generalize.

#### 4. ELIMINATION OF ULTRAIMAGINARIES

One cannot avoid the non-tame ultrimaginaries of Example 2.6 which do not satisfy finite character and hence cannot be eliminated. Similarly, on a type of rank  $\omega$  we cannot eliminate the relation of having mutually finite rank over each other (Example 2.7), since the rank over a class modulo such a relation is not defined. We thus content ourselves with elimination of supersimple ultrimaginaries in a simple theory (and in particular of quasi-finitary ultrimaginaries in a supersimple theory) up to rank of lower order of magnitude. This seems to be optimal, given the examples cited.

**Definition 4.1.** Let  $e$  be ultrimaginary. We shall say that  $SU(a/e) < \omega^\alpha$  if for all representatives  $b$  of  $e$  we have  $SU(a/b) < \omega^\alpha$ . Conversely,  $SU(e/a) < \omega^\alpha$  if there is a representative  $b$  with  $SU(b/a) < \omega^\alpha$ .

**Remark 4.2.** This does not mean that we define the value of  $SU(a/e)$  or of  $SU(e/a)$ . In fact, one might define

$$SU(e/a) = \min\{SU(b/a) : b \text{ a representative of } e\},$$

but this suggests a precision I am not sure exists.

**Lemma 4.3.** *Let  $e$  be ultrimaginary.  $SU(e/a) < \omega^0$  if and only if  $e \in \text{bdd}^u(a)$ , and  $SU(a/e) < \omega^0$  if and only if  $a \in \text{bdd}(e)$ .*

*Proof:* If  $b$  is a representative of  $e$  with  $SU(b/a) < \omega^0$ , then  $b \in \text{bdd}(a)$ , so  $e \in \text{bdd}^u(a)$ . If  $e \in \text{bdd}^u(a)$ , then  $e \in \text{dcl}^u(\text{bdd}(a))$ , so  $b = \text{bdd}(a)$  is a representative of  $e$  with  $SU(b/a) < \omega^0$ .

If  $a \notin \text{bdd}(e)$ , then there are arbitrarily many  $e$ -conjugates of  $a$ . Then for any representative  $b$  of  $e$  there is some  $e$ -conjugate  $a'$  of  $a$  which is not in  $\text{bdd}(b)$ . Let  $b'$  be the image of  $b$  under an  $e$ -automorphism mapping  $a'$  to  $a$ . Then  $b'$  is a representative of  $e$ , and  $SU(a/b') \geq \omega^0$ . On the other hand, if  $a \in \text{bdd}(e)$ , then  $a \in \text{bdd}(b)$  for any representative  $b$  of  $e$ , whence  $SU(a/b) < \omega^0$ .  $\square$

**Definition 4.4.** An ultrimaginary  $e$  can be  $\alpha$ -eliminated if there is a representative  $a$  with  $SU(a/e) < \omega^\alpha$ . A supersimple theory has *feeble elimination of ultrimaginaries* if for all ordinals  $\alpha$ , all quasi-finitary ultrimaginaries of rank  $< \omega^{\alpha+1}$  can be  $\alpha$ -eliminated.

**Remark 4.5.** 0-elimination is usually called *weak* elimination; in the presence of imaginaries this equals full elimination. I do not know what the definition of feeble elimination of ultrimaginaries should be in general for simple theories — but then their whole theory is much more problematic.

**Theorem 4.6.** *If  $e$  is ultrimaginary with  $SU(e) < \omega^{\alpha+1}$ , then  $e$  can be  $\alpha$ -eliminated. A supersimple theory has feeble elimination of ultrimaginaries; a supersimple theory of finite rank has elimination of quasi-finitary ultrimaginaries.*

*Proof:* Let  $a$  be a representative of  $e$  of minimal rank. Since  $SU(e) < \omega^{\alpha+1}$  we have  $SU(a) < \omega^{\alpha+1}$ . Suppose  $SU(a/e) \geq \omega^\alpha$ . Then there is some representative  $b$  of  $e$  with  $SU(a/b) \geq \omega^\alpha$ ; we choose it such that  $SU(a/b) \geq \omega^\alpha \cdot n$  for some maximal  $n \geq 1$ . Consider  $a' \equiv_b^{\text{lstp}} a$  with  $a' \perp_b a$ . Since  $e \in \text{dcl}^u(b)$  we have  $e \in \text{dcl}^u(a')$ . By maximality of  $n$ ,

$$SU(a/a') < \omega^\alpha \cdot (n+1) = SU(a/b) + \omega^\alpha = SU(a/a'b) + \omega^\alpha.$$

Hence, if

$$\text{cl}_\alpha(A) = \{c : SU(c/A) < \omega^\alpha\}$$



denotes the  $\alpha$ -closure of  $A$ , we have

$$a \downarrow_{\text{cl}_\alpha(a')} \text{cl}_\alpha(b).$$

On the other hand,  $a \downarrow_b a'$  implies

$$a \downarrow_{\text{cl}_\alpha(b)} \text{cl}_\alpha(a'),$$

so

$$c = \text{Cb}(a/\text{cl}_\alpha(b)\text{cl}_\alpha(a')) \subseteq \text{cl}_\alpha(b) \cap \text{cl}_\alpha(a').$$

Then  $a \downarrow_c b$ , so  $e \in \text{bdd}^u(c)$  by Lemma 3.6. On the other hand,  $SU(c/a') < \omega^\alpha$ , and  $SU(a'/c) \geq SU(a'/cb) \geq \omega^\alpha$  since  $SU(a'/b) \geq \omega^\alpha$  and  $SU(c/b) < \omega^\alpha$ . It follows that

$$SU(a) = SU(a') \geq SU(c) + \omega^\alpha.$$

In particular  $\text{bdd}(c)$  is a representative for  $e$  of lower rank, a contradiction.  $\square$

**Remark 4.7.** Let  $p$  be a regular type (or type of weight 1). Then two realizations  $a$  and  $b$  of  $p$  are independent if and only if  $\text{bdd}^{qfu}(a) \cap \text{bdd}^{qfu}(b) = \text{bdd}^{qfu}(\emptyset)$ : One direction is Lemma 3.6, the other follows from the observation that dependence is an invariant equivalence relation on realizations of  $p$ . However, this does not hold for all types: By elimination of quasifinite ultrimaginaries, it is in particular false in non one-based theories of finite rank.

## 5. DECOMPOSITION

In this section we shall give ultrimaginary proofs of some of Chatzidakis' results from [5] around the weak canonical base property, and suitable generalisations to the supersimple case.  $\Sigma$  will be an  $\emptyset$ -invariant family of partial types in a simple theory.

Recall that  $a$  and  $b$  are domination-equivalent over  $A$ , denoted  $a \sqsubseteq_A b$ , if for any  $c$  we have  $c \downarrow_A a \Leftrightarrow c \downarrow_A b$ . The following lemma is folklore, but we give a proof for completeness.

- Lemma 5.1.**
- (1) Suppose  $a \sqsubseteq b$ . If  $c \downarrow a$  and  $c \downarrow b$ , then  $a \sqsubseteq_c b$ .
  - (2) Suppose  $a \sqsubseteq_c b$ . If  $c \downarrow ab$  then  $a \sqsubseteq b$ .
  - (3) Suppose  $a \sqsubseteq_c b$ . If  $\text{tp}(a)$  and  $\text{tp}(b)$  are foreign to  $\text{tp}(c)$ , then  $a \sqsubseteq b$ .

*Proof:*

- (1) Consider any  $d$  with  $d \not\downarrow_c a$ . Then  $cd \not\downarrow a$ , whence  $cd \not\downarrow b$ . Now  $b \downarrow c$  implies  $b \not\downarrow_c d$ . The converse follows by symmetry.
- (2) Consider any  $d$  with  $d \not\downarrow a$ . Clearly we may assume  $d \downarrow_{ab} c$ , whence  $abd \downarrow c$ . Since  $a \downarrow c$  we get  $d \not\downarrow_c a$ , whence  $d \not\downarrow_c b$  and  $cd \not\downarrow b$ . But  $c \downarrow_a b$ , so  $d \not\downarrow b$ ; the converse follows by symmetry.
- (3) Consider any  $d$  with  $d \not\downarrow a$ . Since  $a \downarrow c$  we get  $d \not\downarrow_c a$ , whence  $d \not\downarrow_c b$  and  $cd \not\downarrow b$ . If  $b \downarrow d$ , then  $b \downarrow_a c$  by foreignness, whence  $b \downarrow cd$ , a contradiction. So  $b \not\downarrow d$ ; the converse follows by symmetry.  $\square$

**Proposition 5.2.** *Let  $A, B, a, b$  be (hyperimaginary) sets, such that  $a$  is quasi-finite,  $\text{bdd}^{qfu}(Aa) \cap \text{bdd}^{qfu}(Bb) = \text{bdd}^{qfu}(\emptyset)$ , and  $a$  and  $b$  are domination-equivalent over  $AB$ . Suppose  $\text{bdd}(A) = \ell_\alpha^\Sigma(Aa)$  and  $\text{bdd}(B) = \ell_\alpha^\Sigma(Bb)$  for some ordinal  $\alpha > 0$  or  $\alpha = \infty$ . Then  $a \in \text{bdd}(A)$  and  $b \in \text{bdd}(B)$ .*

*Proof:* Suppose otherwise. Put  $A_0 = \ell_\alpha^\Sigma(a)$  and  $B_0 = \ell_\alpha^\Sigma(b)$ ; note that  $A_0 = \text{bdd}(A) \cap \text{bdd}(a)$  and  $B_0 = \text{bdd}(B) \cap \text{bdd}(b)$ . Then  $\text{tp}(a/A_0)$  and  $\text{tp}(b/B_0)$  are foreign to  $\text{tp}(AB)$  by Lemma 3.19. Lemma 5.1(3) yields that  $a$  and  $b$  are domination-equivalent over  $A_0B_0$ . We may thus assume that  $Aa$  is quasi-finite.

Define an  $\emptyset$ -invariant relation  $E$  on  $\text{lstp}(Aa)$  by

$$E(A'a', A''a'') \iff a' \sqsubseteq_{A'A''} a''.$$

Clearly, this is reflexive and symmetric. Suppose  $E(A'a', A''a'')$  and  $E(A''a'', A'''a''')$ . By Lemma 5.1(1)

$$a' \sqsubseteq_{A'A''A'''} a'' \quad \text{and} \quad a'' \sqsubseteq_{A'A''A'''} a''',$$

whence  $a' \sqsubseteq_{A'A''A'''} a'''$ . Now  $a' \sqsubseteq_{A'A''} a'''$  by Lemma 5.1(3). Thus  $E(A'a', A'''a''')$  holds and  $E$  is transitive.

Let  $A'a' \equiv_{Bb}^{lstp} Aa$  with  $A'a' \downarrow_{Bb} Aa$ . Again by Lemma 5.1(1)

$$a \sqsubseteq_{AA'B} b \sqsubseteq_{AA'B} a',$$

and  $a \sqsubseteq_{AA'} a'$  by Lemma 5.1(3). Thus  $E(Aa, A'a')$  holds. But

$$\text{bdd}^{qfu}(Aa) \cap \text{bdd}^{qfu}(A'a') \subseteq \text{bdd}^{qfu}(Aa) \cap \text{bdd}^{qfu}(Bb) = \text{bdd}^{qfu}(\emptyset).$$

Hence  $(Aa)_E = (A'a')_E \in \text{bdd}^{qfu}(\emptyset)$ , and there is  $A''a'' \downarrow Aa$  with  $E(Aa, A''a'')$ . But then  $a \sqsubseteq_{AA''} a''$  and  $a \downarrow_{AA''} a''$  yield  $a \downarrow_{AA''} a$ , whence  $a \in \text{bdd}(AA'')$  and finally  $a \in \text{bdd}(A)$  as  $a \downarrow_A A''$ . Similarly,  $b \in \text{bdd}(B)$ .  $\square$

**Corollary 5.3.** *Let  $A, B, a, b$  be (hyperimaginary) sets, such that  $a$  is quasi-finite,  $\text{bdd}^{qfu}(Aa) \cap \text{bdd}^{qfu}(Bb) = \text{bdd}^{qfu}(\emptyset)$ , and  $a$  and  $b$  are interbounded over  $AB$ . Suppose  $AB$  is  $\Sigma$ -analysable in  $\alpha$  steps, for some ordinal  $\alpha$  or  $\alpha = \infty$ . Then  $Aa$  and  $Bb$  are  $\Sigma$ -analysable in  $\alpha$  steps.*

*Proof:* Clearly we may assume that  $A = \ell_\alpha^\Sigma(Aa)$  and  $B = \ell_\alpha^\Sigma(Bb)$ . Since  $a$  and  $b$  are interbounded over  $AB$ , they are domination-equivalent. So  $a \in \text{bdd}(A)$  and  $b \in \text{bdd}(B)$  by Proposition 5.2.  $\square$

**Remark 5.4.** By Theorem 4.6, if  $SU(Aa)$  or  $SU(Bb)$  is finite, then  $\text{bdd}(Aa) \cap \text{bdd}(Bb) = \text{bdd}(\emptyset)$  implies  $\text{bdd}^{qfu}(Aa) \cap \text{bdd}^{qfu}(Bb) = \text{bdd}^{qfu}(\emptyset)$ , and we recover [5, Lemma 1.17 and Lemma 1.24], putting  $\alpha = 1$  and  $\alpha = \infty$ .

**Fact 5.5.** [8, Theorem 3.4(3)] *Let  $\Sigma'$  be an  $\emptyset$ -invariant subfamily of  $\Sigma$ . Suppose  $\text{tp}(a)$  is  $\Sigma$ -analysable, but foreign to  $\Sigma \setminus \Sigma'$ . Then  $a$  and  $\ell_1^{\Sigma'}(a)$  are domination-equivalent.*

**Corollary 5.6.** *Let  $A \subseteq \text{bdd}(\text{Cb}(B/A))$  consist of quasi-finite hyperimaginaries, with  $\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset)$ . If  $A$  is  $\Sigma$ -analysable and  $\Sigma'$  is the subset of one-based partial types in  $\Sigma$ , then  $A$  is analysable in  $\Sigma \setminus \Sigma'$ .*

*Proof:* Suppose  $A$  is not analysable in  $\Sigma \setminus \Sigma'$ . For every finite tuple  $\bar{a} \in A$  put  $c_{\bar{a}} = \text{Cb}(B/\bar{a})$ , and let  $C = \bigcup \{c_{\bar{a}} : \bar{a} \in A\}$ . Then  $A \downarrow_C B$ , as for any  $\bar{a} \in A$  and  $C$ -indiscernible sequence  $(B_i : i < \omega)$  in  $\text{tp}(B/C)$  the set  $\{\pi(\bar{x}, B_i) : i < \omega\}$  is consistent, where  $\pi(\bar{x}, B) = \text{tp}(\bar{a}/B)$ , since  $\pi(\bar{x}, B)$  does not fork over  $c_{\bar{a}} \subseteq C$ . So  $A \subseteq \text{bdd}(C)$ ; as  $A$  is not analysable in  $\Sigma \setminus \Sigma'$ , neither is  $C$ , and there is  $\bar{a} \in A$  such that  $c = c_{\bar{a}}$  is not analysable in  $\Sigma \setminus \Sigma'$ . Clearly  $c \subseteq \text{bdd}(\bar{a})$  is quasi-finite and  $c = \text{Cb}(B/c)$ . Replacing  $A$  by  $c$  we may thus assume that  $A$  is quasi-finite.

Let  $A' \subseteq \text{bdd}(A)$  and  $B' \subseteq \text{bdd}(B)$  be maximally analysable in  $\Sigma \setminus \Sigma'$ . So  $\text{tp}(A/A')$  and  $\text{tp}(B/B')$  are foreign to  $\Sigma \setminus \Sigma'$ , and  $A \not\subseteq A'$ . Since  $A = \text{Cb}(B/A)$  we get  $A \not\downarrow_{A'} B$ ; as  $A \downarrow_{A'} B'$  by foreignness to  $\Sigma \setminus \Sigma'$ , we obtain  $A \not\downarrow_{A'B'} B$ . In particular  $B \not\subseteq B'$ .

By Fact 5.5 the first  $\Sigma'$ -levels  $a = \ell_1^{\Sigma'}(A/A')$  and  $b = \ell_1^{\Sigma'}(B/B')$  are non-trivial, one-based, and

$$a \sqsubseteq_{A'} A \quad \text{and} \quad b \sqsubseteq_{B'} B.$$

Since  $\text{tp}(Aa/A')$  is foreign to  $\Sigma \setminus \Sigma'$ , we have  $Aa \downarrow_{A'} B'$ , whence  $a \sqsubseteq_{A'B'} A$  by Lemma 5.1(1). Similarly  $b \sqsubseteq_{A'B'} B$ . But  $A \not\downarrow_{A'B'} B$ , and thus

$a \not\perp_{A'B'} b$ . Let  $a_0 = \text{bdd}(A'a) \cap \text{bdd}(A'B'b)$  and  $b_0 = \text{bdd}(B'b) \cap \text{bdd}(A'B'a)$ . By one-basedness of  $\text{tp}(a/A')$  and  $\text{tp}(b/B')$ ,

$$a \downarrow_{A'a_0} B'b \quad \text{and} \quad b \downarrow_{B'b_0} A'a.$$

Hence

$$A'B'a \downarrow_{A'B'a_0} b_0 \quad \text{and} \quad A'B'b \downarrow_{A'B'b_0} a_0.$$

It follows that  $a_0$  and  $b_0$  are interbounded over  $A'B'$ . We can now apply Corollary 5.3 to see that  $a_0$  is analysable in  $\Sigma \setminus \Sigma'$ , whence  $a_0 \in A'$ . But then  $a \downarrow_{A'B'} b$ , a contradiction.  $\square$

**Remark 5.7.** In a theory of finite  $SU$ -rank, due to weak elimination of quasi-finitary ultrimaginaries, we obtain that for any  $A, B$

$$\text{tp}(\text{Cb}(A/B)/\text{bdd}(A) \cap \text{bdd}(B))$$

is analysable in the collection of non one-based types of  $SU$ -rank 1.

**Remark 5.8.** Without the quasi-finite hypothesis in Proposition 5.2, Corollary 5.3 and Corollary 5.6, the conclusions still hold if we assume that the full ultrimaginary bounded closures intersect trivially.

The following Theorem generalizes [5, Proposition 1.16] to super-simple theories of infinite rank, at the price of demanding that the quasifinite ultrimaginary bounded closures intersect trivially, rather than just the bounded closures. The proof is essentially the same, but we have to work with ultrimaginaries at key steps. Of course, in finite rank this is equivalent, due to elimination of quasifinite hyperimaginaries; moreover, the families  $\Sigma_i$  in the Theorem are just different orthogonality classes of regular types of rank 1.

**Definition 5.9.** Two  $\emptyset$ -invariant families  $\Sigma$  and  $\Sigma'$  are *perpendicular* if no realization of a type in  $\Sigma$  can fork with a realisation of a type in  $\Sigma'$ .

**Example 5.10.** If  $p$  and  $p'$  are two orthogonal types of  $SU$ -rank 1 non-orthogonal to  $\emptyset$  (or whose  $\emptyset$ -conjugates remain orthogonal), then the families of  $\emptyset$ -conjugates of  $p$  and of  $p'$  are perpendicular.

**Theorem 5.11.** *Let  $T$  be supersimple. Suppose  $A \subseteq \text{bdd}(\text{Cb}(B/A))$  and  $B \subseteq \text{bdd}(\text{Cb}(A/B))$ , with  $\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset)$ . Let  $(\Sigma_i : i \in I)$  be a family of pairwise perpendicular  $\emptyset$ -invariant families of partial types such that  $A$  is analysable in  $\bigcup_{i \in I} \Sigma_i$ . For  $i \in I$  let  $A_i$  and  $B_i$  be the maximal  $\Sigma_i$ -analysable subset of  $\text{bdd}(A)$  and  $\text{bdd}(B)$ , respectively. Then  $A \subseteq \text{bdd}(A_i : i < \alpha)$  and  $B \subseteq \text{bdd}(B_i : i < \alpha)$ ; moreover  $A_i = \text{bdd}(\text{Cb}(B_i/A))$  and  $B_i = \text{bdd}(\text{Cb}(A_i/B))$ . If  $\Sigma_i$  is one-based, then  $A_i = B_i = \text{bdd}(\emptyset)$ .*

**Remark 5.12.** If  $C \perp AB$ , then  $\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset)$  implies  $\text{bdd}^{qfu}(AC) \cap \text{bdd}^{qfu}(BC) = \text{bdd}^{qfu}(C)$  by Lemma 3.23. Hence Theorem 5.11 applies over  $C$ ; this can serve to refine the decomposition.

*Proof:* Since  $\text{Cb}(A_i/B)$  is  $\text{tp}(A_i)$ -analysable and hence  $\Sigma_i$ -analysable, we have  $\text{Cb}(A_i/B) \subseteq B_i$ ; similarly  $\text{Cb}(B_i/A) \subseteq A_i$ . As the families in  $(\Sigma_i : i \in I)$  are perpendicular, we obtain

$$(A_i : i \in I) \perp_{(B_i : i \in I)} B \quad \text{and} \quad (B_i : i \in I) \perp_{(A_i : i \in I)} A.$$

Suppose  $A \subseteq \text{bdd}(A_i : i \in I)$ . Then  $B = \text{Cb}(A/B) \subseteq \text{bdd}(B_i : i \in I)$ ; moreover

$$\begin{aligned} \text{bdd}(A) &= \text{bdd}(\text{Cb}(B/A)) = \text{bdd}(\text{Cb}(B_i/A) : i \in I) \\ &= \text{bdd}(\text{Cb}(B_i/A_i) : i \in I) \subseteq \text{bdd}(A_i : i \in I) = \text{bdd}(A) \end{aligned}$$

again by perpendicularity. Hence  $\text{bdd}(\text{Cb}(B_i/A_i)) = A_i$ , and similarly  $\text{bdd}(\text{Cb}(A_i/B_i)) = B_i$ . But if  $\Sigma_i$  is one-based, then

$$B_i = \text{bdd}(\text{Cb}(A_i/B_i)) \subseteq \text{bdd}(A_i) \cap \text{bdd}(B_i) = \text{bdd}(\emptyset) ;$$

similarly  $A_i = \text{bdd}(\emptyset)$ .

Put  $\bar{A} = \text{bdd}(A_i : i \in I)$  and  $\bar{B} = \text{bdd}(B_i : i \in I)$ . It remains to show that  $A \subseteq \bar{A}$ . So suppose not. As in the proof of Corollary 5.6 put  $c_{\bar{a}} = \text{Cb}(B/\bar{a})$  for every finite tuple  $\bar{a} \in A$ , and let  $C = \bigcup \{c_{\bar{a}} : \bar{a} \in A\}$ . Then again  $A \perp_C B$  and  $A \subseteq \text{bdd}(C)$ ; moreover  $c_{\bar{a}} = \text{Cb}(B/c_{\bar{a}})$ . Since  $A$  is not contained in  $\bar{A}$ , neither is  $C$ . Hence there is  $\bar{a} \in A$  such that  $c = c_{\bar{a}} \notin \bar{A}$ . As the maximal  $\Sigma_i$ -analysable subset of  $\text{bdd}(c)$  is equal to  $\text{bdd}(c) \cap A_i$  we may replace  $A$  by  $c$  and thus assume that  $A$  is quasi-finite. Similarly, we may assume that  $B$  is quasi-finite.

Since  $A = \text{Cb}(B/A) \not\subseteq \bar{A}$ , we have  $A \not\perp_{\bar{A}} B$ ; as  $A \perp_{\bar{A}} \bar{B}$  we obtain  $A \not\perp_{\bar{A}\bar{B}} B$ . Let  $(b_j : j < \alpha)$  be an analysis of  $B$  over  $\bar{B}$  such that for every  $j < \alpha$  the type  $\text{tp}(b_j/\bar{B}, b_\ell : \ell < j)$  is  $\Sigma_{i_j}$ -analysable for some  $i_j \in I$ . Let  $k$  be minimal with  $A \not\perp_{\bar{A}\bar{B}}(b_j : j \leq k)$ . Then  $A \perp_{\bar{A}} \bar{B}$ ,  $(b_j : j < k)$  and  $\text{Cb}(\bar{B}, (b_j : j \leq k)/A)$  is almost  $\Sigma_{i_k}$ -internal over  $\bar{A}$ . Put  $A' = \ell_1^{\Sigma_{i_k}}(A/\bar{A})$  and  $B' = \ell_1^{\Sigma_{i_k}}(B/\bar{B})$ . Then  $A' \not\subseteq \bar{A}$ , and  $\text{Cb}(A'/B) \subseteq B'$  since  $\bar{A} \perp_{\bar{B}} B$ . Similarly  $\text{Cb}(B'/A) \subseteq A'$ . Moreover  $A' \not\perp_{\bar{A}\bar{B}} B$ , whence  $A' \not\perp_{\bar{A}\bar{B}} B'$ . Replacing  $A$  by  $\text{Cb}(B'/A) = \text{Cb}(B'/A')$  and  $\bar{B}$  by  $\text{Cb}(A'/B) = \text{Cb}(A'/B')$  we may assume that  $\text{tp}(A/\bar{A})$  and  $\text{tp}(B/\bar{B})$  are both almost  $\Sigma_k$ -internal (where we write  $k$  instead of  $i_k$  for ease of notation).

**Claim.**  $\text{bdd}^{qfu}(AB_k) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(B_k)$ .

*Proof of Claim:* Suppose not. As  $B$  is analysable in  $\bigcup_{i \in I} \Sigma_i$ , Corollary 3.21 yields some  $i \in I$  and

$$d \in (\text{bdd}^{qfu}(AB_k) \cap \text{bdd}^{qfu}(B)) \setminus \text{bdd}^{qfu}(B_k)$$

such that  $d$  is almost  $\Sigma_i$ -internal over  $B_k$ ; since  $\text{tp}(B/B_k)$  is foreign to  $\Sigma_k$  we have  $i \neq k$ . Hence  $A \downarrow_{\bar{A}B_k} d$ , whence  $d \in \text{bdd}^{qfu}(\bar{A}B_k)$  by Lemma 3.6. But  $\bar{A} = \text{bdd}(A_i : i \in I)$  and  $d \downarrow_{A_i B_k} \bar{A}$  by almost  $\Sigma_i$ -internality of  $d$  over  $B_k$ , whence  $d \in \text{bdd}^{qfu}(A_i B_k)$ . If  $B_k d \downarrow A_i$ , then  $d \downarrow_{B_k} A_i$  and  $d \in \text{bdd}^{qfu} B_k$  by Lemma 3.6, contradicting the choice of  $d$ . Therefore  $B_k d \not\downarrow A_i$ ; by Corollary 3.21 there is almost  $\Sigma_i$ -internal

$$d' \in \text{bdd}^{qfu}(B_k d) \setminus \text{bdd}^{qfu}(\emptyset).$$

Note that  $d' \in \text{bdd}^{qfu}(A_i B_k) \cap \text{bdd}^{qfu}(B)$ . But then  $d' A_i \downarrow B_k$ , whence  $d' \downarrow_{A_i} B_k$  and

$$d' \in \text{bdd}^{qfu}(A_i) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset),$$

a contradiction.  $\square$

**Claim.** *We may assume  $B_k = \text{bdd}(\emptyset)$ .*

*Proof of Claim:* Put  $A' = \text{Cb}(B/AB_k)$ . Then  $B_k \subset A' = \text{Cb}(B/A')$ , and  $\text{bdd}(A')^{qfu} \cap \text{bdd}(B)^{qfu} = \text{bdd}^{qfu}(B_k)$ . If  $B' = \text{Cb}(A'/B) = \text{Cb}(A'/B')$ , then  $A' \downarrow_{B'} B$  and  $A \downarrow_{A'} B$  yield  $B \downarrow_{B'} A$  by transitivity, since  $B' \subseteq \text{bdd}(B)$ . Thus  $B \subset \text{bdd}(B')$ . We add  $\bar{B}_k$  to the language; note that  $B_k \neq \text{bdd}(\emptyset)$  implies  $B \not\downarrow B_k$ , whence  $SU(B'/B_k) < SU(B)$ . By induction it is thus sufficient to show that  $A', B'$  is still a counterexample over  $B_k$ .

So suppose not, and let  $\text{bdd}(A') = \text{bdd}(A'_i : i \in I)$  and  $\text{bdd}(B') = \text{bdd}(B'_i : i \in I)$  be decompositions, where  $A'_i$  and  $B'_i$  are maximally  $\Sigma_i$ -analysable over  $B_k$  in  $\text{bdd}(A')$  and  $\text{bdd}(B')$ , respectively. So  $B'_k$  is  $\Sigma_k$ -analysable, whence  $B'_k = \bar{B}_k \subseteq \bar{B}$  by maximality. Since  $B \subseteq \text{bdd}(B')$  is almost  $\Sigma_k$ -internal over  $\bar{B}$  and  $(B'_i : i \neq k)$  is foreign to  $\Sigma_k$ , we get  $B \downarrow_{\bar{B}} (B'_i : i \neq k)$ , whence  $B \subset \bar{B}$ , a contradiction.  $\square$

By symmetry, we may also assume  $A_k = \text{bdd}(\emptyset)$ .

Put  $B' = \text{Cb}(B/A\bar{B})$ . Then  $\bar{B} \subseteq \text{bdd}(B')$ , and since  $B$  is almost  $\Sigma_k$ -internal over  $\bar{B}$ , so is  $B'$ . If  $A' = \text{Cb}(B'/A)$ , then  $B' \downarrow_{A'} A$  and  $A \downarrow_{B'} B$  yield  $A \downarrow_{A'} B$ , since  $A' \subseteq \text{bdd}(A)$ . Thus  $A \subseteq \text{bdd}(A')$ . Put  $B'' = \text{Cb}(A/B') = \text{Cb}(A/B'')$ . Then

$$B'' \subseteq \text{bdd}(B') \subseteq \text{bdd}(A\bar{B}),$$

and  $B''$  is almost  $\Sigma_k$ -internal over  $\bar{B}$ . Moreover,  $A \perp_{B''} B'$  implies

$$A \subseteq \text{bdd}(\text{Cb}(B'/A)) \subseteq \text{bdd}(\text{Cb}(B''/A)).$$

**Claim.**  $\text{bdd}^{qfu}(A') \cap \text{bdd}^{qfu}(B'') = \text{bdd}^{qfu}(\emptyset)$ .

*Proof of Claim:* Suppose not. By Corollary 3.21 there is  $i \in I$  and

$$d \in (\text{bdd}^{qfu}(A') \cap \text{bdd}^{qfu}(B'')) \setminus \text{bdd}^{qfu}(\emptyset)$$

which is almost  $\Sigma_i$ -internal; since  $A'$  is foreign to  $\Sigma_k$  we have  $i \neq k$ . As  $B''$  is almost  $\Sigma_k$ -internal over  $\bar{B}$  we have  $d \perp_{\bar{B}} B''$ , whence

$$d \in \text{bdd}^{qfu}(A') \cap \text{bdd}^{qfu}(\bar{B}) \subseteq \text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset),$$

a contradiction.  $\square$

Thus  $A', B''$  is another counterexample; by induction on  $SU(AB)$  we may assume  $\text{bdd}(A'B'') = \text{bdd}(AB)$ . But then

$$B \subset \text{bdd}(A'B'') \subseteq \text{bdd}(AB') \subseteq \text{bdd}(A\bar{B}).$$

By symmetry  $A \subset \text{bdd}(B\bar{A})$ . Since  $\bar{A}\bar{B}$  are analysable in  $\bigcup_{i \neq k} \Sigma_i$ , so are  $A$  and  $B$  by Corollary 5.3. But  $\text{tp}(A/\bar{A})$  is almost  $\Sigma_k$ -internal, whence foreign to  $\bigcup_{i \neq k} \Sigma_i$ , yielding the final contradiction.  $\square$

**Remark 5.13.** In the finite rank context, it is easy to achieve the hypothesis of Theorem 5.11, as it suffices work over  $\text{bdd}(A) \cap \text{bdd}(B)$ . In general, however, if

$$\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) \not\supseteq \text{bdd}^{qfu}(\text{bdd}(A) \cap \text{bdd}(B)),$$

there is no hyperimaginary set  $C$  with

$$\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(C),$$

as this equality implies  $\text{bdd}(C) = \text{bdd}(A) \cap \text{bdd}(B)$ . Thus, we cannot work over  $\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B)$ , which is not eliminable. Feeble elimination nevertheless yields

$$\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) \subset \text{bdd}^{qfu}(\text{cl}_\alpha(A) \cap \text{cl}_\alpha(B))$$

if  $SU(A/\text{bdd}(A) \cap \text{bdd}(B)) < \omega^{\alpha+1}$ , so we can work over  $\alpha$ -closed sets, as is done in [8, Theorem 5.4].

**Corollary 5.14.** *Let  $T$  be supersimple, and  $\Sigma_1$  and  $\Sigma_2$  two perpendicular  $\emptyset$ -invariant families of partial types. Suppose  $a$  is quasi-finite,  $\text{tp}(a)$  is analysable in  $\Sigma_1 \cup \Sigma_2$  and  $\text{tp}(a/A)$  is  $\Sigma_1$ -analysable, with  $\text{bdd}^{qfu}(a) \cap \text{bdd}^{qfu}(A) = \text{bdd}^{qfu}(\emptyset)$ . Then  $\text{tp}(a)$  is  $\Sigma_1$ -analysable.*

*Proof:* Clearly we may assume that  $A = \text{Cb}(a/A)$ . If  $a' = \text{Cb}(A/a)$ , then  $A$  is interbounded with  $\text{Cb}(a'/A)$ . Moreover, as  $\text{tp}(a/a')$  is  $\Sigma_1$ -analysable,  $\text{tp}(a')$  is  $\Sigma_1$ -analysable if and only if  $\text{tp}(a)$  is. So we may assume in addition that  $a = \text{Cb}(A/a)$ .

By Theorem 5.11 we have

$$\text{bdd}(a) = \text{bdd}(\ell_{\infty}^{\Sigma_1}(a), \ell_{\infty}^{\Sigma_2}(a)).$$

Hence  $\text{tp}(\ell_{\infty}^{\Sigma_2}(a)/A)$  is  $\Sigma_1$ -analysable. By perpendicularity,

$$\ell_{\infty}^{\Sigma_2}(a) \in \text{bdd}(A) \cap \text{bdd}(a) = \text{bdd}(\emptyset).$$

Hence  $a \in \ell_{\infty}^{\Sigma_1}(a)$  is  $\Sigma_1$ -analysable.  $\square$

For  $SU(a)$  finite, this specialises to [5, Proposition 1.20]

## REFERENCES

- [1] Itai Ben Yaacov. *Discouraging Results for Ultraimaginary Independence Theory*, J. Symb. Logic 68:846–850, 2003.
- [2] Itai Ben Yaacov, Ivan Tomasic and Frank O. Wagner. *The group configuration in simple theories and its applications*, Bull. Symb. Logic 8:283–298, 2002.
- [3] Itai Ben Yaacov, Ivan Tomasic and Frank O. Wagner. *Constructing an almost hyperdefinable group*, J. Math. Logic 4(2):181–212, 2005.
- [4] Enrique Casanovas. *Simple Theories and Hyperimaginaries*, Lecture Notes in Logic 39. Cambridge University Press, Cambridge, GB, 2011.
- [5] Zoé Chatzidakis. *A note on canonical bases and modular types in supersimple theories*, preprint, September 2002, revised December 2011. arXiv 1201.0134
- [6] Daniel Lascar. *Sous groupes d'automorphismes d'une structure saturée*. In: *Logic Colloquium '82*, Stud. Logic Found. Math., vol. 112, North-Holland, Amsterdam, 1984, pp. 123–134.
- [7] Rahim Moosa and Anand Pillay. *On canonical bases and internality criteria*, III. J. Math. 52:901–917, 2008.
- [8] Daniel Palacín and Frank O. Wagner. *Ample thoughts*, preprint 2011.
- [9] Anand Pillay. *The geometry of forking and groups of finite Morley rank*, J. Symb. Logic 60:1251–1259, 1995.
- [10] Anand Pillay. *Geometric stability theory*. Oxford Logic Guides 32. Oxford University Press, Oxford, GB, 1996.
- [11] Frank O. Wagner. *Simple Theories*. Mathematics and Its Applications 503. Kluwer Academic Publishers, Dordrecht, NL, 2000.
- [12] Frank O. Wagner. *Some remarks on one-basedness*, J. Symb. Logic 69:34–38, 2004.

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