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On the stability under convolution of resurgent functions

David Sauzin

January 5, 2012

Abstract

This article contains a self-contained proof of the stability under convolution of the class of resurgent functions associated with a closed discrete subset Ω of \mathbb{C} , under the assumption that Ω , the set of possible singularities, be stable under addition.

1 The Ω -continuable germs

For any closed discrete subset of \mathbb{C} , we introduce the definition Ω -continuity, which can be rephrased as holomorphy on a certain Riemann surface which depends only on Ω . It is a particular case of the notion of endless continuity due to J. Écalle, at the basis of his theory of resurgent functions [Eca81], [Mal85].¹ Our aim is to give a rigorous and self-contained treatment of the stability under convolution of this class of resurgent functions, with more details and more complete explanations than in [Sau06] which was dealing with the particular case $\Omega = 2\pi i\mathbb{Z}$. (The article [Ou10] proposes another approach for the latter case.)

For any $R > 0$ and $\zeta_0 \in \mathbb{C}$ we use the notations $\mathbb{D}_R := \{\zeta \in \mathbb{C} \mid |\zeta| < R\}$, $\mathbb{D}_R^* := \mathbb{D}_R \setminus \{0\}$ and $D(\zeta_0, R) := \{\zeta \in \mathbb{C} \mid |\zeta - \zeta_0| < R\}$. In this article, “path” means a piecewise C^1 function $\gamma: J \rightarrow \mathbb{C}$, where J is a compact interval of \mathbb{R} .

Definition 1.1. *Let Ω be a non-empty closed discrete subset of \mathbb{C} , let $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ be a holomorphic germ at the origin. We say that $\hat{\varphi}$ is Ω -continuable if there exists $R > 0$ such that \mathbb{D}_R^* is contained in the disc of convergence of $\hat{\varphi}$, $\mathbb{D}_R^* \cap \Omega = \emptyset$ and $\hat{\varphi}$ admits analytic continuation along any path of $\mathbb{C} \setminus \Omega$ originating from any point of \mathbb{D}_R^* . We use the notation*

$$\hat{\mathcal{R}}_\Omega := \{ \text{all } \Omega\text{-continuable holomorphic germs} \} \subset \mathbb{C}\{\zeta\}.$$

The theory can be developed in a greater generality (see particularly [Eca81], Vol. 3, or [CNP93], where a function is defined to be resurgent if it can be analytically continued to an “endless” Riemann surface which may depend on the function), but Definition 1.1 already covers interesting cases: one encounters examples of Ω -continuable germs with $\Omega = \mathbb{N}^*$ or $\Omega = \mathbb{Z}$ when dealing with differential equations formally conjugate to the Euler equation (in the study of the saddle-node singularities) [Eca84], [Sau10], or with $2\pi i\mathbb{Z}$ when dealing with certain difference equations like Abel’s equation for parabolic germs (in holomorphic dynamics) [Eca81], [Sau06], [Sau12].

¹In Écalle’s terminology, resurgent functions are defined either in the “convolutive model”, the “formal model” or the “geometric models”. Roughly speaking, the convolutive model of resurgent functions is a subspace of $\mathbb{C}\{\zeta\}$ defined by requiring a certain property of analytic continuation; from it, one defines the formal model as a subspace of $z^{-1}\mathbb{C}[[z^{-1}]]$ via inverse formal Borel transform and the geometric models by Laplace transforms in certain directions—see [Eca81], [Eca92], [Eca93], [CNP93], [Sau06], [Sau12]. Our Ω -continuable germs belong to the convolutive model of resurgent functions.

Remark 1.2. *As is often the case with analytic continuation and Cauchy integrals, the precise parametrisation of γ will usually not matter, in the sense that we shall get the same result from two paths $\gamma: [a, b] \rightarrow \mathbb{C} \setminus \Omega$ and $\gamma': [a', b'] \rightarrow \mathbb{C} \setminus \Omega$ which only differ by a change of parametrisation ($\gamma = \gamma' \circ \sigma$ with $\sigma: [a, b] \rightarrow [a', b']$ piecewise continuously differentiable, increasing and mapping a to a' and b to b').*

If $\hat{\varphi}$ is a holomorphic germ at $\gamma(a)$ which admits an analytic continuation along γ , we denote by $\text{cont}_\gamma \hat{\varphi}$ the resulting holomorphic germ at the endpoint $\gamma(b)$.

Remark 1.3. *Let $\rho := \min \{|\omega|, \omega \in \Omega \setminus \{0\}\}$. Any $\hat{\varphi} \in \hat{\mathcal{R}}_\Omega$ is a holomorphic germ at 0 with radius of convergence $\geq \rho$ and one can always take $R = \rho$ in Definition 1.1. In fact, even if $0 \in \Omega$, for arbitrary $\zeta_0 \in \mathbb{D}_\rho$ we have*

$$\hat{\varphi} \in \hat{\mathcal{R}}_\Omega \iff \left\{ \begin{array}{l} \hat{\varphi} \text{ holomorphic function of } \mathbb{D}_\rho \text{ admitting analytic continuation along} \\ \text{any path } \gamma: [0, 1] \rightarrow \mathbb{C} \text{ such that } \gamma(0) = \zeta_0 \text{ and } \gamma((0, 1]) \subset \mathbb{C} \setminus \Omega \end{array} \right.$$

(even if $\zeta_0 = 0$: there is no need to avoid 0 at the beginning of the path, when we still are in the disc of convergence of $\hat{\varphi}$).

Example 1.1. A trivial example of Ω -continuable germ is any entire function of \mathbb{C} . Other elementary examples of Ω -continuable germs are the functions which are holomorphic in $\mathbb{C} \setminus \Omega$ and regular at 0, like $\frac{1}{(\zeta - \omega)^m}$ with $m \in \mathbb{N}^*$ and $\omega \in \Omega \setminus \{0\}$. But these are still single-valued examples, whereas the interest of the Definition 1.1 is to authorize multivaluedness when following the analytic continuation. Elementary examples of multivalued continuation are provided by $\sum_{n \geq 1} \frac{\zeta^n}{n} = -\log(1 - \zeta)$ (principal branch of the logarithm), which is $\{1\}$ -continuable, and $\sum_{n \geq 0} \frac{\zeta^n}{n+1} = -\frac{1}{\zeta} \log(1 - \zeta)$, which is $\{0, 1\}$ -continuable.

Example 1.2. If $\omega \in \mathbb{C}^*$ and $m \in \mathbb{N}^*$, then $(\log(\zeta - \omega))^m \in \hat{\mathcal{R}}_{\{\omega\}}$; if moreover $\omega \neq -1$, then $(\log(\zeta - \omega))^{-m} \in \hat{\mathcal{R}}_{\{\omega, \omega+1\}}$.

Example 1.3. If Ω is a closed discrete subset of \mathbb{C} , $0 \notin \Omega$, $\omega \in \Omega$ and $\hat{\psi}$ is holomorphic in $\mathbb{C} \setminus \Omega$, then $\hat{\varphi}(\zeta) = \hat{\psi}(\zeta) \log(\zeta - \omega)$ defines a germ of $\hat{\mathcal{R}}_\Omega$ whose monodromy around ω is given by $2\pi i \hat{\psi}$.

2 The convolution of holomorphic germs at the origin

The convolution in $\mathbb{C}\{\zeta\}$ is defined by the formula

$$\hat{\varphi} * \hat{\psi}(\zeta) := \int_0^\zeta \hat{\varphi}(\xi) \hat{\psi}(\zeta - \xi) d\xi$$

for any $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$: the formula makes sense for $|\zeta|$ close enough and defines a holomorphic germ at 0 whose disc of convergence contains the intersection of the discs of convergence of $\hat{\varphi}$ and $\hat{\psi}$. The convolution law $*$ is commutative and associative.²

The question we address in this article is the question of the stability of $\hat{\mathcal{R}}_\Omega$ under convolution. This amounts to inquiring about the analytic continuation of the germ $\hat{\varphi} * \hat{\psi}$ when Ω -continuity is assumed for $\hat{\varphi}$ and $\hat{\psi}$. Let us first mention an easy case:

²It is induced by the pull-back of the Cauchy product in $z^{-1}\mathbb{C}[[z^{-1}]]$ by the formal Borel transform and corresponds, via the Laplace transform, to the ordinary product of analytic functions.

Lemma 2.1. *Let Ω be any non-empty closed discrete subset of \mathbb{C} and suppose \hat{A} is an entire function of \mathbb{C} . Then, for any $\hat{\varphi} \in \hat{\mathcal{R}}_\Omega$, the convolution product $\hat{A} * \hat{\varphi}$ belongs to $\hat{\mathcal{R}}_\Omega$; its analytic continuation along a path γ of $\mathbb{C} \setminus \Omega$ starting from a point ζ_0 close enough to 0 and ending at a point ζ_1 is the holomorphic germ at ζ_1 explicitly given by*

$$\text{cont}_\gamma(\hat{A} * \hat{\varphi})(\zeta) = \int_0^{\zeta_0} \hat{A}(\zeta - \xi) \hat{\varphi}(\xi) d\xi + \int_\gamma \hat{A}(\zeta - \xi) \hat{\varphi}(\xi) d\xi + \int_{\zeta_1}^{\zeta} \hat{A}(\zeta - \xi) \hat{\varphi}(\xi) d\xi \quad (1)$$

for ζ close enough to ζ_1 .

The proof is left as an exercise, but we wish to emphasize that formulas such as (1) require a word of caution: the value of $\hat{A}(\zeta - \xi)$ is unambiguously defined whatever ζ and ξ are, but in the notation “ $\hat{\varphi}(\xi)$ ” it is understood that we are using the appropriate branch of the possibly multivalued function $\hat{\varphi}$; in such a formula, what branch we are using is clear from the context:

- $\hat{\varphi}$ is unambiguously defined in its disc of convergence D_0 (centred at 0) and the first integral thus makes sense for $\zeta_0 \in D_0$;
- in the second integral ξ is moving along γ which is a path of analytic continuation for $\hat{\varphi}$, we thus consider the analytic continuation of $\hat{\varphi}$ along the piece of γ between its origin and ξ ;
- in the third integral, “ $\hat{\varphi}$ ” is to be understood as $\text{cont}_\gamma \hat{\varphi}$, the germ at ζ_1 resulting from the analytic continuation of $\hat{\varphi}$ along γ , this integral thus makes sense for any ζ at a distance from ζ_1 less than the radius of convergence of $\text{cont}_\gamma \hat{\varphi}$.

Using a parametrisation $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \Omega$, with $\gamma(0) = \zeta_0$ and $\gamma(1) = \zeta_1$, and introducing the truncated paths $\gamma_s := \gamma|_{[0, s]}$ for any $s \in [0, 1]$, the interpretation of the last two integrals in (1) is

$$\begin{aligned} \int_\gamma \hat{A}(\zeta - \xi) \hat{\varphi}(\xi) d\xi &:= \int_0^1 \hat{A}(\zeta - \gamma(s)) (\text{cont}_{\gamma_s} \hat{\varphi})(\gamma(s)) \gamma'(s) ds, \\ \int_{\zeta_1}^{\zeta} \hat{A}(\zeta - \xi) \hat{\varphi}(\xi) d\xi &:= \int_{\zeta_1}^{\zeta} \hat{A}(\zeta - \xi) (\text{cont}_\gamma \hat{\varphi})(\xi) d\xi. \end{aligned}$$

3 Main result

We now wish to consider the case of the convolution of two Ω -continuable holomorphic germs at 0, $\hat{\varphi}$ and $\hat{\psi}$, without assuming that any of them extends to an entire function. The main result of this article is

Theorem 3.1. *Let Ω be a non-empty closed discrete subset of \mathbb{C} which is stable by addition. Then the space $\hat{\mathcal{R}}_\Omega$ is stable by convolution.*

The hypothesis on Ω is satisfied by the typical examples \mathbb{Z} or $2\pi i\mathbb{Z}$, but also by \mathbb{N}^* , $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{N}^* \times \mathbb{N}$ or $\{m + n\sqrt{2} \mid m, n \in \mathbb{N}^*\}$ for instance. An elementary example will convince us that the stability of Ω under addition is a necessary condition for the conclusion:

Example 3.1 ([CNP93]). Let us consider $\omega_1, \omega_2 \in \mathbb{C}^*$ and $\hat{\varphi}(\zeta) = \frac{1}{\zeta - \omega_1}$, $\hat{\psi}(\zeta) = \frac{1}{\zeta - \omega_2}$ and study

$$\hat{\chi}(\zeta) = \hat{\varphi} * \hat{\psi}(\zeta) = \int_0^{\zeta} \frac{1}{(\xi - \omega_1)(\zeta - \xi - \omega_2)} d\xi, \quad |\zeta| < \min\{|\omega_1|, |\omega_2|\}.$$

The formula

$$\frac{1}{(\xi - \omega_1)(\zeta - \xi - \omega_2)} = \frac{1}{\zeta - \omega_1 - \omega_2} \left(\frac{1}{\xi - \omega_1} + \frac{1}{\zeta - \xi - \omega_2} \right)$$

shows that, for any $\zeta \neq \omega_1 + \omega_2$ of modulus $< \min\{|\omega_1|, |\omega_2|\}$, one can write

$$\hat{\chi}(\zeta) = \frac{1}{\zeta - \omega_1 - \omega_2} (L_1(\zeta) + L_2(\zeta)), \quad L_j(\zeta) := \int_0^\zeta \frac{d\xi}{\xi - \omega_j} \quad (2)$$

(with the help of the change of variable $\xi \mapsto \zeta - \xi$ in the case of L_2). Removing the half-lines $\omega_j[1, +\infty)$ from \mathbb{C} , we obtain a cut plane Δ in which $\hat{\chi}$ has a meromorphic continuation (since $[0, \zeta]$ avoids the points ω_1 and ω_2 for all $\zeta \in \Delta$). We can in fact follow the meromorphic continuation of $\hat{\chi}$ along any path which avoids ω_1 and ω_2 , because

$$L_j(\zeta) = - \int_0^{\zeta/\omega_j} \frac{d\xi}{1 - \xi} = \log \left(1 - \frac{\zeta}{\omega_j} \right) \in \hat{\mathcal{R}}_{\{\omega_j\}}.$$

We used the words “meromorphic continuation” and not “analytic continuation” because of the factor $\frac{1}{\zeta - \omega_1 - \omega_2}$. The conclusion is thus only $\hat{\chi} \in \hat{\mathcal{R}}_\Omega$, with $\Omega := \{\omega_1, \omega_2, \omega_1 + \omega_2\}$.

In fact, if $\omega := \omega_1 + \omega_2 \in \Delta$, the principal branch of $\hat{\chi}$ (*i.e.* its meromorphic continuation to Δ) has a removable singularity³ at ω , because $(L_1 + L_2)(\omega) = \int_0^\omega \frac{d\xi}{\xi - \omega_1} + \int_0^\omega \frac{d\xi}{\xi - \omega_2} = 0$ in that case (by the change of variable $\xi \mapsto \omega - \xi$ in one of the integrals). But it is easy to see that this does not happen for all the branches of $\hat{\chi}$: when considering all the paths γ going from 0 to ω and avoiding ω_1 and ω_2 , the number $\text{cont}_\gamma L_1(\omega) + \text{cont}_\gamma L_2(\omega)$ takes all the values of the form $2\pi i k$ with $k \in \mathbb{Z}$; whenever this value is nonzero, the corresponding branch of $\hat{\chi}$ does have a pole at ω .

This example thus shows that $\hat{\mathcal{R}}_{\{\omega_1, \omega_2\}}$ is *not* stable by convolution (it contains $\hat{\varphi}_1$ and $\hat{\varphi}_2$ but not $\hat{\varphi}_1 * \hat{\varphi}_2$).

4 Proof of the main result: Analytic part

We now fix Ω as in the assumption of Theorem 3.1 and start the proof. We begin with a new definition:

Definition 4.1. *Let $I = [0, 1]$. A continuous map $H: I \times J \rightarrow \mathbb{C}$, where J is a compact interval of \mathbb{R} , is called a symmetric Ω -homotopy if, for each $t \in J$,*

$$H_t: s \in I \mapsto H_t(s) := H(s, t)$$

defines a path which satisfies

- i) $H_t(0) = 0$,*
- ii) $H_t((0, 1]) \subset \mathbb{C} \setminus \Omega$,*
- iii) $H_t(1) - H_t(s) = H_t(1 - s)$ for every $s \in I$.*

³This is consistent with the well-known fact that the space of holomorphic functions of an open set Δ which is star-shaped with respect to 0 is stable under convolution.

We then call endpoint path of H the path

$$\Gamma_H: t \in J \mapsto H_t(1).$$

Writing $J = [a, b]$, we call H_a (resp. H_b) the initial path of H (resp. its final path).

The first two conditions imply that each path H_t is a path of analytic continuation for any $\hat{\varphi} \in \hat{\mathcal{R}}_\Omega$, in view of Remark 1.3. Here is a technical statement we shall use:

Lemma 4.2. *For a symmetric Ω -homotopy H defined on $I \times J$, using the notation $H_{t|s}$ for the truncated path $(H_t)_{|[0,s]}$ (analogously to what we did when commenting Lemma 2.1), there exists $\delta > 0$ such that, for any $\hat{\varphi} \in \hat{\mathcal{R}}_\Omega$ and $(s, t) \in I \times J$, the radius of convergence of the holomorphic germ $\text{cont}_{H_{t|s}} \hat{\varphi}$ at $H_t(s)$ is at least δ .*

Proof. Let ρ be as in Remark 1.3. Consider $U := H^{-1}(\mathbb{D}_{\rho/2})$: this is an open subset of $I \times J$ which contains $\{0\} \times J$; we define U_0 to be the connected component of U which contains $\{0\} \times J$, $K := I \times J \setminus U_0$ and $\delta := \min \{ \text{dist}(H(K), \Omega), \rho/2 \}$.

We have $\delta > 0$ because $H(K)$ is compact (since K is compact) and $H(K) \subset \mathbb{C} \setminus \Omega$ (since $K \subset (0, 1] \times J$). Now, for any s and t ,

- either $(s, t) \in U_0$, then $[0, s] \times \{t\} \subset U_0$, the truncated path $H_{t|s}$ lies in $H(U_0) \subset \mathbb{D}_{\rho/2}$, hence $\text{cont}_{H_{t|s}} \hat{\varphi}$ is a holomorphic germ at $H_t(s)$ with radius of convergence $\geq \delta$;
- or $(s, t) \in K$, and then $\text{dist}(H_t(s), \Omega) \geq \delta$, which yields the same conclusion for the germ $\text{cont}_{H_{t|s}} \hat{\varphi}$.

□

The third condition in Definition 4.1 means that each path H_t is symmetric with respect to its midpoint $\frac{1}{2}H_t(1)$. Here is the motivation behind this requirement:

Lemma 4.3. *Suppose H is a symmetric Ω -homotopy whose initial path is contained in \mathbb{D}_ρ , with ρ as in Remark 1.3. Then its endpoint path Γ_H is a path of analytic continuation for any convolution product $\hat{\varphi} * \hat{\psi}$ with $\hat{\varphi}, \hat{\psi} \in \hat{\mathcal{R}}_\Omega$.*

Proof. We assume that H is defined on $I \times J$ and we set $\gamma := \Gamma_H$. Let $\hat{\varphi}, \hat{\psi} \in \hat{\mathcal{R}}_\Omega$ and, for $t \in J$, consider the formula

$$\hat{\chi}_t(\zeta) = \int_{H_t} \hat{\varphi}(\xi) \hat{\psi}(\zeta - \xi) d\xi + \int_{\gamma(t)}^{\zeta} \hat{\varphi}(\xi) \hat{\psi}(\zeta - \xi) d\xi \quad (3)$$

(recall that $\gamma(t) = H_t(1)$). We shall check that $\hat{\chi}_t$ is a well-defined holomorphic germ at $\gamma(t)$ and that it provides the analytic continuation of $\hat{\varphi} * \hat{\psi}$ along γ .

a) The idea is that when ξ moves along H_t , $\xi = H_t(s)$ with $s \in I$, we can use for “ $\hat{\varphi}(\xi)$ ” the analytic continuation of $\hat{\varphi}$ along the truncated path $H_{t|s}$; correspondingly, if ζ is close to $\gamma(t)$, then $\zeta - \xi$ is close to $\gamma(t) - \xi = H_t(1) - H_t(s) = H_t(1 - s)$, thus for “ $\hat{\psi}(\zeta - \xi)$ ” we can use the analytic continuation of $\hat{\psi}$ along $H_{t|1-s}$. In other words, setting $\zeta = \gamma(t) + \sigma$, we wish to interpret (3) as

$$\begin{aligned} \hat{\chi}_t(\gamma(t) + \sigma) := & \int_0^1 (\text{cont}_{H_{t|s}} \hat{\varphi})(H_t(s)) (\text{cont}_{H_{t|1-s}} \hat{\psi})(H_t(1 - s) + \sigma) H_t'(s) ds \\ & + \int_0^1 (\text{cont}_{H_t} \hat{\varphi})(\gamma(t) + u\sigma) \hat{\psi}((1 - u)\sigma) \sigma du \quad (4) \end{aligned}$$

(in the last integral, we have performed the change variable $\xi = \gamma(t) + u\sigma$; it is the germ of $\hat{\psi}$ at the origin that we wish to use there).

Lemma 4.2 provides $\delta > 0$ such that, by regular dependence of the integrals upon the parameter σ , the right-hand side of (4) is holomorphic for $|\sigma| < \delta$. We thus have a family of analytic elements $(\hat{\chi}_t, D_t)$, $t \in J$, with $D_t := \{\zeta \in \mathbb{C} \mid |\zeta - \gamma(t)| < \delta\}$.

b) For t small enough, the path H_t is contained in \mathbb{D}_ρ which is open and simply connected; then, for $|\zeta|$ small enough, the rectilinear segment $[0, \zeta]$ and the concatenation of H_t and $[\gamma(t), \zeta]$ are homotopic in \mathbb{D}_ρ , hence the Cauchy theorem implies $\hat{\chi}_t(\zeta) = \hat{\varphi} * \hat{\psi}(\zeta)$.

c) By uniform continuity, there exists $\varepsilon > 0$ such that, for any $t_0, t \in J$,

$$|t - t_0| \leq \varepsilon \implies |H_t(s) - H_{t_0}(s)| < \delta/2 \quad \text{for all } s \in I. \quad (5)$$

To complete the proof, we check that, for any $t_0 < t_1$ in J such that $|t_1 - t_0| \leq \varepsilon$, we have $\hat{\chi}_{t_0} \equiv \hat{\chi}_{t_1}$ in $D_{t_0} \cap D_{t_1}$.

Because of (5), the assumption $|t_1 - t_0| \leq \varepsilon$ implies that, for $t \in [t_0, t_1]$, $s \in I$ and $|\xi - H_{t_0}(s)| < \delta$,

$$\text{cont}_{H_t|_s} \hat{\varphi}(\xi) = \text{cont}_{H_{t_0}|_s} \hat{\varphi}(\xi), \quad \text{cont}_{H_t|_{1-s}} \hat{\psi}(\xi) = \text{cont}_{H_{t_0}|_{1-s}} \hat{\psi}(\xi).$$

Thus, as soon as $t \in [t_0, t_1]$ and $|\zeta - \gamma(t_0)| < \delta/2$, we can write

$$\begin{aligned} \hat{\chi}_t(\zeta) &= \int_0^1 (\text{cont}_{H_t|_s} \hat{\varphi})(H_t(s)) (\text{cont}_{H_t|_{1-s}} \hat{\psi})(\zeta - H_t(s)) H_t'(s) ds \\ &\quad + \int_{\gamma(t)}^\zeta (\text{cont}_{H_t} \hat{\varphi})(\xi) \hat{\psi}(\zeta - \xi) d\xi \end{aligned}$$

and we get $\hat{\chi}_{t_0}(\zeta) = \hat{\chi}_t(\zeta)$ from the Cauchy theorem by means of the homotopy induced by H between the concatenation of H_{t_0} and $[\gamma(t_0), \zeta]$ and the concatenation of H_t and $[\gamma(t), \zeta]$. \square

5 Proof of the main result: Geometric part

5.1 The key lemma

In view of Lemma 4.3, the proof of Theorem 3.1 immediately follows from the following purely geometric result:

Lemma 5.1. *For any path $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \Omega$ such that $\gamma(0) \in \mathbb{D}_\rho^*$ and the left and right derivatives γ'_\pm do not vanish on $[0, 1]$, there exists a symmetric Ω -homotopy H whose endpoint path is γ and whose initial path is a rectilinear segment, i.e. $\Gamma_H = \gamma$ and $H_0(s) \equiv s\gamma(0)$.*

The proof is very simple when γ does not pass through 0, which is automatic if we assume $0 \in \Omega$. The general case requires an extra work which is technical and involves a quantitative version of the simpler case. With a view to helping the reader to grasp the mechanism of the proof, we thus begin with the case when $0 \in \Omega$.

5.2 Proof of the key lemma when $0 \in \Omega$

Assume that γ is given as in the hypothesis of Lemma 5.1. We are looking for a symmetric Ω -homotopy whose initial path is imposed: it must be

$$s \in I \mapsto H_0(s) := s\gamma(0),$$

which satisfies the three requirements of Definition 4.1 at $t = 0$:

- (i) $H_0(0) = 0$,
- (ii) $H_0((0, 1]) \subset \mathbb{C} \setminus \Omega$,
- (iii) $H_0(1) - H_0(s) = H_0(1 - s)$ for every $s \in I$.

The idea is to define a family of maps $(\Psi_t)_{t \in [0, 1]}$ so that

$$H_t(s) := \Psi_t(H_0(s)), \quad s \in I, \quad (6)$$

yield the desired homotopy. For that, it is sufficient that $(t, \zeta) \in [0, 1] \times \mathbb{C} \mapsto \Psi_t(\zeta)$ be continuously differentiable (for the structure of real two-dimensional vector space of \mathbb{C}), $\Psi_0 = \text{Id}$ and, for each $t \in [0, 1]$,

- (i') $\Psi_t(0) = 0$,
- (ii') $\Psi_t(\mathbb{C} \setminus \Omega) \subset \mathbb{C} \setminus \Omega$,
- (iii') $\Psi_t(\gamma(0) - \zeta) = \Psi_t(\gamma(0)) - \Psi_t(\zeta)$ for all $\zeta \in \mathbb{C}$,
- (iv') $\Psi_t(\gamma(0)) = \gamma(t)$.

In fact, the properties (i')–(iv') ensure that any initial path H_0 satisfying (i)–(iii) and ending at $\gamma(0)$ produces through (6) a symmetric Ω -homotopy whose endpoint path is γ . Consequently, we may assume without loss of generality that γ is C^1 on $[0, 1]$ (then, if γ is only piecewise C^1 , we just need to concatenate the symmetric Ω -homotopies associated with the various pieces).

The maps Ψ_t will be generated by the flow of a non-autonomous vector field $X(\zeta, t)$ associated with γ that we now define. We view $(\mathbb{C}, |\cdot|)$ as a real 2-dimensional Banach space and pick⁴ a C^1 function $\eta: \mathbb{C} \rightarrow [0, 1]$ such that

$$\{\zeta \in \mathbb{C} \mid \eta(\zeta) = 0\} = \Omega.$$

Observe that $D(\zeta, t) := \eta(\zeta) + \eta(\gamma(t) - \zeta)$ defines a C^1 function of (ζ, t) and that

$$D(\zeta, t) > 0 \quad \text{for all } \zeta \in \mathbb{C} \text{ and } t \in [0, 1].$$

⁴For instance pick a C^1 function $\varphi_0: \mathbb{R} \rightarrow [0, 1]$ such that $\{x \in \mathbb{R} \mid \varphi_0(x) = 1\} = \{0\}$ and $\varphi_0(x) = 0$ for $|x| \geq 1$, and a bijection $\omega: \mathbb{N} \rightarrow \Omega$; then set $\delta_k := \text{dist}(\omega(k), \Omega \setminus \{\omega(k)\}) > 0$ and $\sigma(\zeta) := \sum_k \varphi_0(\frac{4|\zeta - \omega(k)|^2}{\delta_k^2})$: for each $\zeta \in \mathbb{C}$ there is at most one non-zero term in this series (because $k \neq \ell$, $|\zeta - \omega(k)| < \delta_k/2$ and $|\zeta - \omega(\ell)| < \delta_\ell/2$ would imply $|\omega(k) - \omega(\ell)| < (\delta_k + \delta_\ell)/2$, which would contradict $|\omega(k) - \omega(\ell)| \geq \delta_k$ and δ_ℓ), thus σ is C^1 , takes its values in $[0, 1]$ and satisfies $\{\zeta \in \mathbb{C} \mid \sigma(\zeta) = 1\} = \Omega$, therefore $\eta := 1 - \sigma$ will do. Other solution: adapt the proof of Lemma 5.2.

Indeed, $D(\zeta, t) = 0$ would imply $\zeta \in \Omega$ and $\gamma(t) - \zeta \in \Omega$, hence $\gamma(t) \in \Omega$, which would contradict our assumptions (it is here that we use the stability under addition of Ω). Therefore, the formula

$$X(\zeta, t) := \frac{\eta(\zeta)}{\eta(\zeta) + \eta(\gamma(t) - \zeta)} \gamma'(t)$$

defines a non-autonomous vector field, which is continuous in (ζ, t) on $\mathbb{C} \times [0, 1]$, C^1 in ζ and has its partial derivatives continuous in (ζ, t) . The Cauchy-Lipschitz theorem on the existence and uniqueness of solutions to the first-order differential equation $\frac{d\zeta}{dt} = X(\zeta, t)$ applies: for every $\zeta \in \mathbb{C}$ and $t_0 \in [0, 1]$ there is a unique solution $t \mapsto \Phi^{t_0, t}(\zeta)$ such that $\Phi^{t_0, t_0}(\zeta) = \zeta$. The fact that the vector field X is bounded implies that $\Phi^{t_0, t}(\zeta)$ is defined for all $t \in [0, 1]$ and the classical theory guarantees that $(t_0, t, \zeta) \mapsto \Phi^{t_0, t}(\zeta)$ is C^1 on $[0, 1] \times [0, 1] \times \mathbb{C}$.

Let us set $\Psi_t := \Phi^{0, t}$ for $t \in [0, 1]$. We easily check that this family of maps satisfies (i')–(iv'), as a consequence of the identities

$$X(\omega, t) = 0 \quad \text{for all } \omega \in \Omega, \tag{7}$$

$$X(\gamma(t) - \zeta, t) = \gamma'(t) - X(\zeta, t) \quad \text{for all } \zeta \in \mathbb{C} \tag{8}$$

valid for all $t \in [0, 1]$ ((7) and (8) directly follow from our definition of X ; here we use $0 \in \Omega$ to ensure $\eta(0) = 0$):

- (i') and (ii') follow from (7) which yields $\Phi^{t_0, t}(\omega) = \omega$ for every t_0 and t , and from the non-autonomous flow property $\Phi^{t_0, 0} \circ \Phi^{0, t} = \text{Id}$ (hence $\Psi_t(\zeta) = \omega$ implies $\zeta = \Phi^{t, 0}(\omega) = \omega$);
- (iv') follows from the fact that $X(\gamma(t), t) = \gamma'(t)$, by (8) with $\zeta = 0$, hence $t \mapsto \gamma(t)$ is a solution of X ;
- (iii') follows from (8): for any solution $t \mapsto \zeta(t)$, the curve $t \mapsto \xi(t) := \gamma(t) - \zeta(t)$ satisfies $\xi(0) = \gamma(0) - \zeta(0)$ and $\xi'(t) = \gamma'(t) - X(\zeta(t), t) = X(\xi(t), t)$, hence it is a solution: $\xi(t) = \Psi_t(\gamma(0) - \zeta(0))$.

5.3 Proof of the key lemma when $0 \notin \Omega$

From now on, we suppose $0 \notin \Omega$ and we use the notation

$$\Omega_\varepsilon := \{ \zeta \in \mathbb{C} \mid \text{dist}(\zeta, \Omega) < \varepsilon \}$$

for any $\varepsilon > 0$. We shall require the following technical

Lemma 5.2. *For any $\varepsilon > 0$ there exists a C^1 function $\eta: \mathbb{C} \rightarrow [0, 1]$ such that*

$$\{ \zeta \in \mathbb{C} \mid \eta(\zeta) = 0 \} = \{0\} \cup \overline{\Omega_\varepsilon}.$$

Proof. Pick a C^1 function $\chi: \mathbb{R} \rightarrow [0, 1]$ such that $\{x \in \mathbb{R} \mid \chi(x) = 0\} = [-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$ and $\chi(x) = 1$ for $|x| \geq \sqrt{1 + \varepsilon}$, and a bijection $\omega: \mathbb{N}^* \rightarrow \Omega$. For each $k \in \mathbb{N}^*$, $\eta_k(\zeta) := \chi(|\zeta - \omega(k)|^2)$ defines a C^1 function on \mathbb{C} such that $\eta_k^{-1}(0) = \overline{D(\omega(k), \varepsilon)}$ and $\eta_k \equiv 1$ on $\mathbb{C} \setminus D(\omega(k), 1 + \varepsilon)$. Consider the infinite product

$$\eta_*(\zeta) := \prod_{k \in \mathbb{N}^*} \eta_k(\zeta). \tag{9}$$

For any bounded open subset U of \mathbb{C} , the set $\mathcal{F}_U := \{k \in \mathbb{N}^* \mid U \cap D(\omega(k), 1 + \varepsilon) \neq \emptyset\}$ is finite (because Ω is discrete), thus almost all the factors in (9) are equal to 1 when $\zeta \in U$: $(\eta_*)|_U = \prod_{k \in \mathcal{F}_U} (\eta_k)|_U$, hence η_* is C^1 , takes its values in $[0, 1]$ and

$$\eta_*^{-1}(0) \cap U = \bigcup_{k \in \mathcal{F}_U} \overline{D(\omega(k), \varepsilon)} \cap U,$$

whence it follows that $\eta_*^{-1}(0) = \overline{\Omega_\varepsilon}$.

If $0 \in \overline{\Omega_\varepsilon}$, then one can take $\eta = \eta_*$. If not, then one can take the product $\eta = \eta_0 \eta_*$ with $\eta_0(\zeta) := \chi_0(|\zeta|^2)$, where χ_0 is any C^1 function on \mathbb{R} which takes its values in $[0, 1]$ and such that $\chi_0^{-1}(0) = \{0\}$. \square

We now repeat the work of the previous section replacing Ω with $\{0\} \cup \Omega$, adding quantitative information (we still assume that we are given a path which does not pass through 0 but we want to control the way the corresponding symmetric Ω -homotopy approaches the points of Ω) and authorizing a more general initial path than a rectilinear one.

Lemma 5.3. *Let $\delta, \delta' > 0$ with $\delta' < \delta/2$. Suppose that $J = [a, b]$ is a compact interval of \mathbb{R} and $\gamma: J \rightarrow \mathbb{C}$ is a path such that*

$$0 \notin \gamma(J) \quad \text{and} \quad \gamma(J) \subset \mathbb{C} \setminus \Omega_\delta.$$

Suppose that $h: I \rightarrow \mathbb{C}$ is a C^1 path such that

- (i) $h(0) = 0$,
- (ii) $h(I) \subset \mathbb{C} \setminus \Omega_{\delta'}$,
- (iii) $h(1-s) = h(1) - h(s)$ for all $s \in I$,
- (iv) $h(1) = \gamma(a)$.

Then there exists a symmetric Ω -homotopy H defined on $I \times J$, whose initial path is h , whose endpoint path is γ , which satisfies $H(I \times J) \subset \mathbb{C} \setminus \Omega_{\delta'}$ and whose final path is C^1 .

Proof. We may assume without loss of generality that γ is C^1 on J (if γ is only piecewise C^1 , we just need to concatenate the symmetric Ω -homotopies associated with the various pieces). We shall define a family of maps $(\Psi_t)_{t \in J}$ so that

$$H_t(s) := \Psi_t(h(s)), \quad s \in I, \tag{10}$$

yield the desired homotopy. For that, it is sufficient that $(t, \zeta) \in J \times \mathbb{C} \mapsto \Psi_t(\zeta)$ be continuously differentiable, $\Psi_0 = \text{Id}$ and, for each $t \in J$,

- (i') $\Psi_t(0) = 0$,
- (ii') $\Psi_t(\mathbb{C} \setminus \Omega_{\delta'}) \subset \mathbb{C} \setminus \Omega_{\delta'}$,
- (iii') $\Psi_t(\gamma(0) - \zeta) = \Psi_t(\gamma(0)) - \Psi_t(\zeta)$ for all $\zeta \in \mathbb{C}$,
- (iv') $\Psi_t(\gamma(a)) = \gamma(t)$.

As in Section 5.2, our maps Ψ_t will be generated by a non-autonomous vector field.

Lemma 5.2 allows us to choose a C^1 function $\eta: \mathbb{C} \rightarrow [0, 1]$ such that

$$\{\zeta \in \mathbb{C} \mid \eta(\zeta) = 0\} = \{0\} \cup \Omega_{\delta'}.$$

We observe that $D(\zeta, t) := \eta(\zeta) + \eta(\gamma(t) - \zeta)$ defines a C^1 function of (ζ, t) and that

$$D(\zeta, t) > 0 \quad \text{for all } \zeta \in \mathbb{C} \text{ and } t \in [0, 1].$$

Indeed, $D(\zeta, t) = 0$ would imply that both ζ and $\gamma(t) - \zeta$ lie in $\{0\} \cup \Omega_{\delta'}$, hence $\gamma(t) \in \{0\} \cup \Omega_{2\delta'}$, which would contradict our assumption $\gamma(J) \subset \mathbb{C} \setminus (\{0\} \cup \Omega_{\delta})$. Therefore,

$$X(\zeta, t) := \frac{\eta(\zeta)}{\eta(\zeta) + \eta(\gamma(t) - \zeta)} \gamma'(t), \quad (\zeta, t) \in \mathbb{C} \times J,$$

defines a non-autonomous vector field whose flow $(\Phi^{t_0, t})_{t_0, t \in J}$ allows one to conclude the proof exactly as in Section 5.2, setting $\Psi_t := \Phi^{a, t}$ and replacing (7) with

$$X(\omega, t) = 0 \quad \text{for all } \omega \in \{0\} \cup \Omega_{\delta'}.$$

□

We now consider the case of a path γ which entirely lies close to 0.

Lemma 5.4. *Let $\varepsilon, \delta' > 0$ with $0 < \varepsilon < \delta'$. Suppose that $K = [a, b]$ is a compact interval of \mathbb{R} and $\gamma: K \rightarrow \mathbb{C}$ is a path such that*

$$\gamma(K) \subset \overline{\mathbb{D}}_{\varepsilon/2}.$$

Suppose that $h: I \rightarrow \mathbb{C}$ is a C^1 path such that

$$(i) \quad h(0) = 0,$$

$$(ii) \quad h(I) \subset \mathbb{C} \setminus \Omega_{\delta'},$$

$$(iii) \quad h(1 - s) = h(1) - h(s) \text{ for all } s \in I,$$

$$(iv) \quad h(1) = \gamma(a).$$

Then there exists a symmetric Ω -homotopy H defined on $I \times K$, whose initial path is h , whose endpoint path is γ , which satisfies

$$H(I \times K) \subset \mathbb{C} \setminus \Omega_{\delta''} \quad \text{with } \delta'' := \delta' - \varepsilon$$

and whose final path is C^1 .

Proof. Define $H(s, t) := h(s) + s(\gamma(t) - \gamma(a))$. This way $H(s, a) = h(s)$, $H(1, t) = \gamma(t)$ and H is a symmetric Ω -homotopy as required: $H(0, t) = 0$, $H(s, t) + H(1 - s, t) = h(s) + h(1 - s) + \gamma(t) - \gamma(a) = \gamma(t)$, $\text{dist}(H(s, t), \Omega) \geq \text{dist}(h(s), \Omega) - |\gamma(t) - \gamma(a)| \geq \delta' - \varepsilon$. □

Proof of Lemma 5.1 under the standing assumption $0 \notin \Omega$. Let γ be as in the hypothesis of Lemma 5.1. Without loss of generality, we can assume $\gamma(1) \neq 0$ (if not, view γ as the restriction of a path $\tilde{\gamma}: [0, 2] \rightarrow \mathbb{C} \setminus \Omega$ such that $\tilde{\gamma}(2) \neq 0$, with which is associated a symmetric Ω -homotopy \tilde{H} defined on $I \times [0, 2]$, and restrict \tilde{H} to $I \times [0, 1]$). Let $\delta := \text{dist}(\Omega, \gamma([0, 1]))$.

The set $Z := \{t \in [0, 1] \mid \gamma(t) = 0\}$ is closed; it is also discrete because of the non-vanishing of the derivatives of γ , thus it has a finite cardinality $N \in \mathbb{N}$. If $N = 0$, then we can apply Lemma 5.3 with $J = [0, 1]$ and $h(s) \equiv s\gamma(0)$ and the proof is complete.

From now on we suppose $N \geq 1$. Let us write

$$Z = \{t_1, \dots, t_N\} \quad \text{with } 0 < t_1 < \dots < t_N < 1.$$

We define

$$\delta_0 := \frac{1}{2} \min \left\{ \frac{\delta}{2}, \rho - |\gamma(0)| \right\} \quad \text{and} \quad \varepsilon := \min \left\{ |\gamma(0)|, |\gamma(1)|, \frac{\delta_0}{N+1} \right\}.$$

The uniform continuity of γ allows us to find pairwise disjoint closed intervals of positive lengths K_1, \dots, K_N such that

$$t_j \in K_j \quad \text{and} \quad \gamma(K_j) \subset \overline{\mathbb{D}}_{\varepsilon/2}, \quad j = 1, \dots, N.$$

By considering the connected components of $[0, 1] \setminus \bigcup K_j$ and taking their closures, we get adjacent closed subintervals of positive lengths of $[0, 1]$,

$$J_0, K_1, J_1, K_2, \dots, J_{N-1}, K_N, J_N$$

with $J_j = [a_j, b_j]$, $K_j = [b_{j-1}, a_j]$, $a_0 = 0$, $b_N = 1$. Observe that

$$0 \notin \gamma(J_j) \quad \text{and} \quad \gamma(J_j) \subset \mathbb{C} \setminus \Omega_{\delta}, \quad j = 0, \dots, N.$$

- We apply Lemma 5.3 with $J = J_0 = [0, b_0]$, $h(s) \equiv s\gamma(0)$ and $\delta' = \delta_0$ (which is allowed by the choice of δ_0): we get a symmetric Ω -homotopy H defined on $I \times J_0$ whose initial path is the rectilinear segment $[0, \gamma(0)]$, whose endpoint path is $\gamma|_{J_0}$ and whose final path H_{b_0} is C^1 and lies in $\mathbb{C} \setminus \Omega_{\delta_0}$.
- We apply Lemma 5.4 with $K = K_1$, $\delta' = \delta_0$ and $h = H_{b_0}$: we get an extension of our symmetric Ω -homotopy H to $I \times K_1$, in which the endpoint path is extended by $\gamma|_{K_1}$ and the final path is now H_{a_1} , a C^1 path contained in $\mathbb{C} \setminus \Omega_{\delta_1}$ with $\delta_1 := \delta_0 - \varepsilon$.
- And so on: we apply alternatively Lemma 5.4 on K_j and Lemma 5.3 on J_j : we get an extension of the symmetric Ω -homotopy H to $I \times K_j$ or $I \times J_j$ such that both $H_{a_j}(I)$ and $H_{b_j}(I)$ are contained in $\mathbb{C} \setminus \Omega_{\delta_j}$ with $\delta_j := \delta_0 - j\varepsilon$.

When we reach $j = N$, the proof is complete. □

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