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PLUS ULTRA

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ABSTRACT. We develop some basic simplicity-theoretic facts for quasi-finitary ultrimaginaries, i.e. classes of finite tuples modulo \emptyset -invariant equivalence relations, in a supersimple theory. We also show that they are geometrically eliminable in a weak sense: If e is an ultrimaginary definable over a tupe a with $SU(a) < \omega^{\alpha+1}$, then e is eliminable up to rank $< \omega^{\alpha}$.

1. INTRODUCTION

This paper arose out of an attempt to understand and generalize [2, Proposition 1.10 and Lemma 1.11]. In doing so, we realized that certain stability-theoretic phenomena were best explained using ultrimaginaries. Although Ben Yaacov [1] has shown that no satisfactory independence theory can exist for all ultrimaginaries, as there are problems both with the finite character and with the extension axiom for independence, at least finite character can be salvaged if one restricts to quasi-finitary ultrimaginaries in a supersimple theory. This enables us to recover certain tools from simplicity theory, even though, due to the lack of extension, canonical bases are not available.

2. ULTRAIMAGINARIES

Definition 2.1. An *ultrimaginary* is the class a_E of a tuple a under an \emptyset -invariant equivalence relation E .

Clearly, we may assume that a is a countable tuple.

Definition 2.2. An ultrimaginary a_E is *definable* over a set A if any automorphism of the monster model fixing A stabilises the E -class of a . It is *bounded* over A if the orbit of a under the group of automorphisms

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of the monster model which fix A is contained in boundedly many E -classes. A *representative* of an ultraimaginary e is any tuple a such that e is definable over a .

Remark 2.3. As usual, if $E_A(x, y)$ is an A -invariant equivalence relation, one considers the \emptyset -invariant relation $E(xX, yY)$ given by

$$(X = Y \wedge X \equiv A \wedge E_X(x, y)) \vee (X = Y \wedge x = y).$$

This is an equivalence relation, and $(aA)_E$ is interdefinable over A with a_{E_A} .

Remark 2.4. As any \emptyset -invariant relation, E is given by a union of types over \emptyset .

Ultraimaginaries arise quite naturally in stability and simplicity theory.

Example. Let $p_A \in S(A)$ be a regular type. For $A', A'' \models \text{tp}(A)$ put $E(A', A'')$ if $p_{A'} \not\perp p_{A''}$. Then E is an \emptyset -invariant equivalence relation, and A_E codes the non-orthogonality class of p_A .

Contrary to the case of *hyperimaginaries* (classes modulo type-definable equivalence relations) or even *almost hyperimaginaries* (ultraimaginaries which are interbounded with a hyperimaginary), the work with ultraimaginaries requires caution, as we lose compactness when we consider them as some kind of elements in our structure. For two ultraimaginaries a and b in a simple theory we define $a \downarrow b$ if they have representatives which are independent. This notion has to be handled with care, as not all of the usual properties hold.

Example. [1] Let E be the \emptyset -invariant equivalence relation on infinite sequences which holds if they differ on only finitely many elements. Consider a sequence $I = (a_i : i < \omega)$ of elements such that no finite subtuple is bounded over the remaining elements. Then every finite tuple $\bar{a} \in I$ can be moved to a disjoint conjugate over I_E , but I cannot. Similarly, if I is a Morley sequence in a simple theory, then $\bar{a} \downarrow I_E$ for any finite $\bar{a} \in I$, but $I \not\downarrow I_E$. If I is a sequence of independent realizations of pairwise orthogonal (or even perpendicular) regular types, then I_E is orthogonal (or perpendicular) to any finite subset of them, but not to all of them simultaneously.

Even in the ω -stable context, for classes of finite tuples, the theory is not smooth.

Example. Let T be the theory of a cycle-free graph (forest) of infinite valency, with predicates $P_n(x, y)$ for couples of points of distance n for all $n < \omega$. It is easy to see by back-and-forth that T eliminates quantifiers and is ω -stable of rank ω ; the formula $P_n(x, a)$ has rank n over a . Let E be the \emptyset -invariant equivalence relation of being in the same connected component. Then existence of non-forking extensions fails over a_E , as any two points in the connected component of a have some finite distance n , and hence rank n over one another, but rank $\geq k$ over a_E for all $k < \omega$, since a_E is definable over a point of distance k .

The behaviour of Example 2 is inconvenient and signifies that we shall avoid working *over* an ultrimaginary. The behaviour of Example 2 is outright vexatious; we shall restrict the class of ultrimaginaries under consideration in order to preserve the finite character of independence.

Definition 2.5. Let T be simple. An ultrimaginary e is *tame* if for all hyperimaginary A, B we have $e \downarrow_A B$ if and only if $e \downarrow_A \bar{b}$ for all finite $\bar{b} \in B$. It is *quasi-finitary* if there is hyperimaginary A in $\text{bdd}(e)$ and a finite tuple a such that e is bounded over Aa .

The set of all / all tame / all quasi-finite ultrimaginaries definable over A will be denoted by $\text{dcl}^u(A)$ / $\text{dcl}^{tu}(A)$ / $\text{dcl}^{qfu}(A)$, respectively. Similarly, $\text{bdd}^u(A)$ / $\text{bdd}^{tu}(A)$ / $\text{bdd}^{qfu}(A)$ will denote the corresponding bounded closures.

Recall that two tuples a and b have the same *Lascar strong type* over A , denoted $a \equiv_A^{\text{lstp}} b$ or $b \models \text{lstp}(a/A)$, if they lie in the same class modulo all A -invariant equivalence relations with only boundedly many classes. This is the finest bounded A -invariant equivalence relation, so $\text{bdd}^u(A)$ is bounded by the number of Lascar strong types over A .

Proposition 2.6. *The following are equivalent:*

- (1) $\text{bdd}^u(Aa) \cap \text{bdd}^u(Ab) = \text{bdd}^u(A)$.
- (2) *For any $a' \models \text{lstp}(a/A)$ there is $n < \omega$ and a sequence $(a_i, b_i : i \leq n)$ such that*

$$a_0 = a, \quad b_0 = b, \quad a_n = a'$$

and for each $i < n$

$$b_{i+1} \models \text{lstp}(b_i/Aa_i) \quad \text{and} \quad a_{i+1} \models \text{lstp}(a_i/Ab_{i+1}).$$

If a and b are finite, this is also equivalent to $\text{bdd}^{qfu}(Aa) \cap \text{bdd}^{qfu}(Ab) = \text{bdd}^{qfu}(A)$.

Proof: (1) \Rightarrow (2) Suppose $\text{bdd}^u(Aa) \cap \text{bdd}^u(Ab) = \text{bdd}^u(A)$, and define an A -invariant relation on $\text{lstp}(ab/A)$ by $E(xy, x'y')$ if there is a sequence $(x_i, y_i : i \leq n)$ such that

$$ab \equiv_A^{\text{lstp}} x_0 y_0, \quad x_0 y_0 = xy, \quad x_n y_n = x' y'$$

and for each $i < n$

$$y_{i+1} \models \text{lstp}(y_i/Ax_i) \quad \text{and} \quad x_{i+1} \models \text{lstp}(x_i/Ay_{i+1}).$$

Clearly E is an equivalence relation. Now if $b' \models \text{lstp}(b/Aa)$, then $\models E(ab, ab')$. Hence $(ab)_E \in \text{bdd}^u(Aa)$. Similarly $(ab)_E \in \text{bdd}^u(Ab)$, whence $(ab)_E \in \text{bdd}^u(A)$. But for any $a' \models \text{lstp}(a/A)$ there is b' with $ab \equiv_A^{\text{lstp}} a'b'$. Then $\models E(ab, a'b')$; in particular (2) holds.

(2) \Rightarrow (1) Suppose not, and consider $e \in (\text{bdd}^u(Aa) \cap \text{bdd}^u(Ab)) \setminus \text{bdd}^u(A)$. As $e \notin \text{bdd}^u(A)$ there is $a' \models \text{lstp}(a/A)$ with $e \notin \text{bdd}^u(Aa) \cap \text{bdd}^u(Aa')$. Consider a sequence $(a_i, b_i : i \leq n)$ as in (2). Since $b_{i+1} \models \text{lstp}(b_i/Aa_i)$ and $a_{i+1} \models \text{lstp}(a_i/Ab_{i+1})$ we have

$$\begin{aligned} \text{bdd}^u(Aa_i) \cap \text{bdd}^u(Ab_i) &= \text{bdd}^u(Aa_i) \cap \text{bdd}^u(Ab_{i+1}) \\ &= \text{bdd}^u(Aa_{i+1}) \cap \text{bdd}^u(Ab_{i+1}). \end{aligned}$$

In particular,

$$\begin{aligned} e \in \text{bdd}^u(Aa) \cap \text{bdd}^u(Ab) &= \text{bdd}^u(Aa_0) \cap \text{bdd}^u(Ab_0) \\ &\subseteq \text{bdd}^u(Aa) \cap \text{bdd}^u(Aa'), \end{aligned}$$

a contradiction.

The last assertion follows from the fact that for finite ab the ultraimaginary $(ab)_E$ in the proof of (1) \Rightarrow (2) is quasi-finitary. \square

3. ULTRAIMAGINARIES IN SIMPLE THEORIES

From now on the ambient theory will be simple. Our notation is standard and follows [6]. We shall be working in a sufficiently saturated model of the ambient theory. Tuples are tuples of hyperimaginaries, and closures (definable, algebraic and bounded closures) will include hyperimaginaries.

Remark 3.1. Since in a simple theory Lascar strong type equals Kim-Pillay strong type, we have $\text{bdd}^u(A) = \text{dcl}^u(\text{bdd}(A))$. But of course, as with real and imaginary algebraic closures, $\text{bdd}(A) \cap \text{bdd}(B) = \text{bdd}(\emptyset)$ does not imply $\text{bdd}^u(A) \cap \text{bdd}^u(B) = \text{bdd}^u(\emptyset)$.

We shall first see that ultraimaginary bounded closures of independent sets intersect trivially.

Lemma 3.2. *If $A \downarrow_B C$, then $\text{bdd}^u(A) \cap \text{bdd}^u(C) \subseteq \text{bdd}^u(B)$.*

Proof: Consider $a_E \in \text{bdd}^u(A) \cap \text{bdd}^u(C)$. We may assume $a \downarrow_{AB} C$, whence $Aa \downarrow_B C$. Let $(a_i : i < \omega)$ be a Morley sequence in $\text{lstp}(a/BC)$. Then $E(a_i, a_j)$ for all $i, j < \omega$. But $a_i \downarrow_B a_j$ for $i \neq j$, so $\pi(x, a_j) = \text{tp}(a_i/a_j)$ does not fork over B , and neither does $\pi(x, a)$.

Now suppose $a_E \notin \text{bdd}^u(B)$. We can then find a long sequence $(a'_i : i < \alpha)$ of B -conjugates of a such that $\neg E(a'_i, a'_j)$ for $i \neq j$. By the Erdős-Rado theorem and compactness there is an infinite B -indiscernible sequence $(a''_i : i < \omega)$ whose 2-type over B is among the 2-types of $(a'_i : i < \alpha)$. In particular $\neg E(a''_i, a''_j)$ for $i \neq j$, and $(\pi(x, a''_i) : i < \omega)$ is 2-inconsistent. Since $a''_0 \models \text{tp}(a/B)$, we see that $\pi(x, a)$ divides over B , a contradiction. \square

Definition 3.3. We shall say that an ultraimaginary e is *independent* over A to B , denoted $e \downarrow_A B$, if it has a representative which is.

Even if not all ultraimaginarys satisfy finite character of independence, transitivity still holds.

Lemma 3.4. *Let T be simple, a and b hyperimaginary, and e ultraimaginary. If $a \downarrow e$ and $b \downarrow_a e$, then $ab \downarrow e$.*

Proof: Suppose $e = c_E$. By hypothesis there are $c' \equiv c \equiv c''$ with $c_E = c'_E = c''_E$ and $a \downarrow c'$ as well as $b \downarrow_a c''$. Consider $c''' \equiv_{ac''} c'$ with $c''' \downarrow_{ac''} b$. Then $b \downarrow_a c'''$; as $c' \downarrow a$ implies $c''' \downarrow a$ we get $c''' \downarrow ab$, and clearly $E(c', c'')$ yields $E(c''', c'')$, whence $c'''_E = c''_E = e$. \square

Corollary 3.5. *Supposer a_E is an ultraimaginary with $SU(a) < \infty$. Then a_E is tame. In a supersimple theory, quasifinite ultraimaginarys are tame.*

Proof: Consider sets A and B . As $SU(a) < \infty$ there is a finite $b \in B$ with $a \downarrow_{Ab} B$. So $a_E \downarrow_A B$ if and only if $a_E \downarrow_A b$ by Lemma 3.4.

If e is quasifinite, $A \subseteq \text{bdd}(e)$ and a is a finite tuple with $e \in \text{bdd}^u(Aa)$, then for any B, C we have $e \downarrow_B C$ if and only if $A \downarrow_B C$ and $e \downarrow_{AB} C$. The result follows from the previous paragraph, working over A . \square

In particular, in a supersimple theory quasifinite ultraimaginarys are the correct ones to consider: Due to elimination of hyperimaginarys all parameters consist of imaginarys of ordinal SU -rank; as canonical bases of such imaginarys are finite, we can always reduce to a quasifinite situation.

The following two Propositions tells us how to obtain invariant equivalence relations, and hence ultrimaginaries.

Proposition 3.6. *Let T be stable. For algebraically closed A and an \emptyset -invariant equivalence relation E on $\text{tp}(b)$, consider the relation $R(X, Y)$ given by*

$$\exists xy [Xx \equiv Yy \equiv Ab \wedge x \underset{X}{\downarrow} Y \wedge y \underset{Y}{\downarrow} X \wedge E(x, y)].$$

Then R is an \emptyset -invariant equivalence relation.

Proof: Clearly, R is \emptyset -invariant, reflexive and symmetric. So suppose that $R(A, A')$ and $R(A', A'')$ both hold, and let this be witnessed by b, b' and b^*, b'' . Let $b_1 \models \text{tp}(b'/A) = \text{tp}(b^*/A')$ with $b' \underset{A'}{\downarrow} AA''$. Since A' is algebraically closed, $b' \underset{A'}{\downarrow} A$ and $b^* \underset{A'}{\downarrow} A''$ we have $b_1 \equiv_{AA'} b'$ and $b_1 \equiv_{A'A''} b^*$. Hence there are b_0, b_2 with $bb' \equiv_{AA'} b_0b_1$ and $b^*b'' \equiv_{A'A''} b_1b_2$. In particular $E(b_0, b_1)$ and $E(b_1, b_2)$ hold, and so does $E(b_0, b_2)$. Moreover, we may assume $b_0 \underset{AA'b_1}{\downarrow} A''$ and $b_2 \underset{A'A''b_1}{\downarrow} A'$. Now $b_1 \underset{A'}{\downarrow} AA''$ implies $b_0 \underset{AA'}{\downarrow} A''$ and $b_2 \underset{A'A''}{\downarrow} A$. Then $b_0 \underset{A}{\downarrow} A'$ and $b_2 \underset{A''}{\downarrow} A'$ imply $b_0 \underset{A}{\downarrow} A''$ and $b_2 \underset{A''}{\downarrow} A$, whence $R(A, A'')$ holds. So R is transitive. \square

Proposition 3.7. *Let T be simple. Suppose R is an \emptyset -invariant, reflexive, symmetric and generically transitive relation on $\text{lstp}(a)$, i.e. for $a_1, a_2, a_3 \models \text{lstp}(a)$ with $a_1 \underset{a_2}{\downarrow} a_3$ and $R(a_1, a_2)$ and $R(a_2, a_3)$ we also have $R(a_1, a_3)$. Let E be the transitive closure of R , and suppose $a_E \in \text{bdd}^u(\emptyset)$. Suppose further that p is a regular type such that $SU_p(a)$ is finite. Then there is $a' \underset{\text{cl}_p(\emptyset)}{\downarrow} a$ with $R(a, a')$.*

Proof: Let $a' \equiv^{\text{lstp}} a$ with $a' \underset{a}{\downarrow} a$. Then $a_E = a'_E$, so there is $n < \omega$ and a chain $a = a_0, a_1, \dots, a_n = a'$ such that $R(a_i, a_{i+1})$ holds for all $i < n$. Put $a'_0 = a_0$, and for $0 < i < n$ let

$$a'_i \equiv_{a_i}^{\text{lstp}} a'_{i-1} \quad \text{with} \quad a'_i \underset{a_i}{\downarrow} a_{i+1}.$$

Claim. $\text{bdd}^u(a'_i) \cap \text{bdd}^u(a_{i+1}) \subseteq \text{bdd}^u(a_0)$.

Proof of Claim: For $i = 0$ this is trivial. For $i > 0$, as $a'_i \equiv_{a_i}^{\text{lstp}} a'_{i-1}$ and $\text{bdd}^u(a_i) = \text{dcl}^u(\text{bdd}(a_i))$, we get

$$\text{bdd}^u(a'_i) \cap \text{bdd}^u(a_i) = \text{bdd}^u(a'_{i-1}) \cap \text{bdd}^u(a_i).$$

Next, $a'_i \underset{a_i}{\downarrow} a_{i+1}$ implies

$$\text{bdd}^u(a'_i a_i) \cap \text{bdd}^u(a_i a_{i+1}) = \text{bdd}^u(a_i)$$

by Lemma 3.2. Hence inductively

$$\begin{aligned} \text{bdd}^u(a'_i) \cap \text{bdd}^u(a_{i+1}) &\subseteq \text{bdd}^u(a'_i) \cap \text{bdd}^u(a_i) \\ &= \text{bdd}^u(a'_{i-1}) \cap \text{bdd}^u(a_i) \\ &\subseteq \text{bdd}^u(a_0). \quad \square \end{aligned}$$

Now by generic transitivity and induction, $R(a'_i, a_{i+1})$ holds for all $i < n$. In particular $R(a'_{n-1}, a_n)$ holds, and by Lemma 3.2

$$\text{bdd}^u(a'_{n-1}) \cap \text{bdd}^u(a_n) \subseteq \text{bdd}^u(a_0) \cap \text{bdd}^u(a_n) = \text{bdd}^u(\emptyset).$$

Choose a'' with $R(a'', a_n)$ such that $SU_p(a''/a_n)$ is maximal possible. We may choose it such that $a'' \downarrow_{a_n} a'_{n-1}$. Then

$$\text{bdd}^u(a'') \cap \text{bdd}^u(a'_{n-1}) \subseteq \text{bdd}^u(a_n) \cap \text{bdd}^u(a'_{n-1}) = \text{bdd}^u(\emptyset)$$

and

$$SU_p(a''/a'_{n-1}) \geq SU_p(a''/a'_{n-1}a_n) = SU_p(a''/a_n).$$

Put $\pi(x, y) = \text{tp}(a'', a'_{n-1})$. Then for $a_1a_2 \models \pi$ we have $R(a_1, a_2)$, $\text{bdd}^u(a_1) \cap \text{bdd}^u(a_2) = \text{bdd}^u(\emptyset)$ and $SU_p(a_1/a_2)$ is maximal possible among couples satisfying R . Note that

$$SU_p(a_2/a_1) = SU_p(a_1a_2) - SU_p(a_1) = SU_p(a_1a_2) - SU_p(a_2) = SU_p(a_1/a_2),$$

so this is also maximal.

Choose $a_3 \downarrow_{a_2} a_1$ with $a_3 \equiv_{a_2}^{\text{lstp}} a_1$. By generic transitivity $R(a_1, a_3)$ holds. Moreover,

$$SU_p(a_3/a_1) \geq SU_p(a_3/a_1a_2) = SU_p(a_3/a_2),$$

so equality holds. Similarly,

$$SU_p(a_1/a_3) = SU_p(a_1/a_2a_3) = SU_p(a_1/a_2).$$

For a set A let

$$\text{cl}_p(A) = \{a : SU_p(a/A) = 0\}$$

denote the p -closure of A . Then $SU_p(a_i/a_j) = SU_p(a_i/a_ja_k)$ for $\{i, j, k\} = \{1, 2, 3\}$ means that

$$\text{cl}_p(a_i) \downarrow_{\text{cl}_p(a_j)} \text{cl}_p(a_k).$$

In particular,

$$\text{cl}_p(a_i) \cap \text{cl}_p(a_k) = \text{cl}_p(a_1) \cap \text{cl}_p(a_2) \cap \text{cl}_p(a_3).$$

Let $b = \text{cl}_p(a_1) \cap \text{cl}_p(a_2) \cap \text{bdd}(a_1a_2)$. Then $\text{cl}_p(a_1) \cap \text{cl}_p(a_2) = \text{cl}_p(b)$ by [3, Lemma 3.18]. Let $F(x, y)$ be the \emptyset -invariant equivalence relation on $\text{lstp}(b)$ given by $\text{cl}_p(x) = \text{cl}_p(y)$. As b_F is fixed by the $\text{bdd}(a_2)$ -automorphism moving a_1 to a_3 and $a_1 \downarrow_{a_2} a_3$, we get $b_F \in \text{bdd}^u(a_2)$

by Lemma 3.2. Similarly, considering an $a'_3 \downarrow_{a_1} a_2$ with $a'_3 \equiv_{a_1}^{lstp} a_2$ we obtain $b_F \in \text{bdd}^u(a_1)$, whence

$$b_F \in \text{bdd}^u(a_1) \cap \text{bdd}^u(a_2) = \text{bdd}^u(\emptyset).$$

So if $b' \downarrow b$ satisfies $\text{lstp}(b)$, then $b'_F = b_F$ and $\text{cl}_p(b') = \text{cl}_p(b) = \text{cl}_p(\emptyset)$. But now

$$\text{Cb}(a_3/\text{cl}_p(a_1)\text{cl}_p(a_2)) \subseteq \text{cl}_p(a_1) \cap \text{cl}_p(a_2) = \text{cl}_p(b) = \text{cl}_p(\emptyset),$$

so $a_3 \downarrow_{\text{cl}_p(\emptyset)} a_2$, as required. \square

Remark 3.8. We cannot generalize [6, Lemma 3.3.1] and strengthen Proposition 3.7 to say that if R is \emptyset -invariant, reflexive, symmetric and generically transitive on a Lascar strong type, then the transitive closure E of R equals the 2-step iteration of R . Consider on the forest of Example 2 the relation $R(a, b)$ which holds if 3 divides the distance between a and b . This is generically transitive, as for $a' \downarrow_a a''$ the distance between a' and a'' is the sum of the distances between a' and a and between a and a'' . However, two points of distance 2 are easily seen to be R^2 -related, so the transitive closure E of R is just the relation of being in the same connected component. But no two points of distance 1 are R^2 -related.

Definition 3.9. We shall say that an ultramaginary e is (*almost*) Σ -internal, or is Σ -analysable, if it has a representative which is. Similarly, we shall say that e has finite or ordinal rank over some set A if it has a representative which does. Finally, e is orthogonal over A to some type p if for all $B \downarrow_A e$ such that p is over B and for any realization $b \models p|B$ we have $e \downarrow_A Bb$.

Remark 3.10. This definition does not imply that we define the notion of an analysis of an ultramaginary, or the rank of an ultramaginary. Moreover, e orthogonal to p over A does not imply that e has a representative which is orthogonal to p .

Let Σ be an \emptyset -invariant family of partial types, and recall that the first Σ -level of a over A is the set

$$\ell_1^\Sigma(a/A) = \{b \in \text{bdd}(aA) : \text{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Lemma 3.11. *If $c = \ell_1^\Sigma(b)$ and $\text{tp}(c'/A)$ is almost Σ -internal, where $A \downarrow b$, then $b \downarrow_c Ac'$.*

Proof: $\text{Cb}(Acc'/b)$ is definable over a Morley sequence in $\text{lstp}(Acc'/b)$ and thus almost Σ -internal, as $A \downarrow b$. So $\text{Cb}(Acc'/b) \subseteq c$ by definition. \square

Proposition 3.12. *Let T be simple. Suppose b_E is an ultraimaginary non-orthogonal to some regular type p , and $SU_p(\ell_1^p(b)) < \omega$. Then there is almost p -internal $e \in \text{bdd}^u(b_E) \setminus \text{bdd}^u(\emptyset)$. Moreover, $e \in \text{bdd}^u(\ell_1^p(b))$.*

Proof: Let $c = \ell_1^p(b)$. Define an \emptyset -invariant relation R on $\text{tp}(c)$ by

$$R(c', c'') \iff \exists b' b'' \ b' c' \equiv b'' c'' \equiv bc \wedge E(b', b'').$$

This is reflexive and symmetric; moreover for $c' \downarrow_{c''} c'''$ with $\models R(c', c'')$ and $\models R(c'', c''')$ we can find b', b'', b^*, b''' with

$$b' c' \equiv b'' c'' \equiv b^* c'' \equiv b''' c'''$$

and $\models E(b', b'')$ and $\models E(b^*, b''')$. Since c'' is boundedly closed, $b'' \equiv_{c''}^{lstp} b^*$, and $b'' \downarrow_{c''} c'$ and $b^* \downarrow_{c''} c'''$ by Lemma 3.11. By the Independence Theorem we can assume $b'' = b^*$, whence $E(b', b''')$ holds and $\models R(c', c''')$. So R is generically transitive; let F be its transitive closure. The class c_F is clearly almost p -internal. Moreover, if $\models E(b', b)$ and $b' c' \equiv bc$, then $\models F(c', c)$, so c_F is bounded over b_E .

Finally, suppose $c_F \in \text{bdd}(\emptyset)$. By Proposition 3.7 there is $c' \downarrow_{\text{cl}_p(\emptyset)} c$ with $\models R(c', c)$. Hence there are b', b^* with $b' c' \equiv b^* c \equiv bc$ and $\models E(b', b^*)$. Applying a c -automorphism (and moving c'), we may assume $b = b^*$. Let $A \downarrow bcb'c'$ be some parameters and a some realizations of p over A with $a \not\downarrow_A b_E$; we may assume $b' c' \downarrow_{bc} Aa$. Then $b \downarrow_{Ac} a$ by Lemma 3.11, whence $b' c' \downarrow_{Ac} a$ and $b' c' \downarrow_c Aa$. Thus $b' c' \downarrow_{\text{cl}_p(c)} Aa$. Now $c' \downarrow_{\text{cl}_p(\emptyset)} c$ yields $c' \downarrow_{\text{cl}_p(\emptyset)} \text{cl}_p(c)$, and hence $c' \downarrow_{\text{cl}_p(\emptyset)} Aa$. As $a \downarrow_A \text{cl}_p(\emptyset)$ we get $a \downarrow_A c'$. Now $b' \downarrow_{Ac'} a$ by Lemma 3.11, whence $b' \downarrow_A a$. As $b_E = b'_E$ we obtain $a \downarrow_A b_E$, a contradiction. \square

Corollary 3.13. *Let T be supersimple, and e be a quasifinitary ultraimaginary. Suppose $\text{tp}(e/B)$ is non-orthogonal to some regular type p . Then there is an almost p -internal $e' \in \text{bdd}^{qfu}(Be) \setminus \text{bdd}^{qfu}(B)$, of finite SU_p -rank over B .*

Proof: Let $A \subseteq \text{bdd}(e)$ be hyperimaginary and a a finite tuple such that $e \in \text{bdd}^u(Aa)$. If $\text{tp}(A/B)$ is non-foreign to p , we are done by [6, Proposition 3.4.12]. But if $\text{tp}(A/B)$ is foreign to p , then $\ell_1^p(aA/B) \downarrow_B A$, so $\ell_1^p(aA/B) = \ell_1^p(a/AB)$. Since $\ell_1^p(a/AB)$ is p -simple over AB , we get

$$SU_p(\ell_1^p(aA/B)/B) = SU_p(\ell_1^p(a/AB)/AB) = w_p(\ell_1^p(a/AB)/AB),$$

which is finite by supersimplicity.

By Proposition 3.12 applied over B there is an almost p -internal hyperimaginary $e_0 \in \text{bdd}^u(Be) \setminus \text{bdd}^u(B)$; moreover $e_0 \in \text{bdd}^u(\ell_1^p(aA/B))$.

By supersimplicity, there is a finite $b \in A$ with $a \downarrow_{Bb\ell_1^p(aA/B)} A$. As $A \downarrow_{Bb} \ell_1^p(aA/B)$, we get $A \downarrow_{Bb} a\ell_1^p(aA/B)$, whence $A \downarrow_{Bba} \ell_1^p(aA/B)$. Thus

$$\ell_1^p(aA/B) = \ell_1^p(ab/B) \subseteq \text{bdd}(abB),$$

and $e' = e_0B$ is quasifinite of finite SU_p -rank over B . \square

Remark 3.14. For hyperimaginary e in a simple theory, the proof of Corollary 3.13 uses a canonical base over e . As we cannot consider types over ultrimaginaries, this does not make sense in our context.

Proposition 3.15. *Let T be supersimple. If $AB \downarrow D$ and $\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset)$, then $\text{bdd}^{qfu}(AD) \cap \text{bdd}^{qfu}(BD) = \text{bdd}^{qfu}(D)$.*

Proof: We may assume that A , B and D are boundedly closed. Consider

$$e \in (\text{bdd}^{qfu}(AD) \cap \text{bdd}^{qfu}(BD)) \setminus \text{bdd}^{qfu}(D).$$

Let p be a regular type of least SU -rank non-orthogonal to e over D . By Corollary 3.13 we may assume that e is almost p -internal of finite SU_p -rank over D . Suppose $C \subseteq \text{bdd}(e)$ is hyperimaginary. Then

$$C \subseteq \text{bdd}(AD) \cap \text{bdd}(BD) = D$$

by [4, Fact 2.4]; replacing e by eD we may assume $D \in \text{dcl}(e)$. If c is finite with $e \in \text{bdd}(Dc)$, there is finite $a \in A$ with $c \downarrow_{aD} A$, and $e \in \text{bdd}^{qfu}(aD)$ by Lemma 3.2. Similarly, if c' is almost p -internal of finite SU_p -rank over D with $e \in \text{bdd}^{qfu}(c'D)$, then $a' = \text{Cb}(c'D/a)$ is almost p -internal of finite SU_p -rank, since $D \downarrow a$ and $a' \in \text{bdd}(a)$. Moreover, $c' \downarrow_{a'D} a$ implies $e \in \text{bdd}^{qfu}(a'D)$. So we may assume that $A = \text{bdd}(a')$ and $SU_p(A) < \omega$.

Let $(A_i : i < \omega)$ be a Morley sequence in $\text{lstp}(A/BD)$ with $A_0 = A$, and put $B' = \text{bdd}(A_1A_2)$. Then B' is almost p -internal of finite SU_p -rank. Since $e \in \text{bdd}^{qfu}(BD)$ we have $e \in \text{bdd}^{qfu}(A_iD)$ for all $i < \omega$. Let e' be the set of $B'D$ -conjugates of e , again a quasifinite ultrahyperimaginary. Since any $B'D$ -conjugate of e is again in

$$\text{bdd}^{qfu}(A_1D) \cap \text{bdd}^{qfu}(A_2D) = \text{bdd}^{qfu}(BD) \cap \text{bdd}^{qfu}(A_1D) = \text{bdd}^{qfu}(BD) \cap \text{bdd}^{qfu}(AD),$$

we have $e' \in \text{dcl}^{qfu}(B'D) \cap \text{bdd}^{qfu}(AD)$. Moreover, $B' \downarrow_{BD} A$, whence $B' \downarrow_B A$ and

$$\text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B') \subseteq \text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B) = \text{bdd}^{qfu}(\emptyset).$$

Choose $A' \equiv_{AD}^{lstp} B'$ with $A' \downarrow_{AD} B'$. Then $e \in \text{dcl}^{qfu}(A'D) \cap \text{dcl}^{qfu}(B'D)$. Furthermore, $A' \downarrow_A D$, whence $A' \downarrow_A B'$ and

$$\text{bdd}^{qfu}(A') \cap \text{bdd}^{qfu}(B') \subseteq \text{bdd}^{qfu}(A) \cap \text{bdd}^{qfu}(B') = \text{bdd}^{qfu}(\emptyset).$$

We may assume $e' = (A'D)_E$ for some \emptyset -invariant equivalence relation E . Define a \emptyset -invariant reflexive and symmetric relation R on $\text{lstp}(A')$ by

$$R(X, Y) \Leftrightarrow \exists Z [XZ \equiv YZ \equiv A'D \wedge Z \downarrow XY \wedge E(XZ, YZ)].$$

By the independence theorem, for $A_1 \downarrow_{A_2} A_3$ with $R(A_1, A_2)$ and $R(A_2, A_3)$ we have $R(A_1, A_3)$. Hence R is generically transitive; let E' be the transitive closure of R . Clearly $A'_{E'}$ is quasifinite.

Next, consider $A'' \equiv_{B'} A'$ with $A'' \downarrow_{B'} A'$. By the independence theorem there is D' with $A'D \equiv_{B'} A'D' \equiv_{B'} A''D'$ and $D' \downarrow_{B'} A'A''$. Then $D' \downarrow B'$, whence $D' \downarrow A'A''$ and $(A'D')_E = (A''D')_E \in \text{dcl}^{qfu}(B'D')$. Therefore $E'(A', A'')$ holds and $A'_{E'} \in \text{dcl}^{qfu}(B')$. Thus

$$A'_{E'} \in \text{dcl}^{qfu}(A') \cap \text{dcl}^{qfu}(B') \subseteq \text{bdd}^{qfu}(\emptyset).$$

By Proposition 3.7 there is $A'' \downarrow_{\text{cl}_p(\emptyset)} A'$ with $R(A', A'')$. Let D' witness $R(A', A'')$. Then $D' \equiv_{A'} D$, so we may assume $D' = D$. Since $\text{cl}_p(D) \downarrow \text{cl}_p(A'A'')$ and $\text{cl}_p(A') \downarrow_{\text{cl}_p(\emptyset)} \text{cl}_p(A'')$ we obtain

$$\text{cl}_p(A') \downarrow_{\text{cl}_p(\emptyset)} \text{cl}_p(A'')\text{cl}_p(D)$$

and hence $A' \downarrow_{\text{cl}_p(D)} A''$. But now

$$e' = (A'D)_E = (A''D)_E \in \text{dcl}^{qfu}(AD) \cap \text{dcl}^{qfu}(A'D) \subseteq \text{bdd}^{qfu}(\text{cl}_p(D))$$

by Lemma 3.2. Since e' is orthogonal over D to $\text{cl}_p(D)$ by minimality of $SU(p)$, we have $e' \in \text{bdd}^{qfu}(D)$, and $e \in \text{bdd}^{qfu}(e') \subseteq \text{bdd}^{qfu}(D)$. \square

Remark 3.16. Again, the proof of the hyperimaginary analogue of Proposition 3.15 for simple theories uses canonical bases and does not generalize.

4. ELIMINATION OF ULTRAIMAGINARIES

One cannot avoid the non-tame ultraimaginarys of Example 2 which do not satisfy finite character and hence cannot be eliminated. We thus content ourselves with the elimination of quasi-finitary ultraimaginarys in supersimple theories.

Definition 4.1. Let e be ultrimaginary. We shall say that $SU(a/e) < \omega^\alpha$ if for all representatives b of e we have $SU(a/b) < \omega^\alpha$. Conversely, $SU(e/a) < \omega^\alpha$ if there is a representative b with $SU(b/a) < \omega^\alpha$.

Remark 4.2. This does not mean that we define the type of a over e or of e over a , nor the value of $SU(a/e)$ or of $SU(e/a)$.

Lemma 4.3. $U(e/a) < \omega^0$ if and only if $e \in \text{bdd}^u(a)$, and $U(a/e) < \omega^0$ if and only if $a \in \text{bdd}(e)$.

Proof: If b is a representative of e with $SU(b/a) < \omega^0$, then $b \in \text{bdd}(a)$, so $e \in \text{bdd}^u(a)$. If $e \in \text{bdd}^u(a)$, then $e \in \text{dcl}^u(\text{bdd}(a))$, so $b = \text{bdd}(a)$ is a representative of e with $SU(b/a) < \omega^0$.

If $a \notin \text{bdd}(e)$, then there are arbitrarily many e -conjugates of a . Then for any representative b of e there is some e -conjugate a' of a which is not in $\text{bdd}(b)$. Let b' be the image of b under an e -automorphism mapping a' to a . Then b' is a representative of e , and $SU(a/b') \geq \omega^0$. On the other hand, if $a \in \text{bdd}(e)$, then $a \in \text{bdd}(b)$ for any representative b of e , whence $SU(a/b) < \omega^0$. \square

Definition 4.4. An ultrimaginary e can be α -eliminated if there is a hyperimaginary a with $SU(a/e) < \omega^\alpha$ and $SU(e/a) < \omega^\alpha$. A theory has α -elimination of ultrimaginaries if all quasi-finitary ultrimaginaries can be α -eliminated.

0-elimination is usually called *geometric elimination*.

Theorem 4.5. *If e is ultrimaginary with $SU(e) < \omega^{\alpha+1}$, then e can be α -eliminated. A supersimple theory of finite rank has geometric elimination of quasifinite ultrimaginaries.*

Proof: Let a be a tuple of minimal rank such that $e \in \text{bdd}^u(a)$. Since $SU(e) < \omega^{\alpha+1}$ we have $SU(a) = \omega^\alpha n + \beta$ with $\beta < \omega^\alpha$; we shall use induction on n . By [6, Lemma 5.1.7] there is $a_0 \in \text{bdd}(a)$ with $SU(a_0) = \beta$ and $SU(a/a_0) = \omega^\alpha n$. By [6, Proposition 5.1.12] there is $B \downarrow_{a_0} a$ containing a_0 and c with $SU(c/B) = \omega^\alpha$ and $c \not\downarrow_B a$. Then $SU(a/Bc) < \omega^\alpha n$; by inductive hypothesis there is b such that $SU(e/Bbc) < \omega^\alpha$ and $SU(b/Bce) < \omega^\alpha$. In particular there is a representative b' of e with $SU(b'/Bbc) < \omega^\alpha$, whence

$$SU(b'/Bce) \leq SU(b'/Bcb) \oplus SU(b/Bce) < \omega^\alpha.$$

Replacing b by b' we may thus assume that b is a representative of e . Let $a' = \text{Cb}(Bbc/a)$. Since $Bbc \downarrow_{a'} a$ we get $e \in \text{bdd}^u(a')$ by Lemma 3.2. Hence $SU(a') = SU(a)$ by minimality of rank, and $SU(a/a') < \omega^\alpha$ by the Lascar inequalities.

- **Case 1:** $SU(c/Be) < \omega^\alpha$. Then $SU(bc/Be) < \omega^\alpha$. Moreover, if $(B_i b_i c_i : i < \omega)$ is a Morley sequence in $\text{lstp}(Bbc/a)$ with $B_0 b_0 c_0 = Bbc$, then $B \downarrow_{a_0} a$ implies that $(B_i : i < \omega) \downarrow_{a_0} a$, whence $(B_i : i < \omega) \downarrow_B a$ and $(B_i : i < \omega) \downarrow_{\text{cl}_\alpha(B)} \text{cl}_\alpha(Ba)$. As $SU(bc/Ba) < \omega^\alpha$ we get $a \downarrow_{\text{cl}_\alpha(Bbc)} (B_i : i < \omega)$, and therefore $a \downarrow_{\text{cl}_\alpha(Bbc)} \text{cl}_\alpha(b, B_i : i < \omega)$. However, $SU(b_i c_i / B_i e) < \omega^\alpha$; as b is a representative of e we get $SU(b_i c_i / B_i b) < \omega^\alpha$ for all $i < \omega$, and

$$a' \downarrow_{\text{cl}_\alpha(Bb)} (B_i b_i c_i : i < \omega).$$

As $a' \in \text{dcl}(B_i b_i c_i : i < \omega)$, we obtain $a' \in \text{cl}_\alpha(Bb)$, so

$$SU(a/Be) \leq SU(a/a') \oplus SU(a'/Bb) \oplus SU(b/Be) < \omega^\alpha.$$

Suppose $SU(a/e) \geq \omega^\alpha$. Then there is a representative b'' of e with $SU(a/b'') \geq \omega^\alpha$. Consider a Morley sequence $(a'_i : i < \alpha)$ in $\text{lstp}(a/b'')$ with $a'_0 = a$. Since $a \downarrow_{a_0} B$ we may assume that $(a'_i : i < \alpha)$ is indiscernible over B (possibly moving b''). But then $SU(a'_i / Bb'') < \omega^\alpha$ for all $i < \alpha$, whence $a'_i \not\downarrow_{b''} B$ for all $i < \alpha$, contradicting boundedness of weight of $\text{tp}(B/b'')$.

- **Case 2:** $SU(c/Be) \geq \omega^\alpha$. There is a representative b_0 of e with $SU(c/Bb_0) \geq \omega^\alpha$, whence $c \downarrow_B b_0$. Choose $b'c' \equiv_{Bb_0} bc$ with $c' \downarrow_{Bb_0} a$. Then $c' \downarrow_B a$, whence $Bc' \downarrow_{a_0} a$, and $SU(b'/Bc'e) < \omega^\alpha$. Moreover $SU(a/Bb'c') \geq \omega^\alpha$, as otherwise modulo ω^α

$$SU(a/B) \geq SU(bc/B) = SU(b'c'/B) = SU(ab'c'/B) \geq SU(ac'/B) > SU(a/B).$$

Let $(B_i b_i c_i : i < \omega)$ be a Morley sequence in $\text{lstp}(Bb'c'/a)$ with $B_0 b_0 c_0 = Bb'c'$, and $a' = \text{Cb}(Bb'c'/a)$. Then $a' \in \text{dcl}(B_i b_i c_i : i < \omega)$. Now $SU(b_i / B_i c_i e) < \omega^\alpha$, whence $SU(b_i / B_i c_i Bb'c') < \omega^\alpha$. As $Bc' \downarrow_{a_0} a$, we get $(B_i c_i : i < \omega) \downarrow_{a_0} a$. So $(B_i c_i : i < \omega) \downarrow_{\text{cl}_\alpha(Bc')} \text{cl}_\alpha(Bc'a)$, whence $\text{cl}_\alpha(Bb'c', B_i c_i : i < \omega) \downarrow_{\text{cl}_\alpha(Bb'c')} a$ and $(B_i b_i c_i : i < \omega) \downarrow_{\text{cl}_\alpha(Bb'c')} a$. It follows that $a' \in \text{cl}_\alpha(Bb'c') \cap \text{bdd}(a)$; since $SU(a/Bb'c') \geq \omega^\alpha$ we get $SU(a) \geq SU(a') + \omega^\alpha$. However, $Bb'c' \downarrow_{a'} a$ implies $e \in \text{bdd}^u(a_0)$ by Lemma 3.2, contradicting minimality of $SU(a)$. \square

5. DECOMPOSITION

In this section Σ will be an \emptyset -invariant family of partial types.

- Lemma 5.1.** (1) Suppose $a \sqsubseteq b$. If $c \downarrow a$ and $c \downarrow b$, then $a \sqsubseteq_c b$.
(2) Suppose $a \sqsubseteq_c b$. If $c \downarrow ab$ then $a \sqsubseteq b$.

- (3) Suppose $a \sqsubseteq_c b$. If $\text{tp}(a)$ and $\text{tp}(b)$ are foreign to $\text{tp}(c)$, then $a \sqsubseteq b$.

Proof:

- (1) Consider any d with $d \not\sqsubseteq_c a$. Then $cd \not\sqsubseteq a$, whence $cd \not\sqsubseteq b$. Now $b \perp c$ implies $b \not\sqsubseteq_c d$. The converse follows by symmetry.
- (2) Consider any d with $d \not\sqsubseteq a$. Clearly we may assume $d \perp_{ab} c$, whence $abd \perp c$. Since $a \perp c$ we get $d \not\sqsubseteq_c a$, whence $d \not\sqsubseteq_c b$ and $cd \not\sqsubseteq b$. But $c \perp_d b$, so $d \not\sqsubseteq b$; the converse follows by symmetry.
- (3) Consider any d with $d \not\sqsubseteq a$. Since $a \perp c$ we get $d \not\sqsubseteq_c a$, whence $d \not\sqsubseteq_c b$ and $cd \not\sqsubseteq b$. If $b \perp d$, then $b \perp_d c$ by foreignness, whence $b \perp cd$, a contradiction. So $b \not\sqsubseteq d$; the converse follows by symmetry. \square

Here are two versions of [2, Lemma 1.11].

Proposition 5.2. *Let A, B, a, b be such that $\text{bdd}^u(Aa) \cap \text{bdd}^u(Bb) = \text{bdd}^u(\emptyset)$, and a and b are domination-equivalent over AB . Suppose AB is Σ -analysable, and $\text{tp}(a/A)$ and $\text{tp}(b/B)$ are foreign to Σ . Then $a \in \text{bdd}(A)$ and $b \in \text{bdd}(B)$.*

Proof: Suppose otherwise. Put $A_0 = \text{Cb}(A/a)$. Then A_0 is Σ -analysable, so $a \perp_A A_0$. It follows that $A_0 = \text{bdd}(A) \cap \text{bdd}(a)$; similarly for $B_0 = \text{Cb}(b/B)$ we get $B_0 = \text{bdd}(B) \cap \text{bdd}(b)$. Lemma 5.1(3) yields that a and b are domination-equivalent over A_0B_0 . We may thus assume that AB is quasi-finitary.

Define an \emptyset -invariant relation E on $\text{lstp}(Aa)$ by

$$E(A'a', A''a'') \iff a' \sqsubseteq_{A'A''} a''.$$

Clearly, this is reflexive and symmetric. Suppose $E(A'a', A''a'')$ and $E(A''a'', A'''a''')$. By Lemma 5.1(1)

$$a' \sqsubseteq_{A'A''A'''} a'' \quad \text{and} \quad a'' \sqsubseteq_{A'A''A'''} a''',$$

whence $a' \sqsubseteq_{A'A''A'''} a'''$. Now $a' \sqsubseteq_{A'A'''} a'''$ by Lemma 5.1(3). Thus $E(A'a', A'''a''')$ holds and E is transitive.

Let $A'a' \equiv_{Bb}^{\text{lstp}} Aa$ with $A'a' \perp_{Bb} Aa$. Again by Lemma 5.1(1)

$$a \sqsubseteq_{AA'B} b \sqsubseteq_{AA'B} a',$$

and $a \sqsubseteq_{AA'} a'$ by Lemma 5.1(3). Thus $E(Aa, A'a')$ holds. But

$$\text{bdd}^u(Aa) \cap \text{bdd}^u(A'a') \subseteq \text{bdd}^u(Aa) \cap \text{bdd}^u(Bb) = \text{bdd}^u(\emptyset).$$

Hence $(Aa)_E = (A'a')_E \in \text{bdd}^u(\emptyset)$, and there is $A''a'' \downarrow_A Aa$ with $E(Aa, A''a'')$. But then $a \sqsubseteq_{AA''} a''$ and $a \downarrow_{AA''} a''$ yield $a \downarrow_{AA'} a$, whence $a \in \text{bdd}(AA')$ and finally $a \in \text{bdd}(A)$ as $a \downarrow_A A'$. Similarly, $b \in \text{bdd}(B)$. \square

Corollary 5.3. *Let A, B, a, b be such that $\text{bdd}^u(Aa) \cap \text{bdd}^u(Bb) = \text{bdd}^u(\emptyset)$, and a and b are interbounded over AB . Suppose AB is Σ -analysable. Then Aa and Bb are Σ -analysable.*

Proof: Clearly we may assume that $A \subseteq \text{bdd}(Aa)$ and $B \subseteq \text{bdd}(Bb)$ are maximal Σ -analysable subsets. Hence $\text{tp}(a/A)$ and $\text{tp}(b/B)$ are foreign to Σ . Since a and b are interbounded over AB , they are domination-equivalent, contradicting Proposition 5.2. \square

Remark 5.4. By Theorem 4.5, if $SU(Aa)$ or $SU(Bb)$ is finite, then $\text{bdd}(Aa) \cap \text{bdd}(Bb) = \text{bdd}(\emptyset)$ implies $\text{bdd}^u(Aa) \cap \text{bdd}^u(Bb) = \text{bdd}^u(\emptyset)$, and we recover [2, Lemma 1.11].

Fact 5.5. [3, Theorem 3.4(3)] *Let Σ' be an \emptyset -invariant subfamily of Σ . Suppose $\text{tp}(a)$ is foreign to $\Sigma \setminus \Sigma'$. Then a and $\ell_1^{\Sigma'}(a)$ are domination-equivalent.*

Corollary 5.6. *Let $A = \text{Cb}(B/A)$ and $B = \text{Cb}(A/B)$, with $\text{bdd}^u(A) \cap \text{bdd}^u(B) = \text{bdd}^u(\emptyset)$. If A is Σ -analysable and Σ' is the subset of one-based partial types in Σ , then A is analysable in $\Sigma \setminus \Sigma'$.*

Proof: Let $A' \subseteq A$ and $B' \subseteq B$ be maximally analysable in $\Sigma \setminus \Sigma'$. So $\text{tp}(A/A')$ and $\text{tp}(B/B')$ are foreign to $\Sigma \setminus \Sigma'$.

Suppose for a contradiction that $A \not\subseteq A'$. Then $B \not\subseteq B'$, since A is $\text{tp}(B)$ -internal and not analysable in $\Sigma \setminus \Sigma'$. By Lemma 5.5 the first Σ' -levels $a = \ell_1^{\Sigma'}(A/A')$ and $b = \ell_1^{\Sigma'}(B/B')$ are non-trivial, one-based, and

$$a \sqsubseteq_{A'} A \quad \text{and} \quad b \sqsubseteq_{B'} B.$$

Hence $a \not\downarrow_{A'B'} b$ by Lemma 5.1(1). Let $a_0 = \text{Cb}(B'b/A'a)$ and $b_0 = \text{Cb}(A'a/B'b)$. Then $a \downarrow_{A'B'a_0} b$ and $a \downarrow_{A'B'b_0} b$, so

$$\text{bdd}(A'B'a_0) \cap \text{bdd}(A'B'b_0) \supseteq \text{bdd}(A'B'a) \cap \text{bdd}(A'B'b).$$

It follows from one-basedness that

$$\text{bdd}(A'B'a_0) = \text{bdd}(A'B'b_0) = \text{bdd}(A'B'a) \cap \text{bdd}(A'B'b).$$

We can now apply Corollary 5.3 to see that $a \in \text{bdd}(A')$, a contradiction. \square

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