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The tree model of a meromorphic plane curve

Abdallah Assi*

Abstract.¹ We associate with a plane meromorphic curve f a tree model $T(f)$ based on its contact structure. Then we give a complete description of the y -derivative of f (resp. the Jacobien $J(f, g)$) in terms of $T(f)$ (resp. $T(fg)$). We also characterize the regularity of f in terms of its tree and we give a bound for the number of its irregular values.

Introduction

Let \mathbf{K} be an algebraically closed field of characteristic 0, and let f, g be two monic reduced polynomial of $\mathbf{K}((x))[y]$ of degrees n, m . Let f_x, g_x (resp. f_y, g_y) be the x -derivative (resp. the y -derivative) of f, g , and let $J(f, g) = f_x g_y - f_y g_x$. Let, by Newton Theorem,

$$f(x, y) = \prod_{i=1}^n (y - y_i(x)), \quad g(x, y) = \prod_{j=1}^m (y - z_j(x))$$

where $(y_i(x))_{1 \leq i \leq n}$ and $(z_j(x))_{1 \leq j \leq m}$ are meromorphic fractional series in x .

The main objective of this paper is to give a complete description of f_y (resp. $J(f, g)$) when the contact structure of f (resp. fg) is given. This description is based on a tree model defined in the following way: let f be as above, and let M be a rational number. Assume that M is the x -order, denoted O_x , of $y_i - y_j$ for some $1 \leq i \neq j \leq n$, and let E_M be the set of irreducible components of f such that $H \in E_M$ if and only if the contact $c(H, \bar{H}) \geq M$ for some $\bar{H} \in E_M$ (where if $H(x, y) = \prod_{i=1}^a (y - Y_i(x))$ and $\bar{H}(x, y) = \prod_{j=1}^b (y - Z_j(x))$, then $c(H, \bar{H}) = \max_{i,j} O_x(Y_i - Z_j)$). Let $H_1, H_2 \in E_M$. We say that $H_1 R_M H_2$ if the contact of H_1 with H_2 is $\geq M$. The set of points of the tree of f at the level M is defined to be the set of equivalence classes of R_M (see Section 6). Let P_i^M be a point of the tree of f at the level M , and let \bar{f} be a monic polynomial of $\mathbf{K}((x))[y]$. We denote by $Q_{\bar{f}}(M, i)$ the product of irreducible components of \bar{f} whose contact with any element of $P(M, i)$ is M . By [6], we have

*Université d'Angers, Mathématiques, 49045 Angers ceded 01, France, e-mail:assi@univ-angers.fr

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$\deg_y Q_{f_y}(M, i) > 1$. Let $f = f_1 \dots f_{\xi(f)}$ be the factorization of f into irreducible components in $\mathbf{K}((x))[y]$. In Section 7., based on the results of Section 5., we describe the properties of $Q_{f_y}(M, i)$ for all P_i^M (degree in y , intersection and contact with f_i for all $i = 1, \dots, \xi(f)$). This result gives a generalization of Merle Theorem ($\xi(f) = 1$) (see Theorem 7.1.) and Delgado Theorem ($\xi(f) = 2$) (see Example 7.11.). These two results use the arithmetic of the semigroup associated with f , which, as shown by Delgado, does not seem to suffice when $\xi(f) \geq 3$.

Let $T(fg)$ be the tree of fg . A point P_i^M of $T(fg)$ is said to be an f -point (resp. g -point) if P_i^M does not contain irreducible components of f (resp. g). In Section 9, based on the results of Sections 4. and 5., we prove that if P_i^M is an f -point (resp. a g -point), then $\deg_y Q_{J(f,g)}(M, i) > 1$. We also describe, as for f_y , the different properties of $Q_{J(f,g)}(M, i)$ in terms of those of fg .

Let the notations be as above, and assume that $f, g \in \mathbf{K}[x^{-1}][y]$. Let $F(x, y) = f(x^{-1}, y), G(x, y) = g(x^{-1}, y)$. For all $\lambda \in \mathbf{K}$, we denote by F_λ the polynomial $F - \lambda$. We say that the family $(F_\lambda)_{\lambda \in \mathbf{K}}$ is regular if the rank of the \mathbf{K} -vector space $\frac{\mathbf{K}[x, y]}{(F_\lambda, F_y)}$, denoted $\text{Int}(F - \lambda, F_y)$, does not depend on $\lambda \in \mathbf{K}$. When $(F_\lambda)_{\lambda \in \mathbf{K}}$ is not regular, there exists a finite number $\lambda_1, \dots, \lambda_s \in \mathbf{K}$ such that $\text{Int}(F - \lambda, F_y) < \text{Int}(F - \lambda_i, F_y)$ for λ generic and $1 \leq i \leq s$.

The regularity of a family of affine curves is related to many problems in affine geometry, in particular the plane Jacobian problem. If $(F_\lambda)_\lambda$ is regular and smooth, then F is equivalent to a coordinate of \mathbf{K}^2 . If $(F_\lambda)_\lambda$ is smooth with only one irregular value λ_1 , then $F - \lambda_1$ is reducible in $\mathbf{K}[x, y]$ and one of its irreducible components is equivalent to a coordinate of \mathbf{K}^2 . In general, nothing is known when $(F_\lambda)_\lambda$ has more than two irregular values (see [4] and references).

In Section 10. we characterize the notion of regularity in terms of the tree $T(f)$. More precisely we divide $T(f)$ into good and bad points. In particular, f is regular if the set of bad points of $T(f)$ is empty. This, with the results of Section 2. is used in Section 11. in order to prove that the set of irregular values of f contains at most $\xi(f) - 1$ elements (see Proposition 10.3.). We give an example where this bound is attained.

1 Characteristic sequences

In this Section we shall recall some well known results about the theory of meromorphic curves (see [2] for example). Let

$$f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$$

be an irreducible monic polynomial of $\mathbf{K}((x))[y]$, where $\mathbf{K}((x))$ denotes the field of meromorphic series over \mathbf{K} . Let, by Newton Theorem, $y(t) \in \mathbf{K}((t))$ such that $f(t^n, y(t)) = 0$. If w is a primitive n th root of unity, then we have:

$$f(t^n, y) = \prod_{k=1}^n (y - y(w^k t)).$$

Write $y(t) = \sum_i a_i t^i$, and let $\text{supp}(y(t)) = \{i; a_i \neq 0\}$. Clearly $\text{supp}(y(t)) = \text{supp}(y(w^k t))$ for all $1 \leq k \leq n-1$. We denote this set by $\text{supp}(f)$ and we recall that $\gcd(n, \text{supp}(f)) = 1$. If we write $x^{\frac{1}{n}}$ for t , then $y(x^{\frac{1}{n}}) = \sum a_i x^{\frac{i}{n}}$ and $f(x, y(x^{\frac{1}{n}})) = 0$, i.e. $y(x^{\frac{1}{n}})$ is a root of $f(x, y) = 0$. By Newton Theorem, there are n distinct roots of $f(x, y) = 0$, given by $y(w^k x^{\frac{1}{n}})$, $1 \leq k \leq n$. We denote the set of roots of f by $\text{Root}(f)$.

We shall associate with f its characteristic sequences $(m_k^f)_{k \geq 0}$, $(d_k^f)_{k \geq 1}$ and $(r_k^f)_{k \geq 0}$ defined by:

$$\begin{aligned} |m_0^f| = d_1^f = |r_0^f| = n, \quad m_1^f = r_1^f = \inf(\{i \in \text{supp}(f) | \gcd(i, n) < \min(i, n)\}), \text{ and for all } k \geq 2, \\ d_k^f = \gcd(m_0^f, \dots, m_{k-1}^f) = \gcd(d_{k-1}^f, m_{k-1}^f), \\ m_k^f = \inf\{i \in \text{supp}f \mid i \text{ is not divisible by } d_k^f\}, \end{aligned}$$

$$\text{and } r_k^f = r_{k-1}^f \frac{d_{k-1}^f}{d_k^f} + m_k^f - m_{k-1}^f.$$

Since $\gcd(n, \text{supp}(f)) = 1$, then there is $h_f \in \mathbb{N}$ such that $d_{h_f+1} = 1$. We denote by convention $m_{h_f+1}^f = r_{h_f+1}^f = +\infty$. The sequence $(m_k)_{0 \leq k \leq h_f}$ is also called the set of Newton-Puiseux exponents of f . We finally set $e_k^f = \frac{d_k^f}{d_{k+1}^f}$ for all $1 \leq k \leq h_f$.

Let H be a polynomial of $\mathbf{K}((x))[y]$. We define the intersection of f with H , denoted $\text{int}(f, H)$, by $\text{int}(f, H) = O_t H(t^n, y(t)) = n \cdot O_x H(x, y(x^{\frac{1}{n}}))$, where O_t (resp. O_x) denotes the order in t (resp. in x).

Let $p, q \in \mathbb{N}$, and let $\alpha(x) \in \mathbf{K}((x^{\frac{1}{p}})), \beta(x) \in \mathbf{K}((x^{\frac{1}{q}}))$. We set

$$c(\alpha, \beta) = O_x(\alpha(x) - \beta(x))$$

and we call $c(\alpha, \beta)$ the contact of α with β . We define the contact of f with $\alpha(x)$ to be

$$c(f, \alpha) = \max_{1 \leq i \leq n} O_x(y_i(x) - \alpha(x))$$

where $\{y_1, \dots, y_n\} = \text{Root}(f)$.

Let $g = y^m + b_1(x)y^{m-1} + \dots + b_m(x)$ be a monic irreducible polynomial of $\mathbf{K}((x))[y]$ and let $\text{Root}(g) = \{z_1, \dots, z_m\}$. We define the contact of f with g to be

$$c(f, g) = c(f, z_1(x)).$$

Note that $c(f, g) = c(f, z_j(x)) = c(g, y_i(x))$ for all $1 \leq j \leq m$ and for all $1 \leq i \leq n$.

Remark 1.1 (see [1]) i) Let $f \in \mathbf{K}[[x]][y]$ (resp. $f \in \mathbf{K}[x^{-1}][y]$). The set of $\text{int}(f, g)$, $g \in \mathbf{K}[[x]][y]$ (resp. $g \in \mathbf{K}[x^{-1}][y]$) is a subsemigroup of \mathbb{Z} . We denote it by $\Gamma(f)$ and we call it the semigroup associated with f . With the notations above, $r_k^f > 0$ (resp. $r_k^f < 0$) for all $k = 0, \dots, h_f$, and $r_0^f, r_1^f, \dots, r_{h_f}^f$ generate $\Gamma(f)$. We write $\Gamma(f) = \langle r_0^f, r_1^f, \dots, r_{h_f}^f \rangle$.

ii) For all $1 \leq k \leq h_f$, e_k^f is the minimal integer such that $e_k^f r_k^f \in \langle r_0^f, r_1^f, \dots, r_{k-1}^f \rangle$.

iii) For all $1 \leq k \leq h_f$, there is an irreducible monic polynomial $g_k \in \mathbf{K}((x))[y]$ of degree $\frac{n}{d_k^f}$ in y such that $c(f, g_k) = \frac{m_k^f}{n}$ and $\text{int}(f, g_k) = r_k^f$. Furthermore, $\Gamma(g_k) = \langle \frac{r_0^f}{d_k^f}, \frac{r_1^f}{d_k^f}, \dots, \frac{r_{k-1}^f}{d_k^f} \rangle$.

Lemma 1.2 (see [1]) Let $y(x) = \sum_i a_i x^{\frac{i}{n}} \in \text{Root}(f)$. Given $s \in \mathbf{N}^*$, let U_s denotes the group of the s th roots of unity in \mathbf{K} . Set

$$R(i) = \{w \in U_n | c(y(x), y(wx)) = O_x(y(x) - y(wx)) \geq \frac{m_i^f}{n}\}$$

$$S(i) = \{w \in U_n | c(y(x), y(wx)) = O_x(y(x) - y(wx)) = \frac{m_i^f}{n}\}.$$

We have the following:

i) For all $1 \leq i \leq h_f + 1$, $R(i) = U_{d_i^f}$. In particular, $\text{card}(R(i)) = d_i^f$.

ii) For all $1 \leq i \leq h_f$, $S(i) = R(i) - R(i+1) = U_{d_i^f} - U_{d_{i+1}^f}$. In particular, $\text{card}(S(i)) = d_i^f - d_{i+1}^f$.

Proof. Let $w \in U_n$, then $y(x) - y(wx) = \sum_k a_k (1 - w^k) x^{\frac{k}{n}}$. In particular, $O_x(y(x) - y(wx)) \geq \frac{m_i^f}{n}$ if and only if $w^k = 1$ for all $k < m_i^f$. This holds if and only if $w \in U_{d_i^f}$. ■

Remark 1.3 i) Let F be a nonzero monic polynomial of $\mathbf{K}((x))[y]$. Assume that F is reduced and let $F = F_1 \dots F_{\xi(F)}$ be the factorization of F into irreducible polynomials of $\mathbf{K}((x))[y]$. We define $\text{Root}(F)$ to be the union of $\text{Root}(F_i)$, $i = 1, \dots, \xi(F)$. Given a polynomial $G \in \mathbf{K}((x))[y]$, we set $\text{int}(F, G) = \sum_{i=1}^{\xi(F)} \text{int}(F_i, G)$.

ii) Let $p \in \mathbf{N}^*$, and let F be a nonzero monic polynomial of $\mathbf{K}((x^{\frac{1}{p}}))[y]$. Assume that F is reduced and let $x = X^p, y = Y$, and $\bar{F}(X, Y) = F(X^p, Y)$. The polynomial \bar{F} is a monic reduced polynomial of $\mathbf{K}((X))[Y]$. Let $\text{Root}(\bar{F}) = \{Y_1(X), \dots, Y_N(X)\}$. The set of roots of $F(x, y) = 0$ is $\{Y_1(x^{\frac{1}{p}}), \dots, Y_s(x^{\frac{1}{p}})\}$.

Let M be a given real number and consider the sequence $(m_k^f)_{1 \leq k \leq h_f + 1}$ of Newton-Puiseux exponents of f . We define the function $S(m^f, M)$ by putting

$$S(m^f, M) = \begin{cases} r_k^f d_k^f + (nM - m_k^f) d_{k+1}^f & \text{if } \frac{m_1^f}{n} \leq \frac{m_k^f}{n} \leq M < \frac{m_{k+1}^f}{n} \\ Md_1 & \text{if } M < \frac{m_1^f}{n} \end{cases}$$

Proposition 1.4 (see [1] or [8]) Let $g = y^m + b_1(x)y^{m-1} + \dots + b_m(x)$ be a monic irreducible polynomial of $\mathbf{K}((x))[y]$. We have the following:

$$\begin{aligned} c(f, g) = M & \quad \text{iff} \quad \text{int}(f, g) = S(m^f, M) \frac{m}{n} \\ c(f, g) < M & \quad \text{iff} \quad \text{int}(f, g) < S(m^f, M) \frac{m}{n} \\ c(f, g) > M & \quad \text{iff} \quad \text{int}(f, g) > S(m^f, M) \frac{m}{n} \end{aligned}$$

Let g_1, g_2 be two irreducible monic polynomials of $\mathbf{K}((x))[y]$ of degrees q_1 and q_2 respectively and let $(m_k^{g_i})_{1 \leq k \leq h_{g_i}}$ be the set of Newton-Puiseux exponents of g_i , $i = 1, 2$.

Lemma 1.5 (see [1]) Let $M = \min(c(f, g_2), c(f, g_1))$. We have the following:

- (i) $c(g_1, g_2) \geq M$.
- (ii) if $c(f, g_2) \neq c(f, g_1)$ then $c(g_1, g_2) = M$.

Lemma 1.6 Let the notations be as above and let $(m_k^g)_{1 \leq k \leq h_g+1}$ be the set of Newton-Puiseux exponents of g . Let $M = c(f, g)$ and assume that $M \geq \frac{m_1^f}{n}$. Let k be the greatest integer such

that $\frac{m_k^f}{n} = \frac{m_k^g}{m} \leq M$. We have the following:

- i) $\frac{n}{d_i^f} = \frac{m}{d_i^g}$ for all $i = 1, \dots, k+1$.
- ii) $\frac{n}{d_{k+1}^f}$ divides m . In particular, if $k = h$ then n divides m .

Proof. ii) results from i), since by i), $m = \frac{n}{d_{k+1}^f} d_{k+1}^g$. On the other hand, let $1 \leq i \leq k$ and remark that $m \cdot n = n \cdot m, m \cdot m_1^f = n \cdot m_1^g, \dots, m \cdot m_{i-1}^f = n \cdot m_{i-1}^g$, in particular $m \cdot d_i^f = m \cdot \gcd(n, m_1^f, \dots, m_{i-1}^f) = n \cdot \gcd(q, m_1^g, \dots, m_{i-1}^g) = n \cdot d_i^g$. This proves i). ■

Lemma 1.7 Let the notations be as in Lemma 1.6. and let $y(x) \in \text{Root}(f)$ (resp. $z(x) \in \text{Root}(g)$) such that $c(y(x), z(x)) = M$. Write $y(x) = \sum_i c_i^f x^{\frac{i}{n}}$ and $z(x) = \sum_j c_j^g x^{\frac{j}{m}}$. If $M = \frac{m_{h_f}^f}{n}$ and $n \geq m$, then either c_{mM}^g -the coefficient of x^M in $z(x)$ - is 0, or $m = n$.

Proof. If $c_{mM}^g \neq 0$, then $M = \frac{m^g}{m}$, hence n divides m . This, with the hypotheses implies that $m = n$. ■

As a corollary we get the following:

Lemma 1.8 Let g_1, g_2 be two monic polynomials of $\mathbf{K}((x))[y]$ of degrees q_1, q_2 respectively, and assume that $c(g_1, f) = c(g_2, f) = \frac{m_{h(f)}^f}{n}$. If $q_1 < n$ and $q_2 < n$, then $c(g_1, g_2) > \frac{m_{h_f}^f}{n}$.

Proof. Let $y(x) \in \text{Root}(f)$ (resp. $z_1(x) \in \text{Root}(g_1), z_2(x) \in \text{Root}(g_2)$) such that $c(y(x), z_1(x)) = c(y(x), z_2(x)) = \frac{m_{h_f}^f}{n}$. In particular $c(z_1(x), z_2(x)) \geq \frac{m_{h(f)}^f}{n}$. By Lemma 1.7., the coefficients $\frac{m_{h_f}^f}{n}$ of x^n in $z_1(x)$ and $z_2(x)$ are 0, which implies that $c(z_1(x), z_2(x)) > \frac{m_{h_f}^f}{n}$. This proves our assertion. ■

2 Equivalent and almost equivalent polynomials

Let f, g be two monic irreducible polynomials of $\mathbf{K}((x))[y]$, of degrees n, m in y . Let $(m_k^f)_{1 \leq k \leq h_f}$, $(d_k^f)_{1 \leq k \leq h_f}$, and $(r_k^f)_{0 \leq k \leq h_f}$ (resp. $(m_k^g)_{1 \leq k \leq h_g}$, $(d_k^g)_{1 \leq k \leq h_g}$, and $(r_k^g)_{0 \leq k \leq h_g}$) be the set of characteristic sequences of f (resp. of g).

Definition 2.1 i) We say that g is equivalent to f if the following holds:

- $h_f = h_g$
- $\frac{m_k^g}{m} = \frac{m_k^f}{n}$ for all $k = 1, \dots, h_f$.
- $c(f, g) \geq \frac{m_{h_f}^f}{n}$.

ii) We say that g is almost equivalent to f if the following holds:

- $h_f = h_g + 1$.
- $\frac{m_k^f}{n} = \frac{m_k^g}{m}$ for all $k = 1, \dots, h_g$.
- $c(f, g) = \frac{m_{h_f}^f}{n}$.

Lemma 2.2 Let the notations be as in Definition 2.1.

- i) If g is equivalent to f , then $m = n$.

ii) If g is almost equivalent to f , then $m = \frac{n}{d_{h_f}^f}$. Furthermore, if $y(x) = \sum_p c_p x^{\frac{p}{m}} \in \text{Root}(g)$, then $c_{\frac{m_{h_f}^f}{n}} = 0$.

Proof. i) results from Lemma 1.6. On the other hand, by the same Lemma, $m = a \frac{n}{d_{h_f}^f}$ for some $a \in \mathbb{N}^*$, but $\gcd(a \frac{n}{d_{h_f}^f}, \frac{a}{d_{h_f}^f} m_1^f, \dots, \frac{a}{d_{h_f}^f} m_{h_f-1}^f) = \frac{a}{d_{h_f}^f} d_{h_f}^f = 1$, hence $a = 1$. This proves the first assertion of ii). Now the least assertion results from Lemma 1.7. ■

Definition 2.3 Let $\{F_1, \dots, F_r\}$ be a set of monic irreducible polynomials of $\mathbf{K}((x))[y]$. Assume that $s > 1$ and let $n_{F_i} = \deg_y F_i$ for all $1 \leq i \leq r$.

i) We say that the sequence (F_1, \dots, F_r) is equivalent if for all $1 \leq i \leq r$, F_i is equivalent to F_1 .

ii) We say that the sequence (F_1, \dots, F_r) is almost equivalent if the following holds:

- The sequence contains an equivalent subsequence of $s - 1$ elements.
- The remaining element is almost equivalent to the elements of the subsequence.

Proposition 2.4 Let the notations be as in Definition 2.3. and let M be a rational number. If $c(F_i, F_j) = M$ for all $i \neq j$, then the sequence (F_1, \dots, F_r) is either equivalent or almost equivalent.

Proof. If $r = 1$, then there is nothing to prove. Assume that $r > 1$, and that $n_{F_1} = \max_{1 \leq k \leq r} n_{F_k}$.

- If $M > m_{h_{F_1}}^{F_1}$, then, by Lemma 1.6., ii), n_{F_1} divides n_{F_k} for all $1 \leq k \leq r$. In particular $n_{F_1} = n_{F_k}$ and F_k is equivalent to F_1 for all $1 \leq k \leq r$.

- Suppose that $M = \frac{m_{h_{F_1}}^{F_1}}{n_{F_1}}$, and that (F_1, \dots, F_r) is not equivalent. Suppose, without loss of generality, that F_2 is not equivalent to F_1 . By hypothesis, $M \geq \frac{m_{h_{F_2}}^{F_2}}{n_{F_2}}$ and $\frac{m_j^{F_1}}{n_{F_1}} = \frac{m_j^{F_2}}{n_{F_2}}$ for all $1 \leq j \leq h_{F_1} - 1$. Let $y(x) = \sum c_p x^{\frac{p}{n_{F_2}}} \in \text{Root}(F_2)$. If the coefficient of x^M in $y(x)$ is non zero, then n_{F_1} divides n_{F_2} , in particular $n_{F_2} = n_{F_1}$, and $\frac{m_{h_{F_1}}^{F_1}}{n_{F_1}} = \frac{m_{h_{F_2}}^{F_2}}{n_{F_2}}$. Hence F_1 is equivalent to F_2 , which is a contradiction. Finally $h_{F_2} = h_{F_1} - 1$, and $n_{F_2} = a \frac{n_{F_1}}{d_{h_{F_1}}^{F_1}}$, but

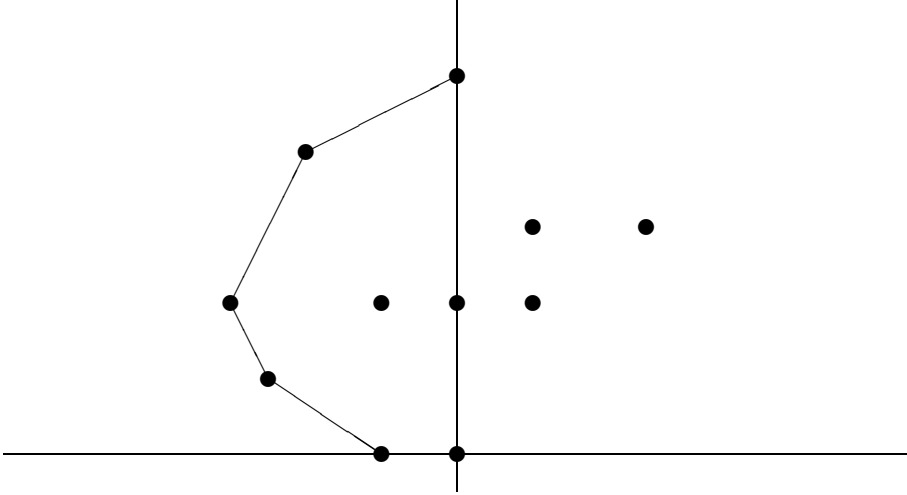
$\gcd(n_{F_2}, m_1^{F_2}, \dots, m_{h_{F_2}}^{F_2}) = 1$, hence $a = 1$ and $n_{F_2} = \frac{n_{F_1}}{d_{h_{F_1}}^{F_1}}$. In particular F_2 is almost equivalent to F_1 . Let $k > 2$. If F_k is not equivalent to F_1 , then $n_{F_k} = n_{F_2} < n_{F_1}$ by the same argument as above. In particular, by Lemma 1.8., $c(F_1, F_2) > M$, which is contradiction. Finally the sequence (F_1, \dots, F_r) is almost equivalent. ■

3 The Newton polygon of a meromorphic plane curve

In this Section we shall recall the notion of the Newton polygon of a meromorphic plane curve. More generally let $p \in \mathbf{N}$ and let $F = y^N + A_1(x)y^{N-1} + \dots + A_{N-1}(x)y + A_N(x)$ be a reduced polynomial of $\mathbf{K}((x^{1/p}))[y]$. For all $i = 0, \dots, N$, let $\alpha_i = O_x A_i(x)$. The Newton boundary of F is defined to be the boundary of the convex hull of $\bigcup_{i=1}^N (\alpha_i, i) + \mathbf{R}_+$.

Write $F(x, y) = \sum_{ij} c_{ij} x^{\frac{i}{p}} y^j$ and let $\text{Supp}(F) = \{(\frac{i}{p}, j) | c_{ij} \neq 0\}$, then the Newton boundary of F is also the boundary of the convex hull of $\bigcup_{(\frac{i}{p}, j) \in \text{Supp}(F)} (\frac{i}{p}, j) + \mathbf{R}_+$.

We define the Newton polygon of F , denoted $N(F)$, to be the union of the compact faces of the Newton boundary of F . Let $\{P_k = (\alpha_{k_j}, k_j), k_0 > k_1 \dots > k_{v_F}\}$ be the set of vertices of $N(F)$. We denote this set by $V(F)$. We denote by $E(F) = \{\Delta_l^F = P_{k_{l-1}} P_{k_l}, l = 1, \dots, v_F\}$ the set of edges of $N(F)$. For all $1 \leq l \leq v_F$ we set $F_{\Delta_l^F} = \sum_{(\frac{i}{p}, j) \in \text{Supp}(F) \cap \Delta_l^F} c_{ij} x^{\frac{i}{p}} y^j$.



Lemma 3.1 Given $1 \leq l \leq v_F$, there are exactly $k_{l-1} - k_l$ elements of $\text{Root}(F)$, $y_j(x)$, $1 \leq j \leq k_{l-1} - k_l$, such that for all $1 \leq j \leq k_{l-1} - k_l$:

i) $O_x(y_j(x)) = \frac{\alpha_{k_{l-1}} - \alpha_{k_l}}{k_{l-1} - k_l}$.

ii) The set of initial coefficients, denoted inco , of $y_1, \dots, y_{(k_{l-1}-k_l)}$ is nothing but the set of nonzero roots of $F_{\Delta_l^F}(1, y)$.

Conversely, given $y(x) \in \text{Root}(F)$, there exists Δ_l^F such that $O_x(y(x)) = \frac{\alpha_{k_{l-1}} - \alpha_{k_l}}{k_{l-1} - k_l}$.

We denote the set of x -orders of $\text{Root}(F)$ by $O(F)$, and we set $\text{Poly}(F) = \{F_{\Delta_l^F}(1, y) | 1 \leq l \leq v_F\}$.

Lemma 3.2 Let F be as above, and let M be a rational number. Define $L_M : \text{Supp}(F) \mapsto \mathbb{Q}$ by $L_M(\frac{i}{p}, j) = \frac{i}{p} + Mj$, and let $a_0 = \inf L_M(\text{Supp}(F))$. Let $\text{in}_M(F) = \sum_{\frac{i}{p} + Mj = a_0} c_{ij} x^{\frac{i}{p}} y^j$. We have the following:

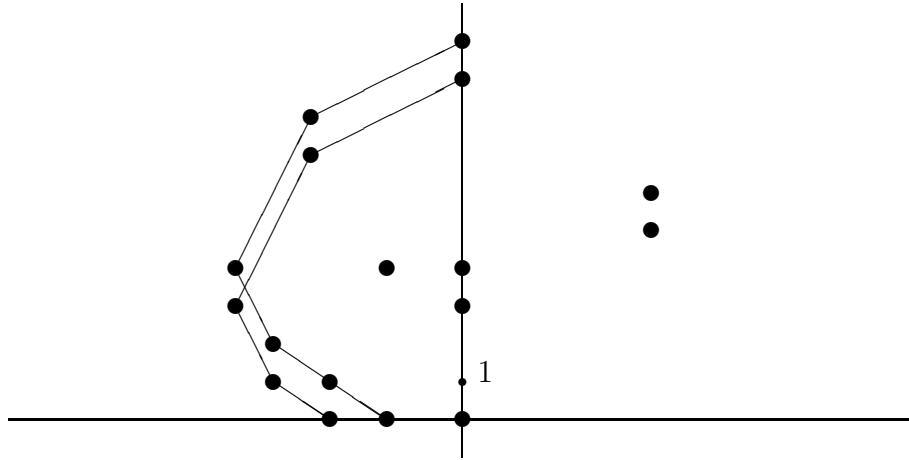
- i) $M \in O(F)$ if and only if $\text{in}_M(F)$ is not a monomial. In this case, $M = \frac{\alpha_{k_{l-1}} - \alpha_{k_l}}{k_{l-1} - k_l}$ for some $1 \leq l \leq v_F$, and $\text{in}_M(F) = F_{\Delta_l^F}$. Furthermore, $(a_0, 0)$ is the point where the line defined by $(\alpha_{k_{l-1}}, k_{l-1})$ and (α_{k_l}, k_l) intersects the x -axis.
- ii) Consider the change of variables $x = X, y = X^M Y$ and let $\bar{F}(X, Y) = F(X, X^M Y)$. We have $\bar{F} = \sum c_{ij} x^{\frac{i}{p} + Mj} = x^{a_0} F_{\Delta_l^F}(1, y) + \sum_{a > a_0} x^a P_a(y)$.

Proof. Easy exercise. ■

Lemma 3.3 Let F be as above and let $N(F)$ be the Newton polygon of F . Let $V(F) = \{P_k = (\alpha_{k_l}, k_l), k_0 > k_1 \dots > k_{v_F}\}$ be the set of vertices of F and assume that $k_{v_F} = 0$, i.e. $N(F)$ meets the x -axis. Assume that $(\alpha^1, 1) \in \text{Supp}(F_{\Delta_{v_F}^F})$ for some $\alpha^1 \in \mathbb{Q}$, and that $(\alpha^1, 1) \notin V(F)$. We have the following

- i) $(\alpha^1, 0) \in V(F_y)$.
- ii) $N(F_y)$ is the translation of $N(F)$ with respect to the vector $(0, -1)$.
- iii) $O(F_y) = O(F)$, $v_F = v_{F_y}$.
- iv) $\deg_y F_{\Delta_{v_F}^F} = \deg_y (F_y)_{\Delta_{v_F}^F} + 1$. In particular, if F has s roots whose order in x is $\frac{\alpha_{k_{v(F)-1}} - \alpha_{k_{v(F)}}}{k_{v_F-1}}$, then F_y has $s - 1$ roots with the same order in x .

Proof. The proof follows immediately from the hypotheses and Lemma 3.1. ■



Lemma 3.4 Let $G = y^m + b_1(x)y^{m-1} + \dots + a_m(x)$ be a reduced polynomial of $\mathbf{K}((x^{\frac{1}{q}}))[y]$ and let $J = J(F, G) = F_x G_y - F_y G_x$ be the Jacobian of F and G . Let $V(G) = \{(\beta_l, l), l_0 > l_1 >$

$\dots > l_{v_G}$ be the set of vertices of $N(G)$ and let $E(G) = \{\Delta_1^G, \dots, \Delta_{v_G}^G\}$ be the set of edges of $N(G)$. Assume that the following holds:

i) $k_{v_F} = l_{v_G} = 0$, $\alpha_{v_F} \neq 0$ and $\beta_{v_F} \neq 0$, i.e. $N(F)$ and $N(G)$ meet the x -axis into points different from the origin.

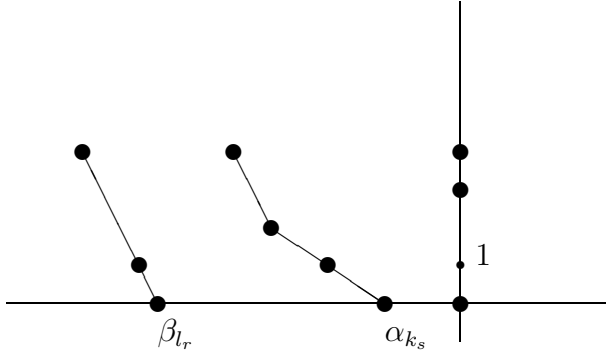
ii) $(\alpha^1, 1) \in \text{Supp}(F_{\Delta_{v_F}^F})$ (resp. $(\beta^1, 1) \in \text{Supp}(G_{\Delta_{v_G}^G})$) for some α^1 (resp. β^1) in \mathbb{Q} , and $(\alpha^1, 1) \notin V(F)$ (resp. $(\beta^1, 1) \notin V(G)$).

iii) $\max(O(F)) > \max(O(G))$.

Then we have:

i) $\max(O(J)) = \max(O(F_y)) = \max(O(F))$.

ii) If $G_{\Delta_{v_G}^G}(x, 0) = ax^{\beta_{l_{v_G}}}$, $a \in \mathbf{K}^*$, then $(\alpha^1 + \beta_{l_{v_G}} - 1, 0) \in V(J)$ and $J_{\Delta_{v_J}^J} = (-F_y G_x)_{\Delta_{v_{F_y G_x}}^{F_y G_x}} = -a\beta_{l_r} \cdot x^{\beta_{l_r} - 1} (F_y)_{\Delta_{v_F}^{F_y}}$.



Proof. It follows from the hypotheses that $(\alpha_{v(F)} - 1, 0) \in V(F_x)$, $(\alpha^1, 0) \in V(F_y)$, $(\beta_{v(G)} - 1, 0) \in V(G_x)$, and $(\beta^1, 0) \in V(G_y)$. In particular $(\alpha_{v(F)} + \beta^1 - 1, 0) \in V(F_x G_y)$ and $(\beta_{v(G)} + \alpha^1 - 1, 0) \in V(F_y G_x)$. Since $\max(O(F)) > \max(O(G))$, then $\beta_{l_{v_G}} - \beta^1 < \alpha_{k_{v(F)}} - \alpha^1$, in particular $\beta_{l_{v_G}} + \alpha^1 - 1 < \alpha_{k_{v_F}} + \beta^1 - 1$, and $(\beta_{l_{v_G}} + \alpha^1 - 1, 0) \in V(J)$. A similar argument shows that the last edge of $J = F_x G_y - F_y G_x$ is nothing but the last edge of $-F_y G_x$, and that $(-F_y G_x)_{\Delta_{v_{F_y G_x}}^{F_y G_x}} = -a\beta_{l_r} \cdot x^{\beta_{l_r} - 1} (F_y)_{\Delta_{v_F}^{F_y}}$. ■

4 Deformation of Newton polygons and applications

Let $f = y^n + a_1(x)y^{n-1} + \dots + a_{n-1}(x)y + a_n(x)$ be a monic reduced polynomial of $\mathbf{K}((x))[y]$ and let $\text{Root}(f) = \{y_1, \dots, y_n\}$. Let $f_1, \dots, f_{\xi(f)}$ be the set of irreducible components of f in $\mathbf{K}((x))[y]$.

Definition 4.1 Let N be a nonnegative integer and let $\gamma(x) = \sum_{k \geq k_0} a_k x^{\frac{k}{N}} \in \mathbf{K}((x^{\frac{1}{N}}))$. Let M be a real number. We set

$$\gamma_{<M} = \begin{cases} \sum_{k \geq k_0, \frac{k}{N} < M} a_k x^{\frac{k}{N}} & \text{if } M > \frac{k_0}{N} \\ 0 & \text{otherwise} \end{cases}$$

and we call $\gamma_{<M}$ the $< M$ -truncation of $\gamma(x)$.

Let θ be a generic element of \mathbf{K} . We set

$$\gamma_{<M,\theta} = \begin{cases} \sum_{k \geq k_0, \frac{k}{N} < M} a_k x^{\frac{k}{N}} + \theta \cdot x^M & \text{if } M \geq \frac{k_0}{N} \\ \theta \cdot x^M & \text{otherwise} \end{cases}$$

and we call $\gamma_{<M,\theta}$ the M -deformation of $\gamma(x)$.

Let N be a nonnegative integer and let $\gamma(x) \in \mathbf{K}((x^{\frac{1}{N}}))$. Let M be a real number and let $\gamma_{<M}$ be the $< M$ -truncation of $\gamma(x)$. Consider the change of variables $X = x, Y = y - \gamma_{<M}$. The polynomial $F(X, Y) = f(X, Y + \gamma_{<M})$ is a monic polynomial of degree n in Y whose coefficients are fractional meromorphic series in X . Let $V(F) = \{P_i = (\alpha_{k_i}, k_i) | i = 1, \dots, v_F\}$ and let $E(F) = \{\Delta_1^F, \dots, \Delta_{v_F}^F\}$.

Lemma 4.2 Let the notations be as above. Assume that $\gamma \notin \text{Root}(f)$ and let $M = \max_{1 \leq j \leq n} c(\gamma, y_j)$. We have the following:

- i) $\text{Root}(F(X, Y)) = \{Y_k = y_k - \gamma_{<M}, k = 1, \dots, n\}$.
- ii) $O(F) = \{c(y_k, \gamma) | k = 1, \dots, n\}$.
- iii) There are exactly $k_i - k_{i+1}$ roots $y(x)$ of F whose contact with γ is $\frac{\alpha_i - \alpha_{i-1}}{k_i - k_{i-1}}$.
- iii) The initial coefficients of $\text{Root}(F)$, denoted $\text{inco}(F)$, is $= \{\text{inco}(y_k - \gamma) | k = 1, \dots, n\}$.

In particular, the Newton polygon $N(F)$ gives us the complete information about the relationship between $\gamma(x)$ with the roots of f . We call it the Newton polygon of f with respect to $\gamma(x)$, and we denote it by $N(f, \gamma)$.

Proof. We have

$$F(X, Y) = f(X, Y + \gamma(X)) = \prod_{k=1}^n (Y + \gamma(X) - y_k(X))$$

now use Lemma 3.1. ■

Lemma 4.3 Let $y_i(x)$ be a root of $f(x, y) = 0$ and let

$$M = \max_{j \neq i} c(y_i, y_j)$$

Let $\tilde{y}_i = y_{i < M, \theta} = (y_i)_{< M}(x) + \theta x^M$ be the M -deformation of y_i and consider the change of variables $X = x, Y = y - \tilde{y}_i(X)$. Let $F(X, Y) = f(X, Y + \tilde{y}_i(X))$. We have the following:

- i) $O(F) = \{c(y_j - y_i) | j \neq i\}$.
- ii) $M = \max(O(F))$.
- iii) The last vertex of $N(F)$ belongs to the x -axis.
- iv) Let $\Delta_{v_F}^F$ be the last edge of $N(F)$. We have $(\alpha^1, 1) \in \text{Supp}(F_{\Delta_{v_F}^F})$ for some α^1 . Furthermore, $(\alpha^1, 1) \notin V(F)$.

Proof. We have

$$F(X, Y) = f(X, Y + \tilde{y}_i(X)) = \prod_{k=1}^n (Y + (y_i)_{< M}(X) + \theta X^M - y_k(X))$$

and by hypothesis, $O((y_i)_{< M}(X) + \theta X^M - y_k(X)) = O(y_i(X) - y_k(X))$ for all $k \neq i$. Furthermore, $O((y_i)_{< M}(X) + \theta X^M - y_i(X)) = M = O(y_i(X) - y_j(X))$ for some $j \neq i$. This implies i) and ii). Now $F(X, 0) = \prod_{k=1}^n ((y_i)_{< M}(X) + \theta X^M - y_k(X)) \neq 0$, hence iii) follows. Let $\Delta_{v_F}^F$ be the last edge of $N(F)$ and let y_{j_1}, \dots, y_{j_l} be the set of roots of f such that $c(y_i - y_{j_k}) = M$ for all $k = 1, \dots, l$. Write $y_i = \sum_p c_p^i x^p$ and let $y_i - y_{j_k} = c_{a_k} x^M + \dots$ for all $k = 1, \dots, l$. It follows that $(Y_i)_{< M}(X) + \theta x^M - y_{j_k}(X) = (c_{a_k} + \theta)x^M + \dots$. Finally $F_{\Delta_{v_F}^F} = (y - (c_M + \theta)x^M) \prod_{k=1}^l (y - (c_{a_k} + \theta)x^M)$. Since θ is generic and $l \geq 1$, then iv) follows immediately. ■

In particular, using the results of Section 3., the last vertex of $N(F_Y)$ is $(\alpha^1, 0)$, $O(F) = O(F_y)$, and $\max(O(F_Y)) = M$. But $F_Y(X, Y) = f_y(X, Y + \tilde{y}_i(X))$. This with the above Lemma led to the following Proposition (see also [10], Lemma 3.3.):

Lemma 4.4 For $y_i(x), y_j(x), i \neq j$, there is a root $z_k(x)$ of $f_y(x, y) = 0$ such that

$$c(y_i(x), y_j(x)) = c(y_i(x), z_k(x)) = c(y_j(x), z_k(x)).$$

Conversely, given $y_i(x), z_k(x)$, there is $y_j(x)$ for which the above equality holds. Moreover, given $y_i(x)$ and $M \in \mathbb{R}$,

$$\text{card}\{y_j(x) | c(y_i(x), y_j(x)) = M\} = \text{card}\{z_k(x) | c(y_i(x), z_k(x)) = M\}.$$

Proof. Let $i \neq j$ and let $M = c(y_i - y_j)$. Let $\tilde{y}_i = (y_i)_{< M} + \theta x^M$ be the M -deformation of y_i . Consider, as in Lemma 4.3., the change of variables $X = x, Y = y - \tilde{y}_i(X)$ and let $F(X, Y) = f(X, Y + \tilde{y}_i(X))$. It follows from Lemma 4.3. that $F(X, 0) \neq 0$, and if $\deg_y F_{\Delta_{v_F}^F} = r + 1$, then

there is r roots y_{j_1}, \dots, y_{j_r} of $f(x, y) = 0$ such that $c(y_i - y_{j_k}) = M$ for all $k = 1, \dots, r$. Since $(\alpha^1, 0) \in \text{Supp}(F_{\Delta_{v_F}^F})$ for some α^1 , then the cardinality of $E(F_y)$ is the same as the cardinality of $E(F)$. Furthermore, $N(F_y)$ is a translation of $N(F)$ with respect of the vector $(0, -1)$. Finally, $(F_y)_{\Delta_{v_F}^F} = (F_{\Delta_{v_F}^F})_y$ is a polynomial of degree r in y . In particular, by Lemma 4.3., there is r roots of $f_y(x, y) = 0$ whose contact with y_i is M . This completes the proof of the result. ■

Let $g = y^m + a_1(x)y^{m-1} + \dots + a_m(x)$ be a reduced monic polynomial of $\mathbf{K}((x))[y]$ and denote by z_1, \dots, z_m the set of roots of g . Let $y_i(x) \in \text{Root}(f)$ and let:

$$M = \max(\{c(y_i, y_j) | j \neq i\} \cup \{c(y_i, z_k) | k = 1, \dots, m\})$$

Lemma 4.5 Let the notations be as above, and assume that $M > \max_{1 \leq k \leq m} c(y_i, z_k)$. Let $\tilde{y}_i = (y_i)_{<M} + \theta x^M$ be the M -deformation of y_i and consider the change of variables $X = x, Y = y - \tilde{y}_i(X)$. Let $F(X, Y) = f(X, Y + \tilde{y}_i(X))$, $G(X, Y) = g(X, Y + \tilde{y}_i(X))$. We have the following

- i) $F(X, 0) \neq 0$ and $G(X, 0) \neq 0$, i.e. $N(F)$ and $N(G)$ meet the x -axis.
- ii) $\max(O(F)) = M > \max(O(G))$
- iii) If $\Delta_{v_F}^F$ (resp. $\Delta_{v_G}^G$) denotes the last edge of $N(F)$ (resp. $N(G)$) then $(\alpha^1, 1) \in \text{Supp}(F_{\Delta_{v_F}^F})$ (resp. $(\beta^1, 1) \in \text{Supp}(G_{\Delta_{v_G}^G})$) for some α^1 (resp. β^1), and $(\alpha^1, 1) \notin V(F)$ (resp. $(\beta^1, 1) \notin V(G)$).

Proof. Let $F(X, Y) = \prod_{j=1}^n (Y - Y_j(X))$ and $G(X, Y) = \prod_{k=1}^m (Y - Z_k(X))$. Clearly $Y_j(X) = y_j(X) - (y_i)_{<M}(X) + \theta X^M$, $Z_k(X) = z_k(X) - (y_i)_{<M}(X) + \theta X^M$. In particular, for all $k = 1, \dots, m$, $O(Z_k) = c(y_i, z_k) < M$. On the other hand, for all $j \neq i$, $O(Y_j) = c(y_i, y_j) \leq M$ with equality for at least one j , and $O(Y_i) = M$. This implies i) and ii). Now iii) follows by a similar argument as in Lemma 4.3. ■

Let $J = J(f, g)$, and note that $J(F, G) = J(X, Y)$. In particular, by the results of the previous Section we get the following:

Lemma 4.6 For $y_i(x), y_j(x), i \neq j$, if $c(y_i, y_j) > \max_{1 \leq k \leq m} c(y_i, z_k)$, then there is a root $u_l(x)$ of $J(x, y) = 0$ such that

$$c(y_i(x), y_j(x)) = c(y_i(x), u_l(x))$$

Conversely, given $y_i(x), u_l(x)$, if $c(y_i, u_l) > c(y_i, z_k), k = 1, \dots, m$, there is $y_j(x)$ for which the above equality holds. Moreover, given $y_i(x)$ and $M \in \mathbb{R}$, if $M > \max_{1 \leq k \leq m} c(y_i, z_k)$, then:

$$\text{card}\{y_j(x) | c(y_i(x), y_j(x)) = M\} = \text{card}\{u_l(x) | c(y_i(x), u_l(x)) = M\}.$$

Proof. Let $M = c(y_i, y_j)$ and consider the change of variables $X = x, Y = y - \tilde{y}_i(X)$, where $\tilde{y}_i = (y_i)_{<M} + \theta x^M$ is the M -deformation of y_i . Let $F(X, Y) = f(X, Y + \tilde{y}_i(X))$ and $G(X, Y) =$

$g(X, Y + \tilde{Y}_i(X))$. It follows from the hypotheses that F and G satisfies conditions i), ii), and iii) of Lemma 3.4. In particular $J(X, Y)_{\Delta_{v(J(X, Y))}} = (G_{\Delta_{v(G)}}(X, 0))_X \cdot (F_{\Delta_{v(F)}})_Y$. The proof follows now by a similar argument as in Lemma 4.4. ■

5 Five main results

Let $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ be a reduced monic polynomial of $\mathbf{K}((x))[y]$ and let $f = f_1 \cdot f_2 \cdot \dots \cdot f_{\xi(f)}$ be the decomposition of f into irreducible components of $\mathbf{K}((x))[y]$. Let f_y be the y -derivative of f and let $\text{Root}(f) = \{y_1(x), \dots, y_n(x)\}$.

For all $1 \leq i \leq \xi(f)$, set $n_{f_i} = \deg_y(f_i)$, and let $(m_k^{f_i})_{1 \leq k \leq h_{f_i}+1}, (d_k^{f_i})_{1 \leq k \leq h_{f_i}+1}, (e_k^{f_i})_{1 \leq k \leq h_{f_i}}, (r_k^{f_i})_{1 \leq k \leq h_{f_i}+1}$ be the set of characteristic sequences of f_i . Let finally $z(x)$ denotes an element of $\text{Root}(f_y)$.

Proposition 5.1 Assume that $\xi(f) = 1$, i. e. $f = f_1$ is irreducible. For all $1 \leq k \leq h_f$, we have:

$$\text{card}\{z(x) | c(f, z(x)) = \frac{m_k^f}{n_f}\} = (e_k^f - 1) \frac{n_f}{d_k^f}.$$

Proof. Note that, by Lemma 4.4., $c(f, z(x)) \in \{\frac{m_1^f}{n_f}, \dots, \frac{m_{h_f}^f}{n_f}\}$. Assume first that $k = h_f$ and fix

a root y_p of f . By Lemma 1.2., y_p has the contact $\frac{m_{h_f}^f}{n_f}$ with exactly $d_{h_f}^f - d_{h_f+1}^f = d_{h_f}^f - 1 = e_{h_f}^f - 1$ roots of f , consequently, by Lemma 4.4., there is $e_{h_f}^f - 1$ roots of f_y whose contact with y_p

is $\frac{m_{h_f}^f}{n_f}$. Denote the set of these roots by D_p . Each element of D_p has the contact $\frac{m_{h_f}^f}{n_f}$ with

exactly $d_{h_f}^f$ roots of f (since we have to add y_p). Denote this set by C_p . Let $y_q \notin C_p$ be a root of f . Repeating with y_q what we did for y_p , we construct D_q and C_q in a similar way.

Obviously $C_p \cap C_q = \emptyset$ (otherwise, $c(y_p, y_q) = \frac{m_{h_f}^f}{n_f}$, which is impossible because $y_q \notin C_p$). This

implies that $D_p \cap D_q = \emptyset$ This process divides the n_f roots of f into $\frac{n_f}{d_{h_f}^f}$ disjoint groups

$C_1, \dots, C_{\frac{n_f}{d_{h_f}^f}}$ such that for all $1 \leq p \leq \frac{n_f}{d_{h_f}^f}$, C_p contains the roots of f having the contact $\frac{m_{h_f}^f}{n_f}$

with the elements of D_p . For all $z(x) \in D_p, c(f, z(x)) = \frac{m_{h_f}^f}{n_f}$, in particular

$$\text{card}\{z(x)|c(f, z(x)) = \frac{m_{h_f}^f}{n_f}\} = \sum_{p=1}^{\frac{n_f}{d_{h_f}^f}} \text{card}D_p = (e_{h_f}^f - 1) \frac{n_f}{d_{h_f}^f}.$$

Assume that the equality is true for $k = h_f, \dots, j+1$, then there is exactly $\sum_{i=j+1}^{h_f} (e_i^f - 1) \frac{n_f}{d_j^f} = n_f - \frac{n_f}{d_j^f}$ roots of f_y having the contact $\geq m_j^f$ with f . We now repeat the same argument

with $\frac{n_f}{d_j^f}, \frac{d_j^f}{d_{j+1}^f} - 1$ instead of n_f and $d_{h_f}^f - 1$. ■

Proposition 5.2 Let $M \in \mathbb{Q}$ and let $1 \leq i \leq \xi(f)$. Assume that $M \neq \frac{m_k^{f_i}}{n_{f_i}}$ for all $k = 1, \dots, h_{f_i}$. We have:

$$\text{card}\{z(x)|c(f_i, z(x)) = M\} = \text{card}\{y(x) \in \text{Root}(f)|c(f_i, y(x)) = M\} = \sum_{c(f_i, f_k)=M} n_{f_k}.$$

Proof. Let, without loss of generality, $i = 1$ and let $k > 1$ be such that $c(f_1, f_k) = M$. Fix a root $y_p(x)$ of f_1 . Since $c(y_p(x), f_k) = M$, then there is a root $y(x)$ of f_k such that $c(y_p(x), y(x)) = M$.

Let $\theta \in \{0, \dots, h_{f_1}\}$ be the smallest integer such that $M < \frac{m_{\theta+1}^{f_1}}{n_{f_1}}$ and consider another root $y_j(x)$ of f_1 . We have:

$$c(y_j, y(x)) = O_x(y_j - y(x)) = O_x(y_j - y_p + y_p - y(x)) = \begin{cases} M & \text{if } O_x(y_j - y_p) \geq \frac{m_{\theta+1}^{f_1}}{n_{f_1}} \\ O_x(y_j - y_p) & \text{if } O_x(y_j - y_p) < \frac{m_{\theta+1}^{f_1}}{n_{f_1}} \end{cases}$$

By Lemma 1.2., there is exactly $d_{\theta+1}^{f_1} - 1$ roots of f_1 having a contact $\geq \frac{m_{\theta+1}^{f_1}}{n_{f_1}}$ with y_p , conse-

quently, by the formula above, there is exactly $d_{\theta+1}^{f_1}$ roots of f_1 having the contact M with $y(x)$ (since we have to add y_p). Denote this set by C_p and let D_k^p be the set of roots of f_k having the contact M with y_p . In particular an element of D_k^p has the contact M with every element of C_p .

Let $y_q \notin C_p$ be a root of f_1 and repeat the same construction with y_q instead of y_p . It is clear that $C_p \cap C_q = \emptyset$ (otherwise, if $\bar{y} \in C_p \cap C_q$, then $c(\bar{y}, y_p) = c(\bar{y}, y_q) = M$, in particular $c(y_p, y_q) \geq M$, which is a contradiction since $y_q \notin C_p$), in particular $D_k^p \cap D_k^q = \emptyset$. This process

divides the set of roots of f_k into disjoint $\frac{n_{f_1}}{d_{\theta+1}^{f_1}}$ groups $D_k^1, \dots, D_k^{\frac{n_{f_1}}{d_{\theta+1}^{f_1}}}$: for all $1 \leq p \leq \frac{n_{f_1}}{d_{\theta+1}^{f_1}}, D_k^p$

contains the roots of f having the contact M with the elements of C_p . Repeating what we did with another f_l , $l \neq k$, such that $c(f_1, f_l) = M$, then adding the D_k^p 's, We obtain disjoint $\frac{n_{f_1}}{d_{\theta+1}^{f_1}}$ groups $D_1, \dots, D_{\frac{n_{f_1}}{d_{\theta+1}^{f_1}}}$ such that D_p contains the roots of f having the contact M with the elements of C_p . We have, by Lemma 4.4.

$$\text{card}\{z(x)|c(y_p(x), z(x)) = M\} = \text{card}\{y(x) \in \text{Root}(f)|c(y_p(x), y(x)) = M\} = \text{card}D_p$$

Let $z(x) \in \text{Root}(f_y)$ and assume that $c(z(x), y_p) = M$. If $y_q \in \text{Root}(f)$, $y_q \neq y_p$, since $c(y_p, y_q) \neq M$, then $c(z(x), y_q) \leq M$. In particular $c(f_1, z(x)) = M$. Finally

$$\text{card}\{z(x)|c(f_1, z(x)) = M\} = \sum_{p=1}^{\frac{n_{f_1}}{d_{\theta+1}^{f_1}}} \text{card}D_p = \sum_{c(f_1, f_k)=M} n_{f_k}$$

This proves our assertion. ■

Proposition 5.3 For all $1 \leq i \leq r$ and for all $1 \leq \theta \leq h_{f_i}$, we have:

$$\begin{aligned} \text{card}\{z(x)|c(f_i, z(x)) = \frac{m_{\theta}^{f_i}}{n_{f_i}}\} &= \text{card}\{y(x) \in \text{Root}(f)|c(f_i, y(x)) = \frac{m_{\theta}^{f_i}}{n_{f_i}}\} + (e_{\theta}^{f_i} - 1) \frac{n_{f_i}}{d_{\theta}^{f_i}} \\ &= \sum_{c(f_i, f_k) = \frac{m_{\theta}^{f_i}}{n_{f_i}}} n_{f_k} + (e_{\theta}^{f_i} - 1) \frac{n_{f_i}}{d_{\theta}^{f_i}}. \end{aligned}$$

Proof. Let, without loss of generality, $i = 1$ and assume that $c(f_1, f_k) = \frac{m_{\theta}^{f_1}}{n_{f_1}}$ for at least one $k > 1$. Let y_p be a root of f_1 . Since $c(f_k, y_p) = \frac{m_{\theta}^{f_1}}{n_{f_1}}$, then there is a root $y(x)$ of f_k such that $c(y_p(x), y(x)) = \frac{m_{\theta}^{f_1}}{n_{f_1}}$. By Lemma 1.2., there is exactly $d_{\theta}^{f_1} - 1$ roots of f_1 having a contact $\geq \frac{m_{\theta}^{f_1}}{n_{f_1}}$ with y_p . Let $y_j(x)$ be a root of f_1 such that $c(y_p, y_j) \geq \frac{m_{\theta}^{f_1}}{n_{f_1}}$, then $c(y_j, y(x)) = O_x(y_j - y(x)) = O_x(y_j - y_p + y_p - y(x)) \geq \frac{m_{\theta}^{f_1}}{n_{f_1}}$. On the other hand, $c(y_j(x), y(x)) \leq c(f_1, f_k) = \frac{m_{\theta}^{f_1}}{n_{f_1}}$, hence $c(y_j, y(x)) = \frac{m_{\theta}^{f_1}}{n_{f_1}}$. Consequently, there is exactly $d_{\theta}^{f_1}$ roots of f_1 having the contact

$\frac{m_\theta^{f_1}}{n_{f_1}}$ with $y(x)$. Denote this set by C_p and let D_p^k be the set of roots of f_k such that for all $y(x) \in D_p^k, c(y_p(x), y(x)) = \frac{m_\theta^{f_1}}{n_{f_1}}$. In particular, an element of D_p^k has the contact $\frac{m_\theta^{f_1}}{n_{f_1}}$ with every element of C_p .

Let $y_q \notin C_p$ be a root of f_1 and repeat the same construction with y_q instead of y_p . We have, by a similar argument as in Proposition 5.2., $C_p \cap C_q = \emptyset$ and consequently $D_p^k \cap D_q^k = \emptyset$. This divides the set of roots of f_k into disjoint $\frac{n_{f_1}}{d_\theta^{f_1}}$ groups $D_k^1, \dots, D_k^{\frac{n_{f_1}}{d_\theta^{f_1}}}$. Each element of D_k^p has the contact $\frac{m_\theta^{f_1}}{n_{f_1}}$ with the elements of C_p . Repeating the same argument with the set of f_l such that $c(f_1, f_l) = \frac{m_\theta^{f_1}}{n_{f_1}}$ with f_1 , then adding the D_p^k 's, we obtain disjoint $\frac{n_{f_1}}{d_\theta^{f_1}}$ groups $D_1, \dots, D_{\frac{n_{f_1}}{d_\theta^{f_1}}}$ such that for all $1 \leq p \leq \frac{n_{f_1}}{d_\theta^{f_1}}, D_p$ contains the roots of $\frac{f}{f_1}$ having the contact $\frac{m_\theta^{f_1}}{n_{f_1}}$ with the elements of C_p . We have, by Lemma 2.1. and Lemma 4.4.:

$$\begin{aligned} \text{card}\{z(x) | c(y_p, z(x)) = \frac{m_\theta^{f_1}}{n_{f_1}}\} &= \text{card}\{y(x) | c(y_p, y(x)) = \frac{m_\theta^{f_1}}{n_{f_1}}\} \\ &= \text{card}D_p + (e_\theta^{f_1} - 1) \frac{n_{f_1}}{d_\theta^{f_1}} \end{aligned}$$

And by a similar argument as in Proposition 5.2.,

$$\text{card}\{z(x) | c(f_1, z(x)) = \frac{m_\theta^{f_1}}{n_{f_1}}\} = \left(\sum_{1 \leq p \leq \frac{n_{f_1}}{d_\theta^{f_1}}} \text{card}D_p \right) + (e_\theta^{f_1} - 1) \frac{n_{f_1}}{d_\theta^{f_1}} = \left(\sum_{c(f_1, f_k) = \frac{m_\theta^{f_1}}{n_{f_1}}} n_{f_k} \right) + (e_\theta^{f_1} - 1) \frac{n_{f_1}}{d_\theta^{f_1}}$$

This proves our assertion. ■

Let $g = y^m + b_1(x)y^{m-1} + \dots + b_m(x)$ be a monic reduced polynomial of $\mathbf{K}((x))[y]$ and let $g_1, \dots, g_{\xi(g)}$ be the set of irreducible components of g in $\mathbf{K}((x))[y]$. Set $\text{Root}(g) = \{z_1, \dots, z_m\}$, and let $J = J(f, g)$ be the Jacobian of f and g .

Proposition 5.4 Let $M \in \mathbb{Q}$ and assume that $c(y(x), y'(x)) = M$ for some $y(x), y'(x) \in \text{Root}(f)$, and that $M > \max_{1 \leq j \leq m} c(y(x), z_j(x))$. Let $1 \leq i \leq \xi(f)$, and assume that $M \neq \frac{m_k^{f_i}}{n_{f_i}}$ for all $k = 1, \dots, h_{f_i}$. We have the following

$$\text{card}\{u(x) \in \text{Root}(J) | c(f_i, u(x)) = M\} = \text{card}\{y_j(x) | c(f_i, y_j(x)) = M\} = \sum_{c(f_i, f_k)=M} n_{f_k}.$$

Proof. The proof is similar to the proof of Proposition 5.2., where Lemma 4.4. is replaced by Lemma 4.6. ■

Proposition 5.5 Let $1 \leq \theta \leq h_{f_i}$ and assume that $\frac{m_\theta^{f_i}}{n_{f_i}} > \max_{1 \leq j \leq n, 1 \leq k \leq m} c((y_j(x), z_k(x)))$. We have the following

$$\begin{aligned} \text{card}\{u(x) \in \text{Root}(J) | c(f_i, u(x)) = \frac{m_\theta^{f_i}}{n_{f_i}}\} &= \text{card}\{y(x) \in \text{Root}(f) | c(f_i, y(x)) = \frac{m_\theta^{f_i}}{n_{f_i}}\} + (e_\theta^{f_i} - 1) \frac{n_{f_i}}{d_\theta^{f_i}} \\ &= \sum_{c(f_i, f_k) = \frac{m_\theta^{f_i}}{n_{f_i}}} n_{f_k} + (e_\theta^{f_i} - 1) \frac{n_{f_i}}{d_\theta^{f_i}}. \end{aligned}$$

Proof. The proof is similar to the proof of Proposition 5.3., where Lemma 4.4. is replaced by Lemma 4.6. ■

6 The tree of contacts

Let f be a reduced monic polynomial in $\mathbf{K}((x))[y]$ and let $f = f_1 \dots f_{\xi(f)}$ be the factorization of f into irreducible components of $\mathbf{K}((x))[y]$. We define the set of contacts of f to be the set:

$$C(f) = \{c(f_p, f_q) | 1 \leq p \neq q \leq \xi(f)\} \cup \bigcup_{k=1}^r \left\{ \frac{m_1^{f_k}}{n_{f_k}}, \dots, \frac{m_{h_{f_k}}^{f_k}}{n_{f_k}} \right\}$$

Let $C(f) = \{M_1, \dots, M_{t_f}\}$. The tree associated with f is constructed as follows:

Let $M \in C(f)$ and define $C_M(f)$ to be the set of irreducible components of f such that

$$f_p \in C_M(f) \Leftrightarrow c(f_p, f_q) \geq M \text{ for some } 1 \leq q \leq \xi(f)$$

with the understanding that $c(f_k, f_k) \geq M$ if and only if $\frac{m_i^{f_k}}{n_{f_k}} \geq M$ for some $1 \leq i \leq h_{f_k}$.

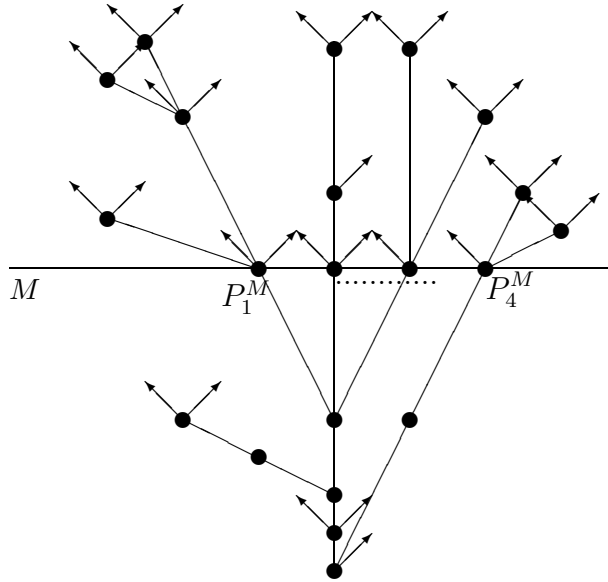
We associate with M the equivalence relation on the set $C_M(f)$, denoted R_M , and defined as follows:

$$f_p R_M f_q \text{ if and only if } c(f_p, f_q) \geq M.$$

We define the points of the tree $T(f)$ at the level M to be the set of equivalence classes of R_M , and we denote this set by $P_1^M, \dots, P_{s_M}^M$. We shall say that P_j^N dominates P_i^M , and we write $P_j^N \geq P_i^M$, if $P_j^N \subseteq P_i^M$. We shall say that P_j^N strictly dominates P_i^M , and we write $P_j^N > P_i^M$, if P_j^N dominates P_i^M , $P_i^M \neq P_j^N$, and $C(f) \cap]M, N[= \emptyset$. This defines an order on the set of points of $T(f)$ with a unique minimal element, denoted $P_1^{M_1}$. A point P_i^M is called a top point of $T(f)$ if it is maximal with respect to this order. We denote by $\text{Top}(f)$ the set of top points of $T(f)$.

Let P_i^M be a point of $T(f)$, and let $P_{i_1}^{N_1}, \dots, P_{i_t}^{N_t}$ be the set of points that strictly dominate P_i^M . We set $D_i^M = P_i^M - \cup_{l=1}^t P_{i_l}^{N_l}$. Clearly $\{P_{i_1}^{N_1}, \dots, P_{i_t}^{N_t}, D_i^M\}$ is a partition of P_i^M . Furthermore, for all $F \in D_i^M$ and for all $F \neq G \in P_i^M$, $c(F, G) = M$. Note that if $F \in D_i^M$, then $M \geq \frac{m_F^F}{n_F}$ and that $P_i^M \in \text{Top}(f)$ if and only if $P_i^M = D_i^M$.

If P_j^N strictly dominates P_i^M , then we link these two points by a line segment. We define the set of edges of $T(f)$ to be the set of these line segments. Given a point P_i^M , if $D_i^M \neq \emptyset$, then we associate with all $F \in D_i^M$ an arrow starting at the point P_i^M . Let $P_1^{M_1}, P_{i_2}^{M_2}, \dots, P_{i_k}^{M_k}$ be a set of points of $T(f)$ such that $P_{i_2}^{M_2}$ strictly dominates $P_1^{M_1}$, $P_{i_k}^{M_k} \in \text{Top}(f)$, and $P_{i_j}^{M_j}$ strictly dominates $P_{i_{j-1}}^{M_{j-1}}$ for all $3 \leq j \leq k$. The union of edges linking these points is called a branch of $T(f)$. Clearly, there are as many branches of $T(f)$ as there are top points of $T(f)$.



Lemma 6.1 Let P_i^M be a point of $T(f)$. We have the following:

- i) For all $F, G \in P_i^M$, $c(F, G) \geq M$.
- ii) For all $F \in P_i^M$ and for all $G \notin P_i^M$, $c(F, G) < M$.

iii) For all $F, G \in P_i^M$ and for all $H \notin P_i^M$, $c(F, H) = c(G, H)$. We denote this rational by $c(H, P_i^M)$.

iv) let $F \in P_i^M$ and let $1 \leq \theta \leq h_F + 1$ be the smallest integer such that $M \leq \frac{m_\theta^F}{n_F}$. If $\theta \geq 2$ then $\frac{m_k^G}{n_G} = \frac{m_k^F}{n_F}$ for all $G \in P_i^M$ and for all $1 \leq k \leq \theta - 1$. We denote this rational number by $\frac{m_k}{n}(P_i^M)$. As a consequence $\frac{n_G}{d_k^G}$ does not depend on $G \in P_i^M$ and $1 \leq k \leq \theta$. We denote this rational number by $\frac{n}{d_k}(P_i^M)$.

Proof. The proof is an easy application of Lemma 1.5. and Lemma 1.6. ■

Let P_i^M be a point of $T(f)$ and define the subsets $X_1(M, i), \dots, X_{s(M, i)}(M, i)$ of P_i^M as follows:

- For all k and for all $F, G \in X_k(M, i)$, $c(F, G) = M$.
- Given $F \in X_k(M, i)$ and $l \neq k$, if $F \notin X_l(M, i)$, then $c(F, G) > M$ for some $G \in X_l(M, i)$ (in particular $F, G \in P_j^N$ for some $P_j^N > P_i^M$).

The sets defined above satisfy the following property:

Lemma 6.2 The cardinality of $X_k(M, i)$ does not depend on $1 \leq k \leq s(M, i)$. We denote this cardinality by $c(M, i)$.

Proof. Assume that $s(M, i) \geq 2$ and let $1 \leq a \neq b \leq s(M, i)$. We shall construct a bijective map from $X_a(M, i)$ to $X_b(M, i)$. Let $F \in X_a(M, i)$. If $F \notin X_b(M, i)$, then there is $\tilde{F} \in X_b(M, i)$ such that $c(F, \tilde{F}) > M$. We claim that \tilde{F} is the only element with this property. In fact, if there is $\tilde{F} \neq G \in X_b(M, i)$ such that $c(F, G) > M$, then $M = c(\tilde{F}, G) \geq \min(c(F, \tilde{F}), c(F, G)) > M$, which is a contradiction. This defines a map

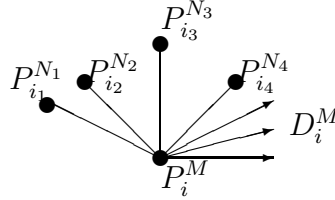
$$\begin{aligned} \phi_{a,b} : X_a(M, i) &\longmapsto X_b(M, i) \\ \phi_{a,b}(F) &= \begin{cases} F & \text{if } F \in X_b(M, i) \\ \tilde{F} & \text{if } F \notin X_b(M, i) \end{cases} \end{aligned}$$

This map is clearly bijective. This completes the proof of the Lemma. ■

Lemma 6.3 Let the notations be as above, and let $P_{i_1}^{N_1}, \dots, P_{i_t}^{N_t}$ be the set of points that strictly dominate P_i^M . We have the following:

- i) $D_i^M \subseteq X_k(M, i)$ for all $1 \leq k \leq s(M, i)$.
- ii) Given $1 \leq k \leq s(M, i)$, $(X_k(M, i) \cap P_{i_1}^{N_1}, \dots, X_k(M, i) \cap P_{i_t}^{N_t}, D_i^M)$ is a partition of $X_k(M, i)$.
- iii) Given $1 \leq k \leq s(M, i)$ and $1 \leq l \leq t$, $X_k(M, i) \cap P_{i_l}^{N_l}$ is reduced to one element.
- iv) $c(M, i) = t + \text{card}(D_i^M)$.

Proof. The first two assertions are clear, on the other hand $3. \implies 4.$ We shall consequently prove 3. Assume, without loss of generality, that $k = 1$, and let $1 \leq l \leq t$. Suppose that $X_1(M, i) \cap P_i^{N_l} = \emptyset$ and let $G \in P_i^{N_l}$. We have $c(F, G) = M$ for all $F \in X_1(M, i)$, in particular $G \in X_1(M, i)$, which is a contradiction. Consequently $X_1(M, i) \cap P_i^{N_l} \neq \emptyset$. Let G_1, G_2 be two polynomials of $X_1(M, i) \cap P_i^{N_l}$. We have $c(G_1, G_2) = M$ and $c(G_1, G_2) \geq N > M$. This is a contradiction if $G_1 \neq G_2$. ■



Let P_i^M be a point of $T(f)$ and assume that $D_i^M \neq \emptyset$. For all $F \in D_i^M$, $c(F, F) \leq M$, in particular $M \geq \frac{m_{h_F}^F}{n_F}$.

Lemma 6.4 Let the notations be as above. We have the following

i) If $M > \frac{m_{h_F}^F}{n_F}$ for all $F \in D_i^M$, then n_F does not depend on $F \in D_i^M$. We denote it by $n(D_i^M)$. We have $\sum_{F \in D_i^M} n_F = (c(M, i) - t)n(D_i^M)$.

ii) If $M = \frac{m_{h_F}^F}{n_F}$ for some $F \in D_i^M$, then one of the following hold

1_{ii}) $M = \frac{m_{h_F}^F}{n_F}$ for all $F \in D_i^M$. In this case, n_F does not depend on $F \in D_i^M$. We denote it by $n(D_i^M)$. We have $\sum_{F \in D_i^M} n_F = (c(M, i) - t)n(D_i^M)$.

1_{iii}) $M > \frac{m_{h_{F'}}^{F'}}{n_{F'}}$ for some $F' \in D_i^M$. In this case, $M = \frac{m_{h_F}^F}{n_F} > \frac{m_{h_{F'}}^{F'}}{n_{F'}}$ for all $F \in D_i^M$, $F \neq F'$, and $n_F, h_F, (d_k^F)_{1 \leq k \leq h_F}$ do not depend on $F \in D_i^M$, $F \neq F'$. We denote these integers by $n(D_i^M), h(D_i^M), d_k^{D_i^M}$. With these notations we have $n(F') = \frac{n(D_i^M)}{d_{h(D_i^M)}^{D_i^M}}$, and $\sum_{F \in D_i^M} n(F) =$

$$(c(M, i) - t - 1) \cdot n(D_i^M) + \frac{n(D_i^M)}{d_{h(D_i^M)}^{D_i^M}}.$$

Proof. By definition, for all $F, G \in D_i^M$, $c(F, G) = M$. Consequently our results follow from Proposition 2.4. ■

Let H be a monic polynomial of $\mathbf{K}((x))[y]$ and let $H_1, \dots, H_{\xi(H)}$ be the set of irreducible components of H in $\mathbf{K}((x))[y]$. Let P_i^M be a point of $T(f)$ and let $F \in P_i^M$. We set:

$$R_{=M}(F; H) = \prod_{c(F, H_k)=M} H_k$$

and

$$R_{>M}(F; H) = \prod_{c(F, H_k)>M} H_k$$

In other words, $R_{=M}(F; H)$ (resp. $R_{>M}(F; H)$) is the product of irreducible components of H whose contact with F is M (resp. $> M$).

Lemma 6.5 Suppose that $P_i^M \notin \text{Top}(f)$ and let $P_{i_1}^{N_1}, \dots, P_{i_t}^{N_t}$ be the set of points that strictly dominate P_i^M . Fix $1 \leq l \leq t$ and let $F \in P_{i_l}^{N_l}$. We have the following

i) For all $G \in P_{i_l}^{N_l}$, $R_{=M}(G; H) = R_{=M}(F; H)$ (resp. $R_{>M}(G; H) = R_{>M}(F; H)$). We denote this polynomial by $R_{=M}(P_{i_l}^{N_l}, H)$ (resp. $R_{>M}(P_{i_l}^{N_l}, H)$).

ii) For all $G \in P_{i_k}^{N_k}, k \neq l$, $R_{>M}(G; H)$ divides $R_{=M}(F; H)$.

iii) For all $G \in D_i^M$, $R_{>M}(G; H)$ divides $R_{=M}(F; H)$.

Proof. Let H_k be an irreducible component of H . If $G \in P_{i_l}^{N_l}$, then $c(F, G) \geq N_k > M$. In particular, by Proposition 1.5., $c(G, H_k) = M$ (resp. $c(G, H_k) > M$) if and only if $c(F, H_k) = M$ (resp. $c(F, H_k) > M$). This proves i). If either $G \in P_{i_k}^{N_k}, k \neq l$ or $G \in D_i^M$, then $c(F, G) = M$. In particular, if $c(G, H_k) > M$, then, by Lemma 1.5. $c(F, H_k) = M$. This proves ii) and iii). ■

Let the notations be as above. It follows from ii), iii) of Lemma 6.5. that that:

$$\prod_{k=2}^t R_{>M}(P_{i_k}^{N_k}, H) \cdot \prod_{F \in D_i^M} R_{>M}(F; H) \text{ divides } R_{=M}(P_{i_1}^{N_1}, H).$$

Set

$$\overline{Q}_H(M, i) = \frac{R_{=M}(P_{i_1}^{N_1}, H)}{\prod_{k=2}^t R_{>M}(P_{i_k}^{N_k}, H) \cdot \prod_{F \in D_i^M} R_{>M}(F; H)}$$

and let

$$Q_H(M, i) = \prod_{c(F, H_k)=M \forall F \in P_i^M} H_k$$

i.e. $Q_H(M, i)$ is the product of the irreducible components of H whose contact with all $F \in P_i^M$ is M .

Lemma 6.6 With the notations above, we have $Q_H(M, i) = \overline{Q}_H(M, i)$.

Proof. Let H_k be an irreducible component of $Q_H(M, i)$. For all $F \in P_i^M$, $c(F, H_k) = M$. In particular, since $\cup_{k=1}^t P_{i_k}^{N_k} \subseteq P_i^M$, then H_k divides $R_{=M}(P_{i_1}^{N_1}; H)$ and H_k does not divide $\prod_{k=2}^t R_{>M}(P_{i_k}^{N_k}; H) \cdot \prod_{F \in D_i^M} R_{>M}(F; H)$. Hence $Q_H(M, i)$ divides $\overline{Q}_H(M, i)$.

Let us prove that $\overline{Q}_H(M, i)$ divides $Q_H(M, i)$. Let $G \in P_i^M$ and let \bar{H} be an irreducible component of $\overline{Q}_H(M, i)$.

- If $G \in P_{i_1}^{N_1}$, then by Lemma 6.5. i), $R_{=M}(G; H) = R_{=M}(P_{i_1}^{N_1}, H)$, in particular $c(G, \bar{H}) = M$.

- If $G \in P_i^M - P_{i_1}^{N_1}$, then by Lemma 6.3., $G \in D_i^M \cup (\cup_{k=2}^t P_{i_k}^{N_k})$. Suppose that $G \in D_i^M$. If $c(G, \bar{H}) > M$, then \bar{H} divides $R_{>M}(G, H) = R_{>M}(D_i^M, H)$. This contradicts the definition of $\overline{Q}_H(M, i)$. In particular $c(G, \bar{H}) = M$. By a similar argument we prove that if $G \in \cup_{k=2}^t P_{i_k}^{N_k}$, then $c(G, \bar{H}) = M$. This implies our assertion. ■

Lemma 6.7 Suppose that $P_i^M \in \text{Top}(f)$, and recall that in this case $P_i^M = D_i^M$. Let F be an element of D_i^M . We have

$$Q_H(M, i) = \frac{R_{=M}(F; H)}{\prod_{G \in D(M, i), G \neq F} R_{>M}(G; H)}$$

Proof. The proof is similar to the proof of Lemma 6.6. ■

7 Factorization of the y -derivative

7.1 The irreducible case

Let f be a monic irreducible polynomial of $\mathbf{K}((x))[y]$ of degree n_f in y and consider the characteristic sequences associated with f as in Section 1. We have the following:

Proposition 7.1 $f_y = P_1 \dots P_{h_f}$ and for all $k = 1, \dots, h_f$:

i) $\deg_y P_k = (e_k^f - 1) \frac{n_f}{d_k^f}$.

ii) $\text{int}(f, P_k) = (e_k^f - 1) r_k^f$.

iii) For all irreducible component P of P_k , $c(f, P) = \frac{m_k^f}{n_f}$.

Proof. i) and iii) result from Proposition 5.1. and ii) results from Proposition 1.4. ■

7.2 The general case

Let the notations be as in Section 5. In particular f is a monic reduced polynomial of $\mathbf{K}((x))[y]$ and $f_1, \dots, f_{\xi(f)}$ are the irreducible components of f in $\mathbf{K}((x))[y]$. Consider the characteristic sequences associated with $f_1, \dots, f_{\xi(f)}$ and let $T(f)$ be the tree of f . Fix a point P_i^M of $T(f)$.

Lemma 7.2 Let the notations be as above and let $P_i^M \in T(f) - \text{Top}(f)$. If $D_i^M \neq \emptyset$, then $R_{>M}(F, f_y) = 1$ for all $F \in D_i^M$.

Proof. Suppose that $P_i^M \notin \text{Top}(f)$, and that $D_i^M \neq \emptyset$. Let $F \in D_i^M$. If $R_{>M}(F, f_y) \neq 1$, then $c(F, H) = N > M$ for some irreducible component H of f_y . In particular, by Lemma 4.4., $c(F, \bar{F}) = N$ for some irreducible component \bar{F} of f , hence $F \in P_j^N$ for some point $P_j^N \in T(f), N > M$. This is a contradiction because $F \in D_i^M$. ■

Lemma 7.3 Suppose that $P_i^M \notin \text{Top}(f)$ and let $P_{i_1}^{N_1}, \dots, P_{i_t}^{N_t}$ be the set of points of $T(f)$ that strictly dominate P_i^M . We have:

$$Q_{f_y}(M, i) = \frac{R_{=M}(P_{i_1}^{N_1}, f_y)}{\prod_{l=2}^t R_{>M}(P_{i_l}^{N_l}, f_y)}$$

Proof. We have, by Lemma 6.6.:

$$Q_{f_y}(M, i) = \frac{R_{=M}(P_{i_1}^{N_1}, f_y)}{\prod_{l=2}^t R_{>M}(P_{i_l}^{N_l}, f_y) \cdot \prod_{F \in D_i^M} R_{>M}(F, f_y)}$$

On the other hand, by Lemma 7.2., if $F \in D_i^M$, then $R_{>M}(F, f_y) = 1$. This proves our assertion. ■

Fix a polynomial $F_l \in P_{i_l}^{N_l}$ for all $1 \leq l \leq t$. By Lemma 6.6., $R_{=M}(P_{i_l}^{N_l}, f_y) = R_{=M}(P_{i_l}, F_l)$ (resp. $R_{>M}(P_{i_l}^{N_l}, f_y) = R_{>M}(P_{i_l}, F_l)$). In particular we have:

$$Q_{f_y}(M, i) = \frac{R_{=M}(F_1, f_y)}{\prod_{l=2}^t R_{>M}(F_l, f_y)}$$

The following Lemmas give the degrees of the two polynomials describing $Q_{f_y}(M, i)$.

Lemma 7.4 Let the notations be as above, and let θ be the smallest integer such that $M \leq \frac{m_\theta^F}{n_F}$ for all $F \in P_i^M$. We have:

$$\deg_y R_{=M}(F_1, f_y) = \begin{cases} \sum_{l=2}^t \left(\sum_{F \in P_{i_l}^{N_l}} n_F \right) + \sum_{F \in D_i^M} n_F & \text{if } M \neq \frac{m_\theta^{F_1}}{n_{F_1}} \\ \sum_{l=2}^t \left(\sum_{F \in P_{i_l}^{N_l}} n_F \right) + \sum_{F \in D_i^M} n_F + (e_\theta^{F_1} - 1) \frac{n_{F_1}}{d_\theta^{F_1}} & \text{if } M = \frac{m_\theta^{F_1}}{n_{F_1}} \end{cases}$$

Proof. This results from Propositions 5.3. and 5.4. ■

Lemma 7.5 Let the notations be as above, and let the hypotheses be as in Lemma 7.4. We have:

$$\deg_y R_{>M}(F_l, f_y) = \sum_{F \in P_{i_l}^{N_l} - \{F_l\}} n_F + \sum_{M < m_j^{F_l}} (e_j^{F_l} - 1) \frac{n_{F_l}}{d_j^{F_l}}$$

Proof. This results from Propositions 5.3. and 5.4. ■

As a corollary we have the following:

Proposition 7.6 Let the notations be as above, and let the hypotheses be as in Lemma 7.4. We have:

$$\deg_y Q_{f_y}(M, i) = \begin{cases} \sum_{F \in D_i^M} n_F + \sum_{l=2}^t [n_{F_l} - \sum_{M < m_j^{F_l}} (e_j^{F_l} - 1) \frac{n_{F_l}}{d_j^{F_l}}] & \text{if } M \neq \frac{m_\theta^{F_1}}{n_{F_1}} \\ \sum_{F \in D_i^M} n_F + \sum_{l=2}^t [n_{F_l} - \sum_{M < m_j^{F_l}} (e_j^{F_l} - 1) \frac{n_{F_l}}{d_j^{F_l}}] + (e_\theta^{F_1} - 1) \frac{n_{F_1}}{d_\theta^{F_1}} & \text{if } M = \frac{m_\theta^{F_1}}{n_{F_1}} \end{cases}$$

Proof. This results from Lemmas 7.4. and 7.5., since $\gcd(R_{=M}(F_1, f_y), R_{>M}(F_l, f_y)) = 1$ for all $2 \leq l \leq t$. ■

Note that

$$n_{F_l} - \sum_{M < m_j^{F_l}} (e_j^{F_l} - 1) \frac{n_{F_l}}{d_j^{F_l}} = \begin{cases} n_{F_l} - \sum_{j=\theta}^{h_{F_l}} (e_j^{F_l} - 1) \frac{n_{F_l}}{d_j^{F_l}} = \frac{n_{F_l}}{d_\theta^{F_l}} & \text{if } M \neq \frac{m_\theta^{F_l}}{n_{F_l}} \\ n_{F_l} - \sum_{j=\theta+1}^{h_{F_l}} (e_j^{F_l} - 1) \frac{n_{F_l}}{d_j^{F_l}} = \frac{n_{F_l}}{d_{\theta+1}^{F_l}} & \text{if } M = \frac{m_\theta^{F_l}}{n_{F_l}} \end{cases}$$

Let A (resp. B) be the set of $1 \leq l \leq t$ for which $M = \frac{m_{\theta}^{F_l}}{n_{F_l}}$ (resp. $M < \frac{m_{\theta}^{F_l}}{n_{F_l}}$). It follows that:

$$\deg_y Q_{f_y}(M, i) = \begin{cases} \sum_{l \in A} \frac{n_{F_l}}{d_{\theta+1}^{F_l}} + \sum_{l \in B - \{1\}} \frac{n_{F_l}}{d_{\theta}^{F_l}} + \sum_{F \in D_i^M} n_F & \text{if } 1 \in A \\ \sum_{l \in A - \{1\}} \frac{n_{F_l}}{d_{\theta+1}^{F_l}} + \sum_{l \in B} \frac{n_{F_l}}{d_{\theta}^{F_l}} + (e_{\theta}^{F_1} - 1) \frac{n_{F_1}}{d_{\theta}^{F_1}} + \sum_{F \in D_i^M} n_F & \text{if } 1 \in B \end{cases}$$

Let $(l_1, l_2) \in A \times B$ and recall that $\frac{n_F}{d_{\theta+1}^F}$ (reps. $\frac{n_F}{d_{\theta}^F}$) does not depend on $F \in \cup_{l \in A} P_i^{N_l}$ (resp. $F \in \cup_{l \in B} P_i^{N_l}$). In particular, if we denote by a (resp. b) the cardinality of A (resp. B), then we have:

$$\deg_y Q_{f_y}(M, i) = \begin{cases} a \frac{n_{F_{l_1}}}{d_{\theta+1}^{F_{l_1}}} + (b-1) \frac{n_{F_{l_2}}}{d_{\theta}^{F_{l_2}}} + \sum_{F \in D_i^M} n_F & \text{if } 1 \in B \\ (a-1) \frac{n_{F_{l_1}}}{d_{\theta+1}^{F_{l_1}}} + b \frac{n_{F_{l_2}}}{d_{\theta}^{F_{l_2}}} + (e_{\theta}^{F_1} - 1) \left(\frac{n_{F_1}}{d_{\theta}^{F_1}} \right) + \sum_{F \in D_i^M} n_F & \text{if } 1 \in A \end{cases}$$

Note that if $B \neq \emptyset$ then $\frac{n_{F_{l_2}}}{d_{\theta}^{F_{l_2}}} = \frac{n_{F_1}}{d_{\theta}^{F_1}}$, on the other hand, if $B = \emptyset$, then $1 \in A$. In particular we get the following:

$$\deg_y Q_{f_y}(M, i) = a \frac{n_{F_{l_1}}}{d_{\theta+1}^{F_{l_1}}} + (b-1) \frac{n_{F_1}}{d_{\theta}^{F_1}} + \sum_{F \in D_i^M} n_F$$

The above results can be stated as follows:

Theorem 7.7 Let P_i^M be a point of $T(f)$ and assume that $P_i^M \notin \text{Top}(f)$. Let $(P_i^{N_l})_{1 \leq l \leq t}$ be the set of points that strictly dominate P_i^M and let θ be the smallest integer such that for all $F \in P_i^M$, $M \leq \frac{m_{\theta}^F}{n_F}$. Let A (resp. B) be the set of $1 \leq l \leq t$ for which $M = \frac{m_{\theta}^F}{n_F}$ (resp. $M < \frac{m_{\theta}^F}{n_F}$) for all $F \in \cup_{l \in A} P_i^{N_l}$ (resp. $F \in \cup_{l \in B} P_i^{N_l}$) and Let $(l_1, l_2) \in A \times B$. If a (resp. b) denotes the cardinality of A (resp. B) then the component $Q_{f_y}(M, i)$ of f_y satisfies the following:

i) $\deg_y Q_{f_y}(M, i) = a \frac{n_{F_{l_1}}}{d_{\theta+1}^{F_{l_1}}} + (b-1) \frac{n_{F_{l_2}}}{d_{\theta}^{F_{l_2}}} + \sum_{F \in D_i^M} n_F$, and $\sum_{F \in D_i^M} n_F$ is given by the formula of

Lemma 6.4., where if $F \in D_i^M$, then h_F is either $\theta - 1$ or θ depending on $M > \frac{m_{\theta}^F}{n_F}$ or $M = \frac{m_{\theta}^F}{n_F}$.

ii) For all irreducible component P of $Q_{f_y}(M, i)$ and for all $F \in P_i^M$, $c(F, P) = M$.

iii) For all irreducible component P of $Q_{f_y}(M, i)$ and for all $F \notin P_i^M$, $c(F, P) = c(F, P_i^M) < M$, where we recall that $c(F, P_i^M)$ is the contact of F with any element of P_i^M .

iv) For all $1 \leq k \leq \xi(f)$:

- If $f_k \in P_i^M$ then $\text{int}(f_k, Q_{f_y}(M, i)) = S(m^{f_k}, M) \frac{\deg_Q f_y(M, i)}{n_{f_k}}$.

- If $f_k \notin P_i^M$ then $\text{int}(f_k, Q_{f_y}(M, i)) = S(m^{f_k}, c(f_k, P_i^M)) \frac{\deg_Q f_y(M, i)}{n_{f_k}}$, where $c(f_k, P_i^M)$ is the contact of f_k with any $F \in P_i^M$. ■

In the following we shall consider the case where P_i^M is a top point of $T(f)$.

Lemma 7.8 Suppose that $P_i^M = D_i^M \in \text{Top}(f)$, and let $F \in P_i^M$. We have the following:

$$Q_{f_y}(M, i) = R_{=M}(F, f_y)$$

Proof. By Lemma 7.2., $R_{>M}(G, f_y) = 1$ for all D_i^M . Our assertion follows from Lemma 6.7. ■

Let $P_i^M = D_i^M = \{F_1, \dots, F_r\}$, and recall that, by Proposition 2.4., the sequence (F_1, \dots, F_r) is either equivalent, or almost equivalent.

Theorem 7.9 Let $P_i^M = \{F_1, \dots, F_r\} \in \text{Top}(f)$ and assume that $n_{F_1} = \max_{1 \leq k \leq r} n_{F_k}$. We have the following:

i) If (F_1, \dots, F_r) is equivalent with $M > \frac{m_{h_{F_1}}^{F_1}}{n_{F_1}}$, then $\deg_y Q_{f_y}(M, i) = (r-1)n_{F_1}$.

ii) If (F_1, \dots, F_r) is equivalent with $M = \frac{m_{h_{F_1}}^{F_1}}{n_{F_1}}$, then $\deg_y Q_{f_y}(M, i) = (r-1)n_{F_1} + (e_{h_{F_1}}^{F_1} - 1) \frac{n_{F_1}}{d_{h_{F_1}}^{F_1}}$.

iii) If (F_1, \dots, F_r) is almost equivalent, then $\deg_y Q_{f_y}(M, i) = (r-1)n_{F_1}$.

Proof. It follows from Lemma 7.8. that $\deg_y Q_{f_y}(M, i) = \deg_y R_{=M}(F_1, f_y)$. Now the hypothesis of i) and ii) implies that $n_{F_k} = n_{F_1}$ for all $k = 2, \dots, r$. Hence i) results from Proposition 5.2. and ii) results from Proposition 5.3. Assume that (F_1, \dots, F_r) is almost equivalent, and that, without loss of generality, (F_1, F_3, \dots, F_r) is equivalent. Since $Q_{f_y}(M, i) = R_{=M}(F_1, f_y) = R_{=M}(F_2, f_y)$, then iii) results from Proposition 5.2. ■

Remark 7.10 When $P_i^M = D_i^M \in \text{Top}(f)$, the numbers a and b of Theorem 7.7. are zero. The reader may verify that the two formulas of Theorem 7.7. and Theorem 7.9. coincide. ■

Example 7.11 i) Delgado's result: Let $f = f_1.f_2$. In [5], in order to generalize Merle's Theorem, F. Delgado uses the arithmetic of the semi-group of f . His result is a particular case of Theorem 7.7. More precisely, let $n_i = \deg_y f_i, i = 1, 2$ and let $M = c(f_1, f_2)$, $I = \text{int}(f_1, f_2)$. Let θ be the smallest integer such that $M \leq \frac{m_\theta^i}{n_i}, i = 1, 2$. We have:

$$f_y = \left(\prod_{k=1}^{\theta-1} Q_{f_y} \left(\frac{m_k^1}{n_{f_1}}, 1 \right) \right) \cdot \bar{f}_y$$

where the properties $Q_{f_y}(\frac{m_k^1}{n_1}, 1)$ are given in the table 0), while those of the components of \bar{f}_y are given in the tables 1), 2), 3), depending on the position of M on $T(f)$. Note that $c(f_j, P)$ means the contact of f_j with an irreducible component of $Q_{f_y}(M, i)$.

0)

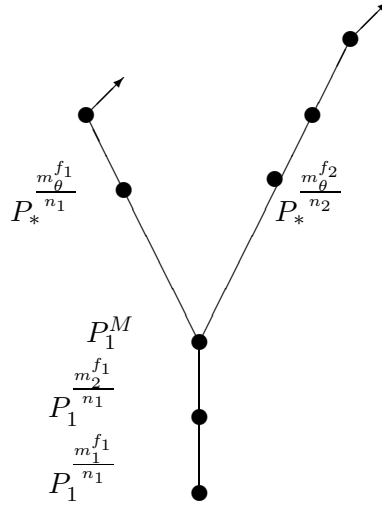
Q	$Q_{f_y}(\frac{m_1^1}{n_1}, 1)$...	$Q_{f_y}(\frac{m_{\theta-1}^1}{n_1}, 1)$
$\deg_y Q$	$(e_1^1 - 1) \frac{n_1}{d_1^{f_1}}$...	$(e_{\theta-1}^1 - 1) \frac{n_1}{d_{\theta-1}^{f_1}}$
$c(f_1, P), \text{int}(f_1, Q)$	$\frac{m_1^{f_1}}{n_1}, (e_1^1 - 1)r_1^1$...	$\frac{m_{\theta-1}^{f_1}}{n_1}, (e_{\theta-1}^1 - 1)r_{\theta-1}^1$
$c(f_2, P), \text{int}(f_2, Q)$	$\frac{m_1^{f_1}}{n_1}, (e_1^1 - 1)r_1^2$...	$\frac{m_{\theta-1}^{f_1}}{n_1}, (e_{\theta-1}^1 - 1)r_{\theta-1}^2$

$$\frac{m_i^1}{n_1}$$

With the notations of Theorem 7.7., for all $1 \leq i \leq \theta - 1$, we have: $P_1^{n_1} = \{f_1, f_2\}, a = 1, b = 0$.

1) $M \neq \frac{m_\theta^i}{n(f_i)}, i = 1, 2$.

Q	$Q_{f_y}(M, 1)$	$Q_{f_y}(\frac{m_k^{f_1}}{n_1}, *), \theta \leq k \leq h_{f_1}$	$Q_{f_y}(\frac{m_k^{f_2}}{n_2}, *), \theta \leq k \leq h_{f_2}$
$\deg_y Q$	$\frac{n_1}{d_\theta^{f_1}} = \frac{n_2}{d_\theta^{f_2}}$	$(e_k^{f_1} - 1) \frac{n_1}{d_k^{f_1}}$	$(e_k^{f_2} - 1) \frac{n_2}{d_k^{f_2}}$
$c(f_1, P), \text{int}(f_1, Q)$	$M, \frac{I}{d_\theta^{f_1}} \frac{n_1}{n_2}$	$\frac{m_k^{f_1}}{n_1}, (e_k^{f_1} - 1)r_k^{f_1}$	$M, (e_k^{f_2} - 1) \frac{I}{d_k^{f_2}}$
$c(f_2, P), \text{int}(f_2, Q)$	$M, \frac{I}{d_\theta^{f_2}} \frac{n_2}{n_1}$	$M, (e_k^{f_1} - 1) \frac{I}{d_k^{f_1}}$	$\frac{m_k^{f_2}}{n_2}, (e_k^{f_2} - 1)r_k^{f_2}$



With the notations of Theorem 7.7., we have:

$$P_1^M = \{f_1, f_2\}, A = \{f_1\}, B = \{f_2\}, a = b = 1$$

$$P_1^{\frac{m_k^{f_1}}{n_1}} = \{f_1\}, \theta \leq k \leq h_{f_1} : a = 1, b = 0, P_1^{\frac{m_k^{f_2}}{n_2}} = \{f_2\}, \theta \leq k \leq h(f_2) : a = 1, b = 0$$

$$2) M = \frac{m_\theta^{f_1}}{n_1} < \frac{m_\theta^{f_2}}{n_2}.$$

Q	$Q_{f_y}(M, 1)$	$Q_{f_y}(\frac{m_k^{f_1}}{n_1}, *), \theta + 1 \leq k \leq h_{f_1}$	$Q_{f_y}(\frac{m_k^{f_2}}{n_2}, *), \theta \leq k \leq h_{f_2}$
$\deg_y Q$	$\frac{n_1}{d_{\theta+1}^{f_1}} = \frac{n_2}{d_\theta^{f_2}} + (e_\theta^{f_1} - 1) \frac{n_1}{d_\theta^{f_1}}$	$(e_k^{f_1} - 1) \frac{n_1}{d_k^{f_1}}$	$(e_k^2 - 1) \frac{n_2}{d_k^{f_2}}$
$c(f_1, P), \text{int}(f_1, P_i)$	$M, e_\theta^{f_1} r_\theta^{f_1}$	$\frac{m_k^{f_1}}{n_1}, (e_k^{f_1} - 1) r_k^{f_1}$	$M, (e_k^2 - 1) \frac{I}{d_k^{f_2}}$
$c(f_2, P), \text{int}(f_2, P_i)$	$M, \frac{I}{d_{\theta+1}^{f_1}} = e_\theta^{f_1} r_\theta^{f_1} \frac{n_2}{n_1}$	$M, (e_k^{f_1} - 1) \frac{I}{d_k^{f_1}}$	$\frac{m_k^{f_2}}{n_2}, (e_k^{f_2} - 1) r_k^{f_2}$

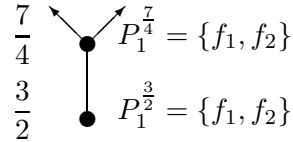
With the notations of Theorem 7.7., we have $P_1^M = \{f_1, f_2\}, P_1^{\frac{m_k^{f_1}}{n_1}} = \{f_1\}$ for all $\theta + 1 \leq k \leq h_{f_1}$, and $P_1^{\frac{m_k^{f_2}}{n_2}} = \{f_2\}$ for all $\theta \leq k \leq h_{f_2}$.

$$3) M = \frac{m_\theta^{f_1}}{n_1} = \frac{m_\theta^{f_2}}{n_2}.$$

Q	$Q_{f_y}(M, 1)$	$Q_{f_y}(\frac{m_k^{f_1}}{n_1}, *), \theta + 1 \leq k \leq h_{f_1}$	$Q_{f_y}(\frac{m_k^{f_2}}{n_2}, *), \theta + 1 \leq k \leq h_{f_2}$
$\deg_y Q$	$\frac{n_1}{d_{\theta+1}^{f_1}} + (e_\theta^{f_1} - 1) \frac{n_1}{d_\theta^{f_1}}$	$(e_k^{f_1} - 1) \frac{n_1}{d_k^{f_1}}$	$(e_k^{f_2} - 1) \frac{n_2}{d_k^{f_2}}$
$c(f_1, P), \text{int}(f_1, Q)$	$M, (2e_\theta^{f_1} - 1) r_\theta^{f_1}$	$\frac{m_k^{f_1}}{n_1}, (e_k^{f_1} - 1) r_k^{f_1}$	$M, (e_k^{f_2} - 1) \frac{I}{d_k^{f_2}}$
$c(f_2, P), \text{int}(f_2, Q)$	$M, (2e_\theta^{f_2} - 1) r_\theta^{f_2}$	$M, (e_k^{f_1} - 1) \frac{I}{d_k^{f_1}}$	$\frac{m_k^{f_2}}{n_2}, (e_k^{f_2} - 1) r_k^{f_2}$

With the notations of Theorem 7.7., we have $P_1^M = \{f_1, f_2\}, P_1^{\frac{m_k^{f_1}}{n_1}} = \{f_1\}$ for all $\theta + 1 \leq k \leq h_{f_1}$, and $P_1^{\frac{m_k^{f_2}}{n_2}} = \{f_2\}$ for all $\theta + 1 \leq k \leq h_{f_2}$.

Example 7.12 i) $f = f_1 \cdot f_2$ and $f_1 = (y^2 - x^3)^2 - x^5 y, f_2 = (y^2 - x^3)^2 + x^5 y$. We have $n_{f_1} = n_{f_2} = n = 4, \underline{r}^{f_1} = \underline{r}^{f_2} = \underline{r} = (4, 6, 13), \underline{d}^{f_1} = \underline{d}^{f_2} = \underline{d} = (4, 2, 1), \underline{m}^{f_1} = \underline{m}^{f_2} = \underline{m} = (4, 6, 7)$, and $c(f_1, f_2) = \frac{7}{4}$. The tree model of f is given by:



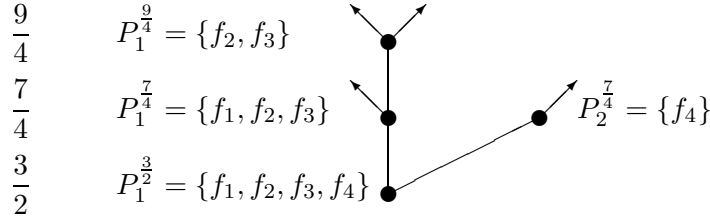
Note that $X(\frac{3}{2}, 1) = \{f_1\}, X(\frac{3}{2}, 2) = \{f_2\}$, and $X(\frac{7}{4}, 1) = P_1^{\frac{7}{4}} = \{f_1, f_2\}$. In particular $c(\frac{3}{2}, 1) = 1$ and $c(\frac{7}{4}, 1) = 2$. With the notations of Theorem 7.7., $f_y = Q(\frac{3}{2}, 1)Q(\frac{7}{4}, 1) = Q_1 Q_2$, where:

$$\deg_y Q_1 = \frac{n}{d_2} - \frac{n}{d_1} = 1 \quad (a = 1, b = 0)$$

$$\deg_y Q_2 = \frac{n}{d_3} - \frac{n}{d_2} + n = 2.4 - 2 = 6 \quad (a = 1, b = 0).$$

Furthermore, for all irreducible component P of Q_1 (resp. Q_2), $c(f_1, P) = c(f_2, P) = \frac{3}{2}$ (resp. $c(f_1, P) = c(f_2, P) = \frac{7}{4}$). Finally, $\text{int}(f_1, Q_1) = (e_1 - 1)r_1 = r_1 = 6 = \text{int}(f_2, Q_1)$ and $\text{int}(f_1, Q_2) = \text{int}(f_1, f_2) + (e_2 - 1)r_2 = 39 = \text{int}(f_2, Q_2)$.

ii) $f = f_1.f_2.f_3.f_4$ and $f_1 = (y^2 - x^3)^2 - x^5y$, $f_2 = (y^2 - x^3)^2 + x^5y$, $f_3 = (y^2 - x^3)^2 + x^5y - x^7$, and $f_4 = (y^2 + x^3)^2 - x^5y$: $\Gamma(f_i) = \langle 4, 6, 13 \rangle = \langle n, r_1, r_2 \rangle$, $i = 1, 2, 3, 4$, $m_1 = 6$, $m_2 = 7$, and $c(f_1, f_2) = c(f_1, f_3) = \frac{7}{4}$, $c(f_1, f_4) = \frac{3}{2}$, $c(f_2, f_3) = \frac{9}{4}$, $c(f_2, f_4) = \frac{3}{2}$, $c(f_3, f_4) = \frac{3}{2}$. The tree model of f is given by:

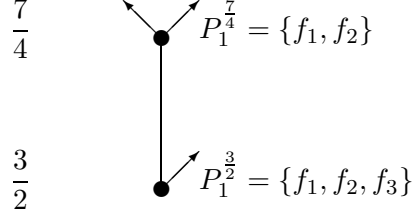


Note that $X_i(\frac{3}{2}, 1) = \{f_i\}$, $i = 1, \dots, 4$, $X_1(\frac{7}{4}, 1) = \{f_1, f_2\}$, $X_2(\frac{7}{4}, 1) = \{f_1, f_3\}$, $D_1^{\frac{7}{4}} = \{f_1\}$, $X_2(\frac{7}{4}, 2) = P_2^{\frac{7}{4}}$ and $X_1(\frac{9}{4}, 1) = \{f_2, f_3\}$. In particular, $c(\frac{3}{2}, 1) = 1$, $c(\frac{7}{4}, 1) = 2 = c(\frac{9}{4}, 1)$, $c(\frac{7}{4}, 2) = 1$, Theorem 7.7. implies that $f_y = Q(\frac{3}{2}, 1)Q(\frac{7}{4}, 1)Q(\frac{7}{4}, 2)Q(\frac{9}{4}, 1) = Q_1Q_2Q_3Q_4$ with the following properties:

$Q_i, \deg_y Q_i$	$Q_1, 3$	$Q_2, 6$	$Q_3, 2$	$Q_4, 4$
$c(f_1, P), \text{int}(f_1, Q_i)$	$\frac{3}{2}, 18$	$\frac{7}{4}, 39$	$\frac{3}{2}, 12$	$\frac{7}{4}, 26$
$c(f_2, P), \text{int}(f_2, Q_i)$	$\frac{3}{2}, 18$	$\frac{7}{4}, 39$	$\frac{3}{2}, 12$	$\frac{9}{4}, 28$
$c(f_3, P), \text{int}(f_3, Q_i)$	$\frac{3}{2}, 18$	$\frac{7}{4}, 39$	$\frac{3}{2}, 12$	$\frac{9}{4}, 28$
$c(f_4, P), \text{int}(f_4, Q_i)$	$\frac{3}{2}, 18$	$\frac{3}{2}, 36$	$\frac{7}{4}, 13$	$\frac{3}{2}, 24$

Where $c(F, P)$ means the contact of F with an irreducible component P of Q_i .

iii) Let $f = f_1.f_2.f_3$, where $f_1 = (y^2 - x^3)^2 - x^5y$, $f_2 = y^2 - x^3$ and $f_3 = y^2 + x^3$. We have $c(f_1, f_2) = \frac{7}{4}$, $c(f_1, f_3) = \frac{3}{2} = c(f_2, f_3)$, $\text{int}(f_1, f_2) = 13$, $\text{int}(f_1, f_3) = 12$ and $\text{int}(f_2, f_3) = 6$. The tree model of f is given by:



With the notations of Theorem 7.7., we have:

$$X(\frac{3}{2}, 1) = \{f_1, f_2\}, X(\frac{3}{2}, 2) = \{f_2, f_3\}, D_1^{\frac{3}{2}} = \{f_3\}, c(\frac{3}{2}, 1) = 2.$$

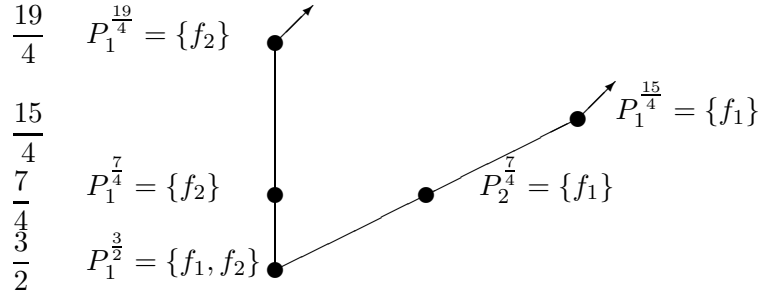
$$X(\frac{7}{4}, 1) = \{f_1, f_2\}, c(\frac{7}{4}, 1) = 2.$$

This gives us the following description:

$Q, \deg_y Q$	$Q_{f_y}(\frac{3}{2}, 1), 5$	$Q_{f_y}(\frac{7}{4}, 1), 2$
$c(f_1, P), \text{int}(f_1, Q)$	$\frac{7}{4}, 26$	$\frac{3}{2}, 18$
$c(f_2, P), \text{int}(f_2, Q)$	$\frac{7}{4}, 13$	$\frac{3}{2}, 9$
$c(f_3, P), \text{int}(f_3, Q)$	$\frac{3}{2}, 12$	$\frac{3}{2}, 9$

Where $c(F, P)$ means the contact of F with an irreducible component P of Q_i .

iv) $f = f_1 \cdot f_2$, where $f_1 = ((y^2 - x^3)^2 - x^5 y)^2 + x^{10}(y^2 - x^3)$ and $f_2 = ((y^2 + x^3)^2 - x^5 y)^2 + x^{22}(y^2 + x^3)$. We have $\Gamma(f_1) = \langle 8, 12, 26, 53 \rangle$, $\Gamma(f_2) = \langle 8, 12, 26, 57 \rangle$, $M = c(f_1, f_2) = \frac{3}{2}$ and $I = \text{int}(f_1, f_2) = 96$. The tree model of f is given by:



With the notations of Theorem 7.7., we have:

$$X(\frac{3}{2}, 1) = \{f_1\}, X(\frac{3}{2}, 2) = \{f_2\}, c(\frac{3}{2}, 1) = 1.$$

$$X(\frac{7}{4}, 1) = \{f_2\}, c(\frac{7}{4}, 1) = 1, X(\frac{7}{4}, 2) = \{f_1\}, c(\frac{7}{4}, 2) = 1$$

$$X(\frac{15}{4}, 1) = \{f_1\}, c(\frac{15}{4}, 1) = 1, X(\frac{19}{4}, 1) = \{f_2\}, c(\frac{19}{4}, 1) = 1$$

This gives us the following description:

$Q, \deg_y Q$	$Q_{f_y}(\frac{3}{2}, 1), 3$	$Q_{f_y}(\frac{7}{4}, 1), 2$	$Q_{f_y}(\frac{7}{4}, 2), 2$	$Q_{f_y}(\frac{15}{4}, 1), 4$	$Q_{f_y}(\frac{19}{4}, 1), 4$
$c(f_1, P), \text{int}(f_1, Q)$	$\frac{3}{2}, 36$	$\frac{3}{2}, 24$	$\frac{7}{4}, 26$	$\frac{15}{4}, 53$	$\frac{3}{2}, 48$
$c(f_2, P), \text{int}(f_2, Q)$	$\frac{3}{2}, 36$	$\frac{7}{4}, 26$	$\frac{3}{2}, 24$	$\frac{3}{2}, 48$	$\frac{19}{4}, 57$

Where $c(F, P)$ means the contact of F with an irreducible component P of Q .

8 Factorization of the Jacobian

Let $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ and $g = y^m + b_1(x)y^{m-1} + \dots + b_m(x)$ be two monic reduced polynomials of $\mathbf{K}((x))[y]$ and consider the Jacobian $J = J(f, g)$ of f and g . The aim of this Section is to give a factorization theorem of J in terms of the tree of $f.g$. Let to this end $T(f.g)$ be the tree of $f.g$ and let $f_1, \dots, f_{\xi(f)}$ (resp. $g_1, \dots, g_{\xi(g)}$) be the irreducible components of f (resp. g) in $\mathbf{K}((x))[y]$.

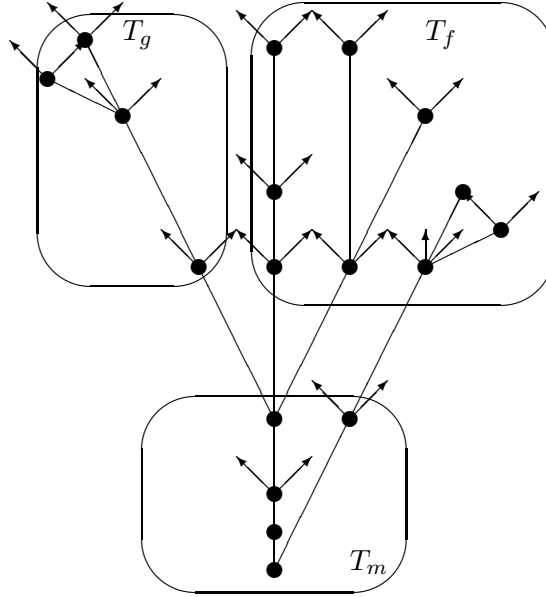
Definition 8.1 Let $P_i^M \in T(f.g)$.

i) We say that P_i^M is an f -point if for all $1 \leq k \leq \xi(g)$, $g_k \notin P_i^M$ (equivalently P_i^M is an f -point if for all $F \in P_i^M$ and for all $1 \leq k \leq \xi(g)$, $c(g_k, F) < M$).

ii) We say that P_i^M is a g -point if for all $1 \leq k \leq \xi(f)$, $f_k \notin P_i^M$ (equivalently P_i^M is a g -point if for all $F \in P_i^M$ and for all $1 \leq k \leq \xi(f)$, $c(f_k, F) < M$).

iii) We say that the point P_i^M is a mixed point if it is neither an f -point nor a g -point.

We denote by T_f (resp. T_g , resp. T_m) the set of f -points (resp. g -points, resp. mixed points) of $T(f.g)$. Clearly $T(f.g) = T_f \cup T_g \cup T_m$.



Lemma 8.2 Let $P_i^M, P_j^N \in T(f.g)$, and assume that $P_j^N \geq P_i^M$.

- i) If $P_i^M \in T_f$ (resp. $P_i^M \in T_g$) then $P_j^N \in T_f$ (resp. $P_j^N \in T_g$).
- ii) if $P_j^N \in T_m$, then $P_i^M \in T_m$.

Proof. Easy exercise. ■

Lemma 8.3 Let the notations be as above. If $T_f \neq \emptyset$ (resp. $T_g \neq \emptyset$), then $\text{Root}(J) \neq \emptyset$.

Proof. Assume that $T_f \neq \emptyset$, and let $P = P_i^M \in T_f$, and let $F \in P$. Let $y_i(x), y_j(x) \in \text{Root}(f)$ such that $c(y_i, y_j) = M$. By hypothesis, $M > \max_{1 \leq k \leq c(y_i, z_k)}$. Now use Lemma 4.6. ■

More generally, assume that $T_f \cup T_g \neq \emptyset$, Propositions 5.4. and 5.5. and similar arguments as in Section 8. led to the following factorization theorem of J .

Theorem 8.4 $J = \bar{J} \cdot \prod_{P_i^M \in T_f} Q_J(M, i) \cdot \prod_{P_i^M \in T_g} Q_J(M, i)$, and $Q_J(M, i), P_i^M \in T_f \cup T_g$ has the following properties:

(*) Assume that $P = P_i^M \in T_f$ and let $(P_i^{N_l})_{1 \leq l \leq t}$ be the set of points that strictly dominate P_i^M . Let θ be the smallest integer such that $M \leq \frac{m_\theta^F}{n(F)}$ for all $F \in P_i^M$. Let A (resp. B) be the set of $1 \leq l \leq t$ for which $M = \frac{m_\theta^F}{n_F}$ (resp. $M < \frac{m_\theta^F}{n_F}$) for all $F \in \cup_{l \in A} P_i^{N_l}$ (resp. $F \in \cup_{l \in B} P_i^{N_l}$) and let $(l_1, l_2) \in A \times B$. If a (resp. b) denotes the cardinality of A (resp. B) then the following hold:

- i) $\deg_y Q_J(M, i) = a \cdot \frac{n_{F_{l_1}}}{d_{\theta+1}^{F_{l_1}}} + (b-1) \cdot \frac{n_{F_1}}{d_\theta^{F_1}} + \sum_{F \in D_i^M} n_F$, and $\sum_{F \in D_i^M} n_F$ is given by the formula of

Lemma 5.6., where if $F \in D_i^M$, then h_F is either $\theta - 1$ or θ depending on $M > \frac{m_{h_F}^F}{n_F}$ or $M = \frac{m_{h_F}^F}{n_F}$.

- ii) For all irreducible component P of $Q_J(M, i)$ and for all $F \in P_i^M$, $c(F, P) = M$.

iii) For all irreducible component P of $Q_J(M, i)$ and for all $F \notin P_i^M$ (this holds in particular when $F = g_k, 1 \leq k \leq \xi(g)$), $c(F, P) = c(F, P_i^M) < M$.

iv) For all $1 \leq k \leq \xi(f)$:

- If $f_k \in P_i^M$ then $\text{int}(f_k, Q_J(M, i)) = S(m(f_k), M) \deg_Q J(M, i) n_{f_k}$.
- If $f_k \notin P_i^M$ then $\text{int}(f_k, Q_J(M, i)) = S(m(f_k), c(f_k, P_i^M)) \deg_Q J(M, i) n_{f_k}$.

(**) Assume that $P = P_i^M \in T_g$ and let $(P_i^{N_l})_{1 \leq l \leq t}$ be the set of points that strictly dominate P_i^M .

Let θ be the smallest integer such that $M \leq \frac{m_\theta^F}{n(F)}$ for all $F \in P_i^M$. Let A (resp. B) be the set of $1 \leq l \leq t$ for which $M = \frac{m_\theta^F}{n_F}$ (resp. $M < \frac{m_\theta^F}{n_F}$) for all $F \in \cup_{l \in A} P_i^{N_l}$ (resp. $F \in \cup_{l \in B} P_i^{N_l}$) and let $(l_1, l_2) \in A \times B$. If a (resp. b) denotes the cardinality of A (resp. B) then the following hold:

i) $\deg_y Q_J(M, i) = a \cdot \frac{n_{F_1}}{d_{\theta+1}^{F_1}} + (b-1) \cdot \frac{n_{F_1}}{d_{\theta}^{F_1}} + \sum_{F \in D_i^M} n_F$, and $\sum_{F \in D_i^M} n_F$ is given by the formula of Lemma 5.6., where if $F \in D_i^M$, then h_F is either $\theta - 1$ or θ depending on $M > \frac{m_{h_F}^F}{n_F}$ or $M = \frac{m_{h_F}^F}{n_F}$.

ii) For all irreducible component P of $Q_J(M, i)$ and for all $F \in P_i^M$, $c(F, P) = M$.

iii) For all irreducible component P of $Q_J(M, i)$ and for all $F \notin P_i^M$ (this holds in particular when $F = f_k, 1 \leq k \leq \xi(f)$), $c(F, P) = c(F, P_i^M) < M$.

iv) For all $1 \leq k \leq \xi(g)$:

- If $g_k \in P_i^M$ then $\text{int}(g_k, Q_J(M, i)) = S(m(g_k), M) \deg Q_J(M, i) n_{g_k}$.

- If $g_k \notin P_i^M$ then $\text{int}(g_k, Q_J(M, i)) = S(m(f_k), c(g_k, P_i^M)) \deg Q_J(M, i) n_{g_k}$.

Proof. The proof is similar to the proof of Theorem 7.7. ■

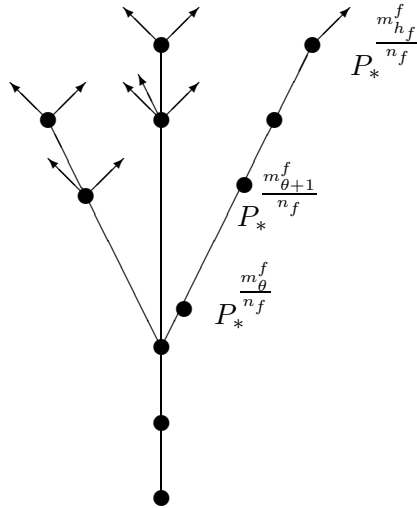
Corollary 8.5 Assume that $\xi(f) = 1$, i.e. $f = f_1$ is an irreducible polynomial of $\mathbf{K}((x))[y]$, and let $M = \max_{k=1}^{\xi(g)} c(f, g_k)$. Let θ be the smallest integer such that $M < \frac{m_{\theta}^f}{n_f}$. If $\theta < h_f$, then $J = J(f, g) = \bar{J} \cdot \prod_{k=\theta}^{h_f} J_k$, where for all $\theta \leq k \leq h_f$,

i) $\deg_y J_k = (e_k^f - 1) \frac{n_f}{d_k^f}$.

ii) $\text{int}(f, J_k) = (e_k^f - 1) r_k^f$.

iii) For all irreducible component P of J_k , $c(f, P) = \frac{m_k^f}{n_f}$.

iv) For all $1 \leq j \leq \xi(g)$ and for all irreducible component P of J_k , $c(g_j, P) = c(g_j, f)$



Proof. In fact, $T_f = \{P_1^{\frac{m_{\theta}^f}{n_f}}, \dots, P_1^{\frac{m_{h_f}^f}{n_f}}\}$. The result is consequently a particular case of Theorem 8.4. ■

9 Bad and good points on the tree of f

Let $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ be a monic reduced polynomial of $\mathbf{K}((x^{-1}))[y]$, and let $f = f_1 \dots f_{\xi(f)}$ be the factorisation of f into irreducible components in $\mathbf{K}((x))[y]$. We shall assume that f is generic in the following sense: for all irreducible component H of f_y , $\text{int}(f, H) \leq 0$.

Definition 9.1 Let F, G be two monic polynomials of $\mathbf{K}((x))[y]$, and let H be an irreducible monic polynomial of $\mathbf{K}((x))[y]$. We say that H is regular (resp. irregular) with respect to F if $\text{int}(F, H) \neq 0$ (resp. $\text{int}(F, H) = 0$). We define $\text{Reg}(G, F)$ (resp. $\text{Irreg}(G, F)$) to be the set of regular (resp. irregular) components of G with respect to F . Let $\gamma(x) \in \mathbf{K}((x^{\frac{1}{p}}))$, $p \in \mathbb{N}$. We say that γ is regular (resp. irregular) with respect to F if $O_x F(x, \gamma(x)) \neq 0$ (resp. $O_x F(x, \gamma(x)) = 0$). If $G = F_y$, then we write $\text{Reg}(F)$ (resp. $\text{Irreg}(F)$) for $\text{Reg}(F_y, F)$ (resp. $\text{Irreg}(F_y, F)$).

Lemma 9.2 We have $\text{Irreg}(f, f_y) = \emptyset$.

Proof. Let $1 \leq j \leq \xi(f)$ and let $y(x) \in \text{Root}(f_j)$. Finally let $M = \max_{k \neq j} c(f_j, f_k)$. By Lemma 4.4., $M = \max c(f_j, H)$, where H runs over the set of irreducible components of f_y . Since f is generic, then $\sum_{y \neq \bar{y} \in \text{Root}(f)} O_x(y - \bar{y}) + M \leq 0$. If $M < 0$, then $O_x(y - \bar{y}) \leq M < 0$ for all $\bar{y} \in \text{Root}(f)$, $\bar{y} \neq y$, in particular $\sum_{y \neq \bar{y} \in \text{Root}(f)} O_x(y - \bar{y}) < 0$. If $M > 0$, then $\sum_{y \neq \bar{y} \in \text{Root}(f)} O_x(y - \bar{y}) \leq -M < 0$. Finally $O_x f_y(x, y(x)) = \sum_{y \neq \bar{y} \in \text{Root}(f)} O_x(y - \bar{y}) < 0$, in particular $\text{int}(f_j, f_y) < 0$. This proves our assertion. ■

Definition 9.3 Let F, G be two monic polynomials of $\mathbf{K}((x))[y]$, and let H be an irreducible component of G . Assume that $H \in \text{Irreg}(G, F)$ and let $\gamma \in \text{Root}(H)$. We have $F(x, \gamma(x)) = \lambda + u(x)$ where $\lambda \in \mathbf{K}^*$ and $u(0) = 0$. In particular, $\text{int}(F - \lambda, H) > 0$, hence $H \in \text{Reg}(G, F - \lambda)$. We say that λ is an irregular value of F with respect to G . We define $\text{irreg}(F, G)$ to be the set of irregular values of F with respect to G . If $G = F_y$, then we write $\text{reg}(F)$ (resp. $\text{irreg}(F)$) for $\text{reg}(F_y, F)$ (resp. $\text{irreg}(F_y, F)$).

Definition 9.4 Let P_i^M be a point of $\text{Top}(f)$.

- i) We say that P_i^M is a good point if $H \in \text{Reg}(f)$ for some irreducible component of $Q_{f_y}(M, i)$.
- ii) We say that P_i^M is a bad point if $H \in \text{Irreg}(f)$ for some irreducible component of $Q_{f_y}(M, i)$.

Lemma 9.5 Let P_i^M be a point of $\text{Top}(f)$.

- i) If P_i^M is a good point, then for all irreducible component H of $Q_{f_y}(M, i)$, $H \in \text{Reg}(f)$.
- ii) If P_i^M is a bad point, then for all irreducible component H of $Q_{f_y}(M, i)$, $H \in \text{Irreg}(f)$.

Proof. i) By hypothesis, there is an irreducible component \bar{H} of $Q_{f_y}(M, i)$ such that $\text{int}(f, \bar{H}) < 0$. Let H be an irreducible component of $Q_{f_y}(M, i)$, and let $\gamma(x)$ (resp. $\bar{\gamma}(x)$) be a root of H (resp. \bar{H}) such that $\max_{i=1}^n c(\gamma, y_i) = M = \max_{i=1}^n c(\bar{\gamma}, y_i)$. We have:

$$O_x f(x, \gamma(x)) = \sum_{i=1}^n c(\gamma(x), y_i(x)) = \sum_{i=1}^n c(\bar{\gamma}(x), y_i(x)) = O_x f(x, \bar{\gamma}(x))$$

in particular $\text{int}(f, H) = \frac{1}{n_H} O_x f(x, \gamma(x)) < 0$.

- ii) The proof is similar to the proof of i). ■

10 The number of irregular values of a meromorphic curve

Let the notations be as in Section 9, and let $P_i^M = \{F_1, \dots, F_r\}$ be a bad point of $\text{Top}(f)$. For all irreducible component H of $Q_{f_y}(M, i)$, $\text{int}(f, H) = 0$, in particular, if $\gamma(x) \in \text{Root}(H)$, then $f(x, \gamma(x)) = \lambda + u(x)$, where $\lambda \in \mathbf{K}^*$ and $u(x)$ is a fractional power series such that $u(0) = 0$. In particular, $\lambda \in \text{irreg}(f)$. Let $\{\lambda_1(M, i), \dots, \lambda_{p(M, i)}(M, i)\}$ be the set of irregular values of f obtained from the components of $Q_{f_y}(M, i)$ as above -more precisely $\{\lambda_1(M, i), \dots, \lambda_{p(M, i)}(M, i)\} = \{\text{inco}(f(x, \gamma(x))) | \gamma(x) \in \text{Root}(Q_{f_y}(M, i))\}$. We have the following:

Proposition 10.1 Assume that $n_{F_1} = \max_{1 \leq i \leq r} n_{F_i}$.

- i) If (F_1, \dots, F_r) is equivalent and $M > \frac{m_{h_{F_1}}^{F_1}}{n_{F_1}}$, then $p(M, i) \leq r - 1$.
- ii) If (F_1, \dots, F_r) is equivalent and $M = \frac{m_{h_{F_1}}^{F_1}}{n_{F_1}}$, then $p(M, i) \leq r$.
- iii) If (F_1, \dots, F_r) is almost equivalent, then $p(M, i) \leq r - 1$.

Proof. i) Let H be an irreducible component of $Q_{f_y}(M, i)$. Since $c(H, F_1) = M > \frac{m_{h_{F_1}}^{F_1}}{n_{F_1}}$, then n_{F_1} divides n_H . On the other hand, by Theorem 7.9., $\deg_y Q_{f_y}(M, i) = (r - 1)n_{F_1}$, In particular, $\xi(Q_{f_y}(M, i)) \leq r - 1$. This proves our assertion.

ii) Let H be an irreducible component of $Q_{f_y}(M, i)$. Since $c(H, F_1) = \frac{m_{h_{F_1}}^{F_1}}{n_{F_1}}$, then $\frac{n_{F_1}}{d_{h_{F_1}}^{F_1}}$ divides n_H . More precisely, let $\gamma(x) = \sum_p c_p x^{\frac{p}{n_H}} \in \text{Root}(H)$, then one of the following holds:

- The coefficient of x^M in $\gamma(x)$ is nonzero, hence n_{F_1} divides n_H . In this case, we say that H is of type I.
- The coefficient of x^M in $\gamma(x)$ is zero, hence $\frac{n_{F_1}}{d_{h_{F_1}}^{F_1}}$ divides n_H . In this case, we say that H is of type II.

Let H_1, H_2 be two irreducible components of type II of $Q_{f_y}(M, i)$. If $\gamma_1(x) \in \text{Root}(H_1)$ (resp. $\gamma_2(x) \in \text{Root}(H_2)$), then $c(y_i, \gamma_1) = c(y_i, \gamma_2)$, and $\text{inco}(y_i - \gamma_1) = \text{inco}(y_i - \gamma_2)$ for all $y_i \in \text{Root}(f)$. In particular, H_1 and H_2 give rise to the same irregular value of f . On the other hand, by Theorem 7.9., $\deg_y Q_{f_y}(M, i) = (r - 1)n_{F_1} + (e_{h_{F_1}}^{F_1} - 1) \frac{n_{F_1}}{d_{h_{F_1}}^{F_1}}$, hence the number of irreducible components of $Q_{f_y}(M, i)$ of type I is bounded by $r - 1$. This proves our assertion.

- iii) The proof is similar to the proof of ii). ■

Corollary 10.2 Let f be as above. The number of irregular values of f is bounded by $\xi(f)$.

Proof. This results from Proposition 10.1. ■

Suppose that $f \in \mathbf{K}[x^{-1}][y]$, and let $f_1, \dots, f_{\xi(f)}$ be the set of irreducible components of f in $\mathbf{K}((x))[y]$. If for all irreducible components of f_y , $\text{int}(f, H) = 0$, then $\text{int}(f - \lambda, f_y) > 0$ for some $\lambda \in \text{irreg}(f)$. This is a contradiction, since $f_y \in \mathbf{K}[x^{-1}][y]$. In particular, the tree $T(f)$ of f contains at least one good point. This leads to the following:

Proposition 10.3 Let the notations be as above. If $f \in \mathbf{K}[x^{-1}][y]$, then the number of irregular values of f is bounded by $\xi(f) - 1$.

Proof. A good point of $T(f)$ contains at least one irreducible component of f . In particular at most $\xi(f) - 1$ irreducible components of f belong to the set of bad points of f . Now use Proposition 10.1. ■

Remark 10.4 Let the notations be as above and assume that $f \in \mathbf{K}[x^{-1}][y]$. If $\text{irreg}(f)$ has exactly $\xi(f) - 1$ elements, then for all $P_i^M \notin \text{Top}(f)$, $D_i^M = \emptyset$. More precisely, it follows from the proof of Proposition 10.1. that the cardinality of $\text{irreg}(f)$ is bounded by

$$\sum_{P_i^M \in \text{Top}(f)} \text{card}(P_i^M)$$

In particular, if $\text{card}(\text{irreg}(f)) = \xi(f) - 1$ then the following holds:

- i) f has exactly one good point in $\text{Top}(f)$ with exactly one element, say $f_{\xi(f)}$.
- ii) For all $1 \leq i \leq \xi(f) - 1$, $f_i \in P_i^M$ for some bad point $P_i^M \in \text{Top}(f)$.

Furthermore, given a bad point $P_i^M = \{F_1, \dots, F_r\}$, (F_1, \dots, F_r) is equivalent, and $M = \frac{M_{h_{F_1}}^{F_1}}{n_{F_1}}$.

Remark 10.5 The polynomial $f = y^4 + x^{-1}y^2 + y + 1$ has two irreducible components. On the other hand, $f_y = 4y^3 + x^{-1} + 1$, and $\text{int}(f, f_y) = -3$, $\text{int}(f - 1, f_y) = -2$. The set of irregular values of f is reduced to one element. In particular the bound of Proposition 8.3. is sharp.

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