

MODELING EXTREME VALUES OF PROCESSES OBSERVED AT IRREGULAR TIME STEPS: APPLICATION TO SIGNIFICANT WAVE HEIGHT. SUPPLEMENTARY MATERIAL

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1. Consistency of MPL_1 estimator. In this supplementary material, we prove the following theorem.

THEOREM 1.1. *Let $\{Z_t\}$ be a Gaussian extreme value process with unit Fréchet margins, as defined in the paper, and dependence parameter $\nu^* \in \Theta = [\nu_-, \nu_+]$ with $0 < \nu_- < \nu_+$. Assume that the process is observed at regular time $\{1 \dots n\}$, and denote by*

$$L_n(\nu) = \prod_{i=1}^{n-1} \frac{\partial^2 F_Z}{\partial z_1 \partial z_2}(Z_i, Z_{i+1}; \nu),$$

the pairwise likelihood function, with F_z defined by

$$F_Z(z_{t_1}, z_{t_2}; \nu) = \mathbb{P}(Z_{t_1} \leq z_{t_1}, Z_{t_2} \leq z_{t_2}) = \exp[-V(z_{t_1}, z_{t_2}; \nu)],$$

and

$$(1) \quad V(z_{t_1}, z_{t_2}; \nu) = \frac{1}{z_{t_1}} \Phi\left(\frac{a}{2} + \frac{1}{a} \log \frac{z_{t_2}}{z_{t_1}}\right) + \frac{1}{z_{t_2}} \Phi\left(\frac{a}{2} + \frac{1}{a} \log \frac{z_{t_1}}{z_{t_2}}\right).$$

We define $\hat{\nu}_n = \operatorname{argmax} L_n(\nu)$ as the corresponding estimator of ν^ . Then, $\hat{\nu}_n$ is a strongly consistent estimator of ν^* , i.e.,*

$$\lim_{n \rightarrow \infty} \hat{\nu}_n = \nu^* \text{ a.s.}$$

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A Gaussian extremal process $\{Z_t\}$ is a moving maxima process as defined in [Stoev \(2008\)](#). Using the results given in that paper, we deduce that $\{Z_t\}$ is a stationary unit Fréchet process, continuous in probability, mixing, and hence ergodic. It allows us to use the following theorem, which is a straightforward generalization of Theorem 1.12 in [Pfanzagl \(1969\)](#):

THEOREM 1.2. *Let $\{Z_i\}_{i=1,\dots,n}$ be a stationary and ergodic process, whose distribution depends on a parameter $\nu^* \in \Theta$, where Θ is a compact subset of \mathbb{R} , and let Q_n be a contrast function defined as*

$$Q_n(\nu) = \frac{1}{n} \sum_{i=1}^{n-1} f(Z_i, Z_{i+1}; \nu),$$

where f is a measurable function with real values and continuous in ν . Suppose that:

- (a) $\mathbb{E} \inf_{\nu \in \Theta} f(Z_1, Z_2; \nu) > -\infty$;
- (b) $\nu \mapsto \mathbb{E} f(Z_1, Z_2; \nu)$ has a unique finite minimum at ν^* .

Then, the minimum contrast estimator $\hat{\nu}_n = \operatorname{argmin}_{\nu \in \Theta} Q_n(\nu)$ is strongly consistent:

$$\lim_{n \rightarrow \infty} \hat{\nu}_n = \nu^* \text{ a.s.}$$

We use this theorem with

$$f(Z_1, Z_2; \nu) = -\log p(Z_1, Z_2; \nu),$$

where $p(z_1, z_2; \nu)$ denotes the joint pdf of (Z_1, Z_2) , $\Theta = (\nu_-, \nu_+)$ with $0 < \nu_- < \nu_+$. An explicit expression for f is given in Section 1.1. Properties (1) and (2) of Theorem 1.2 are verified in Sections 1.2 and 1.3, respectively.

1.1. *Expression of the contrast.* We have

$$p(Z_1, Z_2; \nu) = \frac{\partial^2}{\partial z_1 \partial z_2} F_Z(z_1, z_2; \nu),$$

with $F_Z(z_1, z_2; \nu) = \exp((-V(z_1, z_2; \nu)))$ and V defined in (1) and thus

$$f(Z_1, Z_2; \nu) = V(z_1, z_2; \nu) - \log \left(\frac{\partial V}{\partial z_1}(z_1, z_2) \frac{\partial V}{\partial z_2}(z_1, z_2) + \frac{\partial^2 V}{\partial z_1 \partial z_2}(z_1, z_2) \right).$$

The function V and its derivatives satisfy

$$\begin{aligned}
V(z_1, z_2; \nu) &= \frac{\Phi(a/2 + 1/a \log \frac{z_2}{z_1})}{z_1} + \frac{\Phi(a/2 + 1/a \log \frac{z_1}{z_2})}{z_2} = \frac{\Phi(w)}{z_1} + \frac{\Phi(v)}{z_2}, \\
\frac{\partial V}{\partial z_1}(z_1, z_2; \nu) &= -\frac{\Phi(w)}{z_1^2} - \frac{\varphi(w)}{az_1^2} + \frac{\varphi(v)}{az_1 z_2}, \\
\frac{\partial V}{\partial z_2}(z_1, z_2; \nu) &= -\frac{\Phi(v)}{z_2^2} - \frac{\varphi(v)}{az_2^2} + \frac{\varphi(w)}{az_1 z_2}, \\
\frac{\partial^2 V}{\partial z_1 \partial z_2}(z_1, z_2; \nu) &= -\frac{v\varphi(w)}{a^2 z_1^2 z_2} - \frac{w\varphi(v)}{a^2 z_1 z_2^2},
\end{aligned}$$

with

$$\begin{aligned}
w &= \frac{a}{2} + \frac{1}{a} \log \frac{z_2}{z_1}, \\
v &= a - w = \frac{a}{2} + \frac{1}{a} \log \frac{z_1}{z_2}, \\
\varphi(w) &= \frac{1}{\sqrt{2\pi}} e^{-a^2/8} e^{-\frac{\log^2(z_1/z_2)}{2a^2}} \sqrt{\frac{z_1}{z_2}}, \\
\varphi(v) &= \frac{1}{\sqrt{2\pi}} e^{-a^2/8} e^{-\frac{\log^2(z_1/z_2)}{2a^2}} \sqrt{\frac{z_2}{z_1}}.
\end{aligned}$$

Taking into account that $\frac{\varphi(w)}{az_1^2} - \frac{\varphi(v)}{az_1 z_2} = 0$, the following simplified expressions for the derivatives of V are obtained:

$$\begin{aligned}
\frac{\partial V}{\partial z_1}(z_1, z_2) &= -\frac{\Phi(w)}{z_1^2}, \\
\frac{\partial V}{\partial z_2}(z_1, z_2) &= -\frac{\Phi(v)}{z_2^2}, \\
\frac{\partial^2 V}{\partial z_1 \partial z_2}(z_1, z_2) &= -\frac{\varphi(w)}{az_1^2 z_2} = -\frac{\varphi(v)}{az_2^2 z_1} = \frac{e^{-a^2/8} \exp\left(\frac{\log^2 \frac{z_2}{z_1}}{2a^2}\right)}{(z_1 z_2)^{3/2}}.
\end{aligned}$$

Finally, we deduce that

$$(2) \quad f(z_1, z_2; \nu) = V(z_1, z_2; \nu) - \log \left[\frac{\Phi(w)\Phi(v)}{a^2 z_1 z_2} + \frac{\varphi(w)}{az_1^2 z_2} \right].$$

1.2. *Lower bound (a).* We have

$$\inf_{\nu} \{f(z_1, z_2; \nu)\} \geq \inf_{\nu} \{V(z_1, z_2; \nu)\} + \inf_{\nu} \left\{ -\log \left[\frac{\Phi(w)\Phi(v)}{a^2 z_1 z_2} + \frac{\varphi(w)}{az_1^2 z_2} \right] \right\},$$

and each term on the right-hand side of this expression is treated separately below.

- **Term** $V(z_1, z_2; \nu)$.

V can be bounded using the Fréchet bound ([Fréchet \(1951\)](#)) ,

$$P(Z_1 \leq z_1) + P(Z_2 \leq z_2) - 1 \leq \mathbb{P}(Z_1 \leq z_1, Z_2 \leq z_2) \leq \min \{P(Z_1 \leq z_1), P(Z_2 \leq z_2)\},$$

which implies that

$$\min \left(-\frac{1}{z_1}, -\frac{1}{z_2} \right) \leq V(z_1, z_2; \nu) \leq \exp \left(-\frac{1}{z_1} \right) + \exp \left(\frac{1}{z_2} \right),$$

and thus

$$\inf_{\nu} \{V(z_1, z_2; \nu)\} \geq \min \left(-\frac{1}{z_1}, -\frac{1}{z_2} \right).$$

The right-hand term of the last expression has a finite expectation since $\frac{1}{Z_1}$ and $\frac{1}{Z_2}$ have unit exponential distributions.

- **Term** $-\log \left[\frac{\Phi(w)\Phi(v)}{a^2 z_1 z_2} + \frac{\varphi(w)}{a z_1^2 z_2} \right]$.

We have

$$\begin{aligned} -\log \left[\frac{\Phi(w)\Phi(v)}{z_1 z_2} \nu^2 + \frac{\varphi(w)}{z_1^2 z_2} \nu \right] &\geq 1 - \frac{\Phi(w)\Phi(v)}{z_1 z_2} \nu^2 + \frac{\varphi(w)}{z_1^2 z_2} \nu \\ &\geq 1 - \frac{\nu^2}{z_1 z_2} \\ &\geq 1 - \frac{\nu_+^2}{z_1 z_2}, \end{aligned}$$

and Cauchy's inequality implies that the right-hand term of the last expression has a finite expectation since

$$\mathbb{E} \left[\frac{1}{Z_1 Z_2} \right] \leq \sqrt{\mathbb{E}[1/Z_1^2] \mathbb{E}[1/Z_2^2]} = 1.$$

1.3. Identifiability (b). We must now prove that $\nu \mapsto \mathbb{E}f(Z_1, Z_2; \nu)$ has a unique finite minimum at ν^* on $\Theta = [\nu_- \nu_+]$. To this end we denote by P_ν the distribution with density function $p(z_1, z_2; \nu)$. Then,

$$\mathbb{E}_{\nu^*} \left[-\log \frac{p(Z_1, Z_2; \nu)}{p(Z_1, Z_2; \nu^*)} \right] = K(P_{\nu^*}, P_\nu),$$

is the Kullback-Leibler divergence between P_{ν^*} and P_ν . We have $K \geq 0$ and $K = 0$ iff $P_\nu = P_{\nu^*}$. As the density functions are positive and continuous in (z_1, z_2) , this is equivalent

to $p(z_1, z_2; \nu^*) = p(z_1, z_2; \nu)$ for all (z_1, z_2) . In particular, $K = 0$ implies that for all $z_1 = z_2 = z > 0$, we have

$$\exp \left[-\frac{2}{z} \Phi \left(\frac{1}{2\nu} \right) \right] \left[\frac{\Phi \left(\frac{1}{2\nu} \right)^2}{z^2} \nu^2 + \frac{\varphi \left(\frac{1}{2\nu} \right)}{z^3} \nu \right] = \exp \left[-\frac{2}{z} \Phi \left(\frac{1}{2\nu^*} \right) \right] \left[\frac{\Phi \left(\frac{1}{2\nu^*} \right)^2}{z^2} \nu^{*2} + \frac{\varphi \left(\frac{1}{2\nu^*} \right)}{z^3} \nu^* \right].$$

Letting $z \rightarrow 0$ while $z > 0$ we see that the exponents in the exponential function must be equal, i.e.,

$$\Phi \left(\frac{1}{\nu} \right) = \Phi \left(\frac{1}{\nu^*} \right).$$

Hence, $\nu = \nu^*$. The proof is complete.

References.

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