

# On the stability under convolution of resurgent functions

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January 5, 2012

## Abstract

This article contains a self-contained proof of the stability under convolution of the class of resurgent functions associated with a closed discrete subset  $\Omega$  of  $\mathbb{C}$ , under the assumption that  $\Omega$ , the set of possible singularities, be stable under addition.

## 1 The $\Omega$ -continuable germs

For any closed discrete subset of  $\mathbb{C}$ , we introduce the definition  $\Omega$ -continuity, which can be rephrased as holomorphy on a certain Riemann surface which depends only on  $\Omega$ . It is a particular case of the notion of endless continuity due to J. Écalle, at the basis of his theory of resurgent functions [Eca81], [Mal85].<sup>1</sup> Our aim is to give a rigorous and self-contained treatment of the stability under convolution of this class of resurgent functions, with more details and more complete explanations than in [Sau06] which was dealing with the particular case  $\Omega = 2\pi i\mathbb{Z}$ . (The article [Ou10] proposes another approach for the latter case.)

For any  $R > 0$  and  $\zeta_0 \in \mathbb{C}$  we use the notations  $\mathbb{D}_R := \{\zeta \in \mathbb{C} \mid |\zeta| < R\}$ ,  $\mathbb{D}_R^* := \mathbb{D}_R \setminus \{0\}$  and  $D(\zeta_0, R) := \{\zeta \in \mathbb{C} \mid |\zeta - \zeta_0| < R\}$ . In this article, “path” means a piecewise  $C^1$  function  $\gamma: J \rightarrow \mathbb{C}$ , where  $J$  is a compact interval of  $\mathbb{R}$ .

**Definition 1.1.** *Let  $\Omega$  be a non-empty closed discrete subset of  $\mathbb{C}$ , let  $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$  be a holomorphic germ at the origin. We say that  $\hat{\varphi}$  is  $\Omega$ -continuable if there exists  $R > 0$  such that  $\mathbb{D}_R^*$  is contained in the disc of convergence of  $\hat{\varphi}$ ,  $\mathbb{D}_R^* \cap \Omega = \emptyset$  and  $\hat{\varphi}$  admits analytic continuation along any path of  $\mathbb{C} \setminus \Omega$  originating from any point of  $\mathbb{D}_R^*$ . We use the notation*

$$\hat{\mathcal{R}}_\Omega := \{ \text{all } \Omega\text{-continuable holomorphic germs} \} \subset \mathbb{C}\{\zeta\}.$$

The theory can be developed in a greater generality (see particularly [Eca81], Vol. 3, or [CNP93], where a function is defined to be resurgent if it can be analytically continued to an “endless” Riemann surface which may depend on the function), but Definition 1.1 already covers interesting cases: one encounters examples of  $\Omega$ -continuable germs with  $\Omega = \mathbb{N}^*$  or  $\Omega = \mathbb{Z}$  when dealing with differential equations formally conjugate to the Euler equation (in the study of the saddle-node singularities) [Eca84], [Sau10], or with  $2\pi i\mathbb{Z}$  when dealing with certain difference equations like Abel’s equation for parabolic germs (in holomorphic dynamics) [Eca81], [Sau06], [Sau12].

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<sup>1</sup>In Écalle’s terminology, resurgent functions are defined either in the “convolutive model”, the “formal model” or the “geometric models”. Roughly speaking, the convolutive model of resurgent functions is a subspace of  $\mathbb{C}\{\zeta\}$  defined by requiring a certain property of analytic continuation; from it, one defines the formal model as a subspace of  $z^{-1}\mathbb{C}[[z^{-1}]]$  via inverse formal Borel transform and the geometric models by Laplace transforms in certain directions—see [Eca81], [Eca92], [Eca93], [CNP93], [Sau06], [Sau12]. Our  $\Omega$ -continuable germs belong to the convolutive model of resurgent functions.

**Remark 1.2.** As is often the case with analytic continuation and Cauchy integrals, the precise parametrisation of  $\gamma$  will usually not matter, in the sense that we shall get the same result from two paths  $\gamma: [a, b] \rightarrow \mathbb{C} \setminus \Omega$  and  $\gamma': [a', b'] \rightarrow \mathbb{C} \setminus \Omega$  which only differ by a change of parametrisation ( $\gamma = \gamma' \circ \sigma$  with  $\sigma: [a, b] \rightarrow [a', b']$  piecewise continuously differentiable, increasing and mapping  $a$  to  $a'$  and  $b$  to  $b'$ ).

If  $\hat{\varphi}$  is a holomorphic germ at  $\gamma(a)$  which admits an analytic continuation along  $\gamma$ , we denote by  $\text{cont}_\gamma \hat{\varphi}$  the resulting holomorphic germ at the endpoint  $\gamma(b)$ .

**Remark 1.3.** Let  $\rho := \min \{|\omega|, \omega \in \Omega \setminus \{0\}\}$ . Any  $\hat{\varphi} \in \hat{\mathcal{R}}_\Omega$  is a holomorphic germ at 0 with radius of convergence  $\geq \rho$  and one can always take  $R = \rho$  in Definition 1.1. In fact, even if  $0 \in \Omega$ , for arbitrary  $\zeta_0 \in \mathbb{D}_\rho$  we have

$$\hat{\varphi} \in \hat{\mathcal{R}}_\Omega \iff \left\{ \begin{array}{l} \hat{\varphi} \text{ holomorphic function of } \mathbb{D}_\rho \text{ admitting analytic continuation along} \\ \text{any path } \gamma: [0, 1] \rightarrow \mathbb{C} \text{ such that } \gamma(0) = \zeta_0 \text{ and } \gamma((0, 1]) \subset \mathbb{C} \setminus \Omega \end{array} \right.$$

(even if  $\zeta_0 = 0$ : there is no need to avoid 0 at the beginning of the path, when we still are in the disc of convergence of  $\hat{\varphi}$ ).

**Example 1.1.** A trivial example of  $\Omega$ -continuable germ is any entire function of  $\mathbb{C}$ . Other elementary examples of  $\Omega$ -continuable germs are the functions which are holomorphic in  $\mathbb{C} \setminus \Omega$  and regular at 0, like  $\frac{1}{(\zeta - \omega)^m}$  with  $m \in \mathbb{N}^*$  and  $\omega \in \Omega \setminus \{0\}$ . But these are still single-valued examples, whereas the interest of the Definition 1.1 is to authorize multivaluedness when following the analytic continuation. Elementary examples of multivalued continuation are provided by  $\sum_{n \geq 1} \frac{\zeta^n}{n} = -\log(1 - \zeta)$  (principal branch of the logarithm), which is  $\{1\}$ -continuable, and  $\sum_{n \geq 0} \frac{\zeta^n}{n+1} = -\frac{1}{\zeta} \log(1 - \zeta)$ , which is  $\{0, 1\}$ -continuable.

**Example 1.2.** If  $\omega \in \mathbb{C}^*$  and  $m \in \mathbb{N}^*$ , then  $(\log(\zeta - \omega))^m \in \hat{\mathcal{R}}_{\{\omega\}}$ ; if moreover  $\omega \neq -1$ , then  $(\log(\zeta - \omega))^{-m} \in \hat{\mathcal{R}}_{\{\omega, \omega+1\}}$ .

**Example 1.3.** If  $\Omega$  is a closed discrete subset of  $\mathbb{C}$ ,  $0 \notin \Omega$ ,  $\omega \in \Omega$  and  $\hat{\psi}$  is holomorphic in  $\mathbb{C} \setminus \Omega$ , then  $\hat{\varphi}(\zeta) = \hat{\psi}(\zeta) \log(\zeta - \omega)$  defines a germ of  $\hat{\mathcal{R}}_\Omega$  whose monodromy around  $\omega$  is given by  $2\pi i \hat{\psi}$ .

## 2 The convolution of holomorphic germs at the origin

The convolution in  $\mathbb{C}\{\zeta\}$  is defined by the formula

$$\hat{\varphi} * \hat{\psi}(\zeta) := \int_0^\zeta \hat{\varphi}(\xi) \hat{\psi}(\zeta - \xi) d\xi$$

for any  $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$ : the formula makes sense for  $|\zeta|$  close enough and defines a holomorphic germ at 0 whose disc of convergence contains the intersection of the discs of convergence of  $\hat{\varphi}$  and  $\hat{\psi}$ . The convolution law  $*$  is commutative and associative.<sup>2</sup>

The question we address in this article is the question of the stability of  $\hat{\mathcal{R}}_\Omega$  under convolution. This amounts to inquiring about the analytic continuation of the germ  $\hat{\varphi} * \hat{\psi}$  when  $\Omega$ -continuity is assumed for  $\hat{\varphi}$  and  $\hat{\psi}$ . Let us first mention an easy case:

<sup>2</sup>It is induced by the pull-back of the Cauchy product in  $z^{-1}\mathbb{C}[[z^{-1}]]$  by the formal Borel transform and corresponds, via the Laplace transform, to the ordinary product of analytic functions.

**Lemma 2.1.** *Let  $\Omega$  be any non-empty closed discrete subset of  $\mathbb{C}$  and suppose  $\hat{A}$  is an entire function of  $\mathbb{C}$ . Then, for any  $\hat{\varphi} \in \hat{\mathcal{H}}_\Omega$ , the convolution product  $\hat{A} * \hat{\varphi}$  belongs to  $\hat{\mathcal{H}}_\Omega$ ; its analytic continuation along a path  $\gamma$  of  $\mathbb{C} \setminus \Omega$  starting from a point  $\zeta_0$  close enough to 0 and ending at a point  $\zeta_1$  is the holomorphic germ at  $\zeta_1$  explicitly given by*

$$\text{cont}_\gamma(\hat{A} * \hat{\varphi})(\zeta) = \int_0^{\zeta_0} \hat{A}(\zeta - \xi)\hat{\varphi}(\xi) d\xi + \int_\gamma \hat{A}(\zeta - \xi)\hat{\varphi}(\xi) d\xi + \int_{\zeta_1}^\zeta \hat{A}(\zeta - \xi)\hat{\varphi}(\xi) d\xi \quad (1)$$

for  $\zeta$  close enough to  $\zeta_1$ .

The proof is left as an exercise, but we wish to emphasize that formulas such as (1) require a word of caution: the value of  $\hat{A}(\zeta - \xi)$  is unambiguously defined whatever  $\zeta$  and  $\xi$  are, but in the notation “ $\hat{\varphi}(\xi)$ ” it is understood that we are using the appropriate branch of the possibly multivalued function  $\hat{\varphi}$ ; in such a formula, what branch we are using is clear from the context:

- $\hat{\varphi}$  is unambiguously defined in its disc of convergence  $D_0$  (centred at 0) and the first integral thus makes sense for  $\zeta_0 \in D_0$ ;
- in the second integral  $\xi$  is moving along  $\gamma$  which is a path of analytic continuation for  $\hat{\varphi}$ , we thus consider the analytic continuation of  $\hat{\varphi}$  along the piece of  $\gamma$  between its origin and  $\xi$ ;
- in the third integral, “ $\hat{\varphi}$ ” is to be understood as  $\text{cont}_\gamma \hat{\varphi}$ , the germ at  $\zeta_1$  resulting from the analytic continuation of  $\hat{\varphi}$  along  $\gamma$ , this integral thus makes sense for any  $\zeta$  at a distance from  $\zeta_1$  less than the radius of convergence of  $\text{cont}_\gamma \hat{\varphi}$ .

Using a parametrisation  $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \Omega$ , with  $\gamma(0) = \zeta_0$  and  $\gamma(1) = \zeta_1$ , and introducing the truncated paths  $\gamma_s := \gamma|_{[0, s]}$  for any  $s \in [0, 1]$ , the interpretation of the last two integrals in (1) is

$$\begin{aligned} \int_\gamma \hat{A}(\zeta - \xi)\hat{\varphi}(\xi) d\xi &:= \int_0^1 \hat{A}(\zeta - \gamma(s))(\text{cont}_{\gamma_s} \hat{\varphi})(\gamma(s))\gamma'(s) ds, \\ \int_{\zeta_1}^\zeta \hat{A}(\zeta - \xi)\hat{\varphi}(\xi) d\xi &:= \int_{\zeta_1}^\zeta \hat{A}(\zeta - \xi)(\text{cont}_\gamma \hat{\varphi})(\xi) d\xi. \end{aligned}$$

### 3 Main result

We now wish to consider the case of the convolution of two  $\Omega$ -continuable holomorphic germs at 0,  $\hat{\varphi}$  and  $\hat{\psi}$ , without assuming that any of them extends to an entire function. The main result of this article is

**Theorem 3.1.** *Let  $\Omega$  be a non-empty closed discrete subset of  $\mathbb{C}$  which is stable by addition. Then the space  $\hat{\mathcal{H}}_\Omega$  is stable by convolution.*

The hypothesis on  $\Omega$  is satisfied by the typical examples  $\mathbb{Z}$  or  $2\pi i\mathbb{Z}$ , but also by  $\mathbb{N}^*$ ,  $\mathbb{Z} \times \mathbb{Z}$ ,  $\mathbb{N}^* \times \mathbb{N}$  or  $\{m + n\sqrt{2} \mid m, n \in \mathbb{N}^*\}$  for instance. An elementary example will convince us that the stability of  $\Omega$  under addition is a necessary condition for the conclusion:

**Example 3.1** ([CNP93]). Let us consider  $\omega_1, \omega_2 \in \mathbb{C}^*$  and  $\hat{\varphi}(\zeta) = \frac{1}{\zeta - \omega_1}$ ,  $\hat{\psi}(\zeta) = \frac{1}{\zeta - \omega_2}$  and study

$$\hat{\chi}(\zeta) = \hat{\varphi} * \hat{\psi}(\zeta) = \int_0^\zeta \frac{1}{(\xi - \omega_1)(\zeta - \xi - \omega_2)} d\xi, \quad |\zeta| < \min\{|\omega_1|, |\omega_2|\}.$$

The formula

$$\frac{1}{(\xi - \omega_1)(\zeta - \xi - \omega_2)} = \frac{1}{\zeta - \omega_1 - \omega_2} \left( \frac{1}{\xi - \omega_1} + \frac{1}{\zeta - \xi - \omega_2} \right)$$

shows that, for any  $\zeta \neq \omega_1 + \omega_2$  of modulus  $< \min\{|\omega_1|, |\omega_2|\}$ , one can write

$$\hat{\chi}(\zeta) = \frac{1}{\zeta - \omega_1 - \omega_2} (L_1(\zeta) + L_2(\zeta)), \quad L_j(\zeta) := \int_0^\zeta \frac{d\xi}{\xi - \omega_j} \quad (2)$$

(with the help of the change of variable  $\xi \mapsto \zeta - \xi$  in the case of  $L_2$ ). Removing the half-lines  $\omega_j[1, +\infty)$  from  $\mathbb{C}$ , we obtain a cut plane  $\Delta$  in which  $\hat{\chi}$  has a meromorphic continuation (since  $[0, \zeta]$  avoids the points  $\omega_1$  and  $\omega_2$  for all  $\zeta \in \Delta$ ). We can in fact follow the meromorphic continuation of  $\hat{\chi}$  along any path which avoids  $\omega_1$  and  $\omega_2$ , because

$$L_j(\zeta) = - \int_0^{\zeta/\omega_j} \frac{d\xi}{1 - \xi} = \log \left( 1 - \frac{\zeta}{\omega_j} \right) \in \hat{\mathcal{R}}_{\{\omega_j\}}.$$

We used the words “meromorphic continuation” and not “analytic continuation” because of the factor  $\frac{1}{\zeta - \omega_1 - \omega_2}$ . The conclusion is thus only  $\hat{\chi} \in \hat{\mathcal{R}}_\Omega$ , with  $\Omega := \{\omega_1, \omega_2, \omega_1 + \omega_2\}$ .

In fact, if  $\omega := \omega_1 + \omega_2 \in \Delta$ , the principal branch of  $\hat{\chi}$  (*i.e.* its meromorphic continuation to  $\Delta$ ) has a removable singularity<sup>3</sup> at  $\omega$ , because  $(L_1 + L_2)(\omega) = \int_0^\omega \frac{d\xi}{\xi - \omega_1} + \int_0^\omega \frac{d\xi}{\xi - \omega_2} = 0$  in that case (by the change of variable  $\xi \mapsto \omega - \xi$  in one of the integrals). But it is easy to see that this does not happen for all the branches of  $\hat{\chi}$ : when considering all the paths  $\gamma$  going from 0 to  $\omega$  and avoiding  $\omega_1$  and  $\omega_2$ , the number  $\text{cont}_\gamma L_1(\omega) + \text{cont}_\gamma L_2(\omega)$  takes all the values of the form  $2\pi i k$  with  $k \in \mathbb{Z}$ ; whenever this value is nonzero, the corresponding branch of  $\hat{\chi}$  does have a pole at  $\omega$ .

This example thus shows that  $\hat{\mathcal{R}}_{\{\omega_1, \omega_2\}}$  is *not* stable by convolution (it contains  $\hat{\varphi}_1$  and  $\hat{\varphi}_2$  but not  $\hat{\varphi}_1 * \hat{\varphi}_2$ ).

## 4 Proof of the main result: Analytic part

We now fix  $\Omega$  as in the assumption of Theorem 3.1 and start the proof. We begin with a new definition:

**Definition 4.1.** *Let  $I = [0, 1]$ . A continuous map  $H: I \times J \rightarrow \mathbb{C}$ , where  $J$  is a compact interval of  $\mathbb{R}$ , is called a symmetric  $\Omega$ -homotopy if, for each  $t \in J$ ,*

$$H_t: s \in I \mapsto H_t(s) := H(s, t)$$

*defines a path which satisfies*

- i)  $H_t(0) = 0$ ,
- ii)  $H_t((0, 1]) \subset \mathbb{C} \setminus \Omega$ ,
- iii)  $H_t(1) - H_t(s) = H_t(1 - s)$  for every  $s \in I$ .

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<sup>3</sup>This is consistent with the well-known fact that the space of holomorphic functions of an open set  $\Delta$  which is star-shaped with respect to 0 is stable under convolution.

We then call endpoint path of  $H$  the path

$$\Gamma_H: t \in J \mapsto H_t(1).$$

Writing  $J = [a, b]$ , we call  $H_a$  (resp.  $H_b$ ) the initial path of  $H$  (resp. its final path).

The first two conditions imply that each path  $H_t$  is a path of analytic continuation for any  $\hat{\varphi} \in \hat{\mathcal{R}}_\Omega$ , in view of Remark 1.3. Here is a technical statement we shall use:

**Lemma 4.2.** *For a symmetric  $\Omega$ -homotopy  $H$  defined on  $I \times J$ , using the notation  $H_{t|s}$  for the truncated path  $(H_t)_{|[0,s]}$  (analogously to what we did when commenting Lemma 2.1), there exists  $\delta > 0$  such that, for any  $\hat{\varphi} \in \hat{\mathcal{R}}_\Omega$  and  $(s, t) \in I \times J$ , the radius of convergence of the holomorphic germ  $\text{cont}_{H_{t|s}} \hat{\varphi}$  at  $H_t(s)$  is at least  $\delta$ .*

*Proof.* Let  $\rho$  be as in Remark 1.3. Consider  $U := H^{-1}(\mathbb{D}_{\rho/2})$ : this is an open subset of  $I \times J$  which contains  $\{0\} \times J$ ; we define  $U_0$  to be the connected component of  $U$  which contains  $\{0\} \times J$ ,  $K := I \times J \setminus U_0$  and  $\delta := \min \{ \text{dist}(H(K), \Omega), \rho/2 \}$ .

We have  $\delta > 0$  because  $H(K)$  is compact (since  $K$  is compact) and  $H(K) \subset \mathbb{C} \setminus \Omega$  (since  $K \subset (0, 1] \times J$ ). Now, for any  $s$  and  $t$ ,

- either  $(s, t) \in U_0$ , then  $[0, s] \times \{t\} \subset U_0$ , the truncated path  $H_{t|s}$  lies in  $H(U_0) \subset \mathbb{D}_{\rho/2}$ , hence  $\text{cont}_{H_{t|s}} \hat{\varphi}$  is a holomorphic germ at  $H_t(s)$  with radius of convergence  $\geq \delta$ ;
- or  $(s, t) \in K$ , and then  $\text{dist}(H_t(s), \Omega) \geq \delta$ , which yields the same conclusion for the germ  $\text{cont}_{H_{t|s}} \hat{\varphi}$ .

□

The third condition in Definition 4.1 means that each path  $H_t$  is symmetric with respect to its midpoint  $\frac{1}{2}H_t(1)$ . Here is the motivation behind this requirement:

**Lemma 4.3.** *Suppose  $H$  is a symmetric  $\Omega$ -homotopy whose initial path is contained in  $\mathbb{D}_\rho$ , with  $\rho$  as in Remark 1.3. Then its endpoint path  $\Gamma_H$  is a path of analytic continuation for any convolution product  $\hat{\varphi} * \hat{\psi}$  with  $\hat{\varphi}, \hat{\psi} \in \hat{\mathcal{R}}_\Omega$ .*

*Proof.* We assume that  $H$  is defined on  $I \times J$  and we set  $\gamma := \Gamma_H$ . Let  $\hat{\varphi}, \hat{\psi} \in \hat{\mathcal{R}}_\Omega$  and, for  $t \in J$ , consider the formula

$$\hat{\chi}_t(\zeta) = \int_{H_t} \hat{\varphi}(\xi) \hat{\psi}(\zeta - \xi) d\xi + \int_{\gamma(t)}^{\zeta} \hat{\varphi}(\xi) \hat{\psi}(\zeta - \xi) d\xi \quad (3)$$

(recall that  $\gamma(t) = H_t(1)$ ). We shall check that  $\hat{\chi}_t$  is a well-defined holomorphic germ at  $\gamma(t)$  and that it provides the analytic continuation of  $\hat{\varphi} * \hat{\psi}$  along  $\gamma$ .

a) The idea is that when  $\xi$  moves along  $H_t$ ,  $\xi = H_t(s)$  with  $s \in I$ , we can use for “ $\hat{\varphi}(\xi)$ ” the analytic continuation of  $\hat{\varphi}$  along the truncated path  $H_{t|s}$ ; correspondingly, if  $\zeta$  is close to  $\gamma(t)$ , then  $\zeta - \xi$  is close to  $\gamma(t) - \xi = H_t(1) - H_t(s) = H_t(1 - s)$ , thus for “ $\hat{\psi}(\zeta - \xi)$ ” we can use the analytic continuation of  $\hat{\psi}$  along  $H_{t|1-s}$ . In other words, setting  $\zeta = \gamma(t) + \sigma$ , we wish to interpret (3) as

$$\begin{aligned} \hat{\chi}_t(\gamma(t) + \sigma) := & \int_0^1 (\text{cont}_{H_{t|s}} \hat{\varphi})(H_t(s)) (\text{cont}_{H_{t|1-s}} \hat{\psi})(H_t(1 - s) + \sigma) H_t'(s) ds \\ & + \int_0^1 (\text{cont}_{H_t} \hat{\varphi})(\gamma(t) + u\sigma) \hat{\psi}((1 - u)\sigma) \sigma du \quad (4) \end{aligned}$$

(in the last integral, we have performed the change variable  $\xi = \gamma(t) + u\sigma$ ; it is the germ of  $\hat{\psi}$  at the origin that we wish to use there).

Lemma 4.2 provides  $\delta > 0$  such that, by regular dependence of the integrals upon the parameter  $\sigma$ , the right-hand side of (4) is holomorphic for  $|\sigma| < \delta$ . We thus have a family of analytic elements  $(\hat{\chi}_t, D_t)$ ,  $t \in J$ , with  $D_t := \{\zeta \in \mathbb{C} \mid |\zeta - \gamma(t)| < \delta\}$ .

b) For  $t$  small enough, the path  $H_t$  is contained in  $\mathbb{D}_\rho$  which is open and simply connected; then, for  $|\zeta|$  small enough, the rectilinear segment  $[0, \zeta]$  and the concatenation of  $H_t$  and  $[\gamma(t), \zeta]$  are homotopic in  $\mathbb{D}_\rho$ , hence the Cauchy theorem implies  $\hat{\chi}_t(\zeta) = \hat{\varphi} * \hat{\psi}(\zeta)$ .

c) By uniform continuity, there exists  $\varepsilon > 0$  such that, for any  $t_0, t \in J$ ,

$$|t - t_0| \leq \varepsilon \implies |H_t(s) - H_{t_0}(s)| < \delta/2 \quad \text{for all } s \in I. \quad (5)$$

To complete the proof, we check that, for any  $t_0 < t_1$  in  $J$  such that  $|t_1 - t_0| \leq \varepsilon$ , we have  $\hat{\chi}_{t_0} \equiv \hat{\chi}_{t_1}$  in  $D_{t_0} \cap D_{t_1}$ .

Because of (5), the assumption  $|t_1 - t_0| \leq \varepsilon$  implies that, for  $t \in [t_0, t_1]$ ,  $s \in I$  and  $|\xi - H_{t_0}(s)| < \delta$ ,

$$\text{cont}_{H_t|_s} \hat{\varphi}(\xi) = \text{cont}_{H_{t_0}|_s} \hat{\varphi}(\xi), \quad \text{cont}_{H_t|_{1-s}} \hat{\psi}(\xi) = \text{cont}_{H_{t_0}|_{1-s}} \hat{\psi}(\xi).$$

Thus, as soon as  $t \in [t_0, t_1]$  and  $|\zeta - \gamma(t_0)| < \delta/2$ , we can write

$$\begin{aligned} \hat{\chi}_t(\zeta) &= \int_0^1 (\text{cont}_{H_t|_s} \hat{\varphi})(H_t(s)) (\text{cont}_{H_t|_{1-s}} \hat{\psi})(\zeta - H_t(s)) H_t'(s) ds \\ &\quad + \int_{\gamma(t)}^\zeta (\text{cont}_{H_t} \hat{\varphi})(\xi) \hat{\psi}(\zeta - \xi) d\xi \end{aligned}$$

and we get  $\hat{\chi}_{t_0}(\zeta) = \hat{\chi}_t(\zeta)$  from the Cauchy theorem by means of the homotopy induced by  $H$  between the concatenation of  $H_{t_0}$  and  $[\gamma(t_0), \zeta]$  and the concatenation of  $H_t$  and  $[\gamma(t), \zeta]$ .  $\square$

## 5 Proof of the main result: Geometric part

### 5.1 The key lemma

In view of Lemma 4.3, the proof of Theorem 3.1 immediately follows from the following purely geometric result:

**Lemma 5.1.** *For any path  $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \Omega$  such that  $\gamma(0) \in \mathbb{D}_\rho^*$  and the left and right derivatives  $\gamma'_\pm$  do not vanish on  $[0, 1]$ , there exists a symmetric  $\Omega$ -homotopy  $H$  whose endpoint path is  $\gamma$  and whose initial path is a rectilinear segment, i.e.  $\Gamma_H = \gamma$  and  $H_0(s) \equiv s\gamma(0)$ .*

The proof is very simple when  $\gamma$  does not pass through 0, which is automatic if we assume  $0 \in \Omega$ . The general case requires an extra work which is technical and involves a quantitative version of the simpler case. With a view to helping the reader to grasp the mechanism of the proof, we thus begin with the case when  $0 \in \Omega$ .

## 5.2 Proof of the key lemma when $0 \in \Omega$

Assume that  $\gamma$  is given as in the hypothesis of Lemma 5.1. We are looking for a symmetric  $\Omega$ -homotopy whose initial path is imposed: it must be

$$s \in I \mapsto H_0(s) := s\gamma(0),$$

which satisfies the three requirements of Definition 4.1 at  $t = 0$ :

- (i)  $H_0(0) = 0$ ,
- (ii)  $H_0((0, 1]) \subset \mathbb{C} \setminus \Omega$ ,
- (iii)  $H_0(1) - H_0(s) = H_0(1 - s)$  for every  $s \in I$ .

The idea is to define a family of maps  $(\Psi_t)_{t \in [0, 1]}$  so that

$$H_t(s) := \Psi_t(H_0(s)), \quad s \in I, \quad (6)$$

yield the desired homotopy. For that, it is sufficient that  $(t, \zeta) \in [0, 1] \times \mathbb{C} \mapsto \Psi_t(\zeta)$  be continuously differentiable (for the structure of real two-dimensional vector space of  $\mathbb{C}$ ),  $\Psi_0 = \text{Id}$  and, for each  $t \in [0, 1]$ ,

- (i')  $\Psi_t(0) = 0$ ,
- (ii')  $\Psi_t(\mathbb{C} \setminus \Omega) \subset \mathbb{C} \setminus \Omega$ ,
- (iii')  $\Psi_t(\gamma(0) - \zeta) = \Psi_t(\gamma(0)) - \Psi_t(\zeta)$  for all  $\zeta \in \mathbb{C}$ ,
- (iv')  $\Psi_t(\gamma(0)) = \gamma(t)$ .

In fact, the properties (i')–(iv') ensure that any initial path  $H_0$  satisfying (i)–(iii) and ending at  $\gamma(0)$  produces through (6) a symmetric  $\Omega$ -homotopy whose endpoint path is  $\gamma$ . Consequently, we may assume without loss of generality that  $\gamma$  is  $C^1$  on  $[0, 1]$  (then, if  $\gamma$  is only piecewise  $C^1$ , we just need to concatenate the symmetric  $\Omega$ -homotopies associated with the various pieces).

The maps  $\Psi_t$  will be generated by the flow of a non-autonomous vector field  $X(\zeta, t)$  associated with  $\gamma$  that we now define. We view  $(\mathbb{C}, |\cdot|)$  as a real 2-dimensional Banach space and pick<sup>4</sup> a  $C^1$  function  $\eta: \mathbb{C} \rightarrow [0, 1]$  such that

$$\{\zeta \in \mathbb{C} \mid \eta(\zeta) = 0\} = \Omega.$$

Observe that  $D(\zeta, t) := \eta(\zeta) + \eta(\gamma(t) - \zeta)$  defines a  $C^1$  function of  $(\zeta, t)$  and that

$$D(\zeta, t) > 0 \quad \text{for all } \zeta \in \mathbb{C} \text{ and } t \in [0, 1].$$

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<sup>4</sup>For instance pick a  $C^1$  function  $\varphi_0: \mathbb{R} \rightarrow [0, 1]$  such that  $\{x \in \mathbb{R} \mid \varphi_0(x) = 1\} = \{0\}$  and  $\varphi_0(x) = 0$  for  $|x| \geq 1$ , and a bijection  $\omega: \mathbb{N} \rightarrow \Omega$ ; then set  $\delta_k := \text{dist}(\omega(k), \Omega \setminus \{\omega(k)\}) > 0$  and  $\sigma(\zeta) := \sum_k \varphi_0(\frac{4|\zeta - \omega(k)|^2}{\delta_k^2})$ : for each  $\zeta \in \mathbb{C}$  there is at most one non-zero term in this series (because  $k \neq \ell$ ,  $|\zeta - \omega(k)| < \delta_k/2$  and  $|\zeta - \omega(\ell)| < \delta_\ell/2$  would imply  $|\omega(k) - \omega(\ell)| < (\delta_k + \delta_\ell)/2$ , which would contradict  $|\omega(k) - \omega(\ell)| \geq \delta_k$  and  $\delta_\ell$ ), thus  $\sigma$  is  $C^1$ , takes its values in  $[0, 1]$  and satisfies  $\{\zeta \in \mathbb{C} \mid \sigma(\zeta) = 1\} = \Omega$ , therefore  $\eta := 1 - \sigma$  will do. Other solution: adapt the proof of Lemma 5.2.

Indeed,  $D(\zeta, t) = 0$  would imply  $\zeta \in \Omega$  and  $\gamma(t) - \zeta \in \Omega$ , hence  $\gamma(t) \in \Omega$ , which would contradict our assumptions (it is here that we use the stability under addition of  $\Omega$ ). Therefore, the formula

$$X(\zeta, t) := \frac{\eta(\zeta)}{\eta(\zeta) + \eta(\gamma(t) - \zeta)} \gamma'(t)$$

defines a non-autonomous vector field, which is continuous in  $(\zeta, t)$  on  $\mathbb{C} \times [0, 1]$ ,  $C^1$  in  $\zeta$  and has its partial derivatives continuous in  $(\zeta, t)$ . The Cauchy-Lipschitz theorem on the existence and uniqueness of solutions to the first-order differential equation  $\frac{d\zeta}{dt} = X(\zeta, t)$  applies: for every  $\zeta \in \mathbb{C}$  and  $t_0 \in [0, 1]$  there is a unique solution  $t \mapsto \Phi^{t_0, t}(\zeta)$  such that  $\Phi^{t_0, t_0}(\zeta) = \zeta$ . The fact that the vector field  $X$  is bounded implies that  $\Phi^{t_0, t}(\zeta)$  is defined for all  $t \in [0, 1]$  and the classical theory guarantees that  $(t_0, t, \zeta) \mapsto \Phi^{t_0, t}(\zeta)$  is  $C^1$  on  $[0, 1] \times [0, 1] \times \mathbb{C}$ .

Let us set  $\Psi_t := \Phi^{0, t}$  for  $t \in [0, 1]$ . We easily check that this family of maps satisfies (i')–(iv'), as a consequence of the identities

$$X(\omega, t) = 0 \quad \text{for all } \omega \in \Omega, \tag{7}$$

$$X(\gamma(t) - \zeta, t) = \gamma'(t) - X(\zeta, t) \quad \text{for all } \zeta \in \mathbb{C} \tag{8}$$

valid for all  $t \in [0, 1]$  ((7) and (8) directly follow from our definition of  $X$ ; here we use  $0 \in \Omega$  to ensure  $\eta(0) = 0$ ):

- (i') and (ii') follow from (7) which yields  $\Phi^{t_0, t}(\omega) = \omega$  for every  $t_0$  and  $t$ , and from the non-autonomous flow property  $\Phi^{t_0, 0} \circ \Phi^{0, t} = \text{Id}$  (hence  $\Psi_t(\zeta) = \omega$  implies  $\zeta = \Phi^{t, 0}(\omega) = \omega$ );
- (iv') follows from the fact that  $X(\gamma(t), t) = \gamma'(t)$ , by (8) with  $\zeta = 0$ , hence  $t \mapsto \gamma(t)$  is a solution of  $X$ ;
- (iii') follows from (8): for any solution  $t \mapsto \zeta(t)$ , the curve  $t \mapsto \xi(t) := \gamma(t) - \zeta(t)$  satisfies  $\xi(0) = \gamma(0) - \zeta(0)$  and  $\xi'(t) = \gamma'(t) - X(\zeta(t), t) = X(\xi(t), t)$ , hence it is a solution:  $\xi(t) = \Psi_t(\gamma(0) - \zeta(0))$ .

### 5.3 Proof of the key lemma when $0 \notin \Omega$

From now on, we suppose  $0 \notin \Omega$  and we use the notation

$$\Omega_\varepsilon := \{ \zeta \in \mathbb{C} \mid \text{dist}(\zeta, \Omega) < \varepsilon \}$$

for any  $\varepsilon > 0$ . We shall require the following technical

**Lemma 5.2.** *For any  $\varepsilon > 0$  there exists a  $C^1$  function  $\eta: \mathbb{C} \rightarrow [0, 1]$  such that*

$$\{ \zeta \in \mathbb{C} \mid \eta(\zeta) = 0 \} = \{0\} \cup \overline{\Omega}_\varepsilon.$$

*Proof.* Pick a  $C^1$  function  $\chi: \mathbb{R} \rightarrow [0, 1]$  such that  $\{x \in \mathbb{R} \mid \chi(x) = 0\} = [-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$  and  $\chi(x) = 1$  for  $|x| \geq \sqrt{1 + \varepsilon}$ , and a bijection  $\omega: \mathbb{N}^* \rightarrow \Omega$ . For each  $k \in \mathbb{N}^*$ ,  $\eta_k(\zeta) := \chi(|\zeta - \omega(k)|^2)$  defines a  $C^1$  function on  $\mathbb{C}$  such that  $\eta_k^{-1}(0) = \overline{D(\omega(k), \varepsilon)}$  and  $\eta_k \equiv 1$  on  $\mathbb{C} \setminus D(\omega(k), 1 + \varepsilon)$ . Consider the infinite product

$$\eta_*(\zeta) := \prod_{k \in \mathbb{N}^*} \eta_k(\zeta). \tag{9}$$



For any bounded open subset  $U$  of  $\mathbb{C}$ , the set  $\mathcal{F}_U := \{k \in \mathbb{N}^* \mid U \cap D(\omega(k), 1 + \varepsilon) \neq \emptyset\}$  is finite (because  $\Omega$  is discrete), thus almost all the factors in (9) are equal to 1 when  $\zeta \in U$ :  $(\eta_*)|_U = \prod_{k \in \mathcal{F}_U} (\eta_k)|_U$ , hence  $\eta_*$  is  $C^1$ , takes its values in  $[0, 1]$  and

$$\eta_*^{-1}(0) \cap U = \bigcup_{k \in \mathcal{F}_U} \overline{D(\omega(k), \varepsilon)} \cap U,$$

whence it follows that  $\eta_*^{-1}(0) = \overline{\Omega_\varepsilon}$ .

If  $0 \in \overline{\Omega_\varepsilon}$ , then one can take  $\eta = \eta_*$ . If not, then one can take the product  $\eta = \eta_0 \eta_*$  with  $\eta_0(\zeta) := \chi_0(|\zeta|^2)$ , where  $\chi_0$  is any  $C^1$  function on  $\mathbb{R}$  which takes its values in  $[0, 1]$  and such that  $\chi_0^{-1}(0) = \{0\}$ .  $\square$

We now repeat the work of the previous section replacing  $\Omega$  with  $\{0\} \cup \Omega$ , adding quantitative information (we still assume that we are given a path which does not pass through 0 but we want to control the way the corresponding symmetric  $\Omega$ -homotopy approaches the points of  $\Omega$ ) and authorizing a more general initial path than a rectilinear one.

**Lemma 5.3.** *Let  $\delta, \delta' > 0$  with  $\delta' < \delta/2$ . Suppose that  $J = [a, b]$  is a compact interval of  $\mathbb{R}$  and  $\gamma: J \rightarrow \mathbb{C}$  is a path such that*

$$0 \notin \gamma(J) \quad \text{and} \quad \gamma(J) \subset \mathbb{C} \setminus \Omega_\delta.$$

*Suppose that  $h: I \rightarrow \mathbb{C}$  is a  $C^1$  path such that*

- (i)  $h(0) = 0$ ,
- (ii)  $h(I) \subset \mathbb{C} \setminus \Omega_{\delta'}$ ,
- (iii)  $h(1-s) = h(1) - h(s)$  for all  $s \in I$ ,
- (iv)  $h(1) = \gamma(a)$ .

*Then there exists a symmetric  $\Omega$ -homotopy  $H$  defined on  $I \times J$ , whose initial path is  $h$ , whose endpoint path is  $\gamma$ , which satisfies  $H(I \times J) \subset \mathbb{C} \setminus \Omega_{\delta'}$  and whose final path is  $C^1$ .*

*Proof.* We may assume without loss of generality that  $\gamma$  is  $C^1$  on  $J$  (if  $\gamma$  is only piecewise  $C^1$ , we just need to concatenate the symmetric  $\Omega$ -homotopies associated with the various pieces). We shall define a family of maps  $(\Psi_t)_{t \in J}$  so that

$$H_t(s) := \Psi_t(h(s)), \quad s \in I, \tag{10}$$

yield the desired homotopy. For that, it is sufficient that  $(t, \zeta) \in J \times \mathbb{C} \mapsto \Psi_t(\zeta)$  be continuously differentiable,  $\Psi_0 = \text{Id}$  and, for each  $t \in J$ ,

- (i')  $\Psi_t(0) = 0$ ,
- (ii')  $\Psi_t(\mathbb{C} \setminus \Omega_{\delta'}) \subset \mathbb{C} \setminus \Omega_{\delta'}$ ,
- (iii')  $\Psi_t(\gamma(0) - \zeta) = \Psi_t(\gamma(0)) - \Psi_t(\zeta)$  for all  $\zeta \in \mathbb{C}$ ,
- (iv')  $\Psi_t(\gamma(a)) = \gamma(t)$ .

As in Section 5.2, our maps  $\Psi_t$  will be generated by a non-autonomous vector field.

Lemma 5.2 allows us to choose a  $C^1$  function  $\eta: \mathbb{C} \rightarrow [0, 1]$  such that

$$\{\zeta \in \mathbb{C} \mid \eta(\zeta) = 0\} = \{0\} \cup \Omega_{\delta'}.$$

We observe that  $D(\zeta, t) := \eta(\zeta) + \eta(\gamma(t) - \zeta)$  defines a  $C^1$  function of  $(\zeta, t)$  and that

$$D(\zeta, t) > 0 \quad \text{for all } \zeta \in \mathbb{C} \text{ and } t \in [0, 1].$$

Indeed,  $D(\zeta, t) = 0$  would imply that both  $\zeta$  and  $\gamma(t) - \zeta$  lie in  $\{0\} \cup \Omega_{\delta'}$ , hence  $\gamma(t) \in \{0\} \cup \Omega_{2\delta'}$ , which would contradict our assumption  $\gamma(J) \subset \mathbb{C} \setminus (\{0\} \cup \Omega_{\delta})$ . Therefore,

$$X(\zeta, t) := \frac{\eta(\zeta)}{\eta(\zeta) + \eta(\gamma(t) - \zeta)} \gamma'(t), \quad (\zeta, t) \in \mathbb{C} \times J,$$

defines a non-autonomous vector field whose flow  $(\Phi^{t_0, t})_{t_0, t \in J}$  allows one to conclude the proof exactly as in Section 5.2, setting  $\Psi_t := \Phi^{a, t}$  and replacing (7) with

$$X(\omega, t) = 0 \quad \text{for all } \omega \in \{0\} \cup \Omega_{\delta'}.$$

□

We now consider the case of a path  $\gamma$  which entirely lies close to 0.

**Lemma 5.4.** *Let  $\varepsilon, \delta' > 0$  with  $0 < \varepsilon < \delta'$ . Suppose that  $K = [a, b]$  is a compact interval of  $\mathbb{R}$  and  $\gamma: K \rightarrow \mathbb{C}$  is a path such that*

$$\gamma(K) \subset \overline{\mathbb{D}}_{\varepsilon/2}.$$

*Suppose that  $h: I \rightarrow \mathbb{C}$  is a  $C^1$  path such that*

$$(i) \quad h(0) = 0,$$

$$(ii) \quad h(I) \subset \mathbb{C} \setminus \Omega_{\delta'},$$

$$(iii) \quad h(1 - s) = h(1) - h(s) \text{ for all } s \in I,$$

$$(iv) \quad h(1) = \gamma(a).$$

*Then there exists a symmetric  $\Omega$ -homotopy  $H$  defined on  $I \times K$ , whose initial path is  $h$ , whose endpoint path is  $\gamma$ , which satisfies*

$$H(I \times K) \subset \mathbb{C} \setminus \Omega_{\delta''} \quad \text{with } \delta'' := \delta' - \varepsilon$$

*and whose final path is  $C^1$ .*

*Proof.* Define  $H(s, t) := h(s) + s(\gamma(t) - \gamma(a))$ . This way  $H(s, a) = h(s)$ ,  $H(1, t) = \gamma(t)$  and  $H$  is a symmetric  $\Omega$ -homotopy as required:  $H(0, t) = 0$ ,  $H(s, t) + H(1 - s, t) = h(s) + h(1 - s) + \gamma(t) - \gamma(a) = \gamma(t)$ ,  $\text{dist}(H(s, t), \Omega) \geq \text{dist}(h(s), \Omega) - |\gamma(t) - \gamma(a)| \geq \delta' - \varepsilon$ . □

**Proof of Lemma 5.1 under the standing assumption  $0 \notin \Omega$ .** Let  $\gamma$  be as in the hypothesis of Lemma 5.1. Without loss of generality, we can assume  $\gamma(1) \neq 0$  (if not, view  $\gamma$  as the restriction of a path  $\tilde{\gamma}: [0, 2] \rightarrow \mathbb{C} \setminus \Omega$  such that  $\tilde{\gamma}(2) \neq 0$ , with which is associated a symmetric  $\Omega$ -homotopy  $\tilde{H}$  defined on  $I \times [0, 2]$ , and restrict  $\tilde{H}$  to  $I \times [0, 1]$ ). Let  $\delta := \text{dist}(\Omega, \gamma([0, 1]))$ .

The set  $Z := \{t \in [0, 1] \mid \gamma(t) = 0\}$  is closed; it is also discrete because of the non-vanishing of the derivatives of  $\gamma$ , thus it has a finite cardinality  $N \in \mathbb{N}$ . If  $N = 0$ , then we can apply Lemma 5.3 with  $J = [0, 1]$  and  $h(s) \equiv s\gamma(0)$  and the proof is complete.

From now on we suppose  $N \geq 1$ . Let us write

$$Z = \{t_1, \dots, t_N\} \quad \text{with } 0 < t_1 < \dots < t_N < 1.$$

We define

$$\delta_0 := \frac{1}{2} \min \left\{ \frac{\delta}{2}, \rho - |\gamma(0)| \right\} \quad \text{and} \quad \varepsilon := \min \left\{ |\gamma(0)|, |\gamma(1)|, \frac{\delta_0}{N+1} \right\}.$$

The uniform continuity of  $\gamma$  allows us to find pairwise disjoint closed intervals of positive lengths  $K_1, \dots, K_N$  such that

$$t_j \in K_j \quad \text{and} \quad \gamma(K_j) \subset \overline{\mathbb{D}}_{\varepsilon/2}, \quad j = 1, \dots, N.$$

By considering the connected components of  $[0, 1] \setminus \bigcup K_j$  and taking their closures, we get adjacent closed subintervals of positive lengths of  $[0, 1]$ ,

$$J_0, K_1, J_1, K_2, \dots, J_{N-1}, K_N, J_N$$

with  $J_j = [a_j, b_j]$ ,  $K_j = [b_{j-1}, a_j]$ ,  $a_0 = 0$ ,  $b_N = 1$ . Observe that

$$0 \notin \gamma(J_j) \quad \text{and} \quad \gamma(J_j) \subset \mathbb{C} \setminus \Omega_{\delta}, \quad j = 0, \dots, N.$$

- We apply Lemma 5.3 with  $J = J_0 = [0, b_0]$ ,  $h(s) \equiv s\gamma(0)$  and  $\delta' = \delta_0$  (which is allowed by the choice of  $\delta_0$ ): we get a symmetric  $\Omega$ -homotopy  $H$  defined on  $I \times J_0$  whose initial path is the rectilinear segment  $[0, \gamma(0)]$ , whose endpoint path is  $\gamma|_{J_0}$  and whose final path  $H_{b_0}$  is  $C^1$  and lies in  $\mathbb{C} \setminus \Omega_{\delta_0}$ .
- We apply Lemma 5.4 with  $K = K_1$ ,  $\delta' = \delta_0$  and  $h = H_{b_0}$ : we get an extension of our symmetric  $\Omega$ -homotopy  $H$  to  $I \times K_1$ , in which the endpoint path is extended by  $\gamma|_{K_1}$  and the final path is now  $H_{a_1}$ , a  $C^1$  path contained in  $\mathbb{C} \setminus \Omega_{\delta_1}$  with  $\delta_1 := \delta_0 - \varepsilon$ .
- And so on: we apply alternatively Lemma 5.4 on  $K_j$  and Lemma 5.3 on  $J_j$ : we get an extension of the symmetric  $\Omega$ -homotopy  $H$  to  $I \times K_j$  or  $I \times J_j$  such that both  $H_{a_j}(I)$  and  $H_{b_j}(I)$  are contained in  $\mathbb{C} \setminus \Omega_{\delta_j}$  with  $\delta_j := \delta_0 - j\varepsilon$ .

When we reach  $j = N$ , the proof is complete. □

*Acknowledgements.* The research leading to these results has received funding from the European Community's Seventh Framework Program (FP7/2007–2013) under Grant Agreement n. 236346.

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