HAL open science

On the stability under convolution of resurgent functions

David Sauzin

▶ To cite this version:

David Sauzin. On the stability under convolution of resurgent functions. 2012. hal-00656739v2

HAL Id: hal-00656739

https://hal.science/hal-00656739v2

Preprint submitted on 28 Apr 2012 (v2), last revised 12 Nov 2012 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the stability under convolution of resurgent functions

David Sauzin

April 28, 2012

Abstract

This article introduces, for any closed discrete subset Ω of $\mathbb{C}$, the definition of Ω-continuability, which is a particular case of Écalle's resurgence: Ω-continuable functions are required to be holomorphic near 0 and to admit analytic continuation along any path which avoids Ω. We give a rigorous and self-contained treatment of the stability under convolution of this space of funtions, showing that a necessary and sufficient condition is the stability of Ω under addition.

1 Introduction

In Écalle's theory of resurgent functions, a fundamental notion is that of *endlessly continuable germs*: these are holomorphic germs of one complex variable at the origin which enjoy a certain property of analytic continuation (the possible singularities of their analytic continuation must be isolated, at least locally—[Eca81], [Mal85]); they arise as Borel transforms of possibly divergent formal series which solve certain nonlinear problems.

Since the theory is designed to deal with nonlinear problems, it is an essential fact that the property of endless continuability is stable under convolution (indeed, via Borel transform, the convolution of germs at 0 reflects the Cauchy product of formal series). This allows to define the *algebra* of resurgent functions in the "convolutive model" and then to study certain subalgebras obtained by specifying the location or the nature of the possible singularities that one can encounter in the process of analytic continuation. Écalle then proceeds with defining the "alien calculus", which involves particular derivations of this algebra and is an efficient way of encoding the singularities, and deriving consequences in the "geometric models" obtained by applying the Laplace transform in all possible directions—see [Eca81], [Eca92], [Eca93], [CNP93], [Sau06], [Sau12].

Unfortunately, the proof of the stability under convolution of endlessly continuable germs in full generality is rather complicated. Écalle's argument is based on the "symmetrically contractile" paths, but the fact that one can always find such paths was not proven with full details and complete rigour.

We shall deal in this article with a particular case of endless continuability, which we call Ω-continuability, which corresponds to specifying a priori the possible location of the singularities: they are required to lie in a set Ω that we fix in advance. This means that there is one Riemann surface over $\mathbb{C}$, depending only on Ω, on which every Ω-continuable germs induce a holomorphic function (whereas in the general case of endless continuability there is an "endless" Riemann surface which does depend on the considered germ). Our definition already covers interesting cases: one encounters examples of Ω-continuable germs with $\Omega = \mathbb{N}^*$ or $\Omega = \mathbb{Z}$ when dealing

with differential equations formally conjugate to the Euler equation (in the study of the saddle-node singularities) [Eca84], [Sau10], or with $2\pi\mathrm{i}\mathbb{Z}$ when dealing with certain difference equations like Abel's equation for parabolic germs (in holomorphic dynamics) [Eca81], [Sau06], [Sau12].

Our aim is to give a rigorous and self-contained treatment of the stability under convolution of this space of resurgent functions, with more details and more complete explanations than in [Sau06] which was dealing with the particular case $\Omega = 2\pi\mathrm{i}\mathbb{Z}$. (The article [Ou10] proposes another approach for the latter case.)

For any closed discrete subset of $\mathbb{C}$, we shall thus introduce the definition Ω-continuability in Section 2, recall the definition of convolution in Section 3 and state in Section 4 our main result, Theorem 4.1, which is *the equivalence of the stability under convolution of Ω-continuable germs and the stability under addition of the set Ω*. The rest of the article will be devoted to the proof of this theorem.

A novel feature of our proof (even if we certainly owe a debt to [Eca81] and [CNP93]) is the construction of "symmetric Ω-homotopies" (rather than symmetrically contractile paths) by means of certain non-autonomous vector fields.

2 The Ω-continuable germs

In this article, "path" means a piecewise C^1 function $\gamma\colon J \to \mathbb{C}$, where J is a compact interval of $\mathbb{R}$. For any $R > 0$ and $\zeta_0 \in \mathbb{C}$ we use the notations $D(\zeta_0, R) := \{\, \zeta \in \mathbb{C} \mid |\zeta - \zeta_0| < R \,\}$, $\mathbb{D}_R := D(0, R)$ and $\mathbb{D}_R^* := \mathbb{D}_R \setminus \{0\}$.

Definition 2.1. *Let Ω be a non-empty closed discrete subset of $\mathbb{C}$, let $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ be a holomorphic germ at the origin. We say that $\hat{\varphi}$ is Ω-continuable if there exists $R > 0$ not larger than the radius of convergence of $\hat{\varphi}$ such that $\mathbb{D}_R^* \cap \Omega = \emptyset$ and $\hat{\varphi}$ admits analytic continuation along any path of $\mathbb{C} \setminus \Omega$ originating from any point of $\mathbb{D}_R^*$. We use the notation*

$$\hat{\mathscr{R}}_\Omega := \{\, \textit{all } \Omega\textit{-continuable holomorphic germs} \,\} \subset \mathbb{C}\{\zeta\}.$$

Remark 2.2. Let $\rho := \min\big\{|\omega|,\ \omega \in \Omega \setminus \{0\}\big\}$. Any $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$ is a holomorphic germ at 0 with radius of convergence $\geq \rho$ and one can always take $R = \rho$ in Definition 2.1. In fact, given an arbitrary $\zeta_0 \in \mathbb{D}_\rho$, we have

$$\hat{\varphi} \in \hat{\mathscr{R}}_\Omega \quad \Longleftrightarrow \quad \left| \begin{array}{l} \hat{\varphi} \text{ germ of holomorphic function of } \mathbb{D}_\rho \text{ admitting analytic continuation} \\ \text{along any path } \gamma\colon [0,1] \to \mathbb{C} \text{ such that } \gamma(0) = \zeta_0 \text{ and } \gamma\big((0,1]\big) \subset \mathbb{C} \setminus \Omega \end{array} \right.$$

(even if $\zeta_0 = 0$ and $0 \in \Omega$: there is no need to avoid 0 at the beginning of the path, when we still are in the disc of convergence of $\hat{\varphi}$).

Example 2.3. Trivially, any entire function of $\mathbb{C}$ defines an Ω-continuable germ. Other elementary examples of Ω-continuable germs are the functions which are holomorphic in $\mathbb{C} \setminus \Omega$ and regular at 0, like $\frac{1}{(\zeta-\omega)^m}$ with $m \in \mathbb{N}^*$ and $\omega \in \Omega \setminus \{0\}$. But these are still single-valued examples, whereas the interest of the Definition 2.1 is to authorize multivaluedness when following the analytic continuation. Elementary examples of multivalued continuation are provided by $\sum_{n\geq 1} \frac{\zeta^n}{n} = -\log(1-\zeta)$ (principal branch of the logarithm), which is Ω-continuable if and only if $1 \in \Omega$, and $\sum_{n\geq 0} \frac{\zeta^n}{n+1} = -\frac{1}{\zeta}\log(1-\zeta)$, which is Ω-continuable if and only if $\{0,1\} \subset \Omega$.

Example 2.4. If $\omega \in \mathbb{C}^*$ and $m \in \mathbb{N}^*$, then $\big(\log(\zeta-\omega)\big)^m \in \hat{\mathscr{R}}_{\{\omega\}}$; if moreover $\omega \neq -1$, then $\big(\log(\zeta-\omega)\big)^{-m} \in \hat{\mathscr{R}}_{\{\omega,\omega+1\}}$.

Example 2.5. If Ω is a closed discrete subset of $\mathbb{C}$, $0 \notin \Omega$, $\omega \in \Omega$ and $\hat{\psi}$ is holomorphic in $\mathbb{C} \setminus \Omega$, then $\hat{\varphi}(\zeta) = \hat{\psi}(\zeta)\log(\zeta - \omega)$ defines a germ of $\hat{\mathscr{R}}_\Omega$ whose monodromy around ω is given by $2\pi\mathrm{i}\hat{\psi}$.

Notation 2.6. *Given a path $\gamma\colon [a,b] \to \mathbb{C}$, if $\hat{\varphi}$ is a holomorphic germ at $\gamma(a)$ which admits an analytic continuation along γ, we denote by $\mathrm{cont}_\gamma \hat{\varphi}$ the resulting holomorphic germ at the endpoint $\gamma(b)$.*

As is often the case with analytic continuation and Cauchy integrals, the precise parametrisation of our paths will usually not matter, in the sense that we shall get the same results from two paths $\gamma\colon [a,b] \to \mathbb{C}$ and $\gamma'\colon [a',b'] \to \mathbb{C}$ which ony differ by a change of parametrisation ($\gamma = \gamma' \circ \sigma$ with $\sigma\colon [a,b] \to [a',b']$ piecewise continuously differentiable, increasing and mapping a to a' and b to b').

We identify $\mathbb{C}\{\zeta\}$, the space of power series with positive radius of convergence, with the space of holomorphic germs at 0. Given $\hat{\varphi} \in \mathbb{C}\{\zeta\}$, we shall often use the same symbol $\hat{\varphi}$ to denote the holomorphic function it defines, and even the principal branch of its analytic continuation when such a notion is well-defined.

3 The convolution of holomorphic germs at the origin

The convolution in $\mathbb{C}\{\zeta\}$ is defined by the formula

$$\hat{\varphi} * \hat{\psi}(\zeta) := \int_0^\zeta \hat{\varphi}(\xi)\hat{\psi}(\zeta - \xi)\,\mathrm{d}\xi$$

for any $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$: the formula makes sense for $|\zeta|$ close enough and defines a holomorphic germ at 0 whose disc of convergence contains the intersection of the discs of convergence of $\hat{\varphi}$ and $\hat{\psi}$. The convolution law $*$ is commutative and associative.[1]

The question we address in this article is the question of the stability of $\hat{\mathscr{R}}_\Omega$ under convolution. As already mentioned, this is relevant when dealing with the formal solutions of nonlinear problems and this is absolutely necessary to develop the theory of resurgent functions and alien calculus for Ω-continuable germs.

This amounts to inquiring about the analytic continuation of the germ $\hat{\varphi} * \hat{\psi}$ when Ω-continuability is assumed for $\hat{\varphi}$ and $\hat{\psi}$. Let us first mention an easy case, which is used in [DS12] and [Sau10]:

Lemma 3.1. *Let Ω be any non-empty closed discrete subset of $\mathbb{C}$ and suppose $\hat{A}$ is an entire function of $\mathbb{C}$. Then, for any $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$, the convolution product $\hat{A} * \hat{\varphi}$ belongs to $\hat{\mathscr{R}}_\Omega$; its analytic continuation along a path γ of $\mathbb{C} \setminus \Omega$ starting from a point ζ_0 close enough to 0 and ending at a point ζ_1 is the holomorphic germ at ζ_1 explicitly given by*

$$\mathrm{cont}_\gamma(\hat{A} * \hat{\varphi})(\zeta) = \int_0^{\zeta_0} \hat{A}(\zeta - \xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi + \int_\gamma \hat{A}(\zeta - \xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi + \int_{\zeta_1}^{\zeta} \hat{A}(\zeta - \xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi \qquad (1)$$

for ζ close enough to ζ_1.

[1] Indeed, it is induced by the pull-back of the Cauchy product in $z^{-1}\mathbb{C}[[z^{-1}]]$ by the formal Borel transform and corresponds, via the Laplace transform, to the ordinary product of analytic functions.

The proof is left as an exercise (see *e.g.* the proof of Lemma 5.3 for a formalized proof in a more complicated situation), but we wish to emphasize that formulas such as (1) require a word of caution: the value of $\hat{A}(\zeta-\xi)$ is unambiguously defined whatever ζ and ξ are, but in the notation "$\hat{\varphi}(\xi)$" it is understood that we are using the appropriate branch of the possibily multivalued function $\hat{\varphi}$; in such a formula, what branch we are using is clear from the context:

- $\hat{\varphi}$ is unambiguously defined in its disc of convergence D_0 (centred at 0) and the first integral thus makes sense for $\zeta_0 \in D_0$;
- in the second integral ξ is moving along γ which is a path of analytic continuation for $\hat{\varphi}$, we thus consider the analytic continuation of $\hat{\varphi}$ along the piece of γ between its origin and ξ;
- in the third integral, "$\hat{\varphi}$" is to be understood as $\operatorname{cont}_\gamma \hat{\varphi}$, the germ at ζ_1 resulting form the analytic continuation of $\hat{\varphi}$ along γ, this integral then makes sense for any ζ at a distance from ζ_1 less than the radius of convergence of $\operatorname{cont}_\gamma \hat{\varphi}$.

Using a parametrisation $\gamma\colon [0,1] \to \mathbb{C} \setminus \Omega$, with $\gamma(0) = \zeta_0$ and $\gamma(1) = \zeta_1$, and introducing the truncated paths $\gamma_s := \gamma_{|[0,s]}$ for any $s \in [0,1]$, the interpretation of the last two integrals in (1) is

$$\int_\gamma \hat{A}(\zeta-\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi := \int_0^1 \hat{A}(\zeta-\gamma(s))(\operatorname{cont}_{\gamma_s}\hat{\varphi})(\gamma(s))\gamma'(s)\,\mathrm{d}s,$$

$$\int_{\zeta_1}^{\zeta} \hat{A}(\zeta-\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi := \int_{\zeta_1}^{\zeta} \hat{A}(\zeta-\xi)(\operatorname{cont}_\gamma\hat{\varphi})(\xi)\,\mathrm{d}\xi.$$

4 Main result

We now wish to be able to consider the case of the convolution of two Ω-continuable holomorphic germs at 0, $\hat{\varphi}$ and $\hat{\psi}$, without assuming that any of them extends to an entire function. The main result of this article is

Theorem 4.1. *Let Ω be a non-empty closed discrete subset of $\mathbb{C}$. Then the space $\hat{\mathscr{R}}_\Omega$ is stable under convolution if and only if Ω is stable under addition.*

The necessary and sufficient condition on Ω is satisfied by the typical examples $\mathbb{Z}$ or $2\pi\mathrm{i}\mathbb{Z}$, but also by $\mathbb{N}^*$, $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{N}^*\times\mathbb{N}$ or $\{m+n\sqrt{2} \mid m,n\in\mathbb{N}^*\}$ for instance.

The rest of the article is dedicated to the proof of Theorem 4.1. The necessity of the condition on Ω will follow from the following elementary example:

Example 4.2 ([CNP93])**.** Let us consider $\omega_1,\omega_2\in\mathbb{C}^*$ and $\hat{\varphi}(\zeta)=\frac{1}{\zeta-\omega_1}$, $\hat{\psi}(\zeta)=\frac{1}{\zeta-\omega_2}$ and study

$$\hat{\chi}(\zeta)=\hat{\varphi}*\hat{\psi}(\zeta)=\int_0^\zeta \frac{1}{(\xi-\omega_1)(\zeta-\xi-\omega_2)}\,\mathrm{d}\xi, \qquad |\zeta|<\min\big\{|\omega_1|,|\omega_2|\big\}.$$

The formula

$$\frac{1}{(\xi-\omega_1)(\zeta-\xi-\omega_2)}=\frac{1}{\zeta-\omega_1-\omega_2}\left(\frac{1}{\xi-\omega_1}+\frac{1}{\zeta-\xi-\omega_2}\right)$$

shows that, for any $\zeta\neq\omega_1+\omega_2$ of modulus $<\min\big\{|\omega_1|,|\omega_2|\big\}$, one can write

$$\hat{\chi}(\zeta)=\frac{1}{\zeta-\omega_1-\omega_2}\big(L_1(\zeta)+L_2(\zeta)\big), \qquad L_j(\zeta):=\int_0^\zeta\frac{\mathrm{d}\xi}{\xi-\omega_j} \tag{2}$$

(with the help of the change of variable $\xi \mapsto \zeta - \xi$ in the case of L_2).

Removing the half-lines $\omega_j[1, +\infty)$ from $\mathbb{C}$, we obtain a cut plane Δ in which $\hat{\chi}$ has a meromorphic continuation (since $[0, \zeta]$ avoids the points ω_1 and ω_2 for all $\zeta \in \Delta$). We can in fact follow the meromorphic continuation of $\hat{\chi}$ along any path which avoids ω_1 and ω_2, because

$$L_j(\zeta) = -\int_0^{\zeta/\omega_j} \frac{\mathrm{d}\xi}{1-\xi} = \log\Big(1 - \frac{\zeta}{\omega_j}\Big) \in \hat{\mathscr{R}}_{\{\omega_j\}}.$$

We used the words "meromorphic continuation" and not "analytic continuation" because of the factor $\frac{1}{\zeta-\omega_1-\omega_2}$. The conclusion is thus only $\hat{\chi} \in \hat{\mathscr{R}}_\Omega$, with $\Omega := \{\omega_1, \omega_2, \omega_1 + \omega_2\}$.

– If $\omega := \omega_1 + \omega_2 \in \Delta$, the principal branch of $\hat{\chi}$ (*i.e.* its meromorphic continuation to Δ) has a removable singularity[2] at ω, because $(L_1 + L_2)(\omega) = \int_0^\omega \frac{\mathrm{d}\xi}{\xi-\omega_1} + \int_0^\omega \frac{\mathrm{d}\xi}{\xi-\omega_2} = 0$ in that case (by the change of variable $\xi \mapsto \omega - \xi$ in one of the integrals). But it is easy to see that this does not happen for all the branches of $\hat{\chi}$: when considering all the paths γ going from 0 to ω and avoiding ω_1 and ω_2, we have

$$\operatorname{cont}_\gamma L_j(\omega) = \int_\gamma \frac{\mathrm{d}\xi}{\xi - \omega_j}, \qquad j = 1, 2,$$

hence $\frac{1}{2\pi\mathrm{i}}\big(\operatorname{cont}_\gamma L_1(\omega) + \operatorname{cont}_\gamma L_2(\omega)\big)$ is the sum of the winding numbers around ω_1 and ω_2 of the loop obtained by concatenating γ and the rectilinear segment $[\omega, 0]$; elementary geometry shows that this sum of winding numbers can take all integer values, but whenever this value is non-zero the corresponding branch of $\hat{\chi}$ does have a pole at ω.

– The case $\omega \notin \Delta$ is slightly different. Then we can write $\omega_j = r_j\,\mathrm{e}^{\mathrm{i}\theta}$ with $r_1, r_2 > 0$ and consider the path γ_0 which follows the segment $[0, \omega]$ except that it circumvents ω_1 and ω_2 by small half-circles travelled anti-clockwise (notice that ω_1 and ω_2 may coincide); an easy computation yields

$$\operatorname{cont}_{\gamma_0} L_1(\omega) = \int_{-r_1}^{-1} \frac{\mathrm{d}\xi}{\xi} + \int_1^{r_2} \frac{\mathrm{d}\xi}{\xi} + \int_{\Gamma_0} \frac{\mathrm{d}\xi}{\xi},$$

where Γ_0 is the half-circle from -1 to 1 with radius 1 travelled anti-clockwise, hence $\operatorname{cont}_{\gamma_0} L_1(\omega) = \log\frac{r_2}{r_1} + \mathrm{i}\pi$, similarly $\operatorname{cont}_{\gamma_0} L_2(\omega) = \log\frac{r_1}{r_2} + \mathrm{i}\pi$, therefore $\operatorname{cont}_\gamma L_1(\omega) + \operatorname{cont}_\gamma L_2(\omega) = 2\pi\mathrm{i}$ is non-zero and this again yields a branch of $\hat{\chi}$ with a pole at ω (and infinitely many others by using other paths than γ_0).

In all cases, there are paths from 0 to $\omega_1 + \omega_2$ which avoid ω_1 and ω_2 and which are not paths of analytic continuation for $\hat{\chi}$. This example thus shows that $\hat{\mathscr{R}}_{\{\omega_1,\omega_2\}}$ is *not* stable under convolution: it contains $\hat{\varphi}_1$ and $\hat{\varphi}_2$ but not $\hat{\varphi}_1 * \hat{\varphi}_2$.

Now, whenever Ω is not stable under addition, one can find $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 + \omega_2 \notin \Omega$ and Example 4.2 then yields $\hat{\varphi}_1, \hat{\varphi}_2 \in \hat{\mathscr{R}}_\Omega$ with $\hat{\varphi}_1 * \hat{\varphi}_2 \notin \hat{\mathscr{R}}_\Omega$. This gives the easy part of Theorem 4.1.

5 Proof of the main result: Analytic part

From now on we assume that Ω is stable under addition. Our aim is to prove that $\hat{\mathscr{R}}_\Omega$ is stable under convolution. We begin with a new definition (see Figure 1):

[2] This is consistent with the well-known fact that the space of holomorphic functions of an open set Δ which is star-shaped with respect to 0 is stable under convolution.

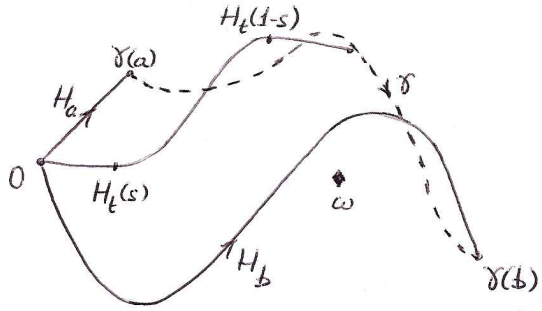

Figure 1: A symmetric Ω-homotopy, with its initial path H_a, its final path H_b and its endpoint-path $\gamma = \Gamma_H$.

Definition 5.1. *A continuous map $H\colon I \times J \to \mathbb{C}$, where $I = [0,1]$ and J is a compact interval of $\mathbb{R}$, is called a symmetric Ω-homotopy if, for each $t \in J$,*

$$s \in I \mapsto H_t(s) := H(s,t)$$

defines a path which satisfies

i) $H_t(0) = 0$,

ii) $H_t\big((0,1]\big) \subset \mathbb{C} \setminus \Omega$,

iii) $H_t(1) - H_t(s) = H_t(1-s)$ *for every* $s \in I$.

We then call endpoint path of H the path

$$\Gamma_H\colon\ t \in J \mapsto H_t(1).$$

Writing $J = [a,b]$, we call H_a (resp. H_b) the initial path of H (resp. its final path).

The first two conditions imply that each path H_t is a path of analytic continuation for any $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$, in view of Remark 2.2.

We shall use the notation $H_{t|s}$ for the truncated paths $(H_t)_{|[0,s]}$, $s \in I$, $t \in J$ (analogously to what we did when commenting Lemma 3.1). Here is a technical statement we shall use:

Lemma 5.2. *For a symmetric Ω-homotopy H defined on $I \times J$, there exists $\delta > 0$ such that, for any $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$ and $(s,t) \in I \times J$, the radius of convergence of the holomorphic germ* $\mathrm{cont}_{H_{t|s}}\,\hat{\varphi}$ *at $H_t(s)$ is at least δ.*

Proof. Let ρ be as in Remark 2.2. Consider $U := H^{-1}(\mathbb{D}_{\rho/2})$: this is an open subset of $I \times J$ which contains $\{0\} \times J$; we define U_0 to be the connected component of U which contains $\{0\} \times J$, $K := I \times J \setminus U_0$ and $\delta := \min\big\{\operatorname{dist}\big(H(K), \Omega\big), \rho/2\big\}$.

We have $\delta > 0$ because $H(K)$ is compact (since K is compact) and $H(K) \subset \mathbb{C} \setminus \Omega$ (since $K \subset (0,1] \times J$). Now, for any s and t,

– either $(s,t) \in U_0$, then $[0,s] \times \{t\} \subset U_0$, the truncated path $H_{t|s}$ lies in $H(U_0) \subset \mathbb{D}_{\rho/2}$, hence $\mathrm{cont}_{H_{t|s}}\,\hat{\varphi}$ is a holomorphic germ at $H_t(s)$ with radius of convergence $\geq \delta$;

– or $(s,t) \in K$, and then $\operatorname{dist}(H_t(s),\Omega) \geq \delta$, which yields the same conclusion for the germ $\operatorname{cont}_{H_{t|s}} \hat{\varphi}$.

□

The third condition in Definition 5.1 means that each path H_t is symmetric with respect to its midpoint $\frac{1}{2}H_t(1)$. Here is the motivation behind this requirement:

Lemma 5.3. *Suppose H is a symmetric Ω-homotopy whose initial path is contained in $\mathbb{D}_\rho$, with ρ as in Remark 2.2. Then its endpoint path Γ_H is a path of analytic continuation for any convolution product $\hat{\varphi} * \hat{\psi}$ with $\hat{\varphi}, \hat{\psi} \in \hat{\mathscr{R}}_\Omega$.*

Proof. We assume that H is defined on $I \times J$ and we set $\gamma := \Gamma_H$. Let $\hat{\varphi}, \hat{\psi} \in \hat{\mathscr{R}}_\Omega$ and, for $t \in J$, consider the formula

$$\hat{\chi}_t(\zeta) = \int_{H_t} \hat{\varphi}(\xi)\hat{\psi}(\zeta - \xi)\,\mathrm{d}\xi + \int_{\gamma(t)}^{\zeta} \hat{\varphi}(\xi)\hat{\psi}(\zeta - \xi)\,\mathrm{d}\xi \tag{3}$$

(recall that $\gamma(t) = H_t(1)$). We shall check that $\hat{\chi}_t$ is a well-defined holomorphic germ at $\gamma(t)$ and that it provides the analytic continuation of $\hat{\varphi} * \hat{\psi}$ along γ.

a) The idea is that when ξ moves along H_t, $\xi = H_t(s)$ with $s \in I$, we can use for "$\hat{\varphi}(\xi)$" the analytic continuation of $\hat{\varphi}$ along the truncated path $H_{t|s}$; correspondingly, if ζ is close to $\gamma(t)$, then $\zeta - \xi$ is close to $\gamma(t) - \xi = H_t(1) - H_t(s) = H_t(1-s)$, thus for "$\hat{\psi}(\zeta - \xi)$" we can use the analytic continuation of $\hat{\psi}$ along $H_{t|1-s}$. In other words, setting $\zeta = \gamma(t) + \sigma$, we wish to interpret (3) as

$$\begin{aligned}\hat{\chi}_t(\gamma(t) + \sigma) := \int_0^1 (\operatorname{cont}_{H_{t|s}} \hat{\varphi})(H_t(s))(\operatorname{cont}_{H_{t|1-s}} \hat{\psi})(H_t(1-s) + \sigma)H_t'(s)\,\mathrm{d}s \\ + \int_0^1 (\operatorname{cont}_{H_t} \hat{\varphi})(\gamma(t) + u\sigma)\hat{\psi}((1-u)\sigma)\sigma\,\mathrm{d}u\end{aligned} \tag{4}$$

(in the last integral, we have performed the change variable $\xi = \gamma(t) + u\sigma$; it is the germ of $\hat{\psi}$ at the origin that we use there).

Lemma 5.2 provides $\delta > 0$ such that, by regular dependence of the integrals upon the parameter σ, the right-hand side of (4) is holomorphic for $|\sigma| < \delta$. We thus have a family of analytic elements $(\hat{\chi}_t, D_t)$, $t \in J$, with $D_t := \{\, \zeta \in \mathbb{C} \mid |\zeta - \gamma(t)| < \delta \,\}$.

b) For t small enough, the path H_t is contained in $\mathbb{D}_\rho$ which is open and simply connected; then, for $|\zeta|$ small enough, the rectilinear segment $[0,\zeta]$ and the concatenation of H_t and $[\gamma(t),\zeta]$ are homotopic in $\mathbb{D}_\rho$, hence the Cauchy theorem implies $\hat{\chi}_t(\zeta) = \hat{\varphi} * \hat{\psi}(\zeta)$.

c) By uniform continuity, there exists $\varepsilon > 0$ such that, for any $t_0, t \in J$,

$$|t - t_0| \leq \varepsilon \quad \Longrightarrow \quad |H_t(s) - H_{t_0}(s)| < \delta/2 \quad \text{for all } s \in I. \tag{5}$$

To complete the proof, we check that, for any $t_0 < t_1$ in J such that $|t_1 - t_0| \leq \varepsilon$, we have $\hat{\chi}_{t_0} \equiv \hat{\chi}_{t_1}$ in $D_{t_0} \cap D_{t_1}$.

Because of (5), the assumption $|t_1 - t_0| \leq \varepsilon$ implies that, for $t \in [t_0, t_1]$, $s \in I$ and $|\xi - H_{t_0}(s)| < \delta$,

$$\operatorname{cont}_{H_{t|s}} \hat{\varphi}(\xi) = \operatorname{cont}_{H_{t_0|s}} \hat{\varphi}(\xi), \quad \operatorname{cont}_{H_{t|1-s}} \hat{\psi}(\xi) = \operatorname{cont}_{H_{t_0|1-s}} \hat{\psi}(\xi).$$

Thus, as soon as $t \in [t_0, t_1]$ and $|\zeta - \gamma(t_0)| < \delta/2$, we can write

$$\hat{\chi}_t(\zeta) = \int_0^1 (\mathrm{cont}_{H_{t_0|s}} \hat{\varphi})(H_t(s))(\mathrm{cont}_{H_{t_0,1-s}} \hat{\psi})(\zeta - H_t(s)) H_t'(s)\,\mathrm{d}s \\ + \int_{\gamma(t)}^{\zeta} (\mathrm{cont}_{H_{t_0}} \hat{\varphi})(\xi)\hat{\psi}(\zeta - \xi)\,\mathrm{d}\xi$$

and we get $\hat{\chi}_{t_0}(\zeta) = \hat{\chi}_t(\zeta)$ from the Cauchy theorem by means of the homotopy induced by H between the concatenation of H_{t_0} and $[\gamma(t_0), \zeta]$ and the concatenation of H_t and $[\gamma(t), \zeta]$. □

Remark 5.4. With the notation of Definition 5.1, when the initial path H_a is a rectilinear segment contained in $\mathbb{D}_\rho$, the final path H_b is what Écalled calls a "symmetrically contractile path" in [Eca81]. The proof of Lemma 5.3 shows that the analytic continuation of $\hat{\varphi} * \hat{\psi}$ until the endpoint $H_b(1) = \Gamma_H(b)$ can be computed by the usual integral taken over H_b (however, it usually be cannot be computed as the same integral over the endpoint path Γ_H, even when the latter integral is well-defined).

6 Proof of the main result: Geometric part

6.1 The key lemma

In view of Lemma 5.3, the proof of Theorem 4.1 will be complete if we prove the following purely geometric result:

Lemma 6.1. *For any path $\gamma\colon I = [0,1] \to \mathbb{C} \setminus \Omega$ such that $\gamma(0) \in \mathbb{D}_\rho^*$ and the left and right derivatives $\gamma'_\pm$ do not vanish on I, there exists a symmetric Ω-homotopy H on $I \times I$ whose endpoint path is γ and whose initial path is a rectilinear segment, i.e. $\Gamma_H = \gamma$ and $H_0(s) \equiv s\gamma(0)$.*

The proof is quite simple when γ does not pass through 0, which is automatic if we assume $0 \in \Omega$. The general case requires an extra work which is technical and involves a quantitative version of the simpler case. With a view to helping the reader to grasp the mechanism of the proof, we thus begin with the case when $0 \in \Omega$.

6.2 Proof of the key lemma when $0 \in \Omega$

Assume that γ is given as in the hypothesis of Lemma 6.1. We are looking for a symmetric Ω-homotopy whose initial path is imposed: it must be

$$s \in I \mapsto H_0(s) := s\gamma(0),$$

which satisfies the three requirements of Definition 5.1 at $t = 0$:

(i) $H_0(0) = 0$,

(ii) $H_0\big((0,1]\big) \subset \mathbb{C} \setminus \Omega$,

(iii) $H_0(1) - H_0(s) = H_0(1-s)$ for every $s \in I$.

The idea is to define a family of maps $(\Psi_t)_{t\in[0,1]}$ so that

$$H_t(s) \coloneqq \Psi_t\big(H_0(s)\big), \qquad s \in I, \tag{6}$$

yield the desired homotopy. For that, it is sufficient that $(t,\zeta) \in [0,1]\times\mathbb{C} \mapsto \Psi_t(\zeta)$ be continuously differentiable (for the structure of real two-dimensional vector space of $\mathbb{C}$), $\Psi_0 = \mathrm{Id}$ and, for each $t \in [0,1]$,

(i') $\Psi_t(0) = 0$,

(ii') $\Psi_t(\mathbb{C}\setminus\Omega) \subset \mathbb{C}\setminus\Omega$,

(iii') $\Psi_t\big(\gamma(0)-\zeta\big) = \Psi_t\big(\gamma(0)\big) - \Psi_t(\zeta)$ for all $\zeta \in \mathbb{C}$,

(iv') $\Psi_t\big(\gamma(0)\big) = \gamma(t)$.

In fact, the properties (i')–(iv') ensure that any initial path H_0 satisfying (i)–(iii) and ending at $\gamma(0)$ produces through (6) a symmetric Ω-homotopy whose endpoint path is γ. Consequently, we may assume without loss of generality that γ is C^1 on $[0,1]$ (then, if γ is only piecewise C^1, we just need to concatenate the symmetric Ω-homotopies associated with the various pieces).

The maps Ψ_t will be generated by the flow of a non-autonomous vector field $X(\zeta,t)$ associated with γ that we now define. We view $(\mathbb{C},|\cdot|)$ as a real 2-dimensional Banach space and pick[3] a C^1 function $\eta\colon \mathbb{C}\to[0,1]$ such that

$$\{\,\zeta\in\mathbb{C} \mid \eta(\zeta)=0\,\} = \Omega.$$

Observe that $D(\zeta,t) \coloneqq \eta(\zeta) + \eta\big(\gamma(t)-\zeta\big)$ defines a C^1 function of (ζ,t) which satisfies

$$D(\zeta,t) > 0 \quad \text{for all } \zeta\in\mathbb{C} \text{ and } t\in[0,1]$$

because Ω is stable under addition; indeed, $D(\zeta,t)=0$ would imply $\zeta\in\Omega$ and $\gamma(t)-\zeta\in\Omega$, hence $\gamma(t)\in\Omega$, which would contradict our assumptions. Therefore, the formula

$$X(\zeta,t) \coloneqq \frac{\eta(\zeta)}{\eta(\zeta)+\eta\big(\gamma(t)-\zeta\big)}\gamma'(t) \tag{7}$$

defines a non-autonomous vector field, which is continuous in (ζ,t) on $\mathbb{C}\times[0,1]$, C^1 in ζ and has its partial derivatives continuous in (ζ,t). The Cauchy-Lipschitz theorem on the existence and uniqueness of solutions to differential equations applies to $\frac{\mathrm{d}\zeta}{\mathrm{d}t} = X(\zeta,t)$: for every $\zeta\in\mathbb{C}$ and $t_0\in[0,1]$ there is a unique solution $t\mapsto\Phi^{t_0,t}(\zeta)$ such that $\Phi^{t_0,t_0}(\zeta)=\zeta$. The fact that the vector field X is bounded implies that $\Phi^{t_0,t}(\zeta)$ is defined for all $t\in[0,1]$ and the classical theory guarantees that $(t_0,t,\zeta)\mapsto\Phi^{t_0,t}(\zeta)$ is C^1 on $[0,1]\times[0,1]\times\mathbb{C}$.

Let us set $\Psi_t \coloneqq \Phi^{0,t}$ for $t\in[0,1]$ and check that this family of maps satisfies (i')–(iv'). We have

$$X(\omega,t) = 0 \quad \text{for all } \omega\in\Omega, \tag{8}$$

$$X\big(\gamma(t)-\zeta,t\big) = \gamma'(t) - X(\zeta,t) \quad \text{for all } \zeta\in\mathbb{C} \tag{9}$$

for all $t\in[0,1]$ (by the very definition of X). Therefore

[3]For instance pick a C^1 function $\varphi_0\colon\mathbb{R}\to[0,1]$ such that $\{\,x\in\mathbb{R}\mid\varphi_0(x)=1\,\}=\{0\}$ and $\varphi_0(x)=0$ for $|x|\geq 1$, and a bijection $\omega\colon\mathbb{N}\to\Omega$; then set $\delta_k \coloneqq \operatorname{dist}\big(\omega(k),\Omega\setminus\{\omega(k)\}\big)>0$ and $\sigma(\zeta)\coloneqq\sum_k\varphi_0\big(\frac{4|\zeta-\omega(k)|^2}{\delta_k^2}\big)$: for each $\zeta\in\mathbb{C}$ there is at most one non-zero term in this series (because $k\neq\ell$, $|\zeta-\omega(k)|<\delta_k/2$ and $|\zeta-\omega(\ell)|<\delta_\ell/2$ would imply $|\omega(k)-\omega(\ell)|<(\delta_k+\delta_\ell)/2$, which would contradict $|\omega(k)-\omega(\ell)|\geq\delta_k$ and δ_ℓ), thus σ is C^1, takes its values in $[0,1]$ and satisfies $\{\,\zeta\in\mathbb{C}\mid\sigma(\zeta)=1\,\}=\Omega$, therefore $\eta\coloneqq 1-\sigma$ will do. Other solution: adapt the proof of Lemma 6.3.

- (i') and (ii') follow from (8) which yields $\Phi^{t_0,t}(\omega) = \omega$ for every t_0 and t, whence $\Psi_t(0) = 0$ since $0 \in \Omega$, and from the non-autonomous flow property $\Phi^{t,0} \circ \Phi^{0,t} = \mathrm{Id}$ (hence $\Psi_t(\zeta) = \omega$ implies $\zeta = \Phi^{t,0}(\omega) = \omega$);
- (iv') follows from the fact that $X\big(\gamma(t), t\big) = \gamma'(t)$, by (8) and (9) with $\zeta = 0$, using again that $0 \in \Omega$, hence $t \mapsto \gamma(t)$ is a solution of X;
- (iii') follows from (9): for any solution $t \mapsto \zeta(t)$, the curve $t \mapsto \xi(t) := \gamma(t) - \zeta(t)$ satisfies $\xi(0) = \gamma(0) - \zeta(0)$ and $\xi'(t) = \gamma'(t) - X\big(\zeta(t), t\big) = X\big(\xi(t), t\big)$, hence it is a solution: $\xi(t) = \Psi_t\big(\gamma(0) - \zeta(0)\big)$.

As explained above, formula (6) thus produces the desired symmetric Ω-homotopy.

Remark 6.2. Our proof of Lemma 6.1, which essentially relies on the use of the flow of the non-autonomous vector field (7), arose as an attempt to understand a related but more complicated construction which can be found in an appendix of the book [CNP93] (however the vector field there was autonomous and we must confess that we were not able to follow completely the arguments of [CNP93]).

6.3 Proof of the key lemma when $0 \notin \Omega$

From now on, we suppose $0 \notin \Omega$ and we use the notation

$$\Omega_\varepsilon := \{\, \zeta \in \mathbb{C} \mid \mathrm{dist}(\zeta, \Omega) < \varepsilon \,\}$$

for any $\varepsilon > 0$, hence $\overline{\Omega}_\varepsilon = \{\, \zeta \in \mathbb{C} \mid \mathrm{dist}(\zeta, \Omega) \le \varepsilon \,\}$. We shall require the following technical

Lemma 6.3. *For any $\varepsilon > 0$ there exists a C^1 function $\eta\colon \mathbb{C} \to [0,1]$ such that*

$$\{\, \zeta \in \mathbb{C} \mid \eta(\zeta) = 0 \,\} = \{0\} \cup \overline{\Omega}_\varepsilon.$$

Proof. Pick a C^1 function $\chi\colon \mathbb{R} \to [0,1]$ such that $\{\, x \in \mathbb{R} \mid \chi(x) = 0 \,\} = [-\varepsilon^2, \varepsilon^2]$ and $\chi(x) = 1$ for $|x| \ge (1+\varepsilon)^2$, and a bijection $\omega\colon \mathbb{N}^* \to \Omega$. For each $k \in \mathbb{N}^*$, $\eta_k(\zeta) := \chi\big(|\zeta - \omega(k)|^2\big)$ defines a C^1 function on $\mathbb{C}$ such that $\eta_k^{-1}(0) = \overline{D(\omega(k), \varepsilon)}$ and $\eta_k \equiv 1$ on $\mathbb{C} \setminus D(\omega(k), 1+\varepsilon)$. Consider the infinite product

$$\eta_*(\zeta) := \prod_{k \in \mathbb{N}^*} \eta_k(\zeta). \tag{10}$$

For any bounded open subset U of $\mathbb{C}$, the set $\mathcal{F}_U := \{\, k \in \mathbb{N}^* \mid U \cap D(\omega(k), 1+\varepsilon) \neq \emptyset \,\}$ is finite (because Ω is discrete), thus almost all the factors in (10) are equal to 1 when $\zeta \in U$: $(\eta_*)_{|U} = \prod_{k \in \mathcal{F}_U} (\eta_k)_{|U}$, hence η_* is C^1, takes its values in $[0,1]$ and

$$\eta_*^{-1}(0) \cap U = \bigcup_{k \in \mathcal{F}_U} \overline{D(\omega(k), \varepsilon)} \cap U,$$

whence it follows that $\eta_*^{-1}(0) = \overline{\Omega}_\varepsilon$.

If $0 \in \overline{\Omega}_\varepsilon$, then one can take $\eta = \eta_*$. If not, then one can take the product $\eta = \eta_0 \eta_*$ with $\eta_0(\zeta) := \chi_0(|\zeta|^2)$, where χ_0 is any C^1 function on $\mathbb{R}$ which takes its values in $[0,1]$ and such that $\chi_0^{-1}(0) = \{0\}$. $\square$

We now repeat the work of the previous section replacing Ω with $\{0\}\cup\Omega$, adding quantitative information (we still assume that we are given a path which does not pass through 0 but we want to control the way the corresponding symmetric Ω-homotopy approaches the points of Ω) and authorizing a more general initial path than a rectilinear one.

Lemma 6.4. *Let $\delta, \delta' > 0$ with $\delta' < \delta/2$. Suppose that $J = [a, b]$ is a compact interval of $\mathbb{R}$ and $\gamma\colon J \to \mathbb{C}$ is a path such that*

$$0 \notin \gamma(J) \quad \text{and} \quad \gamma(J) \subset \mathbb{C} \setminus \Omega_\delta.$$

Suppose that $h\colon I \to \mathbb{C}$ is a C^1 path such that

(i) $h(0) = 0$,

(ii) $h(I) \subset \mathbb{C} \setminus \Omega_{\delta'}$,

(iii) $h(1 - s) = h(1) - h(s)$ *for all* $s \in I$,

(iv) $h(1) = \gamma(a)$.

Then there exists a symmetric Ω-homotopy H defined on $I \times J$, whose initial path is h, whose endpoint path is γ, which satisfies $H(I \times J) \subset \mathbb{C} \setminus \Omega_{\delta'}$ and whose final path is C^1.

Proof. We may assume without loss of generality that γ is C^1 on J (if γ is only piecewise C^1, we just need to concatenate the symmetric Ω-homotopies associated with the various pieces). We shall define a family of maps $(\Psi_t)_{t\in J}$ so that

$$H_t(s) := \Psi_t\big(h(s)\big), \qquad s \in I, \tag{11}$$

yield the desired homotopy. For that, it is sufficient that $(t, \zeta) \in J \times \mathbb{C} \mapsto \Psi_t(\zeta)$ be continuously differentiable, $\Psi_0 = \mathrm{Id}$ and, for each $t \in J$,

(i') $\Psi_t(0) = 0$,

(ii') $\Psi_t(\mathbb{C} \setminus \Omega_{\delta'}) \subset \mathbb{C} \setminus \Omega_{\delta'}$,

(iii') $\Psi_t\big(\gamma(0) - \zeta\big) = \Psi_t\big(\gamma(0)\big) - \Psi_t(\zeta)$ for all $\zeta \in \mathbb{C}$,

(iv') $\Psi_t\big(\gamma(a)\big) = \gamma(t)$.

As in Section 6.2, our maps Ψ_t will be generated by a non-autonomous vector field.

Lemma 6.3 allows us to choose a C^1 function $\eta\colon \mathbb{C} \to [0, 1]$ such that

$$\{\, \zeta \in \mathbb{C} \mid \eta(\zeta) = 0 \,\} = \{0\} \cup \overline{\Omega}_{\delta'}.$$

We observe that $D(\zeta, t) := \eta(\zeta) + \eta\big(\gamma(t) - \zeta\big)$ defines a C^1 function of (ζ, t) which satisfies

$$D(\zeta, t) > 0 \quad \text{for all } \zeta \in \mathbb{C} \text{ and } t \in [0, 1]$$

because Ω is stable under addition; indeed, $D(\zeta, t) = 0$ would imply that both ζ and $\gamma(t)-\zeta$ lie in $\{0\}\cup\overline{\Omega}_{\delta'}$, hence $\gamma(t) \in \{0\}\cup\overline{\Omega}_{2\delta'}$, which would contradict our assumption $\gamma(J) \subset \mathbb{C}\setminus\big(\{0\}\cup\Omega_\delta\big)$. Therefore the formula

$$X(\zeta, t) := \frac{\eta(\zeta)}{\eta(\zeta) + \eta\big(\gamma(t) - \zeta\big)}\gamma'(t), \qquad (\zeta, t) \in \mathbb{C} \times J,$$

defines a non-autonomous vector field whose flow $(\Phi^{t_0,t})_{t_0,t\in J}$ allows one to conclude the proof exactly as in Section 6.2, setting $\Psi_t := \Phi^{a,t}$ and replacing (8) with

$$X(\omega,t) = 0 \quad \text{for all } \omega \in \{0\} \cup \Omega_{\delta'}.$$

□

We now consider the case of a path γ which entirely lies close to 0.

Lemma 6.5. *Let $\varepsilon, \delta' > 0$ with $0 < \varepsilon < \delta'$. Suppose that $K = [a,b]$ is a compact interval of $\mathbb{R}$ and $\gamma\colon K \to \mathbb{C}$ is a path such that*

$$\gamma(K) \subset \overline{\mathbb{D}}_{\varepsilon/2}.$$

Suppose that $h\colon I \to \mathbb{C}$ is a C^1 path such that

(i) $h(0) = 0$,

(ii) $h(I) \subset \mathbb{C} \setminus \Omega_{\delta'}$,

(iii) $h(1-s) = h(1) - h(s)$ *for all* $s \in I$,

(iv) $h(1) = \gamma(a)$.

Then there exists a symmetric Ω-homotopy H defined on $I \times K$, whose initial path is h, whose endpoint path is γ, which satisfies

$$H(I \times K) \subset \mathbb{C} \setminus \Omega_{\delta''} \qquad \text{with } \delta'' := \delta' - \varepsilon$$

and whose final path is C^1.

Proof. Define $H(s,t) := h(s) + s\big(\gamma(t) - \gamma(a)\big)$. This way $H(s,a) = h(s)$, $H(1,t) = \gamma(t)$ and H is a symmetric Ω-homotopy as required: $H(0,t) = 0$, $H(s,t) + H(1-s,t) = h(s) + h(1-s) + \gamma(t) - \gamma(a) = \gamma(t)$, $\operatorname{dist}\big(H(s,t),\Omega\big) \geq \operatorname{dist}\big(h(s),\Omega\big) - |\gamma(t) - \gamma(a)| \geq \delta' - \varepsilon$. □

Proof of the key-lemma when $0 \notin \Omega$. Let γ be as in the hypothesis of Lemma 6.1. Without loss of generality, we can assume $\gamma(1) \neq 0$ (if not, view γ as the restriction of a path $\tilde{\gamma}\colon [0,2] \to \mathbb{C} \setminus \Omega$ such that $\gamma(2) \neq 0$, with which is associated a symmetric Ω-homotopy $\tilde{H}$ defined on $I \times [0,2]$, and restrict $\tilde{H}$ to $I \times [0,1]$). Let $\delta := \operatorname{dist}\big(\Omega, \gamma([0,1])\big)$.

The set $Z := \{\, t \in [0,1] \mid \gamma(t) = 0 \,\}$ is closed; it is also discrete because of the non-vanishing of the derivatives of γ, thus it has a finite cardinality $N \in \mathbb{N}$. If $N = 0$, then we can apply Lemma 6.4 with $J = [0,1]$ and $h(s) \equiv s\gamma(0)$ and the proof is complete.

From now on we suppose $N \geq 1$. Let us write

$$Z = \{t_1, \dots, t_N\} \quad \text{with } 0 < t_1 < \cdots < t_N < 1.$$

We define

$$\delta_0 := \frac{1}{2} \min\Big\{\frac{\delta}{2}, \rho - |\gamma(0)|\Big\} \quad \text{and} \quad \varepsilon := \min\Big\{|\gamma(0)|, |\gamma(1)|, \frac{\delta_0}{N+1}\Big\}.$$

The continuity of γ allows us to find pairwise disjoint closed intervals of positive lengths $K_1, \dots, K_N$ such that

$$t_j \in \mathring{K}_j \quad \text{and} \quad \gamma(K_j) \subset \overline{\mathbb{D}}_{\varepsilon/2}, \qquad j = 1, \dots, N.$$

By considering the connected components of $[0,1] \setminus \bigcup K_j$ and taking their closures, we get adjacent closed subintervals of positive lengths of $[0,1]$,

$$J_0, K_1, J_1, K_2, \dots, J_{N-1}, K_N, J_N$$

with $J_j = [a_j, b_j]$, $K_j = [b_{j-1}, a_j]$, $a_0 = 0$, $b_N = 1$. Observe that

$$0 \notin \gamma(J_j) \quad \text{and} \quad \gamma(J_j) \subset \mathbb{C} \setminus \Omega_\delta, \qquad j = 0, \dots, N.$$

- We apply Lemma 6.4 with $J = J_0 = [0, b_0]$, $h(s) \equiv s\gamma(0)$ and $\delta' = \delta_0$ (which is allowed by the choice of δ_0): we get a symmetric Ω-homotopy H defined on $I \times J_0$ whose initial path is the rectilinear segment $[0, \gamma(0)]$, whose endpoint path is $\gamma_{|J_0}$ and whose final path H_{b_0} is C^1 and lies in $\mathbb{C} \setminus \Omega_{\delta_0}$.
- We apply Lemma 6.5 with $K = K_1$, $\delta' = \delta_0$ and $h = H_{b_0}$: we get an extension of our symmetric Ω-homotopy H to $I \times K_1$, in which the enpoint path is extended by $\gamma_{|K_1}$ and the final path is now H_{a_1}, a C^1 path contained in $\mathbb{C} \setminus \Omega_{\delta_1}$ with $\delta_1 := \delta_0 - \varepsilon$.
- And so on: we apply alternatively Lemma 6.5 on K_j and Lemma 6.4 on J_j: we get an extension of the symmetric Ω-homotopy H to $I \times K_j$ or $I \times J_j$ such that both $H_{a_j}(I)$ and $H_{b_j}(I)$ are contained in $\mathbb{C} \setminus \Omega_{\delta_j}$ with $\delta_j := \delta_0 - j\varepsilon$.

When we reach $j = N$, the proof of Lemma 6.1 is complete. □

Acknowledgements. The research leading to these results has received funding from the European Comunity's Seventh Framework Program (FP7/2007–2013) under Grant Agreement n. 236346.

References

[CNP93] B. Candelpergher, J.-C. Nosmas and F. Pham. *Approche de la résurgence.* Actualités Math., Hermann, Paris, 1993.

[DS12] A. Dudko and D. Sauzin. Écalle-Voronin invariants via resurgence and alein calculus. *In preparation.*

[Eca81] J. Écalle. *Les fonctions résurgentes.* Publ. Math. d'Orsay, Vol. 1: 81-05, Vol. 2: 81-06, Vol. 3: 85-05, 1981, 1985.

[Eca84] J. Écalle. *Cinq applications des fonctions résurgentes.* Publ. Math. d'Orsay 84-62, 1984.

[Eca92] J. Écalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* Actualités Math., Hermann, Paris, 1992.

[Eca93] J. Écalle. Six lectures on Transseries, Analysable Functions and the Constructive Proof of Dulac's conjecture. D.Schlomiuk (ed.), *Bifurcations and Periodic Orbits of Vector Field*, pp. 75–184, Kluwer Ac. Publishers, 1993.

[Mal85] B. Malgrange. Introduction aux travaux de J. Écalle. *L'Enseign. Math.* (2) **31**, 3–4 (1985), 261–282.

[Ou10] Y. Ou. On the stability by convolution product of a resurgent algebra. *Ann. Fac. Sci. Toulouse* (6) **19**, 3–4 (2010), 687–705.

[Sau06] D. Sauzin, Resurgent functions and splitting problems. *RIMS Kokyuroku* **1493** (2005), 48–117.

[Sau10] D. Sauzin, Mould expansions for the saddle-node and resurgence monomials. In *Renormalization and Galois theories. Selected papers of the CIRM workshop, Luminy, France, March 2006*, p. 83–163, A. Connes, F. Fauvet, J.-P. Ramis (eds.), IRMA Lectures in Mathematics and Theoretical Physics 15, Zürich: European Mathematical Society, 2009.

[Sau12] D. Sauzin, Introduction to 1-summability and the resurgence theory. *In preparation.*

David Sauzin

CNRS UMI 3483 - Laboratorio Fibonacci
Collegio Puteano, Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 3, 56126 Pisa, Italy
email: `david.sauzin@sns.it`