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OPERADS & GROTHENDIECK-TEICHMÜLLER GROUPS
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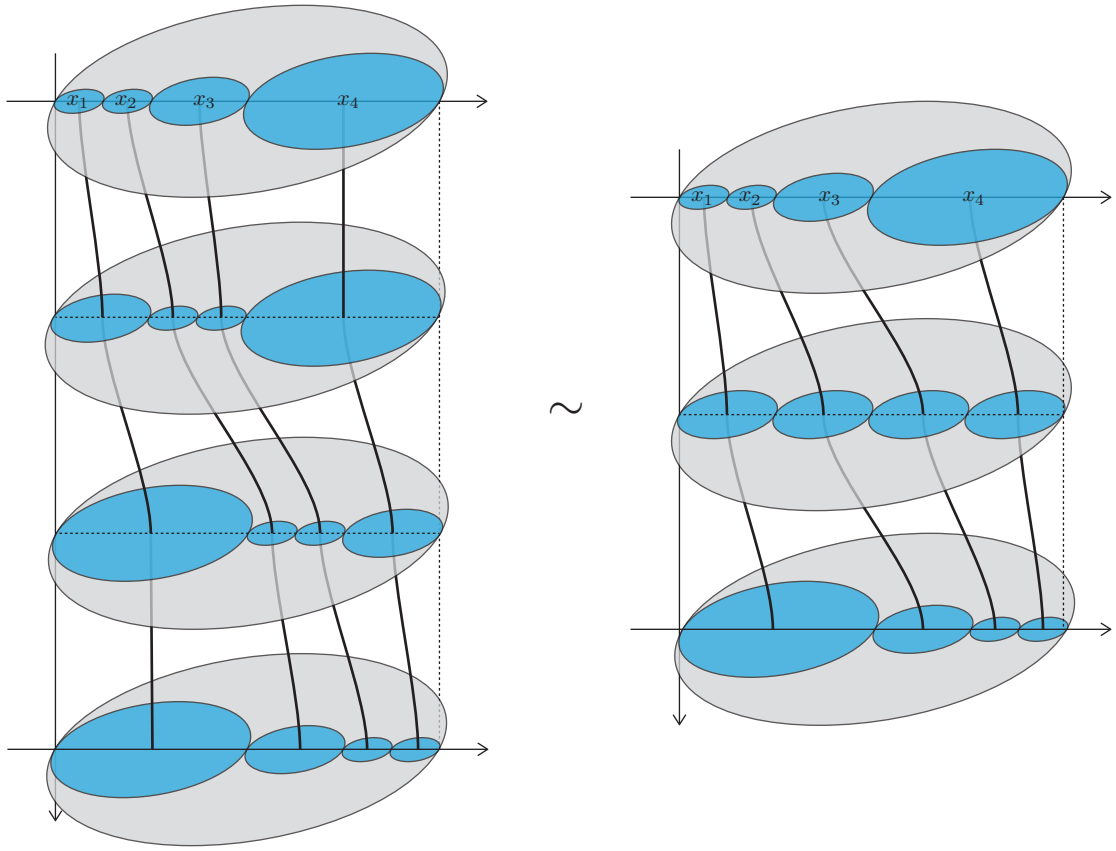
DRAFT DOCUMENT

by

Benoit Fresse



Preprint, 4 January 2012.

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OPERADS & GROTHENDIECK-TEICHMÜLLER GROUPS
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DRAFT DOCUMENT

BENOIT FRESSE

Abstract

This preprint is an extract from a research monograph in preparation on the homotopy of operads and Grothendieck-Teichmüller groups. The ultimate objective of this book is to prove that the Grothendieck-Teichmüller group is the group of homotopy automorphisms of a rational completion of the little 2-discs operad.

The present excerpts include a comprehensive account of the fundamental concepts of operad theory, a survey chapter on little discs operads as well as a detailed account on the connections between little 2-discs, braid groups, and Grothendieck-Teichmüller groups, until the formulation of the main result of the monograph. Most concepts are carefully reviewed in order to make this account accessible to a broad readership, which should include graduate students as well as researchers coming from the various fields of mathematics related to our main topics. This preprint will serve as reference material for a master degree course "Operads 2012", given by the author at université Lille 1, from January until April 2012. See:

`http://math.univ-lille1.fr/~operads/2012courses.html#Lille`

This working draft will not be updated, and the given excerpts should significantly differ from the final version of the monograph in preparation. Nevertheless, a copy with annotated corrections will be made available on the above web-page.

This work has mostly been written during stays at the École Normale Supérieure de Paris, at Northwestern University, and at the Max-Planck-Institut für Mathematik in Bonn. The author is grateful to these institutions for outstanding working conditions, and to numerous colleagues for their warm welcome which has greatly eased this writing task.

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Foreword

Overall Introduction

The first aim of this book is to give an overall reference, starting from scratch, on the application of fine algebraic topology methods to operads. Most definitions, notably fundamental concepts of operad and homotopy theory, are carefully reviewed in order to make our account accessible to a broad readership, including graduate students, as well as researchers coming from the various fields of mathematics related to our core subject.

Ultimately, our objective is to explain, from a homotopical viewpoint, a deep relationship between operads and Grothendieck-Teichmüller groups. This connection, which has arisen from researches on the deformation quantization process in mathematical physics, gives a new approach to understand internal symmetries of structures occurring in various constructions of algebra and topology.

The definition of an operad is reviewed in the first part of the book. For the moment, simply recall that an operad is a structure, formed by collections of abstract operations, which is used to define a category of algebras. In our study, we mainly consider the example of E_n-operads, $n = 1, 2, \ldots, \infty$, used to model a hierarchy of homotopy commutative structures, starting with E_1, fully homotopy associative but not commutative, and ending with E_∞, fully homotopy associative and commutative. The intermediate E_n-operads represent structures, which are more and more homotopy commutative when n increases, but not fully homotopy commutative until $n = \infty$. For the reader, we should mention that the notion of an E_1-operad is synonymous to that of an A_∞-operad, used in the literature when one only deals with purely homotopy associative structures.

The notion of E_n-operad formally refers to a class of operads, rather than to a singled out object. This class consists, in the initial definition, of topological operads which are homotopically equivalent to a reference model, the Boardman-Vogt operad of little n-discs D_n. The operad of little n-cubes, which is a simple variant of the little n-discs operad, is also used in the literature to provide an equivalent definition of the class of E_n-operads. The second part of the book is devoted to detailed recollections on these notions. Nonetheless, as we explain soon, the ultimate objective of the book is not to study E_n-operads themselves, but homotopy automorphisms groups attached to these structures.

Before explaining this goal, we survey some motivating applications of E_n-operads, which are not our main matter (we only give short introductions to these topics in the book), but illustrate our approach of the subject.

The operads of little n-discs D_n were initially introduced to collect operations acting on iterated loop spaces. The first main application, which has motivated the definition of these operads, was the Boardman-Vogt and May recognition theorems asserting, in the most basic outcome, that any connected space equipped with an

action of D_n is homotopy equivalent to an n-fold loop space $\Omega^n X$ (see [16, 17] and [72]).

Recall that the set of connected components of an n-fold loop space $\Omega^n X$ is identified with the nth homotopy group $\pi_n(X)$ of the space X, which is abelian for $n > 1$. The action of D_n on $\Omega^n X$ includes, for any $n > 0$, a product operation $\mu : \Omega^n X \times \Omega^n X \to \Omega^n X$ which, at the level of connected components, gives the composition operation of the group $\pi_n(X)$. The operad D_n carries the homotopy making this product associative, as well as commutative for $n > 1$, and includes further operations, representing fine homotopy constraints, which we need to form a faithful picture of the structure of the n-fold loop space $\Omega^n X$.

Since the initial topological definition, new applications of E_n-operads have been discovered in the fields of algebra and mathematical physics, mostly after the proof of the Deligne conjecture asserting that the Hochschild cochain complex $C^*(A, A)$ of an associative algebra A inherits an action of an E_2-operad. In this context, we use a chain version of the previously considered topological little 2-discs operad D_2.

The cohomology of the Hochschild cochain complex $C^*(A, A)$ is identified in degree 0 with the center $Z(A)$ of the associative algebra A. In a sense, the Hochschild cochain complex represents a derived version of this ordinary center $Z(A)$. From this point of view, the construction of an E_2-structure on $C^*(A, A)$ determines, again, a fine level of homotopical commutativity of the derived center, beyond an apparent commutativity at the cohomology level. The first proofs of the Deligne conjecture have been given by Kontsevich-Soibelman [58] and McClure-Smith [73]. The interpretation in terms of derived centers has been advertised by Kontsevich [56] in order to formulate a natural extension of the conjecture in the context of algebras associated to E_n-operads, for any $n \geq 1$.

The verification of the Deligne conjecture has yielded a second generation of proofs, promoted by Tamarkin [89] and Kontsevich [56], of the Kontsevich formality theorem giving the existence of deformation quantizations. The new approach of this problem also involves Drinfeld's theory of associators, which are used to transport the E_2-structure yielded by the Deligne conjecture on the Hochschild cochain complex to the cohomology. In the final outcome, one obtains that each associator gives rise to a deformation quantization functor. This result has hinted the existence of a deep connection between the deformation quantization problem and the program, initiated in Grothendieck's famous "esquisse" [49], aiming to understand Galois groups through geometric actions on curves. The Grothendieck-Teichmüller groups are devices, introduced in this program, encoding the information which can be captured through the actions considered by Grothendieck. The correspondence between associators and deformation quantizations imply that a rational prounipotent version of the Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$ acts on the moduli space of deformation quantizations. The initial motivation for our work was the desire to understand this connection from a homotopical viewpoint, in terms of homotopical structures associated to E_2-operads. The homotopy automorphisms of operads come in at this point.

Recall again that an operad is a structure encoding a category of algebras. The homotopy automorphisms of an operad P are transformations, defined at the operad level, encoding natural homotopy equivalences on the category of algebras associated to P. In this interpretation, the group of homotopy automorphism classes

of E_2-operads, which we actually aim to determine, represents the internal symmetries of the first level of homotopy commutative structures that E_2-operads encode. In the rational setting, we establish that this group is isomorphic to the prounipotent Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$. This result is new and represents the main outcome of our work. In a more general context, we formulate a conjecture relating the group of homotopy automorphism classes of E_2-operads to a Lie algebra, defined over $\mathbb{Z}$, underlying a graded version of the Grothendieck-Teichmüller group.

Let us focus on the rational case. In this context, we naturally have to consider a rational version of E_2-operads. Thus, to reach our result, we have beforehand to set up a new rational homotopy theory for topological operads and to give a sense to the rationalization of topological operads. We actually define an analogue of Sullivan's model of the rational homotopy of spaces [88] for operads. We simply use cosimplicial commutative algebras instead of Sullivan's differential graded algebras in order to bypass fundamental difficulties arising from the Eilenberg-Zilber equivalence. We also consider cooperads, the dual structures of operads, when we form our model. We precisely show that the rational homotopy of an operad in topological spaces is determined by an associated cooperad in cosimplicial commutative algebras (a cosimplicial commutative Hopf cooperad), and we give a small model of this cooperad, involving the so-called Drinfeld-Kohno Lie algebras, in the case of little 2-discs D_2.

The other main topics considered in our study include the application of Koszul duality techniques, operadic deformation complexes and spectral sequences for the computation of mapping spaces attached to operads. We explain these constructions in details, in a general setting, and from scratch in order to make the methods accessible to a broad readership, as promised at the beginning of this introduction.

In short, this book aims to provide a complete proof of new results together with a transversal account of the background of current researches in operad theory. This plan makes our work complementary of existing references on operads. From a broad angle, our overall aim, as we mention at the beginning of this introduction, is to examine the application of fine algebraic topology methods in the operad context. No existing reference goes as far as we need into this subject, on the aspects we aim to address. By filling this gap in the mathematical literature, we hope to give a solid basis for new researches on applications of operads, and, as a follow-up, for new effective and fruitful interplays between the various fields of mathematics which we mention in this introduction.

Mathematical Objectives

The ultimate goal of this book, as we explained in the overall introduction, is to prove that the Grothendieck-Teichmüller group represents, in the rational setting at least, the group of homotopy automorphism classes of E_2-operads.

The definition of an operad is recalled with full details in the first part of the book. In this introductory section, we only aim to give a rough idea of our main result. Let us simply recall that an operad P in a base category $\mathcal{B}ase$ consists of a collection of objects $\mathsf{P}(r) \in \mathcal{B}ase$, $r \in \mathbb{N}$, which, intuitively, parameterizes operations with r inputs, together with a multiplicative structure, which models the composition of such operations. Together with this notion, we define an operad morphism $f : \mathsf{P} \to \mathsf{Q}$ as a collection of morphisms in the base category $f(r) : \mathsf{P}(r) \to \mathsf{Q}(r)$ preserving the internal structures of the operads. The category of operads in a given base category $\mathcal{B}ase$ is denoted by $\mathcal{B}ase\mathcal{O}p$. For short, we may also use the notation $\mathcal{O}p = \mathcal{B}ase\mathcal{O}p$, when the precision of the base category $\mathcal{B}ase$ can be omitted.

For technical reasons, we have to consider operads P_+ equipped with a distinguished element $* \in \mathsf{P}_+(0)$ (whenever this makes sense), which represents an operation with 0 input (a unitary operation in our terminology). In the set-theoretic context, we moreover assume that $\mathsf{P}_+(0)$ is a one-point set reduced to this element. In the module context, we assume that $\mathsf{P}_+(0)$ is a one dimensional module over the ground ring. In a general setting, we assume that $\mathsf{P}_+(0)$ is the unit object of a tensor structure associated with the base category $\mathcal{B}ase$. We coin the expression of unitary operad to refer to this pointed situation. We also use the notation $\mathcal{O}p_*$, with a lower-script indicating the operad first term added, to refer to the category defined by unitary operads, where we restrict ourselves to morphisms preserving the distinguished element $*$. We usually consider together both a unitary operad P_+ and an associated non-unitary operad P, where the term $\mathsf{P}_+(0)$, spanned by the distinguished element $*$, is removed. We follow the convention to use a lower-script $+$, marking the addition of such a term, in the notation of the unitary operad P_+. We generally perform our constructions on the non-unitary operad P first, and we extend the result to the unitary operad P_+ afterwards, by using the preservation of the additional distinguished element (or unit term) of P_+. We use the expression of unitary extension to refer to this process.

In topology, an E_2-operad usually refers to an operad in the category of spaces which is equivalent to Boardman-Vogt' operad of little 2-discs D_2 in the homotopy category of operads. The spaces $\mathsf{D}_2(r)$ underlying this operad have a trivial homotopy $\pi_* \mathsf{D}_2(r) = 0$ in dimension $* \neq 1$, and we have $\pi_1 \mathsf{D}_2(r) = P_r$ in dimension $* = 1$, where P_r denotes the pure braid group on r strands. Thus, the space $\mathsf{D}_2(r)$ is an Eilenberg-Mac Lane space $K(P_r, 1)$ associated to the pure braid group P_r.

For our purpose, we consider a rational pronilpotent completion of the little 2-discs operad $\widehat{\mathsf{D}}_2$. For this operad, we have $\pi_1\widehat{\mathsf{D}}_2(r) = \widehat{P}_r$, where $\widehat{P}_r$ refers to the Malcev completion of the group P_r, so that each space $\widehat{\mathsf{D}}_2(r)$ is identified with an Eilenberg-Mac Lane space $K(\widehat{P}_r, 1)$. The precise construction of this operad $\widehat{\mathsf{D}}_2$ is addressed, in a general context, in the second part of the book. In applications, we also use a simple model of $\widehat{\mathsf{D}}_2$, obtained by elaborating on the Eilenberg-MacLane space interpretation, and which we explain soon.

Homotopy automorphisms can be defined in the general setting of model categories, which provides a suitable axiomatic framework to apply constructions of homotopy theory in the operad context. To introduce our subject, we explain a basic interpretation of the general definition of a homotopy automorphism in the context of topological operads.

We consider a natural homotopy relation $\simeq$ attached to morphisms of operads in topological spaces: first, to a topological operad Q, we associate the collection of path spaces $\mathsf{Q}^{\Delta^1}(r) = \mathtt{Map}_{\mathcal{T}op}([0,1], \mathsf{Q}(r))$, which inherits an operad structure from Q and defines a path-object associated to Q in the category of topological operads; then we formally define a homotopy between operad morphisms $f, g : \mathsf{P} \to \mathsf{Q}$ as an operad morphism $h : \mathsf{P} \to \mathsf{Q}^{\Delta^1}$ satisfying $d_0 h = f$, $d_1 h = g$, where $d_0, d_1 : \mathsf{Q}^{\Delta^1} \to \mathsf{Q}$ are the natural structure morphisms (evaluation on origin and end points) associated with the path-object Q^{Δ^1}. Intuitively, the homotopy h amounts to giving a continuous family of operad morphisms $h_t : \mathsf{P} \to \mathsf{Q}$ going from $h_0 = f$ to $h_1 = g$.

In a first approximation, we can take the sets of homotopy classes of operad morphisms to form the morphism sets of a homotopy category $\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)$ associated with the category of topological operads $\mathcal{T}op\mathcal{O}p$. The groups of homotopy automorphism classes, which we aim to determine, are precisely the automorphism groups of operads in this homotopy category $\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)$. For a given operad P, the so-defined automorphism group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P})$ explicitly consists of homotopy classes of morphisms $f : \mathsf{P} \to \mathsf{P}$, admitting a homotopy inverse $g : \mathsf{P} \to \mathsf{P}$, so that $fg \simeq id$ and $gf \simeq id$, where we consider, at each level, the operadic homotopy relation $\simeq$.

Now, a topological operad P gives rise to an operad object in the homotopy category of topological spaces $\mathtt{Ho}(\mathcal{T}op)$, and we could also study the automorphism group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op)\mathcal{O}p}(\mathsf{P})$ formed in this category of homotopical operads. But these naive automorphism groups differ from our groups of homotopy automorphisms and do not give the appropriate structure for the homotopy version of the usual constructions of group theory (like homotopy fixed points). Indeed, an automorphism of the operad P in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$ is just a collection of homotopy classes of maps $f(r) \in [\mathsf{P}(r), \mathsf{P}(r)]$, invertible in the homotopy category of spaces, and preserving operadic structures up to homotopy, unlike our homotopy automorphisms which preserve operadic structures strictly. Moreover, actual operad morphisms $f, g : \mathsf{P} \to \mathsf{Q}$ define the same morphism of operads in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$ as soon as we have a homotopy of maps (regardless of operad structures) between $f(r)$ and $g(r)$, for each $r \in \mathbb{N}$. Thus, operad morphisms $f, g : \mathsf{P} \to \mathsf{Q}$ which are homotopic in the strong operadic sense determine the same morphism of operads in the homotopy category of spaces $\mathtt{Ho}(\mathcal{T}op)$ but the converse implication does not hold.

Finally, by associating the collection of homotopy classes of maps $f(r) : \mathsf{P}(r) \to \mathsf{P}(r)$ to any homotopy automorphism $f \in \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P})$, we obtain a mapping $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P}) \to \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op)\mathcal{O}p}(\mathsf{P})$, from the group of homotopy classes of homotopy automorphisms towards the group of automorphisms of the operad in the homotopy category of spaces, but this mapping is neither an injection nor a surjection in general.

To apply homotopy theory methods, we associate to any operad P a whole simplicial set of homotopy automorphisms $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$ so that the group of homotopy automorphism classes $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P})$, which we primarily aim to determine, is identified with the set of connected components of this space $\pi_0(\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P}))$. In the second chapter of the book, we explain the definition of these homotopy automorphism spaces in the general context of simplicial model categories. For the moment, we simply give a short summary of the definition for topological operads.

First, we extend the definition of the operadic homotopy relation, and we consider, for each $n \in \mathbb{N}$, an operad P^{Δ^n} defined by the collection of function spaces $\mathsf{P}^{\Delta^n}(r) = \mathtt{Map}_{\mathcal{T}op}(\Delta^n, \mathsf{P}(r))$ on the n-simplex Δ^n. This operad sequence P^{Δ^n} inherits a simplicial structure from the topological simplices Δ^n. In particular, since we obviously have $\mathsf{P} = \mathsf{P}^{\Delta^0}$, we have a morphism $v^* : \mathsf{P}^{\Delta^n} \to \mathsf{P}$ associated to each vertex v of the n-simplex Δ^n. The simplicial set $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$ is given in dimension n by the morphisms of topological operads $f : \mathsf{P} \to \mathsf{P}^{\Delta^n}$ so that the composites v^*f form homotopy equivalences of the operad $\mathsf{P} = \mathsf{P}^{\Delta^0}$, for all vertices $v \in \Delta^n$. From this definition, we immediately see that the 0-simplices of the simplicial set $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$ are the homotopy equivalences of the operad P, the 1-simplices are the operadic homotopies $h : \mathsf{P} \to \mathsf{P}^{\Delta^1}$ between homotopy equivalences, and therefore, we have a formal identity $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P}) = \pi_0\, \mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$, between our group of homotopy automorphism classes $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P})$ and the set of connected components of $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$.

In what follows, we adopt a common usage of homotopy theory to call space any simplicial set regarded as a combinatorial model of a topological space. So, we use the terminology of homotopy automorphism space for the simplicial set $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$ which we associate to an operad P.

The category of operads in topological spaces, like many categories of operads, has a natural model category structure. The notion of a model category includes the definition of a class of cofibrant objects, generalizing the cell complexes of topology, and which are well suited for the homotopy constructions we aim to address. To be more specific, recall that a map of topological spaces $f : X \to Y$ is a weak-equivalence when this map induces a bijection on connected components $f_* : \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ together with an isomorphism on homotopy groups $f_* : \pi_*(X) \xrightarrow{\sim} \pi_*(Y)$, in every dimension $* > 0$, and for any choice of base point. We define a weak-equivalence of operads as an operad morphism $f : \mathsf{P} \to \mathsf{Q}$ of which underlying maps $f(r) : \mathsf{P}(r) \to \mathsf{Q}(r)$ are weak-equivalences of topological spaces. In what follows, we use the standard notation of model categories $\xrightarrow{\sim}$ to mark the weak-equivalences of any ambient model category. In the context of topological spaces, a classical result asserts that any weak-equivalence between cell complexes is homotopically invertible as a map of topological spaces. In the context of operads, we similarly obtain that any weak-equivalence between cofibrant operads $f : \mathsf{P} \xrightarrow{\sim} \mathsf{Q}$ is homotopically invertible as an operad morphism: we have an operad

morphism in the converse direction $g : \mathsf{Q} \to \mathsf{P}$ so that $fg \simeq id$ and $gf \simeq id$, where we consider the operadic homotopy relation again (as in the definition of homotopy automorphisms).

The proof of the model category axioms for operads includes the construction of a cofibrant replacement functor, which assigns any given operad P to a cofibrant operad Q equipped with a weak-equivalence $\mathsf{Q} \xrightarrow{\sim} \mathsf{P}$. The definition of the homotopy category of operads in terms of homotopy class sets of morphisms is actually the right one when we replace each operad P by such a cofibrant model $\mathsf{Q} \xrightarrow{\sim} \mathsf{P}$. In particular, when we form the group of homotopy automorphism classes of an operad $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P})$, we have to assume that P is cofibrant as an operad, otherwise we tacitely consider that we apply our construction to a cofibrant replacement of P. The general theory of model categories ensures that the obtained group $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathsf{P})$ does not depend, up to isomorphism, on the choice of this cofibrant replacement. We have similar results and we apply similar conventions for the homotopy automorphism spaces $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\mathsf{P})$.

We go back to the little 2-cubes operad. We aim to determine the homotopy groups of the homotopy automorphism space $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+})$ associated to the rational completion of D_2, and in the unitary context, which we mark by the addition of the lower-script $+$ in our notation. Recall that the connected components of this space $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+})$ correspond to homotopy classes of operad homotopy equivalences $f : \widehat{\mathsf{Q}}_{2+} \xrightarrow{\sim} \widehat{\mathsf{Q}}_{2+}$, where $\widehat{\mathsf{Q}}_2$ denotes a cofibrant model of the rationalized little 2-discs operad $\widehat{\mathsf{D}}_2$. In our study, we just focus on the subspace $\mathtt{hAut}^1_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+})$ formed by the connected components of $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+})$ corresponding to morphisms f which induce the identity on homology groups. The whole group $\pi_0\,\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+})$ is actually a semi-direct product of $\pi_0\,\mathtt{hAut}^1_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+})$ with a copy of the multiplicative group $\mathbb{Q}^{\times}$. Our result reads:

THEOREM A. *The automorphism space of the rational pro-nilpotent completion of the little* 2*-discs operad* $\widehat{\mathsf{D}}_{2+}$ *satisfies*

$$\pi_*\,\mathtt{hAut}^1_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+}) = \begin{cases} GT^1(\mathbb{Q}), & \textit{for } * = 0, \\ 0, & \textit{otherwise}, \end{cases}$$

where $GT^1(\mathbb{Q})$ *denotes the rational pro-unipotent version of the Grothendieck-Teichmüller group, defined by V. Drinfeld in [28].*

The identity established in this theorem is a new result. The main goal of this book precisely consists in proving this statement.

The superscript in the notation $GT^1(\mathbb{Q})$ refers, as in the expression of the homotopy automorphism space $\mathtt{hAut}^1_{\mathcal{T}op\mathcal{O}p}(\widehat{\mathsf{D}}_{2+})$, to a version of Drinfeld's prounipotent Grothendieck-Teichmüller group where a scalar factor $\mathbb{Q}^{\times}$ is removed (see [28] for details).

At the beginning of this survey, we explained that the operad of little 2-discs D_2 consists of Eilenberg-Mac Lane spaces $K(P_r, 1)$, where P_r denotes the pure braid group on r strands, and the associated rationalized operad $\widehat{\mathsf{D}}_2$ consists of Eilenberg-Mac Lane spaces $K(\widehat{P}_r, 1)$, where we now consider the Malcev completion of P_r. We have a standard model of the Eilenberg-Mac Lane spaces $K(P_r, 1)$, given by the

classifying spaces of the groups P_r. But these spaces do not form an operad. Nevertheless, we can adapt this classifying space approach to give a simple model of E_2-operad. Instead of the pure braid group P_r, we consider the classifying space of a groupoid of parenthesized braids $PaB(r)$ with, as morphisms, braids on r strands preserving a given coloring on input and output points. The parenthesization refers to an extra structure, added to the input and output sets of braids, which represent the source and target objects of morphisms in our groupoid. Unlike the pure braid groups P_r, the collection $PaB(r)$ forms an operad in the category of groupoids, and the associated collection of classifying spaces $\mathtt{B}(PaB)(r) = \mathtt{B}(PaB(r))$ forms an operad in topological spaces. We check, following an argument of Z. Fiedorowicz, that this operad $\mathtt{B}(PaB)$ is a model of E_2-operad.

For the rationalized operad of little 2-discs $\widehat{\mathsf{D}}_2$, we also have a simple classifying space model $\mathtt{B}(\widehat{PaB})$ obtained by applying a Malcev completion to the groupoids $PaB(r)$. The operad structure associated with this model $\mathtt{B}(\widehat{PaB})$ is still defined at the groupoid level since the collection of completed groupoids obviously inherits an operad composition structure from the uncompleted ones.

The Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$ can actually be identified with an automorphism group associated with (a unitary extension of) this operad in groupoids $\widehat{PaB}$. Consequently, any element $\phi \in GT^1(\mathbb{Q})$ induces an operad automorphism on classifying spaces $\phi_* : \mathtt{B}(\widehat{PaB})_+ \xrightarrow{\simeq} \mathtt{B}(\widehat{PaB})_+$. This automorphism lifts to a homotopy automorphism on any chosen cofibrant model of E_2-operad, so that we have a well-defined homotopy automorphisms of E_2-operad associated to each $\phi \in GT^1(\mathbb{Q})$. Our main theorem precisely asserts that, in the rational setting, this mapping gives exactly all homotopy automorphism classes of E_2-operads.

Most of the book is devoted to the setting up of general theories from which we establish this result.

In short, we gain our result at the level of a category of cosimplicial Hopf cooperads $c\,\mathcal{H}opf\,\mathcal{O}p^c_*$, which we introduce as a suitable analogue of Sullivan's model of rational homotopy for operads. The theorem obtained in this context is also worth recording in view towards algebraic applications of E_2-operads.

The superscript c in the notation of this category $c\,\mathcal{H}opf\,\mathcal{O}p^c_*$ refers to cooperads. The subscript $*$ refers to an adaptation of the definition of unitary structures in the cooperad context. The prefix c marks cosimplicial structures. Simply say, for the moment, that cooperads are structures dual to operads. Basically, a cooperad D consists of a collection of objects of the base category $\mathsf{D}(r) \in \mathcal{B}ase$ together with a comultiplicative structure dual to the multiplicative structure of an operad. The cosimplicial Hopf cooperads D, which we consider in our study, are cooperads in the category of cosimplicial unitary commutative algebras over $\mathbb{Q}$, with an underlying collection $\mathsf{D}(r)$ consisting of objects of this category $\mathcal{B}ase = c\,\mathcal{C}om_+$.

In the usual Sullivan's model for the rational homotopy of topological spaces, we deal with differential graded commutative algebras rather than cosimplicial commutative algebras. In the operadic context, we delay the application of differential graded structures in order to sort out difficulties arising from the Eilenberg-Zilber equivalence.

To an operad in topological spaces P, we can associate the collection of singular complexes $\mathtt{Sing}_\bullet(\mathsf{P}(r))$ of the topological spaces underlying P, which forms an operad in cosimplicial cocommutative coalgebras $\mathtt{Sing}_\bullet(\mathsf{P})$. To define our model

for the rational completion of P, we take $\Bbbk = \mathbb{Q}$ as coefficient ring for the singular complexes, and we form a dual construction assigning a cosimplicial Hopf cooperad $\mathtt{Sing}^{\bullet}(\mathsf{P})$ to P. The mapping $\mathtt{Sing}^{\bullet}(-) : \mathsf{P} \mapsto \mathtt{Sing}^{\bullet}(\mathsf{P})$ defines a contravariant functor $\mathtt{Sing}^{\bullet}(-) : \mathcal{T}op\mathcal{O}p_*^{op} \to c\,\mathcal{H}opf\mathcal{O}p_*^c$, on the category $\mathcal{T}op\mathcal{O}p_*$ formed by unitary operads in topological spaces. We define a functor in the converse direction $\mathbb{G}(-) : c\,\mathcal{H}opf\mathcal{O}p_*^c \to \mathcal{T}op\mathcal{O}p_*^{op}$, and we prove that, under mild finiteness assumptions, the image of the cosimplicial Hopf cooperad $\mathtt{Sing}^{\bullet}(\mathsf{P})$ under a left derived functor of $\mathbb{G}(-)$ returns a topological operad $\widehat{\mathsf{P}} = L\,\mathbb{G}(\mathtt{Sing}^{\bullet}(\mathsf{P}))$ connected to P by a morphism (in the homotopy category of operads) inducing the rationalization on homotopy groups. Thus, the composite construction $\widehat{\mathsf{P}} = L\,\mathbb{G}(\mathtt{Sing}^{\bullet}(\mathsf{P}))$ gives a functorial model for the rationalization process in the category of topological operads.

From this result, we essentially retain that the rational completion of a topological operad is naturally built on its Hopf cooperad counterpart, and this gives our actual reason to address rational homotopy problems about operads in the Hopf cooperad context.

The category $c\,\mathcal{H}opf\mathcal{O}p_*^c$ inherits a model structure, like the category of topological operads, so that we can apply the general theory of model categories to define groups of homotopy automorphism classes $\mathtt{Aut}_{\mathtt{Ho}(c\,\mathcal{H}opf\mathcal{O}p_*^c)}(\mathsf{D})$, as well as homotopy automorphisms spaces $\mathtt{hAut}_{c\,\mathcal{H}opf\mathcal{O}p_*^c}(\mathsf{D})$, for the objects of that category $\mathsf{D} \in c\,\mathcal{H}opf\mathcal{O}p_*^c$. In the case of a topological operad P, our results imply that we have an isomorphism between the group of homotopy automorphisms attached to the Hopf cooperad model $\mathtt{Sing}^{\bullet}(\mathsf{P})$ and the group of homotopy automorphisms attached to the rational completion of P. Besides, we have a homotopy equivalence at the homotopy automorphisms space level inducing this group isomorphism on connected components. Hence, we obtain that homotopy automorphisms of rationalized operads are computable at the level of Hopf cooperads.

For the little 2-discs operad $\mathsf{P} = \mathsf{D}_2$, the object $\mathtt{Sing}^{\bullet}(\mathsf{D}_2)$ gives a reference model of E_2-cooperad in cosimplicial commutative algebras. But, for our study, we consider another model. Indeed, we can use the already considered groupoids of parenthesized braids $PaB(r)$ to form a cosimplicial Hopf E_2-cooperad $C^{\bullet}(\widehat{PaB})$ on which the Grothendieck-Teichmüller group acts (contravariantly). In short, this cooperad is formed by taking continuous duals of the simplicial complexes naturally associated to the groupoids $PaB(r)$. Related to this construction is a cooperad formed from the Drinfeld-Kohno Lie algebras, of which we mentioned the existence in the overall introduction of the book.

In any case, by using this model, we obtain:

THEOREM B. *Let $\mathsf{Q}_2 = C^{\bullet}(\widehat{PaB})$ be our model of cosimplicial Hopf E_2-cooperad, formed from the groupoids of parenthesized braids $PaB(r)$. The homotopy automorphism space associated to this cooperad has trivial homotopy groups*

$$\pi_*(\mathtt{hAut}^1_{c\,\mathcal{H}opf\mathcal{O}p_*^c}(\mathsf{Q}_2)) = 0$$

in dimension $* > 0$, and the action of the Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$ on parenthesized braids lifts to an isomorphism

$$GT^1(\mathbb{Q})^{op} \xrightarrow{\simeq} \pi_0(\mathtt{hAut}^1_{c\,\mathcal{H}opf\mathcal{O}p_*^c}(\mathsf{Q}_2))$$

in dimension $* = 0$.

The assertions of this theorem have been foreseen by M. Kontsevich in [56]. First results in the direction of Theorem B also occur in articles of D. Tamarkin [90] and T. Willwacher [94]. But these authors deal with operads within the category of differential graded modules, forgetting about Hopf structures. Thus, the proof of Theorem B is actually, a new result of this book, like Theorem A, which we essentially deduce from Theorem B by using our rational homotopy theory of operads.

Recall that E_2-operads only give the second layer of a full sequence of homotopy structures, ranging from E_1, fully homotopy associative but non-commutative, until E_∞, fully homotopy associative and commutative. The methods of the present work can easily be applied to determine the group of homotopy automorphism classes of E_1-operads, but the result is trivial in this case. The group of homotopy automorphisms of an E_∞-operad is essentially trivial too (and so does the group of homotopy automorphisms of an E_∞-cooperad). The open question is to define analogues of the Grothendieck-Teichmüller group for E_n-operads when $2 < n < \infty$.

To prove our theorem, we adapt constructions of [24, 22] in order to form a spectral sequence $E^2_s = H^s(\mathtt{HopfDfm}^c_{\mathcal{O}p}(H^*(\mathsf{A}), H^*(\mathsf{B}))^\bullet) \Rightarrow \pi_*(\mathtt{Map}_{c\mathcal{H}opf\mathcal{O}p^c_*}(\mathsf{A},\mathsf{B}))$ computing the homotopy of mapping spaces in the category of cosimplicial Hopf cooperads $\mathtt{Map}_{c\mathcal{H}opf\mathcal{O}p^c_*}(\mathsf{A},\mathsf{B})$ from the cohomology of a deformation complex of graded Hopf cooperads. For the cohomology of the little 2-discs operad $H^*(\mathsf{D}_2)$, the cohomology of this Hopf deformation complex vanishes in degree $* > 0$ and is identified with a graded version of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}$ in degree $* = 0$. We check that all classes of degree $* = 0$ in the E^2-term of our spectral sequence are hit by an actual homotopy automorphism, coming from the Grothendieck-Teichmüller group, to conclude that the spectral sequence degenerates at E^2-stage and to obtain the result of our theorem.

As the reader sees, the proof of our result requires the complete set up of new theories, like the definition of rational models for the homotopy of topological operads. This issue was our first motivation to write a full book. Besides, for mathematicians coming from other domains and graduate students, we have wished to give a comprehensive introduction to our subject, heading to our main theorems as straight as possible and with minimal background.

In a first stage, we heavily use the formalism of Quillen's model categories [75] which we apply to operads in order to form our model for the rational homotopy of topological operads. For background material on Quillen's model categories, we rely on the modern references: Hirschhorn [51] and Hovey [52]. For rational homotopy theory, we mostly refer to Bousfield-Gugenheim' memoir [23] which involves a model category approach close to our needs. Naturally, we also refer to Sullivan's seminal article [88] for the study of homotopy automorphisms in rational homotopy theory. Since we aim to give a self-contained account of the theory, we give a comprehensive introduction to all these subjects – operads, homotopical algebra, and rational homotopy theory – from scratch, before tackling our own constructions. We also fully explain the connection between little 2-discs operads, braided operads, and Grothendieck-Teichmüller theory, arising from works of Fiedorowicz [34], Tamarkin [90, 91], and Kontsevich [56].

In a second stage, we apply operadic deformation complexes to the homotopical study of operads. For our purpose, we need deformation complexes, mixing cooperad and algebra structures. The present book is the first reference explicitly

tackling applications of such mixed deformation complexes to the homotopy of operads. Again, we aim to give a comprehensive introduction, from scratch, to these constructions.

General Conventions

The reader is assumed to be familiar with the language of category theory and to have basic knowledge about fundamental concepts, like adjoint and representable functors, colimits and limits, categorical duality, which we will freely use throughout this monograph. The reader is also assumed to be aware on the applications of colimits and limits in basic examples of categories (including sets, topological spaces, and modules). Nonetheless, we will review some specialized topics, like reflexive coequalizers and filtered colimits, which concern applications of category theory to operads.

We use single script letters (like $\mathcal{C}$, $\mathcal{M}$, ...) as general notation for abstract categories. We use script expressions (like $\mathcal{M}od$, $\mathcal{A}s$, $\mathcal{O}p$, ...) for particular instances of categories (like modules, associative algebras, operads, ...). We also use script expressions for abstract categories, assigned to a specific purpose, and used as a parameter in our constructions. As an example, we use the notation $\mathcal{B}ase$ to refer to an unspecified base category in which we define our higher structures (algebras, operads, ...). We soon explain that the formal definition of the higher structures remains the same in any instance of base category and essentially depends on a *symmetric monoidal structure* given with $\mathcal{B}ase$. We generally assume that the category $\mathcal{B}ase$, to which we assign the role of a base category, is equipped with enriched hom-bifunctors $\mathtt{Hom}_{\mathcal{B}ase}(-,-)$. We give more detail recollections on this notion in §§0.9-0.10.

In practice, we take our base category $\mathcal{B}ase$ among the category of sets $\mathcal{S}et$, the category of simplicial sets $\mathcal{S}imp$, the category of topological spaces $\mathcal{T}op$, a category of $\Bbbk$-modules $\mathcal{M}od$ (where $\Bbbk$ refers to a fixed ground ring), or a variant of these categories. To be precise, besides plain $\Bbbk$-modules, we have to consider categories formed by differential graded modules $dg\,\mathcal{M}od$ (we usually say *dg-modules* for short), graded modules $gr\,\mathcal{M}od$, simplicial modules $s\,\mathcal{M}od$, and cosimplicial modules $c\,\mathcal{M}od$. The first purpose of this preliminary chapter is to quickly recall the definition of these categories (at least, in order to fix our conventions). By the way, we also recall the definition of the category of simplicial sets $\mathcal{S}imp$, which we use along with the familiar category of topological spaces $\mathcal{T}op$.

To complete our account, we will recall the general definition of a symmetric mononoidal category, and we explain some general constructions attached to this structure. The explicit definition of the monoidal category structure on dg-modules, simplicial modules, cosimplicial modules, is put off until we tackle the applications of these categories.

In the module context, we assume that a ground ring $\Bbbk$ is given and fixed once and for all. In certain constructions, we have to assume that this ground ring $\Bbbk$ *is a field of characteristic* 0.

0.1. *Graded and differential graded modules.* The category of differential graded modules $dg\,\mathcal{M}od$ (dg-modules for short) consists of $\Bbbk$-modules equipped with a decomposition $K = \bigoplus_{n\in\mathbb{Z}} K_n$, running over $\mathbb{Z}$, and with a morphism $\delta : K \to K$, the differential of K, such that $\delta^2 = 0$ and $\delta(K_n) \subset K_{n-1}$, for all $n \in \mathbb{Z}$. Naturally, a morphism of dg-modules is a morphism of $\Bbbk$-modules $f : K \to L$ which commutes with differentials and satisfies $f(K_n) \subset L_n$, for all $n \in \mathbb{Z}$.

In textbooks of homological algebra (like [92]), authors mostly deal with the equivalent notion of chain complex, of which components are split off into sequences of $\Bbbk$-modules K_n connected by the differentials $\delta : K_n \to K_{n-1}$ rather than being put together in a single object. The idea of a dg-module (used for instance in [64]) is more natural for our purpose and is also more widely used in homotopy theory. Our convention is to keep the terminology of chain complex for specific constructions, like the normalized chain complex of simplicial sets, or the deformation complex attached an algebraic structure.

The category of graded modules $gr\,\mathcal{M}od$ consists of $\Bbbk$-modules equipped with a decomposition $K = \bigoplus_{n\in\mathbb{Z}} K_n$, running over $\mathbb{Z}$, but no differential. A morphism of graded modules is a morphism of $\Bbbk$-modules $f : K \to L$ such that $f(K_n) \subset L_n$, for all $n \in \mathbb{Z}$.

We have an obvious functor $(-)_\flat : dg\,\mathcal{M}od \to gr\,\mathcal{M}od$ defined by retaining the single graded structure of dg-modules and forgetting about the differential. We consider the underlying graded module of dg-modules, which this forgetful process formalizes, when we address the definition of quasi-free objects. The other way round, we can embed the category of graded modules $gr\,\mathcal{M}od$ into the category of dg-modules $dg\,\mathcal{M}od$, by viewing a graded module as a dg-modules equipped with a trivial differential $\delta = 0$. We use this identification at various places.

Recall that the homology of a dg-module K is defined by the quotient $\Bbbk$-module $\mathtt{H}_*(K) = \ker\delta/\operatorname{im}\delta$ which inherits a natural grading from K. The homology defines a functor $\mathtt{H}_*(-) : dg\,\mathcal{M}od \to gr\,\mathcal{M}od$. The morphisms of dg-modules which induce an isomorphism in homology are the weak-equivalences of the category of dg-modules. We generally use the notation $\xrightarrow{\sim}$ to distinguish the class of weak-equivalence associated to a model category (see §III) and we will naturally use the same convention in the dg-module context. The expression of weak-equivalence actually refers to the general formalism of model categories. In most references of homological algebra, authors use the terminology of quasi-isomorphism rather than the expression of weak-equivalence when they deal with dg-modules.

0.2. *Degrees and signs of dg-algebra.* The component K_n of a dg-module (respectively, graded module) K defines the homogeneous component of degree n of K. To specify the degree of a homogeneous element $x \in K_n$, we use the expression $\deg(x) = n$. We adopt the standard convention of dg-algebra to associate a sign $(-1)^{\deg(x)\deg(y)}$ to each transposition of homogeneous elements (x, y). We do not specify such a sign in general and we simply use the notation $\pm$ to refer to it. We explain soon that the introduction of these signs is forced by the definition of the symmetry isomorphism of the tensor product of dg-modules.

We usually consider lower graded dg-modules, but we also have a standard notion of dg-module equipped with a decomposition in upper graded components $K = \bigoplus_{n\in\mathbb{Z}} K^n$ so that the differential satisfies $\delta(K^n) \subset K^{n+1}$. Certain constructions (like the duality of $\Bbbk$-modules and the conormalized complex of cosimplicial

spaces) naturally produce upper graded dg-modules. In what follows, we apply the relation $K_{-n} = K^n$ to identify an upper graded with a lower graded dg-module.

0.3. *Simplicial and cosimplicial objects, simplicial and cosimplicial modules.* The simplicial category Δ, which models the structure of simplicial and cosimplicial objects, is defined by the collection of finite ordinals $\{0 < \cdots < n\}$, $n \in \mathbb{N}$, as objects together with the non-decreasing maps $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$ as morphisms. Formally, a simplicial object in an ambient category $\mathcal{C}$ is a contravariant functor $X : \Delta^{op} \to \mathcal{C}$ that assigns an object $X_n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u^* : X_n \to X_m$ to each non-decreasing map u. Dually, a cosimplicial object in $\mathcal{C}$ is a covariant functor $X : \Delta \to \mathcal{C}$ which assigns an object $X^n \in \mathcal{C}$ to each $n \in \mathbb{N}$ and a morphism $u^* : X^n \to X^m$ to each non-decreasing map u.

In general, we use the expression $s\,\mathcal{C}$ to denote the category of simplicial objects in a given ambient category $\mathcal{C}$ and the expression $c\,\mathcal{C}$ for the category of cosimplicial objects in $\mathcal{C}$. The only exception is the category of simplicial sets, which can be defined as the category of simplicial objects in sets, but for which we prefer to use the notation $\mathcal{S}imp$ rather than $s\,\mathcal{S}et$.

The fundamental examples of simplicial sets, the simplices Δ^n, $n \in \mathbb{N}$, are the simplicial objects defined by the representable functors $\mathtt{Mor}_\Delta(-, n)$, where we use the notation $\mathtt{Mor}_\Delta(m, n)$ to refer to the morphism sets of the simplicial category Δ. The collection of n-simplices Δ^n, $n \in \mathbb{N}$, forms itself a cosimplicial object in the category of simplicial sets, with the covariant action of non-decreasing maps $u_* : \Delta^m \to \Delta^n$ defined by the composition on the target in the morphism set representation of Δ^n. The standard model of n-simplices in the category of topological spaces

$$\Delta^n_{top} = \{(t_0, \ldots, t_n) | 0 \leq t_i \leq 1 \text{ and } t_0 + \cdots + t_n = 1\}$$

also gives an instance of a cosimplicial object in the category of topological spaces.

The category of simplicial modules $s\,\mathcal{M}od$ is just the category of simplicial objects in the category of $\Bbbk$-modules $\mathcal{M}od$. The category of cosimplicial modules $c\,\mathcal{M}od$ is defined similarly as the category of cosimplicial objects in $\Bbbk$-modules. We will consider the functor $\Bbbk\{-\} : \mathcal{S}imp \to s\,\mathcal{M}od$ which maps a simplicial set X to the simplicial module $\Bbbk\{X\}$ defined in dimension n by the free $\Bbbk$-modules $\Bbbk\{X_n\}$ generated by the sets X_n, $n \in \mathbb{N}$. We take the obvious induced action of the morphisms of the simplicial category on these free $\Bbbk$-modules $\Bbbk\{X_n\}$ to define the simplicial structure of $\Bbbk\{X\}$. We also have a contravariant functor $\Bbbk^{\{-\}} : \mathcal{S}imp^{op} \to c\,\mathcal{M}od$ which maps a simplicial set X to the cosimplicial module $\Bbbk^X$ defined in dimension n by the collection of maps $u : X_n \to \Bbbk$.

0.4. *Faces and degeneracies in a simplicial object.* The maps $d^i : \{0 < \cdots < n-1\} \to \{0 < \cdots < n\}$, $i = 0, \ldots, n$, such that

$$d^i(x) = \begin{cases} x, & \text{for } x < i, \\ x+1, & \text{for } x \geq i, \end{cases} \tag{a}$$

and the maps $s^j : \{0 < \cdots < n\} \to \{0 < \cdots < n+1\}$, $j = 0, \ldots, n$, such that

$$s^j(x) = \begin{cases} x, & \text{for } x \leq j, \\ x-1, & \text{for } x > j, \end{cases} \tag{b}$$

generate the simplicial category in the sense that any non-decreasing map $u : \{0 < \cdots < m\} \to \{0 < \cdots < n\}$ can be written as a composite of maps of that form.

Moreover, any relation between these generating morphisms can be deduced from generating relations

$$\text{(c)}\qquad \begin{aligned} d^j d^i &= d^i d^{j-1}, \text{ for } i<j,\\ s^j d^i &= \begin{cases} d^i s^{j-1}, & \text{for } i<j,\\ id, & \text{for } i=j,j+1,\\ d^{i-1}s^j, & \text{for } i>j,\end{cases}\\ s^j s^i &= s^i s^{j+1}, \text{ for } i\leq j.\end{aligned}$$

As a consequence, the structure of a simplicial object is fully determined by a sequence of objects $X_n \in \mathcal{C}$ together with morphisms $d_i : X_n \to X_{n-1}$, $i = 0, \dots, n$, and $s_j : X_n \to X_{n+1}$, $j = 0, \dots, n$, for which relations opposite to (c) hold. The morphisms $d_i : X_n \to X_{n-1}$, $i = 0, \dots, n$, which represent the image of the maps d^i under the functor defined by X, are the face operators of the simplicial object X (in general, we simply say the faces of X). The morphisms $s_j : X_n \to X_{n+1}$, $j = 0, \dots, n$, which represent the image of the maps s_j are the degeneracy operators of X (or, more simply, the degeneracies of X).

The structure of a cosimplicial object is also fully determined by a sequence of objects $X^n \in \mathcal{C}$ together with morphisms $d^i : X^{n-1} \to X^n$, $i = 0, \dots, n$, and $s^j : X^{n+1} \to X^n$, $j = 0, \dots, n$, for which relations (c) hold. The morphisms $d^i : X^{n-1} \to X^n$, $i = 0, \dots, n$, are called cofaces and the morphisms $s^j : X^n \to X^{n+1}$, $j = 0, \dots, n$, codegeneracies.

0.5. *Topological realization of simplicial sets and singular complex of topological spaces.* Recall that a topological space $|K|$, traditionally called the geometric realization of K, is naturally associated to each simplicial set $K \in \mathcal{S}imp$. This space is defined by the coend

$$|K| = \int^{\Delta} K_n \times \Delta^n_{top}.$$

where each set K_n is viewed as a discrete space and we consider the topological n-simplices Δ^n_{top} (of which definition is recalled in §0.3). The coend process amounts to performing a quotient of the coproduct $\coprod_n K_n \times \Delta^n_{top} = \coprod_n \{\coprod_{\sigma\in K_n}\{\sigma\}\times\Delta^n_{top}\}$ under relations of the form

$$(u^*(\sigma), (t_0, \dots, t_m)) \equiv (\sigma, u_*(t_0, \dots, t_m)),$$

for $u \in \mathtt{Mor}_{\Delta}(m, n)$, $\sigma \in K_n$, and $(t_0, \dots, t_m) \in \Delta^m_{top}$. The definition of the map $u_* : \Delta^m_{top} \to \Delta^n_{top}$ associated to each $u \in \mathtt{Mor}_{\Delta}(m, n)$ involves the cosimplicial structure of the topological n-simplices Δ^m_{top}. One easily checks that the realization of the n-simplex $\Delta^n = \mathtt{Mor}_{\Delta}(-, n)$ is identified with the topological n-simplex Δ^n_{top}.

In the converse direction, we can use the singular complex construction to associate a simplicial set $\mathtt{Sing}_\bullet(X)$ to any topological space X. This simplicial set $\mathtt{Sing}_\bullet(X)$ consists in dimension n of the set of continuous maps $\sigma : \Delta^n_{top} \to X$ going from the topological n-simplex Δ^n_{top} to X. The composition of simplices $\sigma : \Delta^n_{top} \to X$ with the cosimplicial operator $u_* : \Delta^m_{top} \to \Delta^n_{top}$ associated to any $u \in \mathtt{Mor}_{\Delta}(m, n)$ yields a map $u^* : \mathtt{Sing}_n(X) \to \mathtt{Sing}_m(X)$ so that the collection of sets $\mathtt{Sing}_n(X) = \mathtt{Mor}_{\mathcal{T}op}(\Delta^n_{top}, X)$, $n \in \mathbb{N}$, inherits a natural simplicial structure.

The geometric realization $|-| : K \mapsto |K|$ obviously gives a functor $|-| : \mathcal{S}imp \to \mathcal{T}op$. The singular complex construction gives a functor in the converse

direction $\mathtt{Sing}_\bullet : \mathcal{T}op \to \mathcal{S}imp$, which is actually a right adjoint of the geometric realization functor $|-| : \mathcal{S}imp \to \mathcal{T}op$ (see [48, §I.2]).

0.6. *Symmetric monoidal categories and the structure of base categories.* In the introduction of this chapter, we mention that our base categories $\mathcal{B}ase = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \ldots$ are all instances of a symmetric monoidal categories.

By definition, a symmetric monoidal category is a category $\mathcal{M}$ equipped with a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfying natural unit, associativity and symmetry relations. These relations are expressed by structure isomorphisms which have to be given along with the category:

(a) The unit is given by an object $\mathbb{1} \in \mathcal{M}$ together with a natural isomorphism $A \otimes \mathbb{1} \simeq A \simeq \mathbb{1} \otimes A$, associated to each $A \in \mathcal{M}$.
(b) The associativity relation is given by a natural isomorphism $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$, associated to every triple of objects $A, B, C \in \mathcal{M}$, satisfying a pentagonal coherence relation (Mac Lane's pentagon relation) and two additional triangular coherence relations with respect to the unit isomorphism (we refer to [65, §XI.1] for the expression of these constraints).
(c) The symmetry relation is given by a symmetry isomorphism $A \otimes B \simeq B \otimes A$, associated to every pair of objects $A, B \in \mathcal{M}$, satisfying hexagonal coherence relations (Drinfeld's hexagon relation) and two additional triangular coherence relations with respect to the unit isomorphism (see again [65, §XI.1] for details).

In the case of $\Bbbk$-modules $\mathcal{M}od$, the monoidal structure is given by the usual tensor product of $\Bbbk$-modules, taken over the ground ring, together with the ground ring itself as unit object. The definition of the tensor product of dg-modules, simplicial modules, cosimplicial modules is reviewed later on, when we tackle applications of these ground categories. In the category of sets $\mathcal{S}et$ (respectively, topological spaces $\mathcal{T}op$, simplicial sets $\mathcal{S}imp$), the tensor product is simply given by the cartesian product $\otimes = \times$ together with the one-point set $\mathbb{1} = pt$ as unit object. In what follows, we also use the general notation $*$ for the terminal object of a category, and we may write $pt = *$ when we want to stress that the one point-set actually represents the terminal object of the category of sets (respectively, topological spaces, or simplicial sets).

The unit object and the isomorphisms that come with the unit, associativity and commutativity relations of a symmetric monoidal category are part of the structure. Therefore, these morphisms have, in principle, to be given with the definition. But, in our examples, we can assume that the unit and associativity relations are identities, and usually, we just make explicit the definition of the symmetry isomorphism $c = c(A, B) : A \otimes B \xrightarrow{\simeq} B \otimes A$.

In many constructions, we consider symmetric monoidal categories $\mathcal{M}$ equipped with colimits and limits and so that the tensor product of $\mathcal{M}$ preserves colimits on each side. To be explicit, we use:

(d) The canonical morphism $\operatorname{colim}_{\alpha \in \mathcal{I}}(A_\alpha \otimes B) \to (\operatorname{colim}_{\alpha \in \mathcal{I}} A_\alpha) \otimes B$ associated to a diagram $A_\alpha \in \mathcal{M}$, $\alpha \in \mathcal{I}$, is an iso for all $B \in \mathcal{M}$, and similarly as regards the canonical morphism $\operatorname{colim}_{\beta \in \mathcal{J}}(A \otimes B_\beta) \to A \otimes (\operatorname{colim}_{\beta \in \mathcal{J}} B_\beta)$ associated to a diagram $B_\beta \in \mathcal{M}$, $\beta \in \mathcal{J}$, where A is now a fixed object of $\mathcal{M}$.

This requirement is fulfilled by all categories which we take as base categories and is required for the application of categorical constructions to operads and algebras

over operads. The category of coalgebras, of which we recall the definition soon, satisfies (d) whenever the base category does, because colimits of coalgebras are created in the underlying base category (see §II.0.3). On the other hand, we will also consider instances of categories which do not satisfy this colimit condition (d).

Therefore, when we use the notation $\mathcal{B}ase$ (referring to a base category), we assume, by convention, that we deal with a symmetric monoidal category satisfying the base axioms (a-c), as well as the colimit requirement (d). On the other hand, when we use abstract variable notation $\mathcal{M}, \mathcal{N}, \ldots$ (referring to any category), we only assume that the base axioms (a-c) are satisfied, otherwise we will give additional precisions on the internal structures of the considered categories.

0.7. *Symmetric groups and tensor permutations.* We use the notation Σ_r for the group of permutations of $\{1, \ldots, r\}$. Depending on the context, we regard a permutation $s \in \Sigma_r$ as a bijection $s : \{1, \ldots, r\} \to \{1, \ldots, r\}$, or as a sequence $s = (s(1), \ldots, s(r))$, equivalent to an ordering on the set $\{1, \ldots, r\}$. In any case, we will use the notation $id = id_r$ for the identity permutation in Σ_r. We drop the lower-script r, indicating the permutation cardinal, when we do not need to specify this information.

In a symmetric monoidal category equipped with a strictly associative tensor product, we can form r-fold tensor products $T = A_1 \otimes \cdots \otimes A_r$ without care, and drop unnecessary bracketings. Then we also have a natural isomorphism

$$A_1 \otimes \cdots \otimes A_r \xrightarrow{s^*} A_{s(1)} \otimes \cdots \otimes A_{s(r)},$$

associated to each permutation $s \in \Sigma_r$, and so that the standard unit and associativity relations $id^* = id$ and $t^* s^* = (st)^*$ hold. To construct this action, we use the classical presentation of Σ_r, with the transpositions $t_i = (i\ i+1)$ as generating elements, and the identities

$$\text{(a)} \qquad t_i^2 = id, \quad \text{for } i = 1, \ldots, n-1,$$

$$\text{(b)} \qquad t_i t_j = t_j t_i, \quad \text{for } i, j = 1, \ldots, n-1, \text{ with } |i-j| \geq 2,$$

$$\text{(c)} \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad \text{for } i = 1, \ldots, n-2,$$

as generating relations. To begin with, we assign the morphism

$$A_1 \otimes \cdots \otimes A_i \otimes A_{i+1} \otimes \cdots \otimes A_r \xrightarrow{\simeq} A_1 \otimes \cdots \otimes A_{i+1} \otimes A_i \otimes \cdots \otimes A_r,$$

induced by the symmetry isomorphism $c(A_i, A_{i+1}) : A_i \otimes A_{i+1} \xrightarrow{\simeq} A_{i+1} \otimes A_i$, to the transposition $t_i = (i\ i+1)$. The axioms of symmetric monoidal categories imply that these morphisms satisfy the relations (a-c) attached to the elementary transpositions in Σ_r. Hence, we can use the presentation of Σ_r to coherently extend the action of the transpositions $t_i \in \Sigma_r$ on tensor powers to the whole symmetric group.

0.8. *Tensor products over arbitrary finite sets.* In our constructions, we often deal with tensor products $\bigotimes_{i_k \in \underline{r}} A_{i_k}$, running over an arbitrary set $\underline{r} = \{i_1, \ldots, i_r\}$ (not necessarily equipped with a canonical ordering). In fact, we effectively realize such a tensor product $\bigotimes_{i_k \in \underline{r}} A_{i_k}$ as an ordered tensor product $A_{u(1)} \otimes \cdots \otimes A_{u(r)}$, which we associate to the choice of a bijection $u : \{1, \ldots, r\} \xrightarrow{\simeq} \underline{r}$. The tensor products associated to different bijection choices $u, v : \{1, \ldots, r\} \xrightarrow{\simeq} \underline{r}$ differ by a canonical isomorphism $s^* : A_{u(1)} \otimes \cdots \otimes A_{u(r)} \xrightarrow{\simeq} A_{v(1)} \otimes \cdots \otimes A_{v(r)}$ which we

determine from the permutation $s \in \Sigma_r$ such that $v = u \cdot s$, by using the just defined action of symmetric groups on tensors.

In principle, the tensor product $\bigotimes_{i_k \in \underline{r}} A_{i_k}$ is only defined up to these canonical isomorphisms. However, we can adapt the general Kan extension process to make this construction more rigid, at least, when we work in a symmetric monoidal category $\mathcal{M}$ equipped with fixed colimit functors. Formally, we define the unordered tensor product as the colimit $\bigotimes_{i_k \in \underline{r}} A_{i_k} = \operatorname{colim}_{u:\{1,\ldots,r\} \xrightarrow{\simeq} \underline{r}} A_{u(1)} \otimes \cdots \otimes A_{u(r)}$ running over the category formed by the bijections $u : \{1, \ldots, r\} \xrightarrow{\simeq} \underline{r}$ as objects, and the permutations $s \in \Sigma_r$ such that $v = u \cdot s$ as morphisms. The colimit process automatically performs the identification of the tensors associated to different bijection choices.

This construction can be regarded as an instance of a Kan extension process which we will apply to structures, called symmetric sequences, underlying operads (see §A.4).

0.9. *Enriched categories.* The morphism sets of a category $\mathcal{C}$ will always be denoted by $\mathtt{Mor}_{\mathcal{C}}(X, Y)$. But many categories that we consider $\mathcal{C}$, including the base categories themselves, come equipped with a hom-bifunctor $\mathtt{Hom}_{\mathcal{C}}(-, -)$, with values in one of our ground categories $\mathcal{B}ase = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \ldots$, and which provides $\mathcal{C}$ with an additional enriched category structure. In the case of the base categories, we just consider, as we explain soon, an internal hom-bifunctor obtained by adjunction from the tensor product of the category. In the case $\mathcal{B}ase = \mathcal{S}et$, we actually have $\mathtt{Hom}_{\mathcal{S}et}(-, -) = \mathtt{Mor}_{\mathcal{S}et}(-, -)$. In the case $\mathcal{B}ase = \mathcal{T}op$, the hom-objects $\mathtt{Hom}_{\mathcal{T}op}(X, Y)$ are given by the morphism sets $\mathtt{Mor}_{\mathcal{T}op}(X, Y)$ equipped with the usual compact-open topology. In the case $\mathcal{B}ase = \mathcal{M}od$, the hom-objects $\mathtt{Hom}_{\mathcal{M}od}(A, B)$ are similarly given by the morphism sets of the category $\mathtt{Mor}_{\mathcal{M}od}(A, B)$, which come naturally equipped with a module structure (the usual one). In the remaining usual cases $\mathcal{B}ase = \mathcal{S}imp, dg\,\mathcal{M}od, \ldots$, the morphism sets are identified with subsets of the hom-objects. The explicit definition of the hom-objects attached to these categories will be given along with the definition of their monoidal structures.

In general, we deal with two structures, of a different nature, attached to our category $\mathcal{C}$: the underlying plain category structure, with morphism-sets denoted by $\mathtt{Mor}_{\mathcal{C}}(X, Y)$, and an enriched category structure, of which hom-objects are denoted by $\mathtt{Hom}_{\mathcal{C}}(X, Y)$, for any pair of objects $(X, Y) \in \mathcal{C}$. Let us recall the pure definition of an enriched category structure for the moment (we will address the connection between morphism sets and hom-objects with more details in the next paragraph).

In the standard setting of categories, the units of the composition are given by identity morphisms $id_X \in \mathtt{Mor}_{\mathcal{C}}(X, X)$ associated to all objects $X \in \mathcal{C}$. In the enriched category context, we suppose given morphisms

$$\text{(a)} \qquad id_X : \mathbb{1} \to \mathtt{Hom}_{\mathcal{C}}(X, X),$$

for all objects $X \in \mathcal{C}$. and composition products

$$\text{(b)} \qquad \cdot : \mathtt{Hom}_{\mathcal{C}}(Y, Z) \otimes \mathtt{Hom}_{\mathcal{C}}(X, Y) \to \mathtt{Hom}_{\mathcal{C}}(X, Y),$$

for all $X, Y, Z \in \mathcal{C}$, satisfying obvious analogues of the unit and associativity relations of the composition of morphisms in plain categories. These relations are expressed by the commutativity of diagrams.

In the case of a symmetric monoidal category $\mathcal{M}$, the enriched hom-objects come usually equipped with an additional tensor product operation

$$\text{(c)} \qquad \mathtt{Hom}_{\mathcal{M}}(A,B)\otimes \mathtt{Hom}_{\mathcal{M}}(C,D)\xrightarrow{\otimes}\mathtt{Hom}_{\mathcal{M}}(A\otimes C,B\otimes D),$$

where we form the tensor products of the objects (A,C) and (B,D) within $\mathcal{M}$, and the tensor product of the hom objects $\mathtt{Hom}_{\mathcal{M}}(A,B)$ and $\mathtt{Hom}_{\mathcal{M}}(C,D)$ in $\mathcal{B}ase$. This tensor product operation on homomorphisms defines a natural transformation that fulfills natural unit, associativity and symmetry relations involving the unit, associativity and symmetry isomorphisms of $\mathcal{B}ase$ and $\mathcal{M}$. In the case $\mathcal{B}ase = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \dots$, the existence of this structure (as we explain in the next paragraph) automatically follows from the adjunction between tensor products and homomorphisms.

0.10. *Hom-objects extending morphism sets in enriched categories.* The categories that we consider $\mathcal{C}$, are usually equipped, as we just explain, with both a plain category structure, and an enriched one. Recall that we use the notation $\mathtt{Mor}_{\mathcal{C}}(X,Y)$ for the morphism sets of the underlying plain category structure and the notation $\mathtt{Hom}_{\mathcal{C}}(X,Y)$ to refer to the hom-objects of the enriched category structure. In usual cases, we can identify the actual morphisms of $\mathcal{C}$ with particular elements of the hom-objects $\mathtt{Hom}_{\mathcal{C}}(X,Y)$. The latter $f\in\mathtt{Hom}_{\mathcal{C}}(X,Y)$ can be identified with maps $f:A\to B$ satisfying some mild requirements. In this context, we use the terminology of homomorphism to refer to the general elements of the hom-objects $\mathtt{Hom}_{\mathcal{C}}(A,B)$, in order to distinguish them from the actual morphisms in $\mathcal{C}$. We may also use the map style notation $f:A\to B$ when we want to regard such a homomorphism $f\in\mathtt{Hom}_{\mathcal{C}}(X,Y)$ as a map and not as an abstract object.

We proceed as follows to formalize connections between morphisms and homomorphisms in a general setting.

We first temporarily forget about enriched category structures. We assume that $\mathcal{C}$ is any plain category in sets. We consider the objects

$$\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}=\bigoplus_{f\in\mathtt{Mor}_{\mathcal{C}}(X,Y)}\mathbb{1},$$

defined by the coproduct of copies of the unit object of $\mathcal{B}ase$, one for each $f\in\mathtt{Mor}_{\mathcal{C}}(X,Y)$. These objects $\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}$ define the hom-objects of an enriched category structure, naturally associated to $\mathcal{C}$: the unit morphism $\mathbb{1}\xrightarrow{id_X}\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,X)\}$, associated to any $X\in\mathcal{C}$, is given by the embedding of the term $\mathbb{1}$ associated to the identity morphism id_X in $\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,X)\}$; the composition operations $\cdot:\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(Y,Z)\}\otimes\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}\to\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}$ are defined termwise by the isomorphism $\mathbb{1}\otimes\mathbb{1}\xrightarrow{\simeq}\mathbb{1}$ between the copies of $\mathbb{1}$ associated to (f,g) on the source and the copy of $\mathbb{1}$ associated to the composite $f\cdot g$ on the target.

In the case of an enriched category, we precisely assume the existence of morphisms $\iota:\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}\to\mathtt{Hom}_{\mathcal{C}}(X,Y)$, given for every $(X,Y)\in\mathcal{C}$, so that the enriched unit and composition structure on hom-objects $\mathtt{Hom}_{\mathcal{C}}(X,Y)$ extends the unit and composition structure of the enriched morphism sets $\mathbb{1}\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}$. We then have a morphism $\iota_f:\mathbb{1}\to\mathtt{Hom}_{\mathcal{C}}(X,Y)$, associated to any $f\in\mathtt{Mor}_{\mathcal{C}}(X,Y)$, The morphisms ι_{id_X} agrees with the enriched unit $id_X:\mathbb{1}\to\mathtt{Hom}_{\mathcal{C}}(X,Y)$ by assumption on ι. We can combine the morphism ι_f with the enriched composition structure of hom-objects to define morphisms $f_*:\mathtt{Hom}_{\mathcal{C}}(-,X)\to\mathtt{Hom}_{\mathcal{C}}(-,Y)$ and

$f^* : \mathtt{Hom}_{\mathcal{C}}(Y,-) \to \mathtt{Hom}_{\mathcal{C}}(X,-)$, associated to any $f \in \mathtt{Mor}_{\mathcal{C}}(X,Y)$, so that our hom-objects naturally form a bifunctor $\mathtt{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{B}ase$.

In the fundamental examples $\mathcal{C} = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \ldots$, the morphisms can be regarded as particular homomorphisms and we can use this basic identification to obtain all these structures on hom-objects.

0.11. *Closed symmetric monoidal categories.* The base categories considered in this book $\mathcal{B}ase = \mathcal{S}et, \mathcal{T}op, \mathcal{M}od, \ldots$ are examples of closed as symmetric monoidal categories: in each case, we have an internal hom-bifunctor $\mathtt{Hom}_{\mathcal{B}ase}(-,-)$, with values in the category $\mathcal{B}ase$ itself, together with a natural isomorphism

$$\text{(a)} \qquad \mathtt{Mor}_{\mathcal{B}ase}(A \otimes B, C) \simeq \mathtt{Mor}_{\mathcal{B}ase}(A, \mathtt{Hom}_{\mathcal{B}ase}(B,C)),$$

for every $A, B, C \in \mathcal{B}ase$. In good cases (for instance, when we take a category of modules over a field as ground category), we can establish that the category of augmented cocommutative coalgebras $\mathcal{C}om^c_+$, inherits an internal hom as well, and hence, forms another instance of symmetric monoidal category. Nonetheless, in the sequel, we do not really use the closed monoidal structure in this example.

The hom-objects $\mathtt{Hom}_{\mathcal{B}ase}(A,B)$ defined by an internal hom-finctor naturally inherit an evaluation morphism

$$\text{(b)} \qquad \mathtt{Hom}_{\mathcal{B}ase}(A,B) \otimes A \xrightarrow{\epsilon} B,$$

representing the augmentation of the adjunction (a), and which generalizes the usual evaluation of maps in the category of sets. In addition to the evaluation morphism, we have a morphism

$$\text{(c)} \qquad A \xrightarrow{\iota} \mathtt{Hom}_{\mathcal{B}ase}(B, A \otimes B)$$

giving the unit of the adjunction (a).

The internal hom-objects of a closed symmetric monoidal category always inherit a composition operation §0.9(b) and a tensor product operation §0.9(c) which satisfy all expected relations. The composition operation §0.9(b) is given by the right adjoint of the composite evaluation morphism

$$\mathtt{Hom}_{\mathcal{B}ase}(B,C) \otimes \mathtt{Hom}_{\mathcal{B}ase}(A,B) \otimes A \xrightarrow{id \otimes \epsilon} \mathtt{Hom}_{\mathcal{B}ase}(B,C) \otimes B \xrightarrow{\epsilon} C.$$

The composition unit §0.9(a) is given by the right adjoint of the unit isomorphism $\mathbb{1} \otimes A \xrightarrow{\simeq} A$ of the symmetric monoidal structure. The tensor product operation §0.9(c) is given by the adjoint of the morphism

$$\begin{aligned} &\mathtt{Hom}_{\mathcal{B}ase}(A,B) \otimes \mathtt{Hom}_{\mathcal{B}ase}(C,D) \otimes A \otimes C \\ &\qquad\qquad \simeq \mathtt{Hom}_{\mathcal{B}ase}(A,B) \otimes A \otimes \mathtt{Hom}_{\mathcal{B}ase}(C,D) \otimes C \\ &\qquad\qquad \xrightarrow{\epsilon \otimes \epsilon} B \otimes D, \end{aligned}$$

where we apply the symmetry operator of $\mathcal{B}ase$ and we form the tensor product of the evaluation morphisms associated to the hom-objects.

Note that a category $\mathcal{M}$ which fails to satisfy the colimit requirement of §0.6 can not have internal hom-objects. Indeed, the adjunction relation (a) immediately implies that the tensor product of $\mathcal{B}ase$ preserves colimits in A (and hence in B as well by symmetry of the tensor product), as stated in §0.6(d).

In general, the enriched hom-objects $\mathtt{Hom}_{\mathcal{B}ase}(-,-)$ that we attach to the base category $\mathcal{B}ase$ are the internal hom-objects arising from the closed monoidal structure. Later on, we introduce examples of symmetric monoidal categories for which

we do not take an internal hom-bifunctor as enriched structure, but an external hom-bifunctor with value in one of our base categories.

Synopsis

Part 0. The theoretical background. This preliminary part provides an introduction to basic concepts of operad theory and homotopical algebra.

Chapter I. Operads and algebras over operads. In this first chapter, we give an overall introduction to the definition of an operad and of an algebra over an operad (§I.1); we then survey the application of usual categorical constructions to operads (§I.2), and algebras over operads (§I.3); finally, we examine apart the definition of operads associated to algebras equipped with unitary operations (§I.4). An appendix section (§I.5) is devoted to recollections on particular colimits, namely filtered colimits and reflexive coequalizers, which we need in applications to operads.

Chapter II. Operads in monoidal categories. In this second chapter, we examine the definition of operads in general symmetric monoidal categories (§II.1), and we examine in more detail the particular case of operads in augmented cocommutative coalgebras (§II.2). The Hopf cooperads, considered in the description of the book objectives, are actually defined as the dual structures of operads in cocommutative coalgebras. A preliminary section (§II.0) is devoted to recollections on the definition of algebras and coalgebras in general symmetric monoidal categories and an appendix (§II.3) to recollections on the definition of functors preserving symmetric monoidal structures.

Chapter III. Homotopical algebra methods. This fourth preliminary chapter includes introductory sections on: the problem of defining homotopy categories (§2.0); the structure of a model category (§III.1); the homotopy category of a model category (§III.2); mapping spaces and homotopy automorphisms (§III.3); and the definition of model categories associated with operads (§III.4). To conclude this chapter, we also briefly outline the application of operadic homotopy automorphism groups to the study of natural equivalences on homotopy categories of algebras.

Part 1. Models of $\mathbf{E_n}$-operads and Grothendieck-Teichmüller groups. In this part, where we start with our main matter, we aim to explain, with all necessary recollections, the definition of our mapping between the Grothendieck-Teichmüller group and the group of homotopy automorphisms classes of E_2-operads. To reach this objective, we also introduce models of the rational homotopy of operads in the specific case of E_2-operads first, by using a Malcev completion process, and in the general case afterwards.

Chapter 1. Introduction to E_n-operads. This chapter includes: an introductory section on the definition of little n-discs operads and E_n-operads (§1.1); and a survey section on the computation of the cohomology and homology of the little n-discs operads (§1.2). An appendix section (§1.3) is devoted to the statement of our conventions on graded modules.

Chapter 2. Braids and the recognition of E_2-operads. This chapter includes: an introductory section on basic concepts of Artin's braid theory (§2.0); an account of Fiedorowicz's definition of models of E_2-operads from contractible operads endowed with an action of braid groups (§2.1); the definition of the colored braid operad, yielding our working model of E_2-operad (§2.2); a section on the fundamental groupoid of topological operads (§2.3), where we reinterpret the just results obtained; and a concluding section on the recognition of E_n-operads for $n > 2$ (§2.4).

Chapter 3. Malcev completion of E_2-operads and Grothendieck-Teichmüller groups. We begin this chapter with a preliminary section on the Malcev completion of operads in groupoids (§3.1), which we conclude with the definition of a rational model of the little 2-disc operad. We then explain the definition of the prounipotent Grothendieck-Teichmüller group, the main object of our study, as a group of automorphisms of the Malcev completion of an operad in groupoids (§3.2).

In our study of homotopy automorphisms, we consider another categorical model of E_2-operad, yielded by the Malcev completion of a chord diagram operad, of which we also review the definition (section to be added in the final version of this chapter). The notion of chord diagram, in our context, actually refers to a graphical representation of the already mentioned Drinfeld-Kohno Lie algebras. The proof that the enriched categories of colored braids give a rational model of E_2-operad follows from Fiedorowicz's theory of braided operads. To prove the analogous result for chord diagrams, we use the theory of Drinfeld's associators which, by borrowing an idea of Tamarkin, we interpret as isomorphisms of categorical operads between the colored braid and chord diagram operads. By the way, we also recall the definition of a graded version of the Grothendieck-Teichmüller group as a group of operad automorphisms arising from the chord diagram model (section to be added in the final version of this chapter).

For technical reasons, which we explain in this chapter, we consider braid and chord diagrams equipped with an additional structure, a parenthesization, given at the level of the objects of the considered categories (see the previous outline). Because of this refinement, we refer to our categorical operads as the operad of parenthesized braids and the operad of parenthesized chord diagrams respectively. This idea of parenthesization is borrowed from Bar Natan's approach of the theory of Drinfeld's associators. On the other hand, one can observe that the operad of colored (unparenthesized) braids governs the structure of a strict braided monoidal category (this statement is an operadic interpretation of a theorem of Joyal and Street). In the course of our verifications, we establish that the addition of the parenthesization simply gives an operad associated to (general) braided monoidal categories, where associativity constraints are governed by (general) associativity isomorphisms instead of strict identities.

At this stage, we have enough material to formulate our main new theorem: in the rational setting, the identity between the Grothendieck-Teichmüller group and the automorphism group of the Malcev completion of the parenthesized braid operad induces an isomorphism between this Grothendieck-Teichmüller group and the homotopy automorphism group of E_2-operads (§3.3). We conclude the chapter with this statement, whose proof is the matter of the next parts.

Chapter 4. Cochain models of operads. This chapter will include: preliminary technical recollections on the definition of model categories by adjunction; an introduction to cooperads and Hopf cooperads; and the proof that the category of cosimplicial Hopf cooperads gives a suitable model for the rational homotopy theory of operads. By the way, we give the detailed definition of the model categories which we technically need for our study: cosimplicial cocommutative algebras, cosimplicial cooperads, and cosimplicial Hopf cooperads.

Part 2. Operadic deformation complexes. This second core part of the book is devoted to a study of deformation complexes. These objects give an approximation of the operadic mapping spaces which we aim to understand. To begin with, we explain the definition of cosimplicial operadic deformation complexes, which naturally occur in cosimplicial decomposition of our mapping spaces. Then we explain a general reduction process with the aim of computing the homology of these complexes.

For the rational homotopy theory of operads, we are naturally lead to study deformation complexes of Hopf cooperads. But we will review more basic instances of deformation complexes before tackling this case: in each chapter, we examine the deformation complex of commutative algebras first, we address the case of operads and cooperads afterwards, and we only give the case of Hopf cooperads, where both structures are mixed together, at the end of our study. The deformation complexes of commutative algebras, operads, cooperads, are well-known. In contrast, the present book is the first work explicitly dealing with the application of deformation complexes to Hopf cooperads.

Chapter 5. Cosimplicial deformation complexes. In this chapter, we will address the definition of cosimplicial deformation complexes: for commutative algebras; for operads; for cooperads; and, at last, for Hopf cooperads. The deformation complex of Hopf cooperads is obtained by putting together the commutative algebra and cooperad constructions.

Chapter 6. Differential graded reductions. In this chapter, we will address the definition of deformation complexes in the differential graded setting. We proceed in the same order as in the previous chapter: we introduce differential graded deformation complexes for commutative algebras first, for operads and for cooperads afterwards, and for Hopf cooperads at last. In each case, commutative algebras, operads, cooperads, and Hopf cooperads, we define a comparison morphism inducing an isomorphism in homology between the cosimplicial and the differential graded versions of our deformation complexes. This comparison morphism provides a first reduction of the cosimplicial deformation complexes which we aim to compute.

Chapter 7. Koszul reductions and applications to E_n-operads. The deformation complexes of the Hopf cooperads considered in this book inherit fine weight decompositions since the commutative algebras forming our Hopf cooperads are naturally graded, with homogeneous generators in degree 1, and we have an analogous homogeneous structure when we regard the cooperad as a whole. The purpose of this chapter is to explain that, in good cases, the homology of the deformation complexes is located in certain top homogeneous components with respect to the extra weight grading. In this situation, the computation of the homology of our deformation complex reduces to the computation of the homology of a small complex, which is precisely formed from the top components of our weight decomposition. Basically, we explain this process in the context of commutative algebras, operads

and cooperads first, and again, we get the case of Hopf cooperads afterwards, by putting together the commutative algebra and cooperad constructions.

As we mentioned in passing, the reduction to the small homogeneous complex does not give the right result in all cases. The class of good algebras (respectively, operads, cooperads) to which we apply our constructions are called Koszul algebras (respectively, operads, cooperads) in the literature. Therefore, we coin the term of Koszul reduction to refer to this second step of our reduction process, occurring after the differential graded reduction examined in the previous chapter. The cohomology of an E_n-operad gives an instance of a Hopf cooperad formed by a sequence of Koszul commutative algebras, and which is also Koszul as a cooperad.

In the concluding section of the chapter, we prove, by applying the Koszul reduction, that the deformation complex of the cohomology of little 2-discs, viewed as a Hopf cooperad, completely collapses in degree > 0. In the next part, we will use intermediate results on deformation complexes of E_2-operads rather than this latter outcome, because we need fine information to finish the determination of the homotopy automorphism space of the little 2-discs operad. On the other hand, the result obtained in the next part implies that all degree 0 classes in the deformation complex correspond to actual homotopy automorphisms of the little 2-discs operad over the rationals, and this observation is worth recording.

Part 3. Spectral Sequences for operadic mapping spaces. In general, the deformation complexes studied in the previous part only give, as previously mentioned, approximations of the mapping and automorphism spaces that one would aim to determine. In this third part, concluding our main matter, we explain processes, encoded in spectral sequences, to determine the homotopy of automorphism spaces from the computation of deformation complexes. The general background of these constructions is not new, but again, the present book is the first work explicitly dealing with the application of such spectral sequences to Hopf cooperads.

Chapter 8. General obstruction theory for operads in spaces. To begin with, we review a general theory, due to Bousfield and Kan, for the construction of set-theoretic spectral sequences from cosimplicial spaces. By applying this general construction in the operad context, we obtain spectral sequences computing the homotopy of operadic mapping spaces from the cohomology of the operadic deformation complexes considered in the previous part. Recall that our homotopy automorphism spaces consist of invertible connected components of such mapping spaces.

In the case of an E_2-operad, this spectral sequence collapses at the second stage, because the cohomology of the operadic deformation complex vanishes in degree > 0 when we take the full Hopf cooperad structure into account, and we can check that all classes of degree 0 correspond to actual morphisms. This latter verification requires a technical analysis of the correspondence between classes on the E^1-page of the spectral sequence and morphisms on the abutment. For this aim, we need another spectral sequence construction which returns, from the second page, the same outcome as our general spectral sequence. But this construction gives the matter for a new chapter.

In fact, the study of the general spectral sequence can be completely bypassed in the case of E_2-operads, and we could directly proceed with the construction of the next chapter. On the other hand, the general spectral sequence is naturally

functorial, and hence, can more easily be used when we study actions of homotopy automorphisms rather than homotopy automorphisms themselves.

Chapter 9. The Drinfeld-Kohno tower and the associated spectral sequence. In §3, we explain the definition of models of an E_2-operad from categories of colored braids. In the rational context, we can use a rationalized version of the lower central series of braid groups to form a tower of operads with the Malcev completion of the little 2-discs operad as limit term. This tower of operads gives rise to a new spectral sequence, which we analyze completely, and we finish the proof of our result on the homotopy of the space of homotopy automorphisms associated with E_2-operads.

Outlook. The initial motivation for the study of connections between Grothendieck-Teichmüller groups and E_2-operads has been provided by works of D. Tamarkin and M. Kontsevich on the deformation quantization problem. To conclude the book, we will give a short survey of applications of E_2-operads in deformation quantization, following the work of these authors, and we will revisit Tamarkin's and Kontsevich's approaches, based on the theory of Drinfeld associators, for the construction of an action of the Grothendieck-Teichmüller group on the moduli space of deformation quantizations. To be more specific, we will give a homotopy theoretic interpretation of this group action by using our result on the homotopy automorphisms of E_2-operads, and parallel results obtained by T. Willwacher [94] in a purely algebraic setting. Then we will give new motivations, arising from our own works on the cohomology of iterated loop spaces, for further researches on the homotopy automorphisms of E_n-operads, where we now consider any $n \geq 2$.

Appendices. The appendices are devoted to the study of structures that are used all through the book: free operads and cofree cooperads (§§A-D); universal enveloping algebras and group algebras from the viewpoint of Hopf algebra theory (§E).

Appendix A. The definition of operadic composition structures revisited. The definition of an operad in §I is perfectly well suited for studying categories of algebras associated to operads, but to work with operads themselves, we need another definition, giving more insights into the internal structure of operads. The main purpose of this appendix chapter, reached in §A.1, is precisely to give a working definition of the notion of operad. In §§A.2-A.3, we use the approach introduced in this chapter to give a reduced description of the structure of operads associated to algebras equipped with unit operations.

In the definition of §I, we consider that an operad consists of a sequence of terms $\mathsf{P}(r)$ indexed by non-negative integers $r \in \mathbb{N}$. In §A.4, we also explain an extension of this background, where terms $\mathsf{P}(\underline{\mathsf{r}})$ indexed by arbitrary finite sets $\underline{\mathsf{r}} = \{i_1, \ldots, i_r\}$ are considered, the non-negative integers of the initial definition corresponding to the sequence of ordinals $\underline{\mathsf{r}} = \{1 < \cdots < r\}$. This extension enables us to perform constructions invariantly, avoiding the choice of bijections $\{1, \cdots, r\} \xrightarrow{\simeq} \{i_1, \ldots, i_r\}$ when we face general indexing sets $\{i_1, \ldots, i_r\}$, of which elements are not necessarily canonically ordered (this situation occurs in the construction of free operads for instance). By the way, along the chapter, we explain a graphical representation of operads, related to the finite-set indexing extension, and which we heavily use afterwards in order to illustrate our constructions.

Appendix B. The construction of free operads. This appendix includes: a comprehensive account on the formalism of trees (§B.1), which is heavily used in operad theory; the introduction of treewise tensor constructions associated with operads (§B.2); the construction of free operads, in the general case first (§B.3), in the case of connected operad structures (§B.4). and unitary operads (§B.5) afterwards.

Appendix C. The connected free operad monad. In this appendix, we study composition structures associated with the free operad functor. To simplify, we restrict our analysis to connected operads. In §C.1, we first introduce a notion of tree morphism, which we use next, in §§C.2-C.3, to give a description of the two-fold composite of the free operad functor. The free operad functor inherits a monad structure, which, in abstract terms, consists of an associative monoid object structure in the composition category of functors. In §C.4, we give a description of this monadic multiplication, by using the result of the previous section, and we prove that the notion of an operad can be defined in terms of the free operad monad. In the language of category theory, this result asserts that the category of operads is monadic.

Appendix D. The construction of cofree cooperads. In this appendix, we examine a dualization of the constructions of §§B-C with the aim of giving an explicit definition of cofree objects in the category of cooperads. In categorical terms, the dualization process implies the replacement of colimits by limits. This process creates difficulties since the tensor product, involved in all structure definitions, does not commute with all limits, and this problem can hardly be overcome in general. But, under our general connectedness assumption, we still have a simple construction of the cofree cooperad. In short, we observe that the categorical dualization can be performed incompletely when we deal with connected structures: we construct our cofree cooperad with the same underlying functor as the free operad, and with colimits yet, but we provide the obtained object with a cooperad coproduct structure instead of an operadic composition structure. The crux of our argument line lies in the observation that the colimits occurring in our construction reduce to finite coproducts. To be precise, this statement implies that our construction of the cofree cooperad returns the obvious cofree object when the ground category is additive. In general, we still get a structure, which can serve to define cofree objects, but we do not get a cooperad in the obvious sense of the term.

In order to achieve our construction, we devote a first section to a thorough analysis of operadic decomposition of trees, which we use in a second section to define the coproduct structure of the cofree cooperad. Then, to complete our results, we study a natural comonad structure on the cofree cooperad, dual to the monad structure considered in §C for the free operad, and we establish the dual of the result of §C.4: the category of cooperads is comonadic.

Appendix E. The structure of Hopf algebras. In this appendix, we will survey classical results on the structure of cocommutative Hopf algebras and universal enveloping algebras, notably: the equivalence between Lie algebras and cocommutative Hopf algebras given by the Milnor-Moore theorem; and the bijection between primitive and group like elements in complete Hopf algebras. In addition, we will review the definition of the rational completion of groups from the viewpoint of Hopf algebras. The definition of the prounipotent version of the Grothendieck-Teichmüller group, which we consider in this book, heavily relies on this completion process. The structure theorem of cocommutative Hopf algebras are also heavily

used in our study of deformation complexes in Part 2, and in our study of the Drinfeld-Kohno tower in Part 3.

Part 0

Background

CHAPTER I

Operads and Algebras over Operads

The main purpose of this chapter is to explain the general definition of an operad, and of an algebra over an operad. We achieve this program in several stages. We first provide an overall survey of the basic definitions of the theory, and we introduce some fundamental examples of operads which enable us to illustrate the general concepts (§I.1). Afterwards, we explain the application of standard constructions of category theory (like free objects, colimits, limits) to operads (§I.2) and to algebras over operads (§I.3). We use these constructions to identify the usual categories of algebras (associative algebras, commutative algebras, Lie algebras) with categories of algebras associated to operads. We address apart, in a separate section (§I.4), the case of operads associated to algebras equipped with unitary operations. An appendix (§I.5) is devoted to a short survey of the definition of particular colimits (reflexive coequalizers and filtered colimits), which we use in applications to operads.

The basic definition of an operad (and the definition of an algebra over an operad similarly) makes sense in the general setting of symmetric monoidal categories, where we only assume the existence of a tensor product satisfying the fundamental unit, associativity and symmetry axioms §0.6(a-c). This subject, the definition of operads in general symmetric monoidal categories, and the application of symmetric monoidal category concepts to operads, will be examined in the next chapter.

For the moment, we work within a base category $\mathcal{B}ase$, which we take among the category of sets $\mathcal{B}ase = \mathcal{S}et$, simplicial sets $\mathcal{B}ase = \mathcal{S}imp$, topological spaces $\mathcal{B}ase = \mathcal{T}op$, modules over the ground ring $\mathcal{B}ase = \mathcal{M}od$, or among a variant of these categories. The categories of cocommutative coalgebras, which we introduce soon, in §II.0.3, provide other examples of categories to which we can apply all results and constructions of the present chapter, at least when we deal with coalgebras defined over a field, otherwise technical restrictions are needed.

In fact, we essentially use in this chapter that $\mathcal{B}ase$ is equipped with a tensor product $\otimes : \mathcal{B}ase \times \mathcal{B}ase \to \mathcal{B}ase$ preserving colimits on each side, and with an internal-hom $\mathtt{Hom}_{\mathcal{B}ase}(-,-) : \mathcal{B}ase^{op} \times \mathcal{B}ase \to \mathcal{B}ase$ providing $\mathcal{B}ase$ with a closed symmetric monoidal category structure (see §§0.6-0.11).

The preservation of colimits by the tensor product is heavily used in §§I.2-I.3, when we examine the application of categorical constructions to operads and to algebras over operads. Recall that the preservation of colimits is also a necessary condition for the existence of the internal-hom (see §0.11). Therefore, as soon as we deal with internal hom-objects (in §§I.1.10-I.1.14) we implicitly assume that the colimit requirement is fulfilled.

We need to specify some conventions regarding limits and colimits. We generally use the notation 0 for the initial object of the ground category $\mathcal{B}ase$ and the notation $\oplus$ for coproducts. (We do not assume however that the ground category is

additive.) We only go back to set theoretic notation $\emptyset$ and $\amalg$ when we specifically address constructions within the category of sets, topological spaces, or simplicial sets. On the other hand, we use the notation $*$ to refer to the terminal object of categories, except in particular situations where we explicitly assume that our base category has a zero object.

In §§I.2-I.3, we explain that the category of operads and the categories of algebras associated to an operad have all limits and colimits. The limits of operads and algebras are created in the underlying ground category in general, as well as some particular colimits, but not coproducts (see §§I.2-I.3). Therefore, we keep the notation of the ground category for limits in the category of operads and in categories of algebras over operads, but we will adopt another style of notation (the base set notation $\vee$) for coproducts.

This chapter does not include any original idea, apart from our definition of free unitary operads in §I.4. Our purpose is to provide a survey of the existing literature on operads. Besides the monograph [72], we should cite Boardman-Vogt' work [17], leading to another approach of operads, and Ginzburg-Kapranov' work [46], from which we borrow the definition free operads and the definition of operads defined by generators and relations in the non-unitary setting. Reference books, emphasizing various aspects of operads, include [38] (about modules and algebra categories associated to operads), [60] (about operads and higher categories), [63] (focusing on algebraic operads and the Koszul duality theory), [69] (providing an overall introduction to operads and to the Koszul duality), alongside monographs on more focused topics which we will cite when appropriate.

I.1. The definition of an operad and algebras over operads

The purpose of this section, as we just explained, is to introduce the definition of an operad and of an algebra over an operad. We have several approaches available. In this introductory chapter, we mostly deal with May's definitions [72], which has the advantage of giving a direct and simple interpretation of operadic structures in terms of operations acting on algebras. In §§A-C, we will explain the equivalence between May's approach and more combinatorial definitions of operads, involving an interpretation of operadic composition structures in terms of trees. To prepare this subsequent revision of the definition, we give a first introduction to the tree representation of operads in this section. We will more heavily use the formalism of trees in our study of deformation complexes of operads.

Intuitively, an operad P consists of a collection of objects $\mathsf{P}(r)$ collecting abstract operations of r variables $p = p(x_1, \ldots, x_r)$ with a variable number r running over $\mathbb{N}$. The notion of an operad is formally defined as a structure given by such a collection together with composition products modeling the composition of operations. From this viewpoint, an operad can be regarded as a particular instance of analyzer, a notion introduced by Lazard in [59] in order to generalize the power series operations used in the theory of formal Lie groups.

In the literature, the number of variables r in an operation $p = p(x_1, \ldots, x_r)$ (not necessarily related to an operad) is sometimes referred to as the arity of p. Therefore, in the operadic context, we use the term of arity to refer to the number r indexing the terms of an operad and of any related structure. In §A.4, we consider an extension of the definition of an operad where terms $\mathsf{P}(\underline{\mathsf{r}})$ indexed by all finite sets $\underline{\mathsf{r}} = \{i_1, \ldots, i_r\}$ are considered. In this setting, we use the term of arity to refer

to the cardinal of the set $\underline{r} = \{i_1, \ldots, i_r\}$ (either regarded as a non-negative integer, or as a class of finite sets in bijection to each other). But for the moment, we focus on operad terms $\mathsf{P}(r)$ indexed by non-negative integers $r \in \mathbb{N}$, which correspond to finite ordinals $\underline{r} = \{1 < \cdots < r\}$ in this finite set indexing.

The explicit definition of an operad, beyond the intuitive approach, is quite intricate. In fact, this definition is recursive in nature, because it implicitly relies on a primitive operad structure on permutations. In the logical order, we should explicitly define the operations underlying the composition structure of the permutation operad first, and introduce the general definition of an operad afterwards. But, we will proceed differently in order to bring out the ideas underlying the definition. In a first stage, we only define the shape of the structure of an operad. This incomplete account is enough to fully explain the intuitive interpretation of the operad formalism, which we do next. Then we give the missing part of our definition, which amounts to the definition of the alluded-to primitive operad structure on permutation groups.

I.1.1. *The axiomatic definition of an operad.* Formally, an operad in a base category $\mathcal{B}ase$ consists of a sequence of objects $\mathsf{P}(r) \in \mathcal{B}ase$, $r \in \mathbb{N}$, where $\mathsf{P}(r)$ is equipped with an action of the symmetric group on r letters Σ_r, together with

(a) a unit morphism $\eta : \mathbb{1} \to \mathsf{P}(1)$,
(b) and a composition structure, defined by morphisms

$$\mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r) \xrightarrow{\mu} \mathsf{P}(n_1 + \cdots + n_r),$$

for $r \geq 0$ and $n_1, \ldots, n_r \geq 0$,

so that natural equivariance, unit and associativity relations, expressed by the commutativity of the diagrams of Figure I.1, I.2, and I.3, hold. The definition of the permutations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1, \ldots, n_r)$, occurring in the equivariance relations (Figure I.1), is put off until §I.1.6.

The morphism η is referred to as the unit morphism of the operad, and the morphisms μ as the composition products. In what follows, we also use the terminology of full composition product to distinguish these morphisms μ from partial composition operations which we introduce later on.

In general, we specify an operad by the notation of the underlying collection P, and we use the letters η and μ as generic notation for the corresponding unit and product morphisms. We simply add a lower-script $\eta = \eta_{\mathsf{P}}$ (respectively, $\mu = \mu_{\mathsf{P}}$) specifying the operad to which this unit (respectively, product) morphism is attached when necessary.

We have a natural notion of morphism attached to operads. An operad morphism $\phi : \mathsf{P} \to \mathsf{Q}$ consists, to be precise, of a sequence of morphisms in the base category $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$, $r \in \mathbb{N}$, commuting with the action of symmetric groups and preserving the unit and the composition structure of the operads. Accordingly, the operads in a fixed base category $\mathcal{B}ase$ form a category, for which we adopt the notation $\mathcal{O}p$. In what follows, we will simply add a prefix to this notation $\mathcal{O}p = \mathcal{B}ase\mathcal{O}p$ when we need to specify the base category in which our operads are defined. As an example, we may use the notation $\mathcal{T}op\mathcal{O}p$ to refer to the category of topological operads (corresponding to the case of operads in topological spaces).

I.1.2. *Miscellaneous remarks on the definition of an operad.* In the case $r = 0$, the composition product of §I.1.1(b) involves an empty set of factors $\mathsf{P}(n_i)$ and reduces to an endomorphism of $\mathsf{P}(0)$. The (right) unit axiom of Figure I.2 forces this

$$\begin{tikzcd}
\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[r, "id\otimes(t_1\otimes\cdots\otimes t_r)"] \arrow[d, "\mu"'] & \mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[d, "\mu"] \\
\mathsf{P}(n_1+\cdots+n_r) \arrow[r, "t_1\oplus\cdots\oplus t_r"'] & \mathsf{P}(n_1+\cdots+n_r)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[rr, "s\otimes(id\otimes\cdots\otimes id)"] \arrow[d, "id\otimes s^*"', "\simeq"] & & \mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[d, "\mu"] \\
\mathsf{P}(r)\otimes\mathsf{P}(n_{s(1)})\otimes\cdots\otimes\mathsf{P}(n_{s(r)}) \arrow[dr, "\mu"'] & & \mathsf{P}(n_1+\cdots+n_r) \\
& \mathsf{P}(n_{s(1)}+\cdots+n_{s(r)}) \arrow[ur, "s_*(n_1,\ldots,n_r)"'] &
\end{tikzcd}$$

FIGURE I.1. The equivariance axioms of operads, required to hold for all arities $r \geq 0$, $n_1, \ldots, n_r \geq 0$, and all permutations $s \in \Sigma_r$ and $t_1 \in \Sigma_{n_1}, \ldots, t_r \in \Sigma_{n_r}$.

$$\begin{tikzcd}
\mathbb{1}\otimes\mathsf{P}(n) \arrow[r, "\eta\otimes id"] \arrow[dr, "\simeq"'] & \mathsf{P}(1)\otimes\mathsf{P}(n) \arrow[d, "\mu"] \\
& \mathsf{P}(n)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{P}(r)\otimes\mathbb{1}^{\otimes r} \arrow[r, "id\otimes\eta^{\otimes r}"] \arrow[dr, "\simeq"'] & \mathsf{P}(r)\otimes\mathsf{P}(1)^{\otimes r} \arrow[d, "\mu"] \\
& \mathsf{P}(r)
\end{tikzcd}$$

FIGURE I.2. The unit axioms of operads, required to hold for all $r \geq 0$ and all $n \geq 0$.

$$\begin{tikzcd}
\{\mathsf{P}(r)\otimes\mathsf{P}(s_1)\otimes\cdots\otimes\mathsf{P}(s_r)\}\otimes\{\mathsf{P}(n_1^1)\otimes\cdots\otimes\mathsf{P}(n_1^{s_1})\}\otimes\cdots\otimes\{\mathsf{P}(n_r^1)\otimes\cdots\otimes\mathsf{P}(n_r^{s_r})\} \arrow[rr, "\simeq"] \arrow[d, "\mu\otimes id\otimes\cdots\otimes id"'] & & \mathsf{P}(r)\otimes\{\mathsf{P}(s_1)\otimes\mathsf{P}(n_1^1)\otimes\cdots\otimes\mathsf{P}(n_1^{s_1})\}\otimes\cdots\otimes\{\mathsf{P}(s_r)\otimes\mathsf{P}(n_r^1)\otimes\cdots\otimes\mathsf{P}(n_r^{s_r})\} \arrow[d, "id\otimes\mu\otimes\cdots\otimes\mu"] \\
\mathsf{P}(s_1+\cdots+s_r)\otimes\mathsf{P}(n_1^1)\otimes\cdots\otimes\mathsf{P}(n_1^{s_1})\otimes\cdots\otimes\mathsf{P}(n_r^1)\otimes\cdots\otimes\mathsf{P}(n_r^{s_r}) \arrow[dr, "\mu"'] & & \mathsf{P}(r)\otimes\mathsf{P}(n_1^1+\cdots+n_1^{s_1})\otimes\cdots\otimes\mathsf{P}(n_r^1+\cdots+n_r^{s_r}) \arrow[dl, "\mu"] \\
& \mathsf{P}(n_1^1+\cdots+n_1^{s_1}+\cdots+n_r^1+\cdots+n_r^{s_r}) &
\end{tikzcd}$$

FIGURE I.3. The associativity axiom of operads, required to hold for all arities $r \geq 0$, $s_1, \ldots, s_r \geq 0$, and all $n_i^j \geq 0$.

endomorphism to be the identity. Thus, the consideration of a composition product for $r = 0$ does not add anything to the structure. Nevertheless, the formulation

$$\begin{tikzcd}
\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[rr, "s\otimes(t_1\otimes\cdots\otimes t_r)"] \arrow[d, "id\otimes s^*"', "\simeq"] & & \mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[d, "\mu"] \\
\mathsf{P}(r)\otimes\mathsf{P}(n_{s(1)})\otimes\cdots\otimes\mathsf{P}(n_{s(r)}) \arrow[dr, "\mu"'] & & \mathsf{P}(n_1+\cdots+n_r) \\
& \mathsf{P}(n_{s(1)}+\cdots+n_{s(r)}) \arrow[ur, "s(t_1,\ldots,t_r)"'] &
\end{tikzcd}$$

FIGURE I.4. The equivariance axioms of operads, put in a single diagram, where $s(t_1,\ldots,t_r)\in\Sigma_{n_1+\cdots+n_r}$ is actually an operadic composite of the given permutations $s\in\Sigma_r$ and $t_1\in\Sigma_{n_1},\ldots,t_r\in\Sigma_{n_r}$.

of the associativity axiom in full generality in Figure I.3, requires to integrate this degenerate case in the definition of the composition structure of an operad.

The equivariance axioms of Figure I.1 can also be put together in a single equivalent commutative diagram, displayed in Figure I.4. The permutation $s(t_1,\ldots,t_r)$ occurring in this diagram is given by the composite $s(t_1,\ldots,t_r)=t_1\oplus\cdots\oplus t_r\cdot s_*(n_1,\ldots,n_r)$ of the permutations $t_1\oplus\cdots\oplus t_r$ and $s_*(n_1,\ldots,n_r)$, occurring in our initial equivariance axioms. Soon (in Proposition I.1.8), we identify this composite permutation $s(t_1,\ldots,t_r)$ as the outcome of an operadic composition product on permutations.

Intuitively, the object $\mathsf{P}(r)$ collects abstract operations $p=p(x_1,\ldots,x_r)$ in a given arity $r\in\mathbb{N}$ (as we explain in the introduction of this section). The composition structure of §I.1.1(a-b) reflects a natural composition structure attached to operations of this form, and the definition of our operations $t_1\oplus\cdots\oplus t_r$ and $s_*(n_1,\ldots,n_r)$ on permutations reflects this interpretation of the composition structure. Thus, we explain this interpretation first, from the shape of our axioms, and we explicitly define the permutations $t_1\oplus\cdots\oplus t_r$ and $s_*(n_1,\ldots,n_r)$ afterwards.

I.1.3. *The interpretation of an operad structure.* In a point set context, we may use the notation $p(q_1,\ldots,q_r)\in\mathsf{P}(n_1,\ldots,n_r)$ for the image of a tensor $p\otimes(q_1\otimes\cdots\otimes q_r)\in\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r)$ under the composition product §I.1.1(b). The unit morphism of §I.1.1(a) is also equivalent to the definition of a distinguished element $1\in\mathsf{P}(1)$, referred to as the unit of the operad. In many constructions, we consider partial composition operations $\circ_i:\mathsf{P}(m)\otimes\mathsf{P}(n)\to\mathsf{P}(m+n-1)$ determined from the composition product by the formula $p\circ_i q=p(1,\ldots,1,q,1,\ldots,1)$ where the operation $q\in\mathsf{P}(n)$ is plugged in the ith input of $p\in\mathsf{P}(m)$. Operad units are assigned to the remaining inputs of p.

In the intuitive interpretation of elements $p\in\mathsf{P}(r)$ in terms of abstract operations $p=p(x_1,\ldots,x_r)$, the action of permutations $s\in\Sigma_r$ on $\mathsf{P}(r)$ models a permutation of inputs

$$sp=p(x_{s(1)},\ldots,x_{s(r)}),$$

the operadic composition process models the definition of composite operations of the form

$$\begin{aligned} p(q_1,\dots,q_r) = p(&q_1(x_{k_1+1},\dots,x_{k_1+n_1}),\\ &q_2(x_{k_2+1},\dots,x_{k_2+n_2}),\\ &\vdots\\ &q_r(x_{k_r+1},\dots,x_{k_r+n_r})), \end{aligned}$$

where we set $k_i = n_1 + \cdots + n_{i-1}$. Thus, in the expression of the composite $p(q_1,\dots,q_r)$, the variables are split into groupings attached to each plugged operation q_i. Similarly, the operadic unit represents an identity operation (of one variable) $1 = id(x_1)$ and a partial composite $p \circ_i q = p(1,\dots,1,q,1,\dots,1)$ can be identified with a composite operation of the form

$$p \circ_i q = p(x_1,\dots,x_{i-1},q(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{m+n-1}).$$

In these point set representations, the unit axioms read $1(p) = p$, $p(1,\dots,1) = p$, and the associativity axiom reads

$$p(q_1,\dots,q_r)(\varrho_1^1,\dots,\varrho_1^{s_1},\dots,\varrho_r^1,\dots,\varrho_r^{s_r}) = p(q_1(\varrho_1^1,\dots,\varrho_1^{s_1}),\dots,q_r(\varrho_r^1,\dots,\varrho_r^{s_r})),$$

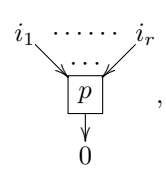
where we assume $p \in \mathsf{P}(r)$, $q_1 \in \mathsf{P}(s_1),\dots,q_r \in \mathsf{P}(s_r)$ and $\varrho_i^j \in \mathsf{P}(n_i^j)$. The equivariance axioms come from the identities

$$\begin{aligned} &p(t_1q_1,\dots,t_rq_r)\\ &\qquad = p(q_1(x_{k_1+t_1(1)},\dots,x_{k_1+t_1(n_1)}),\dots,q_r(x_{k_r+t_r(1)},\dots,x_{k_r+t_r(n_r)})),\\ &sp(q_1,\dots,q_r)\\ &\qquad = p(q_{s(1)}(x_{k_{s(1)}+1},\dots,x_{k_{s(1)}+n_{s(1)}}),\dots,q_{s(r)}(x_{k_{s(r)}+1},\dots,x_{k_{s(r)}+n_{s(r)}})). \end{aligned}$$

The permutations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1,\dots,n_r)$, which we formally define in §I.1.6, correspond to the input permutations occurring in these formulas.

Note that the composition product can be written in terms of partial composites. Indeed, the unit and associativity axioms imply that the composition product satisfies $p(q_1,\dots,q_r) = (\cdots(p \circ_{k_1+1} q_1) \circ_{k_2+1} \cdots) \circ_{k_r+1} q_r$, for any $p \in \mathsf{P}(r)$ and all $q_1 \in \mathsf{P}(n_1),\dots,q_r \in \mathsf{P}(n_r)$, where we set $k_i = n_1 + \cdots + n_{i-1}$ for $i = 1,\dots,r$. This observation is fully developed in §A.1, where we give another definition, in terms of the partial composition operations, of the composition structure of an operad.

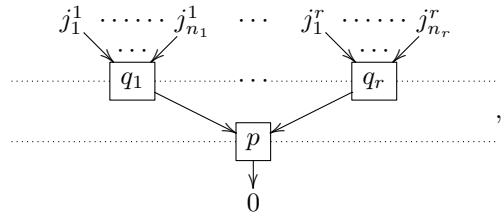
I.1.4. *The graphical representation of operad elements.* To gain intuition, we may also use a black box picture



,

to represent operations of the form collected by an operad $p \in \mathsf{P}(r)$. The ingoing edges of the box represent the inputs of the operation and the outgoing edge is used to symbolize the output.

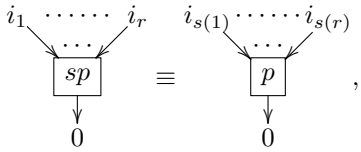
In this picture, the composition products of an operad model composition patterns of the following form



where we plug the outputs of the upper level operations $q_1 \in \mathsf{P}(n_1), \ldots, q_r \in \mathsf{P}(n_r)$ in the inputs of the lower level operation $p \in \mathsf{P}(r)$ to obtain a composite operation $p(q_1, \ldots, q_r) \in \mathsf{P}(n_1 + \cdots + n_r)$ with as much inputs as the upper level operations together and one final output. In the sequel, we use the above picture to represent the tensor $p \otimes (q_1 \otimes \cdots \otimes q_r) \in \mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r)$, to which the operadic composition product is applied, instead of the outcome of the process.

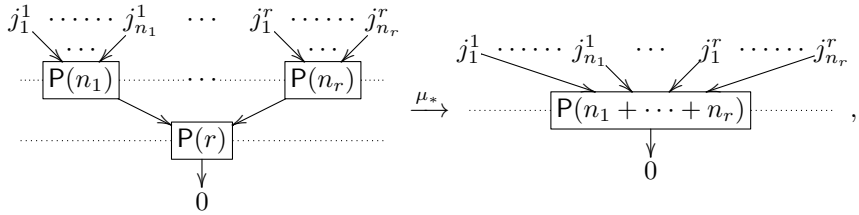
For the moment, as long as we assume that operad terms are indexed by finite ordinals, we can assume that the ingoing edges of a box in such a figure are arranged in the plan according to the input ordering $\underline{\mathsf{r}} = \{1 < \cdots < r\}$ of the corresponding operation $p \in \mathsf{P}(r)$. However, we can make a convenient use of indices attached to ingoing edges of boxes in our figures. Namely, we can use the indices of ingoing edges to mark operations on the inputs of operations. For this purpose, we take the convention that edges represent a bijection, not necessarily the identity one, between an indexing set and the input set of an operation. In the picture of composite operations for instance, we associate the indices $j_k^i = n_1 + \cdots + n_{i-1} + k$, $k = 1, \ldots, n_i$, corresponding to the input indexing of the composite operation $p(q_1, \ldots, q_r)$, with the ingoing edges of the box q_i, $i = 1, \ldots, r$.

To identify equivalent indexings, we simply apply the relation



when we apply a permutation $s \in \Sigma_r$ to the inner operation $p \in \mathsf{P}(r)$. This formalism is fully explained in §A.4, where we consider operad components associated to all finite sets.

I.1.5. *The graphical representation of an operad structure.* The representation of the previous paragraphs can be applied to operad components (instead of elements), and under our conventions, the composition products of an operad can be depicted as morphisms



where the tree-wise arrangement of operad components formally represents the tensor product of these objects.

The unit and associativity relations of operads correspond in the tree-wise representation to the composition schemes of Figure I.5-I.6. In these representations, we identify the application of operadic units and operadic composition products with internal operations on factors forming our tree-wise tensor product. In general, we use the notation η_* and μ_*, symbolizing the performance of internal operations on tree-wise tensors, for these mappings. The factors to which we apply the operation can in principle be identified from the internal structure of the source and target tree of our mapping.

I.1.6. *Fundamental operations on permutations.* We now define the permutations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1, \ldots, n_r)$ occurring in the equivariance relations of Figure I.1. We use the notation $k_i = n_1 + \cdots + n_{i-1} + 1$ introduced in the previous paragraphs. To make our definition more explicit, we use that a permutation of $(1, \ldots, r)$ is equivalent to an ordered sequence $w = (w(1), \ldots, w(n))$ in which each value $k = 1, \ldots, r$ occurs once and only once. In some cases, we can also use the standard table representation

$$w = \begin{pmatrix} 1 & \cdots & n \\ w(1) & \cdots & w(n) \end{pmatrix}.$$

The direct sum of permutations $t_1 \in \Sigma_{n_1}, \ldots, t_r \in \Sigma_{n_r}$ is the permutation of $\{1, \ldots, n_1 + \cdots + n_r\}$ given by the action of t_i on the interval $\{k_i + 1, \ldots, k_i + n_i\} \subset \{1, \ldots, n_1 + \cdots + n_r\}$ through the canonical bijection of ordered sets $\{1 < \cdots < n_i\} \xrightarrow{\simeq} \{k_i + 1 < \cdots < k_i + n_i\}$. This permutation is represented by the sequence

$$t_1 \oplus \cdots \oplus t_r = (k_1 + t_1(1), \ldots, k_1 + t_1(n_1), \ldots, k_r + t_r(1), \ldots, k_r + t_r(n_r))$$

formed by the concatenation of the sequences $t_i = (t_i(1), \ldots, t_i(n_i))$ associated to the permutations t_i, $i = 1, \ldots, r$, together with the index shifts k_i. For instance, in the case of a pair of permutations $s \in \Sigma_m$ and $t \in \Sigma_n$, we obtain the identity:

$$\underbrace{(s(1), \ldots, s(m))}_{s} \oplus \underbrace{(t(1), \ldots, t(n))}_{t} = \begin{pmatrix} 1 & \cdots & m & m+1 & \cdots & m+n \\ s(1) & \cdots & s(m) & m+t(1) & \cdots & m+t(n) \end{pmatrix}.$$

The block permutation $s_*(n_1, \ldots, n_r)$ associated to a permutation $s \in \Sigma_r$, where $n_1, \ldots, n_r \geq 0$ is any collection of natural numbers, is given by the permutation, under s, of the intervals

$$\underline{\mathsf{n}}_i = (k_i + 1, k_i + 2, \ldots, k_i + n_i)$$

in the ambient set $\{1, \ldots, n_1 + \cdots + n_r\}$. In the sequence representation of permutations, the block permutation $s_*(n_1, \ldots, n_r)$ is defined by the sequence

$$s_*(n_1, \ldots, n_r) = (\underline{\mathsf{n}}_{s(1)}, \ldots, \underline{\mathsf{n}}_{s(r)})$$

formed by the concatenation of the blocks $\underline{\mathsf{n}}_i$ ordered according to the permutation s. For instance, the block permutation $t_*(m, n)$ associated to a transposition $t = (1\ 2) \in \Sigma_2$ has the form:

$$\underbrace{(m+1, \ldots, m+n, 1, \ldots, m)}_{t_*(m,n)} = \begin{pmatrix} 1 & \cdots & n & n+1 & \cdots & n+m \\ m+1 & \cdots & m+n & 1 & \cdots & m \end{pmatrix}.$$

The following proposition follows from easy verifications:

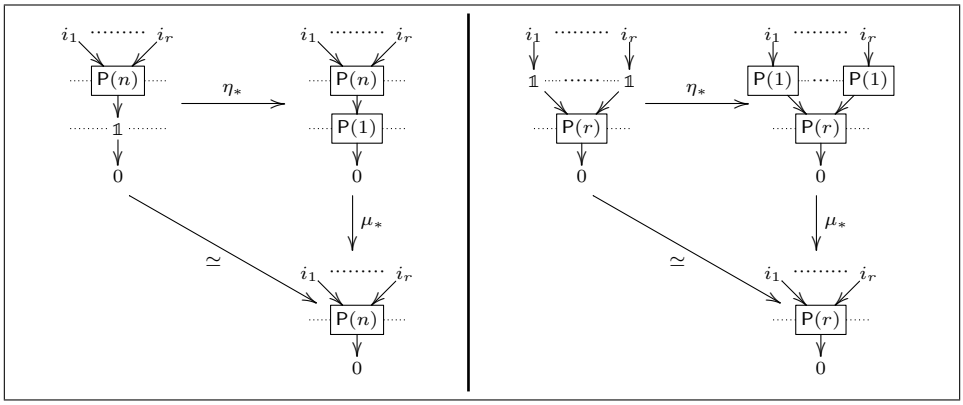


FIGURE I.5. The tree-wise representation of the unit relations of operads

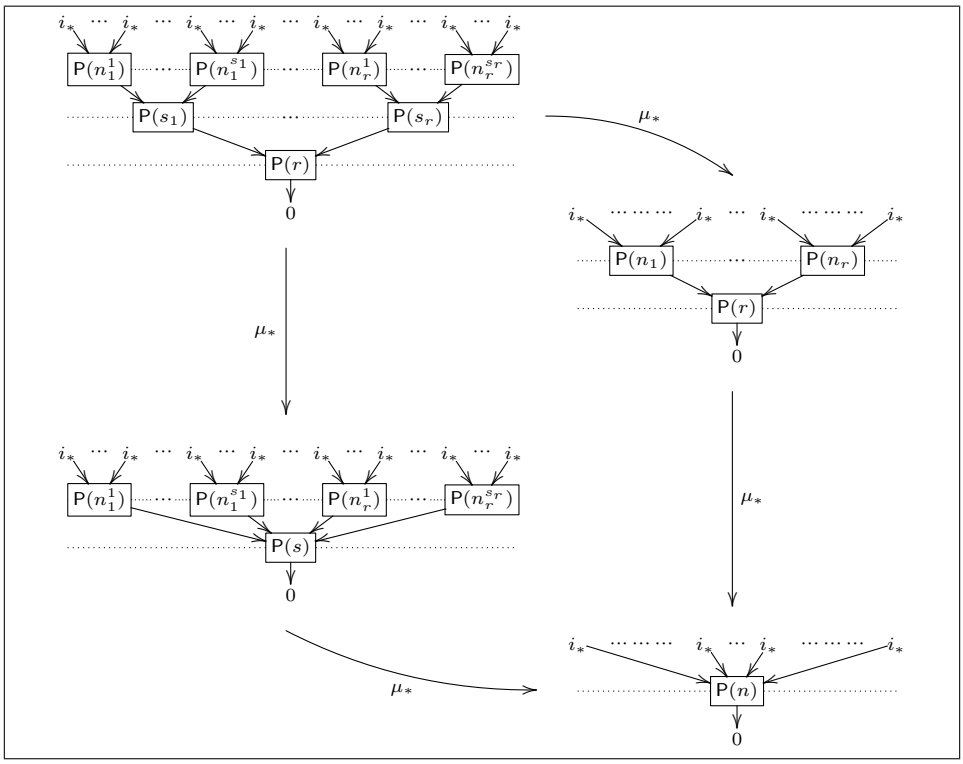


FIGURE I.6. The tree-wise representation of the associativity relations of operads, where, to shorten notation, we set $n_i = n_i^1 + \cdots + n_i^{s_i}$ for $i = 1, \ldots, r$, and $s = s_1 + \cdots + s_r$, $n = n_1 + \cdots + n_r$.

PROPOSITION I.1.7. *Let $n_1, \dots, n_r \geq 0$. In the symmetric group $\Sigma_{n_1+\cdots+n_r}$, we have the relation*

$$s_1 \oplus \cdots \oplus s_r \cdot t_1 \oplus \cdots \oplus t_r = (s_1 t_1) \oplus \cdots \oplus (s_r t_r)$$

for all r-tuples of permutations $(s_1, \dots, s_r), (t_1, \dots, t_r) \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$, the relation

$$s_*(n_1, \dots, n_r) \cdot t_*(n_{s(1)}, \dots, n_{s(r)}) = (st)_*(n_1, \dots, n_r).$$

for every $s, t \in \Sigma_r$, as well as the relation

$$t_1 \oplus \cdots \oplus t_r \cdot s_*(n_1, \dots, n_r) = s_*(n_1, \dots, n_r) \cdot t_{s(1)} \oplus \cdots \oplus t_{s(r)}$$

for every $s \in \Sigma_r$ and all $(t_1, \dots, t_r) \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$. □

Then we obtain:

PROPOSITION I.1.8. *The collection of symmetric groups Σ_n, $n \in \mathbb{N}$, forms an operad in sets so that:*

(a) *the action of the symmetric group on each Σ_n is given by left translations;*
(b) *and the composition product $\mu : \Sigma_r \times (\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}) \to \Sigma_{n_1+\cdots+n_r}$ maps a collection $s \in \Sigma_r$, $(t_1, \dots, t_r) \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$, to the product permutation $s(t_1, \dots, t_r) = t_1 \oplus \cdots \oplus t_r \cdot s_*(n_1, \dots, n_r)$.*

PROOF. Easy verification from the relations of Proposition I.1.7. □

This proposition explains our remark that the operations $t_1 \oplus \cdots \oplus t_r$ and $s_*(n_1, \dots, n_r)$, which occur in the general definition of an operad, come themselves from a primitive operadic composition on the collection of symmetric groups. The definition of the composite $s(t_1, \dots, t_r)$ in Proposition I.1.8 is forced by the equivariance axioms of operads and the requirement $id_r(id_{n_1}, \dots, id_{n_r}) = id_{n_1+\cdots+n_r}$, where we use the notation id_n for the identity permutation, with the lower-script indicating the cardinal of the permutation set (see §0.7). In this sense, the result of Proposition I.1.8 expresses the internal coherence of the definition of an operad.

To give another (more) simple example, we can readily see that:

PROPOSITION I.1.9. *The collection of one-point sets $pt(r) = pt$ form an operad in sets. In this case, the symmetric group action is trivial in each arity, and we take identities of one-point sets to define the composition unit and the composition products of the operad.* □

In §II, we define a generalization of this one-point set operad in the context of symmetric monoidal categories.

Soon (see §§I.1.15-I.1.17), we explain that the permutation operad, as defined in Proposition I.1.8, is identified with the operad of associative monoids, and the one-point set operad is identified with the operad of commutative monoids. But before studying operads associated to basic algebraic structures, we explain the definition of universal operads $\mathtt{End}_A$ associated to the objects A of the base category $\mathcal{B}ase$.

I.1.10. *Endomorphism operads.* The operad $\mathtt{End}_A$ associated to an object $A \in \mathcal{B}ase$ is called the endomorphism operad of A. The definition of this endomorphism operad involves the internal hom-bifunctor of the base category $\mathtt{Hom}_{\mathcal{B}ase}(-,-) : \mathcal{B}ase^{op} \times \mathcal{B}ase \to \mathcal{B}ase$. To simplify the writing, we use the notation $\mathtt{Hom}(-,-) = \mathtt{Hom}_{\mathcal{B}ase}(-,-)$ for this hom bifunctor throughout this paragraph.

Basically, the endomorphism operad of $A \in \mathcal{B}ase$ is defined by the collection of hom-objects

$$\mathtt{End}_A(r) = \mathtt{Hom}(A^{\otimes r}, A),$$

where we form the tensor powers of A in the base category $\mathcal{B}ase$. By functoriality, the hom-objects $\mathtt{End}_A(r) = \mathtt{Hom}(A^{\otimes r}, A)$ inherit an action of symmetric groups from the tensor powers $A^{\otimes r}$ and this gives the symmetric structure of the endomorphism operad.

By adjunction, the composite evaluation morphisms

$$\begin{aligned}\mathtt{Hom}(A^{\otimes r}, A) \otimes \Big\{\bigotimes_{i=1}^{r} \mathtt{Hom}(A^{\otimes n_i}, A)\Big\} \otimes A^{\otimes n} \\ \xrightarrow{\simeq} \mathtt{Hom}(A^{\otimes r}, A) \otimes \Big\{\bigotimes_{i=1}^{r} \mathtt{Hom}(A^{\otimes n_i}, A) \otimes A^{\otimes n_i}\Big\} \\ \xrightarrow{\epsilon} \mathtt{Hom}(A^{\otimes r}, A) \otimes A^{\otimes r} \xrightarrow{\epsilon} A\end{aligned}$$

yield operadic composition operations

$$\mathtt{Hom}(A^{\otimes r}, A) \otimes \Big\{\bigotimes_{i=1}^{r} \mathtt{Hom}(A^{\otimes n_i}, A)\Big\} \xrightarrow{\mu} \mathtt{Hom}(A^{\otimes n}, A),$$

for all $r \geq 0$, $n_1, \ldots, n_r \geq 0$, and where we write $n = n_1 + \cdots + n_r$. These operations define the composition structure of the endomorphism operad. By adjunction too, the symmetric monoidal unit $\mathbb{1} \otimes A \simeq A$ gives a morphism

$$\mathbb{1} \xrightarrow{\eta} \mathtt{Hom}(A, A)$$

providing the collection $\mathtt{End}_A$ with an operadic unit.

The reader can easily check that the axioms of §I.1.1 are fully satisfied in $\mathtt{End}_A$, and hence we have a well defined operad structure on $\mathtt{End}_A$.

I.1.11. *Endomorphism operads in basic ground categories.* In the basic example of sets $\mathcal{B}ase = \mathcal{S}et$, the endomorphism operad of an object $X \in \mathcal{S}et$ consists of the map sets $\mathtt{End}_X(r) = \{f : X^{\times r} \to X\}$ and we have the pointwise formula

$$sf(x_1, \ldots, x_n) = f(x_{s(1)}, \ldots, x_{s(n)})$$

for the action of permutations, where the variables x_k now refer to actual elements of X, as well as the pointwise formula

$$\begin{aligned}f(g_1, \ldots, g_r)(x_1, \ldots, x_{n_1+\cdots+n_r}) = f(&g_1(x_{k_1+1}, \ldots, x_{k_1+n_1}),\\ &g_2(x_{k_2+1}, \ldots, x_{k_2+n_2}),\\ &\vdots\\ &g_r(x_{k_r+1}, \ldots, x_{k_r+n_r}))\end{aligned}$$

for the composition (where we still set $k_i = n_1 + \cdots + n_{i-1}$). The unit is the identity of X.

In the context of topological spaces $\mathcal{B}ase = \mathcal{T}op$, we have the same identification of the endomorphism operad $\mathtt{End}_X$ since the map sets $\mathtt{End}_X(r) = \{f : X^{\times r} \to X\}$ are also equipped with a topology which identify them with the internal hom-objects of the category of spaces $\mathcal{T}op$.

In the module context $\mathcal{B}ase = \mathcal{M}od$, the terms of the endomorphism operad $\mathtt{End}_K$, $K \in \mathcal{M}od$, consist of morphisms $f : K^{\otimes r} \to K$ by construction of hom-objects in $\mathcal{M}od$. Such morphisms $f : K^{\otimes r} \to K$ are equivalent to r-linear maps $f : (x_1, \ldots, x_r) \mapsto f(x_1, \ldots, x_r)$, and the action of permutations and operadic composition structures on such maps are given by the same pointwise formulas as in the context of sets.

I.1.12. *The structure of an algebra over an operad.* An algebra over an operad P (a P-algebra for short) is an object of the ground category $A \in \mathcal{B}ase$ together with morphisms

$$\text{(a)} \qquad \mathsf{P}(r) \otimes A^{\otimes r} \xrightarrow{\lambda} A,$$

given for all $r \geq 0$, and such that equivariance, associativity and unit relations, formalized by the commutative diagrams of Figure I.7-I.9, hold. In applications of this definition, we usually say that the morphisms (a) define the action of the operad P on the object $A \in \mathcal{B}ase$. Once an object A is provided with a fixed P-action, we also say that these morphisms (a) are the evaluation morphisms attached to the P-algebra A.

In general, we refer to a P-algebra by the expression of the underlying object A and we use the letter λ as a generic notation for the morphisms (a) defining the action of the operad on A. As in the operad case (see §I.1.1), we simply add the expression of the algebra as a lower-script to this notation $\lambda = \lambda_A$ when we need to specify it.

The P-algebras form a category with, as morphisms, the morphisms of the ground category $f : A \to B$ which preserve the P-actions on A and B. In what follows, we usually convert the notation of the operad P into calligraphic letters $\mathcal{P}$ in order to get the notation of the category of algebras associated to P. If necessary, then we specify the base category $\mathcal{B}ase$ by adding a prefix to this notation $\mathcal{P} = \mathcal{B}ase\,\mathcal{P}$.

I.1.13. *The interpretation of the structure of an algebra over an operad in the point set context.* In the point set context, we use the notation $p(a_1, \ldots, a_r) \in A$ for the image of a tensor $p \otimes (a_1 \otimes \cdots \otimes a_r) \in \mathsf{P}(r) \otimes A^{\otimes r}$ under the evaluation morphism §I.1.12(a). In the interpretation of operads given in §I.1.3, this evaluation morphism §I.1.12(a) amounts to the evaluation of abstract operations $p = p(x_1, \ldots, x_r)$ on actual elements $a_1, \ldots, a_r \in A$.

In the point-set representation, the unit axiom reads $1(a) = a$ for $a \in A$, the associativity axiom reads

$$p(q_1, \ldots, q_r)(a_1^1, \ldots, a_1^{n_1}, \ldots, a_r^1, \ldots, a_r^{n_r}) = p(q_1(a_1^1, \ldots, a_1^{s_1}), \ldots, q_r(a_r^1, \ldots, a_r^{s_r}))$$

for $p \in \mathsf{P}(r)$, $q_1 \in \mathsf{P}(n_1), \ldots, q_r \in \mathsf{P}(n_r)$ and $a_i^j \in A$, and the equivariance axiom reads

$$sp(a_1, \ldots, a_r) = p(a_{s(1)}, \ldots, a_{s(r)})$$

for $p \in \mathsf{P}(r)$ and $a_1, \ldots, a_r \in A$.

The graphical representation of §I.1.4 can also be applied to depict the action of operads on algebras. In short, we mark the ingoing edges of our black-boxes with algebra elements $a_1, \ldots, a_r \in A$, which we take as inputs for the operation represented by the black-box, and we mark the outgoing edge with the result of the

$$\begin{tikzcd}
\mathsf{P}(r)\otimes A^{\otimes r} \arrow[r, "s\otimes id"] \arrow[d, "id\otimes s^*"', "\simeq"] & \mathsf{P}(r)\otimes A^{\otimes r} \arrow[d, "\lambda"] \\
\mathsf{P}(r)\otimes A^{\otimes r} \arrow[r, "\lambda"'] & A
\end{tikzcd}$$

FIGURE I.7. The equivariance axiom for operad actions, required to hold for all $r \geq 0$, and all permutations $s \in \Sigma_r$.

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "\eta\otimes id"] \arrow[dr, "\simeq"'] & \mathsf{P}(1)\otimes A \arrow[d, "\lambda"] \\
& A
\end{tikzcd}$$

FIGURE I.8. The unit axiom for operad actions.

$$\begin{tikzcd}
\{\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r)\}\otimes A^{\otimes n} \arrow[rr, "\simeq"] \arrow[d, "\mu\otimes id^{\otimes n}"'] & & \mathsf{P}(r)\otimes\{\mathsf{P}(n_1)\otimes A^{\otimes n_1}\}\otimes\cdots\otimes\{\mathsf{P}(n_r)\otimes A^{\otimes n_r}\} \arrow[d, "id\otimes\lambda\otimes\cdots\otimes\lambda"] \\
\mathsf{P}(n_1+\cdots+n_r)\otimes A^{\otimes n} \arrow[dr, "\lambda"'] & & \mathsf{P}(r)\otimes A^{\otimes r} \arrow[dl, "\lambda"] \\
& A &
\end{tikzcd}$$

FIGURE I.9. The associativity axiom for operad actions, required to hold for all $r \geq 0$, $n_1, \ldots, n_r \geq 0$, and where we set $n = n_1 + \cdots + n_r$.

operation $b = p(a_1, \ldots, a_r)$. Thus, we get the following picture:

$$\begin{tikzcd}
a_1 \arrow[dr] & \cdots & a_r \arrow[dl] \\
& \boxed{p} \arrow[d] & \\
& b &
\end{tikzcd} .$$

In the context of a closed monoidal category, the morphisms §I.1.12(a), defining the action of an operad P on an object $A \in \mathcal{B}ase$ are, by adjunction, equivalent to morphisms

$$\phi : \mathsf{P}(r) \to \mathtt{Hom}_{\mathcal{B}ase}(A^{\otimes r}, A)$$

defined for all $r \geq 0$. The equivariance, unit and associativity axioms of operad actions in §I.1.12 are actually equivalent to the observation that these morphisms define an operad morphism from P towards the endomorphism operad associated to A. Hence, we obtain the following result:

PROPOSITION I.1.14. *The action of an operad P on an object $A \in \mathcal{B}ase$ is equivalent to an operad morphism $\phi : \mathsf{P} \to \mathtt{End}_A$, where $\mathtt{End}_A$ is the endomorphism operad of A.* □

The evaluation morphisms $\epsilon : \mathtt{Hom}_{\mathcal{B}ase}(A^{\otimes r}, A) \otimes A^{\otimes r} \to A$ actually give an action of the endomorphism operad $\mathtt{End}_A$ on A. In the equivalence of Proposition I.1.14, this action corresponds to the identity morphism of $\mathtt{End}_A$. The assertion of the proposition can be interpreted as the claim that the endomorphism operad $\mathtt{End}_A$ represents the universal operad acting on A in $\mathcal{B}ase$.

In a point-set context, the morphism $\phi : \mathsf{P} \to \mathtt{End}_A$ associates a map $p : A^{\otimes r} \to A$ to any operation $p \in \mathsf{P}(r)$. In the formalism of §I.1.13, we are simply considering the map $p : (a_1, \ldots, a_r) \mapsto p(a_1, \ldots, a_r)$ associated to a fixed element $p \in \mathsf{P}(r)$. The mapping ϕ is usually omitted in the notation of that map since the expression $p : A^{\otimes r} \to A$ already specifies that we consider a map associated to $p \in \mathsf{P}(r)$ and not the abstract operation itself $p = p(x_1, \ldots, x_r)$.

I.1.15. *Examples of operads associated to basic algebraic structures in sets.* In §I.2, we prove that many usual algebraic structures, including associative algebras and commutative algebras, are governed by operads. Our constructions work in any base category, including sets and $\Bbbk$-modules as most basic examples. The associative operad, the one which we associate to associative algebras, will be denoted by As. The commutative operad, the one which we associate to (associative and) commutative algebras, will be denoted by Com. In general, we do not assume that an algebra is equipped with a unit (unless we explicitly assert the contrary), and, to be precise, we use this notation As (respectively Com) for the version of the associative (respectively, commutative) operad governing the category associative (respectively, commutative) algebras without unit. To refer to the operads governing algebras with unit, we add a lower-script $+$ to the notation and we say that we deal with a unitary version of the operad. The connection between the operads governing the unitary and the non-unitary version of a structure is studied in details in §I.4. For the moment, simply say that As and As_+ agree in arity $r > 0$, but differ in arity $r = 0$, where we have $\mathsf{As}(0) = 0$ (the initial object of the base category) in the non-unitary case, and $\mathsf{As}_+(0) = \mathbb{1}$ (the tensor unit) in the unitary case. In the case of the commutative operad, we obtain the same relation.

We soon give a general and conceptual definition, by generators and relations, of such operads (see §I.2.6, §I.2.8). Nevertheless, we prefer to give a first direct construction of the operad associated to associative (respectively, commutative) algebras in order to complete this introductory account with simple examples. For the moment, we focus on the set-theoretical context, and then we rather use the terminology of associative (respectively, commutative) monoid for the associative (respectively, commutative) algebras. The case of $\Bbbk$-modules, which provides our second basic examples of symmetric monoidal categories (after the category of sets) will be addressed in the next sections.

Previously, we have observed that the collection of symmetric groups Σ_r, $r \in \mathbb{N}$, forms an operad in sets, as well as the collection of one-point sets $pt(r) = pt$. In the next propositions, we precisely prove that the permutation operad has the associative monoids as associated algebras, and the one-point set operad is associated to commutative monoids. In each case, we get structures with or without unit, depending on our choice as regards the term of arity 0 of the operad.

Proposition I.1.16. *The category of associative monoids with unit is isomorphic to the category of algebras over the permutation operad. The category of associative monoids without unit is isomorphic to the category of algebras over the operad formed by dropping the term of arity* 0 *in the permutation operad.*

By dropping the term of arity 0, we mean that we consider a sub-operad of the permutation operad such that $\mathsf{As}(0) = \emptyset$ and $\mathsf{As}(r) = \Sigma_r$ for $r > 0$.

Proof. Let A be an associative monoid with unit. To a permutation $w \in \Sigma_r$, we can associate the operation $w : A^{\times r} \to A$ such that $w(a_1, \ldots, a_r) = a_{w(1)} \cdot \ldots \cdot a_{w(r)}$. In plain terms, this operation is formed by the r-fold product of the sequence of elements $a_{w(1)}, \ldots, a_{w(r)}$ in the monoid A. In the case $r = 0$, we use the unit morphism $\eta : pt \to A$ (equivalent to an empty product) to define the operation assigned to the degenerate permutation $id_0 \in \Sigma_0$. The verification of the axioms of §I.1.12 is the matter of an easy understanding exercise. This process obviously gives a functor between the category of associative monoids with units and the category of algebras over the permutation operad.

In the converse direction, when A is an algebra over the permutation operad, we consider the unit operation $\eta : pt \to A$ associated to the degenerate permutation $id_0 \in \Sigma_0$ and the binary operation $\mu : A \times A \to A$ associated to the identity permutation $id_2 \in \Sigma_2$ in arity $r = 2$. The identity permutation in arity one $1 = id_1 \in \Sigma_1$ defines the unit of the permutation operad and, as such, is supposed to act as the identity operation on A. The unit operation $\eta : pt \to A$ is naturally equivalent to an element $e \in A$ which represents the image of the point pt under η. The identities $id_2(id_0, id_1) = id_1 = id_2(id_1, id_0)$ and $id_2(id_2, id_1) = id_3 = id_2(id_1, id_2)$ in the permutation operation are respectively equivalent to the unit $\mu(e, a) = a = \mu(a, e)$ and associativity relation $\mu(\mu(a_1, a_2), a_3) = \mu(a_1, \mu(a_2, a_3))$ in A. Hence, we have a monoid with unit naturally associated to each algebra over the permutation operad. This correspondence obviously gives a functor which is strictly inverse to the previously considered functor, from associative monoids with units to algebras over the permutation operad. This assertion finishes the proof of the first assertion of the proposition.

The second assertion follows from the same verification (simply drop the consideration of the degenerate permutation id_0 corresponding to the unit operation $\eta : pt \to A$ from our line of arguments). □

Proposition I.1.17. *The category of commutative monoids with unit is isomorphic to the category of algebras over the one-point set operad. The category of commutative monoids without unit is isomorphic to the category of algebras over the operad formed by dropping the term of arity* 0 *in the one-point set operad.*

By dropping the term of arity 0, we mean again that we consider a sub-operad of the one-point set operad such that $\mathsf{Com}(0) = \emptyset$ and $\mathsf{Com}(r) = pt$ for $r > 0$.

In the next section, we establish a generalization of this proposition in the context of symmetric monoidal categories.

Proof. The arguments are the same as in the case of algebras over the permutation operad (Proposition I.1.16). The only difference is the following: the identity $(1\ 2) \cdot pt = pt$ in the one-point set operad implies, according the equivariance axiom of operad actions (diagram of Figure I.7), that the element $pt \in pt(2)$ represents a symmetric operation $\mu : A \times A \to A$, for any algebra over the one-point set operad. This explains that the structures associated to the one-point set operad are commutative. □

I.1.18. *Connected operads.* In subsequent constructions, we have to consider operads P satisfying $\mathsf{P}(0) = 0$ (where 0 denotes the initial object of the base category) and $\mathsf{P}(1) = \mathbb{1}$ (the tensor unit of the base category). We say that the operad P is connected when these conditions are satisfied. We adopt the notation $\mathcal{O}p_{01}$ (with the lower-scripts hinting the operad first terms) for the category of connected operads, regarded as a full subcategory of the category of operads $\mathcal{O}p$.

The unitary versions of the associative As_+ and commutative operads Com_+ are not connected since we have $\mathsf{As}_+(0) = \mathsf{Com}_+(0) = pt$ in the point set context (considered in this section), and $\mathsf{As}_+(0) = \mathsf{Com}_+(0) = \mathbb{1}$ in the more general setting of symmetric monoidal categories. On the other hand, we immediately see that the non-unitary associative operad As, as well as the non-unitary commutative operad Com, formed by dropping these arity 0 terms, are connected.

We go back to connected operads at the end of the next section.

I.2. Categorical constructions on operads

In this section, we explain the definition of free objects in the category of operads, as well as the definition of operads by generators and relations. We also examine the definition of usual categorical constructions, like colimits and limits, in the context of operads.

For these purposes, we naturally have to consider the structure, underlying an operad, formed by a sequence $\mathsf{M} = \{\mathsf{M}(r)\}_{r\in\mathbb{N}}$ such that each $\mathsf{M}(r)$, $r \in \mathbb{N}$, is an object of the base category equipped with an action of the symmetric group Σ_r. We adopt the terminology of symmetric sequence and the notation $\mathcal{S}eq$ to refer to this category of objects $\mathsf{M} = \{\mathsf{M}(r)\}_{r\in\mathbb{N}}$ (another noun occurring in the literature is Σ_*-object). A morphism of symmetric sequences $f : \mathsf{M} \to \mathsf{N}$ is obviously a sequence of morphisms in the ground category $f : \mathsf{M}(r) \to \mathsf{N}(r)$ commuting with the action of symmetric groups.

We have an obvious forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$ mapping an operad P to the underlying sequence $\mathsf{P} = \{\mathsf{P}(r)\}_{r\in\mathbb{N}}$, where only the symmetric group actions of the operad structure are retained. The definition of free operads arises from the following theorem:

THEOREM I.2.1. *The forgetful functor $\omega : \mathcal{O}p \to \mathcal{S}eq$, from the category of operads to the category of symmetric sequences, has a left adjoint $\mathbb{O} : \mathcal{S}eq \to \mathcal{O}p$ mapping any symmetric sequence $\mathsf{M} \in \mathcal{S}eq$ to an associated free operad $\mathbb{O}(\mathsf{M})$.*

This theorem is formally established in §B.3 in an equivalent setting where the category of symmetric sequences $\mathcal{S}eq$ is replaced by the category of symmetric collections $\mathcal{C}oll$.

Intuitively, the free operad is the structure formed by all formal operadic composites of generating elements $\xi \in \mathsf{M}(n)$ with no relation between them apart from the universal relations which can be deduced from the axioms of operads. In §B.3, we use an extension of the tree representation of §§I.1.4-I.1.5 to give an explicit construction of such structures.

By definition of an adjunction, the free operad is characterized by the existence of a functorial bijection

$$(*) \qquad \mathtt{Mor}_{\mathcal{O}p}(\mathbb{O}(\mathsf{M}), \mathsf{P}) \simeq \mathtt{Mor}_{\mathcal{S}eq}(\mathsf{M}, \mathsf{P}),$$

given for any pair (M, P), where $\mathsf{M} \in \mathcal{S}eq$ and $\mathsf{P} \in \mathcal{O}p$. Together with the adjunction relation, we have:

– a morphism of symmetric sequences $\iota : \mathsf{M} \to \mathbb{O}(\mathsf{M})$, the unit of the adjunction, naturally associated to any $\mathsf{M} \in \mathcal{S}eq$, which corresponds to the identity of the free operad $id : \mathbb{O}(\mathsf{M}) \to \mathbb{O}(\mathsf{M})$ under (*);
– an operad morphism $\lambda : \mathbb{O}(\mathsf{P}) \to \mathsf{P}$, called the adjunction augmentation, naturally associated to any operad $\mathsf{P} \in \mathcal{O}p$, and which, under (*), corresponds to the identity of the operad P, viewed as an object of the category of symmetric sequences.

Intuitively, the adjunction augmentation λ is the morphism which applies the formal operadic composites of the free operad $\mathbb{O}(\mathsf{P})$ to their evaluation in P.

In §B.3, we give the explicit construction of the free operad $\mathbb{O}(\mathsf{M})$ and of the morphism $\iota : \mathsf{M} \to \mathbb{O}(\mathsf{M})$ before establishing the adjunction relation. Indeed, our correspondence (a) is defined by associating the composite $\phi \cdot \iota \in \mathtt{Mor}_{\mathcal{S}eq}(\mathsf{M}, \mathsf{P})$ with any operad morphism $\phi \in \mathtt{Mor}_{\mathcal{O}p}(\mathbb{O}(\mathsf{M}), \mathsf{P})$, and the proof that this mapping defines a bijection amounts to the following result:

Proposition I.2.2. *Any morphism of symmetric sequences $f : \mathsf{M} \to \mathsf{P}$, where P is an operad, admits a unique factorization*

$$\begin{tikzcd} \mathsf{M} \arrow[rr, "f"] \arrow[dr, "\iota"'] & & \mathsf{P} \\ & \mathbb{O}(\mathsf{M}) \arrow[ur, dotted, "\exists! \phi_f"'] & \end{tikzcd}$$

such that ϕ_f is an operad morphism.

This proposition, proved in §B.3, expresses the adjunction relation of Theorem I.2.1 in terms of an equivalent universal property, which is usually given in the literature as the definition of a free object (we refer to [65, §IV.1] for the relationship between adjunctions and universals).

I.2.3. *The unit operad.* The purpose of the next paragraphs is to examine the definition of colimits and limits in the context of operads.

To start with, we consider the symmetric sequence

$$\mathsf{I}(r) = \begin{cases} \mathbb{1}, & \text{if } r = 1, \\ 0, & \text{otherwise}, \end{cases}$$

which reduces to a unit object in arity $r = 1$. This symmetric sequence inherits an obvious operad structure: the unit morphism $\mathbb{1} \xrightarrow{\eta} \mathbb{1} = \mathsf{I}(1)$ is the identity morphism of $\mathbb{1}$; the composition products are forced by the unit axiom of Figure I.2.

For a given operad P, we have one and only one operad morphism from I to P, which is simply given by the operadic unit $\mathsf{I}(1) = \mathbb{1} \to \mathsf{P}(1)$ in arity 1. (The preservation of operad unit forces the definition of such a morphism.) Thus, the object I, which we call the unit operad in what follows, defines the initial object of $\mathcal{O}p$. In general, we adopt the notation of the operadic unit η for the initial morphism $\eta : \mathsf{I} \to \mathsf{P}$ attached to this object since this initial morphism essentially reduces to the unit morphism of the considered operad P.

The category of operads has a terminal object too, given by the terminal object of $\mathcal{B}ase$ in each arity.

The category of symmetric sequences, like any category of diagrams, has colimits and limits of any kind, which are formed termwise in the ground category. In the context of operads, we obtain the following general proposition:

Proposition I.2.4.

(a) *The forgetful functor from operads to symmetric sequences creates all kinds of small limits, the filtered colimits, as well as the coequalizers which are reflexive in the category of symmetric sequences.*

(b) *The category of operads admits coproducts too and, as a byproduct, all kinds of small colimits, though the forgetful functor from operads to symmetric sequences does not preserve colimits in general.*

We refer to the appendix section §I.5 for recollections on filtered colimits and reflexive coequalizers.

Proof. Let $\{\mathsf{P}_\alpha\}_{\alpha\in\mathcal{I}}$ be any diagram in the category of operads. The collection

$$(\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha)(r) = \lim_{\alpha\in\mathcal{I}}(\mathsf{P}_\alpha(r)),$$

defined by the termwise limits of the diagrams $\{\mathsf{P}_\alpha(r)\}_{\alpha\in\mathcal{I}}$ in the ground category, inherit a natural operadic composition product

$$\{\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(r)\}\otimes\{\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_1)\}\otimes\cdots\otimes\{\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_r)\}\to\lim_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(n_1+\cdots+n_r)\},$$

for any $r\geq 0$ and $n_1,\ldots,n_r\geq 0$, which is given by the composite of the morphism

$$\lim_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(r)\otimes\mathsf{P}_\alpha(n_1)\otimes\cdots\otimes\mathsf{P}_\alpha(n_r)\}\to\lim_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(n_1+\cdots+n_r)\}$$

induced by the composition products of the operads P_α with the natural morphism

$$\begin{aligned}\{\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(r)\}\otimes\{\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_1)\}\otimes\cdots\otimes\{\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_r)\}&\\ \to\lim_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(r)\otimes\mathsf{P}_\alpha(n_1)\otimes\cdots\otimes\mathsf{P}_\alpha(n_r)\}&\end{aligned}$$

deduced from the universal property of limits. The operad units $\eta:\mathbb{1}\to\mathsf{P}_\alpha(1)$ yield a unit morphism on the limit too. We readily deduce from the uniqueness requirement in the universal property of limits that the axioms of operads are fulfilled in $\lim_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha$, and we also easily check that this operad, formed by a termwise limit, represents the limit of the diagram $\{\mathsf{P}_\alpha\}_{\alpha\in\mathcal{I}}$ in the category of operads.

In the case of colimits, we can not adapt this construction to provide the termwise colimit

$$(\operatorname*{colim}_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha)(r) = \operatorname*{colim}_{\alpha\in\mathcal{I}}(\mathsf{P}_\alpha(r))$$

with an operadic composition structure, at least in general, because the natural morphism goes in the wrong direction:

$$\begin{aligned}\{\operatorname*{colim}_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(r)\}\otimes\{\operatorname*{colim}_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_1)\}\otimes\cdots\otimes\{\operatorname*{colim}_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_r)\}&\\ \leftarrow\operatorname*{colim}_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(r)\otimes\mathsf{P}_\alpha(n_1)\otimes\cdots\otimes\mathsf{P}_\alpha(n_r)\}.&\end{aligned}$$

Nevertheless, the results of Proposition I.5.2 and Proposition I.5.4 imply that this morphism is iso when the diagrams $\{\mathsf{P}_\alpha(n)\}_{\alpha\in\mathcal{I}}$ are shaped on a filtered category or form reflexive coequalizers in the ground category. Hence, in these situations, we can form natural composition products

$$\{\operatorname*{colim}_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(r)\}\otimes\{\operatorname*{colim}_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_1)\}\otimes\cdots\otimes\{\operatorname*{colim}_{\alpha\in\mathcal{I}}\mathsf{P}_\alpha(n_r)\}\to\operatorname*{colim}_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(n_1+\cdots+n_r)\}$$

by composition of the morphisms

$$\operatorname*{colim}_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(r)\otimes\mathsf{P}_\alpha(n_1)\otimes\cdots\otimes\mathsf{P}_\alpha(n_r)\}\to\operatorname*{colim}_{\alpha\in\mathcal{I}}\{\mathsf{P}_\alpha(n_1+\cdots+n_r)\}$$

induced by the composition products of the operads P_α with a colimit comparison isomorphism. The unit morphisms $\eta : \mathbb{1} \to \mathsf{P}_\alpha(1)$, composed with the canonical morphisms $\mathsf{P}_\alpha(1) \to \operatorname{colim}_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha(1)$, also give a canonical unit towards the colimit $\operatorname{colim}_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha(1)$. Then we easily check again that the axioms of operad are fulfilled in $\operatorname{colim}_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha$ and that this operad, formed by a termwise colimit, represents the colimit of the diagram $\{\mathsf{P}_\alpha\}_{\alpha\in\mathcal{I}}$ in the category of operads.

To realize a coproduct of a collection of operads P_α, $\alpha \in \mathcal{I}$, we form a reflexive coequalizer of the form

$$\begin{tikzcd} \mathbb{O}(\bigoplus_{\alpha\in\mathcal{I}} \mathbb{O}(\mathsf{P}_\alpha)) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{O}(\bigoplus_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha) \arrow[l, bend right, "s_0"'] \arrow[r, dotted] & \mathsf{Q} \end{tikzcd}$$

where the morphisms (d_0, d_1) are determined on each generating summand $\mathbb{O}(\mathsf{P}_\alpha)$ of the free operad $\mathsf{Q}_1 = \mathbb{O}(\bigoplus_{\alpha\in\mathcal{I}} \mathbb{O}(\mathsf{P}_\alpha))$ by:

- the morphism $\mathbb{O}(\iota_\alpha) : \mathbb{O}(\mathsf{P}_\alpha) \to \bigoplus_{\alpha\in\mathcal{I}} \mathbb{O}(\mathsf{P}_\alpha)$ induced by the canonical embedding $\iota_\alpha : \mathsf{P}_\alpha \to \bigoplus_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha$ as regards d_0;
- the composite of the adjunction augmentation $\lambda : \mathbb{O}(\mathsf{P}_\alpha) \to \mathsf{P}_\alpha$ with the canonical embedding $\iota_\alpha : \mathsf{P}_\alpha \to \bigoplus_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha$ and the adjunction unit of the free operad $\iota : \bigoplus_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha \to \mathbb{O}(\bigoplus_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha)$ as regards d_1.

The reflection morphism s_0 is given by the adjunction unit of the free operad $\iota : \mathsf{P}_\alpha \to \mathbb{O}(\mathsf{P}_\alpha)$ on each generating summand of $\mathsf{Q}_0 = \mathbb{O}(\bigoplus_{\alpha\in\mathcal{I}} \mathsf{P}_\alpha)$. By the result established in the first part of the proposition, the existence of this reflection s_0 implies the existence of the coequalizer $\operatorname{coker}(d_0, d_1)$ in the category of operads.

By the universal property of sums and free operads, any morphism $\phi_f : \mathsf{Q}_0 \to \mathsf{R}$ towards an operad R is fully determined by a collection of symmetric sequence morphisms $f_\alpha : \mathsf{P}_\alpha \to \mathsf{R}$. Moreover, we have $\phi_f d_0 = \phi_f d_1$ if and only if the diagram

$$\begin{tikzcd} \mathbb{O}(\mathsf{P}_\alpha) \arrow[r, "\lambda"] \arrow[dr, "\phi_{f_\alpha}"'] & \mathsf{P}_\alpha \arrow[d, "f_\alpha"] \\ & \mathsf{R} \end{tikzcd}$$

commutes for every α, where we consider the operad morphism ϕ_{f_α} associated to f_α. We readily see that this assertion amounts to the requirement that f_α preserves operadic composites and operad units, because λ is given by the evaluation of formal operad composites of the free operad in P_α and maps the unit of the free operad to the unit of P_α. Hence we have $\phi_f d_0 = \phi_f d_1$ if and only if each $f_\alpha : \mathsf{P}_\alpha \to \mathsf{R}$ is an operad morphism, and this implies that the definition of an operad morphism $\overline{\phi}_f : \operatorname{coker}(d_0, d_1) \to \mathsf{R}$ is equivalent to giving operad morphisms $f_\alpha : \mathsf{P}_\alpha \to \mathsf{R}$. We conclude that our coequalizer $\mathsf{Q} = \operatorname{coker}(d_0, d_1)$ represents the coproduct of the operads P_α (in the category of operads), which therefore exists, as claimed in the proposition.

The last assertion of the proposition is an application of the observation of Proposition I.5.5. □

I.2.5. *Operads defined by generators and relations.* The existence of free objects and coequalizers enables us to define operads by generators and relations. In this paragraph, we explain this process in the context of sets. Thus, we take $\mathcal{B}ase = \mathcal{S}et$ as base category.

We start with a symmetric sequence $\mathsf{M} \in \mathcal{S}eq$, whose elements $\xi \in \mathsf{M}(r)$ represents generating operations, and a collection of pairs $(w_0^\alpha, w_1^\alpha) \in \mathbb{O}(\mathsf{M})(n_\alpha)^{\times 2}$, $\alpha \in \mathcal{I}$, in order to define generating relations $w_0^\alpha \equiv w_1^\alpha$ within the free operad $\mathbb{O}(\mathsf{M})$.

We first form the free Σ_n-set $\mathsf{R}(n) = \Sigma_n \otimes \{e_\alpha,\ \alpha \in \mathcal{I}\,|n_\alpha = n\}$, where each e_α denotes an abstract generating element associated to the indexing variable $\alpha \in \mathcal{I}$. The expression $G \otimes K$ is our general notation for the free G-object associated to any object K in a ground category. In the context of sets, we can identify $G \otimes K$ with the cartesian product of G and K.

We consider the symmetric sequence R formed by the collection $\mathsf{R}(n)$, $n \in \mathbb{N}$. We have symmetric sequence morphisms $\rho_0, \rho_1 : \mathsf{R} \rightrightarrows \mathbb{O}(\mathsf{M})$ such that $\rho_0(e_\alpha) = w_0^\alpha$ and $\rho_1(e_\alpha) = w_1^\alpha$ respectively. We form the morphisms of symmetric sequences $\delta_0, \delta_1 : \mathsf{M} \amalg \mathsf{R} \to \mathbb{O}(\mathsf{M})$ induced by $\rho_0, \rho_1 : \mathsf{R} \rightrightarrows \mathbb{O}(\mathsf{M})$ on R and by the universal morphism $\iota : \mathsf{M} \to \mathbb{O}(\mathsf{M})$ on M. We consider the morphisms of free operads $d_0, d_1 : \mathbb{O}(\mathsf{M} \amalg \mathsf{R}) \rightrightarrows \mathbb{O}(\mathsf{M})$ induced by these morphisms δ_0 and δ_1. We have an operad morphism in the converse direction $s_0 : \mathbb{O}(\mathsf{M}) \to \mathbb{O}(\mathsf{M} \amalg \mathsf{R})$, yielded by the composite $\mathsf{M} \hookrightarrow \mathsf{M} \amalg \mathsf{R} \xrightarrow{\iota} \mathbb{O}(\mathsf{M} \amalg \mathsf{R})$, and such that $d_0 s_0 = d_1 s_0 = id$. The reflexive coequalizer $\mathsf{P} = \operatorname{coker}\{\mathbb{O}(\mathsf{M} \amalg \mathsf{R}) \overset{\curvearrowleft}{\rightrightarrows} \mathbb{O}(\mathsf{M})\}$, created in the category of sets, defines the operad

$$\mathsf{P} = \mathbb{O}(\mathsf{M} : w_0^\alpha = w_1^\alpha,\ \alpha \in \mathcal{I})$$

associated to the generating symmetric sequence M together with the generating relations $w_0^\alpha \equiv w_1^\alpha$, $\alpha \in \mathcal{I}$.

Intuitively, the formation of the reflexive coequalizer $\operatorname{coker}\{\mathbb{O}(\mathsf{M} \amalg \mathsf{R}) \overset{\curvearrowleft}{\rightrightarrows} \mathbb{O}(\mathsf{M})\}$ in the underlying category of sets amounts to identifying any formal composites involving a subfactor of the form w_0^α with the same formal composite but where w_0^α is replaced by w_1^α.

For an operad morphism $\phi_f : \mathbb{O}(\mathsf{M}) \to \mathsf{Q}$, we have:

$$\begin{aligned} & \phi_f \cdot d_0 = \phi_f \cdot d_1 \\ \Leftrightarrow \quad & \phi_f \cdot \rho_0 = \phi_f \cdot \rho_1 \\ \Leftrightarrow \quad & \phi_f(w_0^\alpha) = \phi_f(w_1^\alpha),\ \forall \alpha \in \mathcal{I}\,. \end{aligned}$$

Hence, the definition of a morphism $\overline{\phi}_f : \mathsf{P} \to \mathsf{Q}$ on $\mathsf{P} = \mathbb{O}(\mathsf{M} : w_0^\alpha = w_1^\alpha,\ \alpha \in \mathcal{I})$ amounts to giving a morphism of symmetric collections $f : \mathsf{M} \to \mathsf{Q}$ so that the extension of this morphism to the free operad $\phi_f : \mathbb{O}(\mathsf{M}) \to \mathsf{Q}$ maps the relations $w_0^\alpha \equiv w_1^\alpha$, $\alpha \in \mathcal{I}$, to actual identities in the target operad Q.

I.2.6. *Basic examples of operads in sets.* The most classical examples of operads are actually defined by a presentation by generators and relations. To give first examples of application of this process in the context of sets, we explain the presentation of the associative operad As, and of the commutative operad Com, the first instances of operad considered in the introductory sections §I.1. We focus on the non-unitary version of these operads for the moment. As we explain in §I.2.8, we will devote a subsequent chapter §A to the definition of unitary operads in a general context, and therefore, we only give short indications on the presentation of the unitary associative (commutative) operad in the present section.

To give a more intuitive interpretation of our construction, we define the generating symmetric sequence of our operads M by giving operations $p = p(x_1, \ldots, x_n)$ which generate the terms of this sequence $\mathsf{M}(n)$ as Σ_n-sets. We use explicit variables

to specify the arity of generating operations, unless this information has already been specified by the context. We may also use variable permutations to denote operations which correspond under the action of symmetric groups, but this indication may not be sufficient to determine the symmetric structure of our generating collection. Hence, we may have to add this precision.

The associative operad admits a presentation of the form

$$\mathsf{As} = \mathbb{O}(\ \mu(x_1, x_2), \mu(x_2, x_1)\ :\ \mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))\ ),$$

with a single generating operation $\mu = \mu(x_1, x_2)$ in arity 2, on which the group Σ_2 operates freely, together with the associativity relation, expressed by the composite identity $\mu(\mu, 1) \equiv \mu(1, \mu)$, as single generating relation. The commutative operad admits a presentation of the form

$$\mathsf{Com} = \mathbb{O}(\ \mu(x_1, x_2)\ :\ \mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))\ ),$$

with a single generating operation $\mu = \mu(x_1, x_2)$ in arity 2, on which the group Σ_2 operates trivially, together with the associativity relation $\mu(\mu, 1) \equiv \mu(1, \mu)$ as generating relation again.

In §I.1, we observed that we can use the permutation operad (respectively, the one-point set operad) to give a direct construction of an operad governing associative (respectively, commutative) algebras. The next proposition establishes the identity between this approach and the definition by generators and relations, and gives, in a sense, an operadic counterpart of the result of Proposition I.1.16-I.1.17.

Proposition I.2.7.

(a) *The set-theoretic operad* As*, as defined in §I.2.6, satisfies*

$$\mathsf{As}(r) = \begin{cases} \emptyset, & \text{if } r = 0, \\ \Sigma_r, & \text{otherwise}, \end{cases}$$

and is identified with the permutation operad of Proposition I.1.8, where we have dropped the term in arity 0*.*

(b) *The set-theoretic operad* Com*, as defined in §I.2.6, satisfies*

$$\mathsf{Com}(r) = \begin{cases} \emptyset, & \text{if } r = 0, \\ pt, & \text{otherwise}, \end{cases}$$

and is identified with the one-point set operad of Proposition I.1.9, where we have dropped the term in arity 0*.*

Proof. We focus on the example of the associative operad (a) as the case of the commutative operad (b) follows from similar arguments. We use the temporary notation Π for the permutation operad. We consider, to be precise, the non-unitary version of this operad with the term in arity 0 withdrawn (as specified in the proposition).

To start with, we observe (as in the proof of Proposition I.1.16) that the permutation $\mu = id \in \Sigma_2$ satisfies the generating relations of the associative operad As in the permutation operad. Hence, we have a well-defined operad morphism $\phi : \mathsf{As} \to \Pi$ mapping the generating operation of As to this permutation. To prove that this morphism is iso, we form a morphism in the converse direction by assigning the composite operation $\psi(w) = w \cdot \mu(\cdots(\mu(\mu, 1), 1), \ldots, 1)$ to any $w \in \Sigma_r$. We

immediately see that $\phi\psi = id$ and we easily obtain that $id \equiv \psi\phi$ from the relations of As. The conclusion follows. $\square$

I.2.8. *The presentation of unitary operads.* To define unitary versions of the commutative operad and of the associative operad, we may simply add a generating operation e in arity 0 and relations of the form $\mu(e,1) = 1 = \mu(1,e)$, expressing the neutral element identities, to our presentations. Thus, we may set

$$\mathsf{As}_+ = \mathbb{O}(\ e, \mu(x_1,x_2), \mu(x_2,x_1)\ :\ \mu(\mu,1) = \mu(1,\mu)\ ,\ \mu(e,1) = 1 \equiv \mu(1,e)\ ),$$
$$\mathsf{Com}_+ = \mathbb{O}(\ e, \mu(x_1,x_2)\ :\ \mu(\mu,1) = \mu(1,\mu)\ ,\ \mu(e,1) = 1 = \mu(1,e)\ ),$$

to define these operads. The result of Proposition I.2.7 also extends to the unitary version of our operads, so that $\mathsf{As}_+(r) = \Sigma_r$ (respectively, $\mathsf{Com}_+(r) = pt$), for all r (including $r = 0$), and similarly in the $\Bbbk$-module setting.

From these identifications, we see that the generating unitary operation e is special (at least in our examples). Indeed, in the outcome of the presentation process, the terms of arity $r > 0$ of the operad As_+ (respectively, Com_+) agrees with the terms of the non-unitary operad As, formed by dropping the unitary operation e from the presentation.

We need to put arity 0 terms apart in certain constructions. We therefore do not use the general approach of operads defined by generators and relations in the unitary case, and we put off further studies of unitary operads until a subsequent section (§I.4), where we will specifically devote to that subject.

I.2.9. *Operad ideals and presentations of operads in module categories.* The construction of §I.2.5 has an analogue in the context of modules over a ground ring: we simply have to replace the set $\{e_\alpha,\ \alpha \in \mathcal{I}\,|n_\alpha = n\}$ by the associated free $\Bbbk$-module $\Bbbk\{e_\alpha,\ \alpha \in \mathcal{I}\,|n_\alpha = n\}$, and we replace all set-theoretic constructions by their analogue in $\Bbbk$-modules. The purpose of this paragraph is to explain that, in the setting of module categories, we can use an operadic version of the notion of ideal in order to obtain another approach to the construction of operads by generators and relations. In the next part of the book, we apply an extension of this construction in the graded context. For the moment, we focus on the plain module environment.

In brief, an ideal of an operad in $\Bbbk$-modules P is a collection of submodules $\mathsf{S}(n) \subset \mathsf{P}(n)$, each of which preserved by the action of the symmetric group on $\mathsf{P}(n)$, and so that any composite $p(q_1,\ldots,q_r) \in \mathsf{P}(n_1 + \cdots + n_r)$ involving at least one factor in S remains in S. Equivalently, the collection S forms a sub-object of P in the category of symmetric sequences, and we have:

$$\begin{aligned} &p(q_1,\ldots,q_r) \in \mathsf{S}(n_1 + \cdots + n_r),\\ &\qquad\qquad \text{for all } p \in \mathsf{P}(r), \text{ and } q_1 \in \mathsf{P}(n_1),\ldots,q_r \in \mathsf{P}(n_r),\\ &\qquad\qquad\qquad\qquad \text{as soon as } p \in \mathsf{S}(r) \text{ or } q_i \in \mathsf{S}(n_i) \text{ for some } i. \end{aligned}$$

We immediately see that the collection $\mathsf{P}\,/\,\mathsf{S}(n) = \mathsf{P}(n)/\,\mathsf{S}(n)$ obtained by forming the quotient of an operad P over an ideal S inherits an operad structure from P.

To a collection of elements $z^\alpha \in \mathsf{P}(n_\alpha)$, $\alpha \in \mathcal{I}$, in an operad P, we associate the symmetric sequence $< z^\alpha, \alpha \in \mathcal{I} > \subset \mathsf{P}$ generated by the composites of the form $p(1,\ldots,z^\alpha(q_1,\ldots,q_{n_\alpha}),\ldots,1)$, where the factors p and $q_1,\ldots,q_{n_\alpha}$ runs over the whole operad P. We easily check, by using the axioms of operads, that this symmetric sequence $\mathsf{S} =< z^\alpha, \alpha \in \mathcal{I} >$ forms an ideal in P and is actually the smallest ideal including the elements z^α, $\alpha \in \mathcal{I}$. We can also easily check that an

operad morphism $\phi : \mathsf{P} \to \mathsf{Q}$ factors through the quotient $\mathsf{P} \,/ < z^\alpha, \alpha \in \mathcal{I} >$ if and only if we have $\phi(z^\alpha) = 0$ in Q, for all $\alpha \in \mathcal{I}$. Accordingly, in the case of a free operad $\mathsf{P} = \mathbb{O}(\mathsf{M})$, any operad morphism $\overline{\phi}_f : \mathbb{O}(\mathsf{M}) \,/ < z^\alpha, \alpha \in \mathcal{I} > \to \mathsf{Q}$ is uniquely determined by a morphism of symmetric collections $f : \mathsf{M} \to \mathsf{Q}$ so that the extension of this morphism to the free operad $\phi_f : \mathbb{O}(\mathsf{M}) \to \mathsf{Q}$ cancels the generating elements of the ideal z^α, $\alpha \in \mathcal{I}$. From this observation, we conclude that, in the module context, we can define operads by generators and relations as quotients

$$\mathbb{O}(\mathsf{M} : w_0^\alpha = w_1^\alpha,\ \alpha \in \mathcal{I}) = \mathbb{O}(\mathsf{M}) \,/ < w_0^\alpha - w_1^\alpha, \alpha \in \mathcal{I} >,$$

where we rewrite the given relations as differences $w_0^\alpha \equiv w_1^\alpha \Leftrightarrow w_0^\alpha - w_1^\alpha \equiv 0$ before forming the ideal $< w_0^\alpha - w_1^\alpha, \alpha \in \mathcal{I} >$.

I.2.10. *Basic examples of operads in module categories.* We can adapt the construction of §I.2.6 to define the module version of the associative (respectively, commutative) operad. We simply replace the generating sets of §I.2.6 by associated free modules (as explained in §I.2.9). We have

$$\begin{aligned}
\mathsf{As} &= \mathbb{O}(\ \Bbbk\,\mu(x_1, x_2) \oplus \Bbbk\,\mu(x_2, x_1)\ :\ \mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))\ ),\\
\mathsf{Com} &= \mathbb{O}(\ \Bbbk\,\mu(x_1, x_2)\ :\ \mu(\mu(x_1, x_2), x_3) = \mu(x_1, \mu(x_2, x_3))\ ),
\end{aligned}$$

with, regarding the notation of generating operations, the same conventions as in the set-theoretic context.

Formally, the generating symmetric sequence of the associative operad is defined by $\mathsf{M}_{\mathsf{As}}(2) = \Bbbk\{\mu(x_1, x_2), \mu(x_2, x_1)\} = \Bbbk\{\Sigma_2\}$ and $\mathsf{M}_{\mathsf{As}}(n) = 0$ for $n \neq 2$. The generating symmetric sequence of the commutative operad in $\Bbbk$-modules is defined by $\mathsf{M}_{\mathsf{Com}}(2) = \Bbbk\{\mu(x_1, x_2)\} = \Bbbk$ and $\mathsf{M}_{\mathsf{Com}}(n) = 0$ for $n \neq 2$. Then, according to §I.2.9, we can identify the associative operad and the commutative operad with quotients $\mathsf{As} = \mathbb{O}(\mathsf{M}_{\mathsf{As}}) \,/ < \mu(\mu, 1) - \mu(1, \mu) >$ and $\mathsf{Com} = \mathbb{O}(\mathsf{M}_{\mathsf{Com}}) \,/ < \mu(\mu, 1) - \mu(1, \mu) >$, where we consider the ideal generated by the difference $\mu(\mu(x_1, x_2), x_3) - \mu(x_1, \mu(x_2, x_3))$ to implement the associativity relation.

The next classical example of operad which we consider is the Lie operad, defined by the presentation

$$\mathsf{Lie} = \mathbb{O}(\ \Bbbk\,\lambda(x_1, x_2)\ :\ \lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) = 0\ ),$$

where we have a single generating operation $\lambda = \lambda(x_1, x_2)$ in arity 2 together with the symmetric group action such that $(1\ 2) \cdot \lambda = -\lambda$. Formally, the generating symmetric sequence of the Lie operad is defined by $\mathsf{M}_{\mathsf{Lie}}(n) = 0$ for $n \neq 2$ and $\mathsf{M}_{\mathsf{Lie}}(2) = \Bbbk\{\lambda(x_1, x_2)\} = \Bbbk^{\pm}$, where $\pm$ refers to a twist of the action of permutations by the signature. We can also realize this operad as a quotient of the free operad $\mathbb{O}(\mathsf{M}_{\mathsf{Lie}})$ under the ideal generated by the element $\lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2)$. This expression corresponds to the classical Jacobi identity of Lie algebras and the quotient by the associated operadic ideal implements this relation in the Lie operad.

In Proposition I.2.7, we have established that the non-unitary associative (respectively, commutative) operad in sets is identified with the non-unitary version of the permutation (respectively, one-point set) operad. In §II.1, we will explain that the free $\Bbbk$-module functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$ induces a functor on operads, and we can use this process to associate an operad in $\Bbbk$-modules to any operad in sets. We can adapt the arguments of Proposition I.2.7 to identify the operad As (respectively, Com), defined by generators and relations, with an operad in $\Bbbk$-modules

associated with the non-unitary permutation (respectively, one-point set) operad. Consequently, we have

$$\mathsf{As}(r) = \begin{cases} 0, & \text{if } r = 0, \\ \Bbbk\{\Sigma_r\}, & \text{otherwise}, \end{cases}$$

for the $\Bbbk$-module version of the associative operad and

$$\mathsf{Com}(r) = \begin{cases} 0, & \text{if } r = 0, \\ \Bbbk, & \text{otherwise}, \end{cases}$$

for the $\Bbbk$-module version of the commutative operad. In the next paragraph, we give another interpretation of these relations by going back to the representation of operad elements $p \in \mathsf{P}(r)$ as abstract operations $p = p(x_1, \ldots, x_r)$.

I.2.11. *The underlying symmetric sequence of classical operads.* The purpose of this paragraph is to review the interpretation of operad elements as abstract operations in the case of the usual operads $\mathsf{As}, \mathsf{Com}, \mathsf{Lie}$. To simplify, we still focus on the non-unitary version of the associative and commutative operads. The generating operations of these operads lie in arity $r > 0$, and similarly in the case of the Lie operad. This implies $\mathsf{Com}(0) = \mathsf{As}(0) = \mathsf{Lie}(0) = 0$. Thus we focus on the terms of arity $r > 0$.

In the case of the associative operad As, an element $p(x_1, \ldots, x_r) \in \mathsf{As}(r)$ is obtained by a multiple composition of an associative product $\mu(x_1, x_2) = x_1 \cdot x_2$ together with an appropriate variable shift ensuring that each variable x_i occurs once in the expression of $p(x_1, \ldots, x_r)$. Consequently, the arity r term of the associative operad $\mathsf{As}(r)$ is identified with the module spanned by all monomials $p(x_1, \ldots, x_r)$ of non-commutative variables $(x_1, \ldots, x_r)$ which have degree one with respect to each variable x_i, $i = 1, \ldots, r$. In standard mathematical notation, such a monomial is written $p(x_1, \ldots, x_r) = x_{i_1} \cdot \ldots \cdot x_{i_r}$, and the degree requirement amounts to the assumption that the sequence $(i_1, \ldots, i_r)$ forms a permutation of $(1, \ldots, r)$. Hence, we obtain

$$\mathsf{As}(r) = \bigoplus_{s \in \Sigma_r} \Bbbk\{x_{s(1)} \cdot \ldots \cdot x_{s(r)}\} = \Bbbk\{\Sigma_r\}, \quad \text{for all } r > 0,$$

and we retrieve the observation that $\mathsf{As}(r)$ is the regular representation of the symmetric group Σ_r.

Similarly, the arity r term of the commutative operad $\mathsf{Com}(r)$ is identified with the module spanned by all monomials $p(x_1, \ldots, x_r)$ formed from a formal composite of products of commutative variables $(x_1, \ldots, x_r)$ so that each variable x_i occurs once and only once in $p(x_1, \ldots, x_r)$. In standard algebraic language, this requirement amounts to assuming that $p(x_1, \ldots, x_r)$ is a monomial of r commutative variables $(x_1, \ldots, x_r)$ which has degree one with respect to each variable x_i, $i = 1, \ldots, r$. In standard mathematical notation, such a monomial is written $p(x_1, \ldots, x_r) = x_1 \cdot \ldots \cdot x_r$. Hence, we immediately obtain

$$\mathsf{Com}(r) = \Bbbk\{x_1 \cdot \ldots \cdot x_r\} = \Bbbk, \quad \text{for all } r > 0,$$

from which we retrieve the identity between $\mathsf{Com}(r)$ and the free $\Bbbk$-module of rank one equipped with the trivial action of the symmetric group.

In the case of the Lie operad Lie, we consider the module spanned by all Lie monomials $p(x_1, \ldots, x_r)$ which have degree one with respect to each variable x_i. The determination of the module structure of $\mathsf{Lie}(r)$ is more intricate than in

the case of the commutative and associative operads. Nevertheless one can prove (see [78, §5.6.2] for instance) that $\mathsf{Lie}(r)$ has a basis of the form

$$\mathsf{Lie}(r) = \bigoplus_{\substack{s\in\Sigma_r\\ s(1)=1}} \Bbbk\{[\cdots[[x_{s(1)}, x_{s(2)}], x_{s(3)}], \ldots, x_{s(r)}]\}, \quad \text{for all } r > 0,$$

where we use the Lie bracket notation $[-,-]$ for the generating operation of Lie. Hence, the $\Bbbk$-module $\mathsf{Lie}(r)$ is free of rank $(r-1)!$. In the case $\mathbb{Q}[e^{2i\pi/r}] \subset \Bbbk$, we also have an identity between $\mathsf{Lie}(r)$ and the induced representation $\mathsf{Lie}(r) = \mathrm{Ind}_{C_r}^{\Sigma_r}\chi$ where C_r denotes the cyclic group generated by the r-cycle $(1\ 2\ \cdots\ r) \in \Sigma_r$ and χ denotes the one-dimensional representation of C_r associated to the character $\chi(1\ 2\ \cdots\ r) = e^{2i\pi/r}$ (see also [78, §8.2] for an overall reference on this subject).

I.2.12. *The example of the Poisson operad.* To complete our examples, we examine the definition of the Poisson operad, of which a graded version plays a significant role in the study of E_n-operads. This operad is defined by the presentation

$$\begin{aligned}\mathsf{Pois} = \ \mathbb{O}(\ \Bbbk\,\mu(x_1,x_2) \oplus \Bbbk\,\lambda(x_1,x_2)\ :\ & \\ & \mu(\mu(x_1,x_2),x_3) = \mu(x_1,\mu(x_2,x_3)),\\ & \lambda(\lambda(x_1,x_2),x_3) + \lambda(\lambda(x_2,x_3),x_1) + \lambda(\lambda(x_3,x_1),x_2) = 0,\\ & \lambda(\mu(x_1,x_2),x_3) = \mu(\lambda(x_1,x_3),x_2) + \mu(x_1,\lambda(x_2,x_3))\ ),\end{aligned}$$

where $\mu = \mu(x_1,x_2)$ is a symmetric generating operation, fixed by the action of the transposition $(1\ 2)\cdot\mu = \mu$, and $\lambda = \lambda(x_1,x_2)$ is an antisymmetric generating operation, which the transposition carries to its opposite $(1\ 2)\cdot\lambda = -\lambda$. From this construction, we see that the Poisson operad is a combination of the commutative operad $\mathsf{Com} = \mathbb{O}(\ \Bbbk\,\mu(x_1,x_2)\ :\ \mu(\mu(x_1,x_2),x_3) = \mu(x_1,\mu(x_2,x_3))\ )$ and of the Lie operad $\mathsf{Lie} = \mathbb{O}(\ \Bbbk\,\lambda(x_1,x_2)\ :\ \lambda(\lambda(x_1,x_2),x_3)+\lambda(\lambda(x_2,x_3),x_1)+\lambda(\lambda(x_3,x_1),x_2) = 0\ )$, together with an additional distribution relation

$$\lambda(\mu(x_1,x_2),x_3) = \mu(\lambda(x_1,x_3),x_2) + \mu(x_1,\lambda(x_2,x_3)),$$

called the Poisson relation, mixing both operads.

The commutative operad (respectively, the Lie operad) can be identified with the suboperad of the Poisson operad generated by the element $\mu \in \mathsf{Pois}(2)$ (respectively, $\lambda \in \mathsf{Pois}(2)$). The Poisson relation implies that each composite of a Lie operation with commutative operations (in that order) can be rewritten as a composite of a commutative operation with Lie operations. One can prove by elaborating on this remark that $\mathsf{Pois}(r)$ is identified with the $\Bbbk$-module spanned by formal products

$$p(x_1,\ldots,x_r) = p_1(x_{11},\ldots,x_{1r_1})\cdot\ldots\cdot p_m(x_{m1},\ldots,x_{mr_m}),$$

of which factors $p_i(x_{i1},\ldots,x_{ir_i})$ run over Lie monomials on r_i variables x_{ik}, each variable x_{ik} occurring once and only once in $p_i(x_{i1},\ldots,x_{ir_i})$ (in other word, this Lie monomial has degree one with respect to each variable), and so that the variable subsets $\{x_{i1},\ldots,x_{ir_i}\}$ form a partition of the total set $\{x_1,\ldots,x_r\}$. (Thus, each variable x_i also occurs once and only once in the complete expression $p(x_1,\ldots,x_r)$.)

I.2.13. *Connected operads.* Recall that a connected operad in a base category $\mathcal{B}ase$ is an operad P such that $\mathsf{P}(0) = 0$ and $\mathsf{P}(1) = \mathbb{1}$.

If the base category is pointed, in the sense that initial and terminal objects coincide in $\mathcal{B}ase$, then any connected operad P inherits a natural augmentation $\epsilon : \mathsf{P} \to \mathsf{I}$, given by the identity in arity 1 and the terminal morphism otherwise.

This augmentation obviously defines a morphism in the category of operads, and accordingly, the unit operad I gives a terminal object in the category of connected operads, in addition to form the initial object. Thus, the category of connected operads is pointed (unlike the whole category of operads) whenever the base category is so.

To a connected operad P, we associate the symmetric sequence $\overline{\mathsf{P}}$ such that

$$\overline{\mathsf{P}}(r) = \begin{cases} 0, & \text{if } r = 0, 1, \\ \mathsf{P}(r), & \text{otherwise.} \end{cases}$$

We call this symmetric sequence the augmentation ideal of P, because we can identify it with the kernel of the augmentation morphism $\epsilon : \mathsf{P} \to \mathsf{I}$ when the base category is pointed. We should mention, however, that we may consider this symmetric sequence associated to an operad $\overline{\mathsf{P}}$ outside the pointed category context, where the above expression makes sense but not the augmentation ideal interpretation.

The category of connected operads is denoted by $\mathcal{O}p_{01}$, with added lower-scripts to mark the operad first terms $\mathsf{P}(0) = 0$, $\mathsf{P}(1) = \mathbb{1}$. We similarly use the expression $\mathcal{S}eq_{00}$ to denote the category formed by symmetric sequences such that $\mathsf{M}(0) = \mathsf{M}(1) = 0$. We say that a symmetric sequence satisfying these conditions is connected (as a symmetric sequence). The mapping $\overline{\omega} : \mathsf{P} \mapsto \overline{\mathsf{P}}$ gives a functor, denoted by $\overline{\omega} : \mathcal{O}p_{01} \to \mathcal{S}eq_{00}$, from the category of connected operads $\mathcal{O}p_{01}$ towards the category of connected symmetric sequences $\mathcal{S}eq_{00}$. In the connected context, we will use the following interpretation of the free operad construction:

Theorem I.2.14. *The free operad $\mathbb{O}(\mathsf{M})$ associated to a connected sequence $\mathsf{M} \in \mathcal{S}eq_{00}$ is connected as an operad and the map $\mathbb{O} : \mathsf{M} \mapsto \mathbb{O}(\mathsf{M})$ defines a left adjoint of the functor $\overline{\omega} : \mathcal{O}p_{01} \to \mathcal{S}eq_{00}$ mapping a connected operads $\mathsf{P} \in \mathcal{O}p_{01}$ to its augmentation ideal $\overline{\mathsf{P}} \in \mathcal{S}eq_{00}$.*

This theorem, which is essentially a follow-up of Theorem I.2.1, is formally established in §B.4 (by using the notion of symmetric collection, equivalent to the notion of symmetric sequence, again).

In general, an operad defined by generators and relations $\mathsf{P} = \mathbb{O}(\mathsf{M} : w_0^\alpha = w_1^\alpha, \alpha \in \mathcal{J})$ is connected (in our sense) if and only if the generating sequence M vanishes in arity $r = 0, 1$, essentially because this result holds for free operads. We retrieve (for instance) that the (non-unitary) associative operad As is connected, like the (non-unitary) commutative operad Com, and the Lie operad Lie.

I.3. Algebras over operads

In the previous section, we focused on the application of categorical constructions to operads. We now study the applications of such constructions at the level of algebra categories associated to operads. We first explain, in the next paragraph, that the construction of operads by generators and relations reflects the definition of usual algebra categories in terms of generating operations $\xi : A^{\otimes r} \to A$ satisfying given relations.

We also give the version of the standard categorical constructions of §I.2 (free objects, colimits and limits) in categories of algebras associated to operads. We will observe (following [72]) that the categories of algebras associated to operads can be characterized as categories of algebras equipped which free objects of a

particular form. One can use this observation to retrieve results of the previous section concerning the terms of the usual operads. By the way, we establish that any operad morphism give rise to adjoint extension and restriction functors at the level of algebra categories. Examples include the standard functors connecting the categories of associative, commutative, and Lie algebras.

I.3.1. *Basic examples of algebra categories associated to operads.* Recall (see Proposition I.1.14) that defining an action of an operad P on an object $A \in \mathcal{B}ase$ amounts to giving an operad morphism $\phi : \mathsf{P} \to \mathtt{End}_A$, where $\mathtt{End}_A$ denotes the endomorphism operad of A. In the case of an operad defined by generators and relations

$$\mathsf{P} = \mathbb{O}(\mathsf{M} : w_0^\alpha = w_1^\alpha,\ \alpha \in \mathcal{I}),$$

we deduce, from the observations of §I.2.5, that such a morphism $\phi_f : \mathsf{P} \to \mathtt{End}_A$ amounts to giving a morphism of symmetric sequences $f : \mathsf{M} \to \mathtt{End}_A$, mapping the abstract generating operations $\xi \in \mathsf{M}(r)$ to actual maps $\xi : A^{\otimes r} \to A$, and so that the identities $w_0^\alpha \equiv w_1^\alpha$ hold in $\mathtt{End}_A$.

For the basic examples of (non-unitary) operads $\mathsf{P} = \mathsf{Com}, \mathsf{As}, \mathsf{Lie}$, we obtain (in the module context):

(a) an algebra over the commutative operad Com is a module A equipped with a product $\mu : A \otimes A \to A$ which satisfies the symmetry relation

$$\mu(a_1, a_2) = \mu(a_2, a_1),$$

for all $a_1, a_2 \in A$, and the associativity relation

$$\mu(\mu(a_1, a_2), a_3) = \mu(a_1, \mu(a_2, a_3)),$$

for all $a_1, a_2, a_3 \in A$;

(b) an algebra over the associative operad As is a module A equipped with a product $\mu : A \otimes A \to A$ which satisfies the associativity relation

$$\mu(\mu(a_1, a_2), a_3) = \mu(a_1, \mu(a_2, a_3))$$

for all $a_1, a_2, a_3 \in A$ (but no symmetry requirement);

(c) an algebra over the Lie operad Lie is a module $\mathfrak{g}$ equipped with an operation $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which satisfies the antisymmetry relation

$$\lambda(x_1, x_2) = -\lambda(x_2, x_1),$$

for all $x_1, x_2 \in \mathfrak{g}$, and the Jacobi identity

$$\lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) = 0,$$

for all $x_1, x_2, x_3 \in \mathfrak{g}$.

We can similarly identify the category of algebras associated to the Poisson operad from the presentation of §I.2.12.

In characteristic 2, we do not necessarily assume that the generating operation of a Lie-algebra $\mathfrak{g}$ satisfies the relation $\lambda(x, x) = 0$ in contrast to the usual definition of a Lie algebra. To associate a category of algebras satisfying this condition to the Lie operad Lie, we have to modify the definition of an algebra over an operad following a process explained in [37, §§1.2.12-1.2.16].

The result of Proposition I.1.16-I.1.17 (in the non-unitary context) is equivalent to the combination of the assertions of (a-b) with Proposition I.2.7.

I.3.2. *The category of algebras associated to an operad and free algebras.* Recall that the algebras associated to a given operad P form a category $\mathcal{P}$ with, as morphisms, the morphisms of the ground category $f : A \to B$ which preserve the P-actions on A and B. We have an obvious forgetful functor $\omega : \mathcal{P} \to \mathcal{B}ase$ from the category of P-algebras $\mathcal{P}$ towards the base category $\mathcal{B}ase$.

We can form a functor in the converse direction by considering a generalized symmetric tensor object

$$S(\mathsf{P}, X) = \bigoplus_{n=0}^{\infty} (\mathsf{P}(n) \otimes X^{\otimes n})_{\Sigma_n},$$

associated to any $X \in \mathcal{B}ase$, where the notation $(-)_{\Sigma_n}$ refers to a coinvariant quotient, identifying the natural Σ_n-action on the tensor power $X^{\otimes n}$ with the internal Σ_n-structure of $\mathsf{P}(n)$. Let $p \in \mathsf{P}(n)$ and $x_1 \otimes \cdots \otimes x_n \in X^{\otimes n}$. In the point-wise context, we formally set

$$p \otimes (x_{s(1)} \otimes \cdots \otimes x_{s(n)}) \equiv s \cdot p \otimes (x_1 \otimes \cdots \otimes x_n),$$

and $(\mathsf{P}(n) \otimes X^{\otimes n})_{\Sigma_n}$ is the quotient under these relations, where $s \in \Sigma_n$.

We have natural evaluation morphisms

$$\mathsf{P}(r) \otimes S(\mathsf{P}, X)^{\otimes r} \xrightarrow{\lambda} S(\mathsf{P}, X)$$

given termwise by morphisms

$$\begin{aligned} \mathsf{P}(r) \otimes (\mathsf{P}(n_1) \otimes X^{\otimes n_1})_{\Sigma_{n_1}} \otimes \cdots \otimes (\mathsf{P}(n_r) \otimes X^{\otimes n_r})_{\Sigma_{n_r}} \\ \to (\mathsf{P}(n_1 + \cdots + n_r) \otimes X^{\otimes n_1 + \cdots + n_r})_{\Sigma_{n_1 + \cdots + n_r}} \end{aligned}$$

induced by the composition products of the operad. We easily check that the axioms of operads imply that these morphisms satisfy the equivariance, associativity and unit axioms of operad actions. We therefore obtain that the object $S(\mathsf{P}, X) \in \mathcal{B}ase$ forms a P-algebra, naturally associated to $X \in \mathcal{B}ase$, so that the mapping $S(\mathsf{P}) : X \mapsto S(\mathsf{P}, X)$ defines a functor $S(\mathsf{P}) : \mathcal{B}ase \to \mathcal{P}$.

For a P-algebra A, the evaluation morphisms of A induce morphisms $\lambda : (\mathsf{P}(n) \otimes A^{\otimes n})_{\Sigma_n} \to A$ for all $n \geq 0$ by equivariance. These morphisms can be patched into a single natural morphism $\lambda : S(\mathsf{P}, A) \to A$ by the universal property of the sum $S(\mathsf{P}, A) = \bigoplus_{n=0}^{\infty} (\mathsf{P}(n) \otimes X^{\otimes n})_{\Sigma_n}$. From the associativity axiom of operad actions, we easily check that $\lambda : S(\mathsf{P}, A) \to A$ preserves P-algebra structures and hence, defines a morphism in the category of P-algebras. In the converse direction, for any $X \in \mathcal{B}ase$, we have a natural morphism $\iota : X \to S(\mathsf{P}, X)$ given by the composite

$$X \xrightarrow{\simeq} \mathbb{1} \otimes X \xrightarrow{\eta \otimes X} \mathsf{P}(1) \otimes X = (\mathsf{P}(1) \otimes X)_{\Sigma_1} \hookrightarrow \bigoplus_{n=0}^{\infty} (\mathsf{P}(n) \otimes X^{\otimes n})_{\Sigma_n},$$

where η refers to the unit morphism of the operad P.

One checks that:

Proposition I.3.3. *The functor $S(\mathsf{P}) : \mathcal{B}ase \to \mathcal{P}$ is left adjoint to the forgetful functor $\omega : \mathcal{P} \to \mathcal{B}ase$. The morphism $\iota : X \to S(\mathsf{P}, X)$ (respectively, $\lambda : S(\mathsf{P}, A) \to A$) defines the unit (respectively, the augmentation) of this adjunction relation.*

Explicitly, this proposition asserts the existence of a bijection

$$\mathtt{Mor}_{\mathcal{P}}(S(\mathsf{P}, X), A) = \mathtt{Mor}_{\mathcal{B}ase}(X, A),$$

for any $X \in \mathcal{B}ase$ and any $A \in \mathcal{P}$. In one direction, to a morphism of P-algebras $\phi : S(\mathsf{P}, X) \to A$ we associate the morphism $f = \phi \cdot \iota$ in the base category. In the other direction, to a morphism in the base category $f : X \to A$ we associate the morphism $\phi_f = \lambda \cdot S(\mathsf{P}, f)$ in the category of P-algebras. The adjunction augmentation itself $\lambda : S(\mathsf{P}, A) \to A$ is the morphism of P-algebras ϕ_{id} associated to the identity of A, regarded as an object of the base category $\mathcal{B}ase$.

Proof. By a general result of category theory (see [65, §IV.1]), we essentially have to check that the composites

$$A \xrightarrow{\iota} S(\mathsf{P}, A) \xrightarrow{\lambda} A \quad \text{and} \quad S(\mathsf{P}, X) \xrightarrow{S(\mathsf{P},\iota)} S(\mathsf{P}, S(\mathsf{P}, X)) \xrightarrow{\lambda} S(\mathsf{P}, X)$$

are both identity morphisms to conclude that our mappings are converse to each other, and hence gives an adjunction relation well. This result follows from the unit axiom of operad actions in the first case and from the unit axiom of operads in the second one. □

The result of Proposition I.3.3 has, like Theorem I.2.1, an equivalent formulation in terms of universal properties. In this point of view, the functor $S(\mathsf{P}) : \mathcal{B}ase \to \mathcal{P}$ is defined by the mapping which associates a free object in the category of P-algebras to any $X \in \mathcal{B}ase$.

Basically, the morphism of P-algebras ϕ_f associated to a given morphism f in the base category is characterized by the equation $\phi_f \cdot \iota = f$ since our adjunction is a bijection. Thus, for a fixed $X \in \mathcal{B}ase$, the result of Proposition I.3.3 amounts to the following proposition:

Proposition I.3.4. *Any morphism in the base category $f : X \to A$, where A is a P-algebra, admits a unique factorization*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & B \\ & S(\mathsf{P}, X) \arrow[ur, dotted, "\exists! \phi_f"'] & \end{tikzcd}$$

such that ϕ_f is a morphism of P-algebras. □

This statement gives the expression of the universal property of free object satisfied by the P-algebra $S(\mathsf{P}, X)$.

I.3.5. *Basic examples of free algebras.* The structure of the basic operads $\mathsf{P} = \mathsf{Com}, \mathsf{As}, \mathsf{Lie}$ can be retrieved from known expansions of free objects in the categories of algebras associated to these operads:

(a) The free commutative algebra (without unit) is identified with (the augmentation ideal of) the symmetric algebra

$$\overline{\mathbb{S}}(K) = \bigoplus_{n=1}^{\infty} (K^{\otimes n})_{\Sigma_n}$$

together with a commutative product yielded by the process of tensor concatenations. From this observation, we immediately retrieve the identity $\mathsf{Com}(n) = \Bbbk$.

(b) The free associative algebra (without unit) is identified with (the augmentation ideal of) the tensor algebra

$$\overline{\mathbb{T}}(K) = \bigoplus_{n=1}^{\infty} K^{\otimes n}$$

together with an associative product defined by the concatenation of tensors. We can retrieve the identity $\mathsf{As}(n) = \Bbbk\{\Sigma_n\}$ from this observation since $K^{\otimes n} = (\Bbbk\{\Sigma_n\} \otimes K^{\otimes n})_{\Sigma_n}$.

(c) The structure of free Lie algebras is more intricate. Nevertheless, in characteristic 0, we can apply the Milnor-Moore theorem to identify the free Lie algebra $\mathbb{L}(K)$ with the primitive part of the tensor algebra $\mathbb{T}(K)$, where we use the formula $\Delta(x) = x \otimes 1 + 1 \otimes x$ to define the coproduct of any generating element $x \in K$. Moreover, we have versions of the Milnor-Moore theorem which enable us to deduce an expansion of the form

$$\mathbb{L}(K) = \bigoplus_{n=1}^{\infty} (\mathsf{Lie}(n) \otimes K^{\otimes n})_{\Sigma_n}$$

from the relation $\mathbb{L}(K) = \mathbb{P}\,\mathbb{T}(K)$. More details on this construction are given in §E.

We keep focusing on non-unitary algebras, but the identifications of (a-b) obviously extend to the unitary setting.

Proposition I.2.4 has the following analogue for the category of algebras associated to an operad:

Proposition I.3.6. *Let* P *be any operad.*

(a) *The forgetful functor from* P*-algebras to the ground category creates all kinds of small limits, the filtered colimits, and the coequalizers which are reflexive in the category of symmetric sequences.*

(b) *The category of* P*-algebras admits coproducts too and, as a byproduct, all kinds of small colimits, though the forgetful functor towards the ground category does not preserve colimits in general.*

Recall that we devote an appendix section §I.5 to recollections on filtered colimits and reflexive coequalizers.

Proof. Same argument line as in the proof of Proposition I.2.4. See also [38, §3.3] or [79, Proposition 2.3.5] for this proposition. □

I.3.7. *Restriction functors.* If an operad morphism $\phi : \mathsf{P} \to \mathsf{Q}$ is given, then we can compare the category of P-algebras and the category of Q-algebras. First, we immediately observe that any Q-algebra B inherits a natural P-algebra structure since the operad P acts on B through Q by way of the morphism $\phi : \mathsf{P} \to \mathsf{Q}$. Thus we have a natural functor $\phi^* : \mathcal{Q} \to \mathcal{P}$, referred to as the restriction functor associated with ϕ, from the category of Q-algebras to the category of P-algebras. The existence of reflexive coequalizers can be used to define a morphism in the converse direction, so that:

Proposition I.3.8. *The restriction functor* $\phi^* : \mathcal{Q} \to \mathcal{P}$*, associated to any operad morphism* $\phi : \mathsf{P} \to \mathsf{Q}$*, has a left adjoint* $\phi_! : \mathcal{P} \to \mathcal{Q}$*, referred to as the extension functor associated with* $\phi : \mathsf{P} \to \mathsf{Q}$*.*

Proof. Let $A \in \mathcal{P}$. Let $\phi_! A$ be the Q-algebra defined by the reflexive coequalizer such that

$$\begin{tikzcd} S(\mathsf{Q}, S(\mathsf{P}, A)) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & S(\mathsf{Q}, A) \arrow[l, bend right, "s_0"'] \arrow[r, dotted] & \phi_! A \end{tikzcd} ,$$

where:

- the morphism d_0 is the morphism of free Q-algebras induced by the adjunction augmentation $\lambda : S(\mathsf{P}, A) \to A$ associated to the P-algebra A;
- the morphism d_1 is induced by $S(\phi, A) : S(\mathsf{P}, A) \to S(\mathsf{Q}, A)$, by using the functoriality of the generalized symmetric algebra construction with respect to the coefficients;
- and the reflection s_0 is the morphism of free Q-algebras induced by the universal morphism $\iota : A \to S(\mathsf{P}, A)$ of the free P-algebra $S(\mathsf{P}, A)$.

We can easily check, by using the universal property of free Q-algebras, that the definition of a morphism of Q-algebras $g : \phi_! A \to B$ amounts to the definition of a morphism $f : A \to B$ commuting with P-actions. Therefore the mapping $\phi_! : A \mapsto \phi_! A$ defines a left adjoint of the restriction functor $\phi^* : B \mapsto \phi^* B$. □

I.3.9. *Basic examples of extension and restriction functors.* The commutative, associative and Lie operads are connected by morphisms

$$\mathsf{Lie} \xrightarrow{\iota} \mathsf{As} \xrightarrow{\alpha} \mathsf{Com}$$

determined on generating operations $\lambda \in \mathsf{Lie}(2)$, $\mu \in \mathsf{As}(2)$ and $\mu \in \mathsf{Com}(2)$, by the expressions $\iota(\lambda) = \mu - (1\ 2) \cdot \mu$ and $\alpha(\mu) = \mu$.

The restriction functor $\alpha^* : \mathcal{C}om \to \mathcal{A}s$ is identified with the obvious embedding of the category of commutative algebras into the category of associative algebras. The restriction functor $\iota^* : \mathcal{A}s \to \mathcal{L}ie$ is identified with the classical functor mapping an associative algebra A to the Lie algebra $\iota^* A = A_-$ with the same underlying module as A and the commutator $\lambda(a_1, a_2) = \mu(a_1, a_2) - \mu(a_2, a_1)$ as Lie bracket. Throughout this paragraph, we use the notation of the generating operadic operation λ and μ instead of the more usual notation $\lambda(a_1, a_2) = [a_1, a_2]$ and $\mu(a_1, a_2) = a_1 a_2$ to mark the relationship of the constructions with our operad morphisms.

The extension functor $\alpha_! : \mathcal{A}s \to \mathcal{C}om$, defined as the left adjoint of $\alpha^* : \mathcal{C}om \to \mathcal{A}s$, can be identified with the functor mapping an associative algebra A to the quotient $A/ < \lambda(A, A) >$, where $< \lambda(A, A) >$ refers to the two-sided ideal of A generated by the commutators $\lambda(a_1, a_2) = \mu(a_1, a_2) - \mu(a_2, a_1)$, $a_1, a_2 \in A$. The extension functor $\iota_! : \mathcal{L}ie \to \mathcal{A}s$, defined as the left adjoint of $\iota^* : \mathcal{A}s \to \mathcal{L}ie$, can be identified with the functor mapping a Lie algebra $\mathfrak{g}$ to the augmentation ideal of the standard enveloping algebra $\mathbb{U}(\mathfrak{g})$, the quotient of the tensor algebra $\mathbb{T}(\mathfrak{g})$ by the two-sided ideal generated by the differences $\mu(a_1, a_2) - \mu(a_2, a_1) - \lambda(a_1, a_2)$, $a_1, a_2 \in \mathfrak{g}$, where μ refers to the product of $\mathbb{T}(\mathfrak{g})$ and λ to the Lie bracket of $\mathfrak{g}$. In all cases, we can easily check that the functors defined by these basic constructions satisfy the adjunction relation of extension functors and hence is isomorphic to the operadic extension functor of Proposition I.3.8.

I.3.10. *Algebras over connected operads.* The structure of an algebra over the unit operad I (see §I.2.3) reduces to an identity operation, and hence, the category of I-algebras is simply nothing but the base category $\mathcal{B}ase$. In the context of a

pointed category (see §I.2.13), the existence of an augmentation $\epsilon : \mathsf{P} \to \mathsf{I}$, when P is a connected operad, implies that any $X \in \mathcal{B}ase$ inherits the structure of an algebra over P, simply given by a trivial action in arity $r > 1$. In the context of $\Bbbk$-modules, the application of this construction to the classical examples $\mathsf{P} = \mathsf{Com}, \mathsf{As}, \mathsf{Lie}$ identifies a module with an abelian commutative (respectively, associative, Lie) algebra, on which the structure product (respectively, Lie bracket) is identically zero.

The extension functor $\epsilon_! : \mathsf{P} \to \mathcal{B}ase$ associated to an augmentation $\epsilon : \mathsf{P} \to \mathsf{I}$ is identified with an indecomposable functor which, in the module context, amounts to killing the non-trivial operations $p(a_1, \ldots, a_r)$, $r > 1$, occuring in a given P-algebra A. In the case $\mathsf{P} = \mathsf{As}$ (and in the case $\mathsf{P} = \mathsf{Com}$ similarly), this indecomposable functor $\epsilon_! : \mathcal{A}s \to \mathcal{B}ase$ can be defined by the standard construction $\epsilon_! A = A/A^2$ where A^2 refers to the submodule of A spanned by the products $\mu(a, b)$, for $a, b \in A$. In the case $\mathsf{P} = \mathsf{Lie}$, we obtain $\epsilon_! \, \mathfrak{g} = \mathfrak{g} / \Gamma_2(\mathfrak{g})$, where $\Gamma_2(\mathfrak{g})$ refers to the submodule of $\mathfrak{g}$ spanned by the bracket $\lambda(a, b)$, for $a, b \in \mathfrak{g}$.

I.3.11. *Further remarks: operads and monads.* The use of the functor $S(\mathsf{P})$ in operad theory goes back to [72], where it is observed that (a pointed space variant of) this functor $S(\mathsf{P})$ defines a monad on the base category. The category of P-algebras is defined in terms of this monad $S(\mathsf{P})$ in [72]. This definition is formally equivalent to the definition of §I.1.12 where we just consider (in the point of view of [72]) an expansion of the action of the monad $S(\mathsf{P})$ on A. In the approach of [72], the result of Proposition I.3.3 is a consequence of a general statement about algebras over monads (see [65, §VI.2]).

In fact, the definition of $S(\mathsf{M}) : X \mapsto S(\mathsf{M}, X)$ as a functor from the base category to itself makes sense for any symmetric collection M, and not only for operads. The category of symmetric sequence comes also equipped with structures, like a composition product, that reflect pointwise operations on functors (see [38] for an overall reference on this subject). These observations are the source of abstract categorical definitions for the notion of an operad. These definitions are not used in this book, but we can give a sketch of the ideas.

In the point of view of [72], the operads are exactly the symmetric sequences P such that $S(\mathsf{P})$ inherits a monad structure. On the other hand, the category of functors $F : \mathcal{B}ase \to \mathcal{B}ase$ is equipped with a natural monoidal structure, defined by the pointwise composition operation, and monads can be defined abstractly as monoid objects in that category. In parallel, we can interpret the definition of the composition structure of an operad in §I.1.1, as the definition of an abstract monoid structure in the category of symmetric sequences with respect to the composition operation reflecting the composition structure of functors. In that respect, the correspondence between operads and monads follows from the relationship between the composition of symmetric sequences and the composition of functors (we refer to [83] for the introduction of this idea, to the book [38] for an overall account of operad theory based on this approach and further references).

This monadic approach of operads supposes that the tensor product of the base category commutes with colimits. But we soon consider categories for which this requirement and hence, the monadic approach, fail.

I.4. The definition of unitary operads

In §I.1.15, we introduced the expression of unitary operad for operads P satisfying $\mathsf{P}(0) = pt$ (in the set-theoretic context). The terminology refers to the observation that the evaluation morphism of a P-algebra gives a morphism $\lambda : \mathsf{P}(0) \to A$ in arity $r = 0$, and if we assume $\mathsf{P}(0) = pt$, then this morphism is equivalent to a the definition of a distinguished unit element in A. Because of this interpretation, we also use the expression of unitary operation to refer to the arity 0 elements of an operad (in general), and, as a follow-up, we use the noun of non-unitary operad to refer to operads which have nothing in arity 0 (in contrast to unitary operads) so that $\mathsf{P}(0) = \emptyset$ (in the set-theoretic context yet).

The operad of unitary associative monoids As_+ and the operad of unitary commutative monoids Com_+, defined in §I.1.15 in the set-theoretic context, are basic examples of unitary operads in sets. The operads As and Com, formed by dropping the arity 0 terms of these unitary operads As_+ and Com_+, give basic instances of non-unitary operads.

In the general context of a symmetric monoidal category, we say that an operad P is unitary when we have $\mathsf{P}(0) = \mathbb{1}$, the tensor unit of the considered category (which the one-point set represents in the set-theoretic context), and that an operad is non-unitary when we have $\mathsf{P}(0) = 0$, the initial object. For instance, we have a unitary version of the associative operad in $\Bbbk$-modules As_+, satisfying $\mathsf{As}_+(0) = \Bbbk$, as well as a non-unitary version As (already considered in §I.2), satisfying $\mathsf{As}(0) = \Bbbk$, and similarly in the case of the commutative operad.

We consider the category formed by the unitary operads as objects and the operad morphisms $\phi : \mathsf{P} \to \mathsf{Q}$ which are the identity of $\mathbb{1}$ in arity 0 as morphisms. We adopt the convention to mark the consideration of fixed terms in operad categories by adding lower-scripts to our notation. We therefore use the expression $\mathcal{O}p_0$ to refer to the category of non-unitary operads, and the expression $\mathcal{O}p_1$ for the just defined category of unitary operads. We should note that non-unitary operads form a full subcategory of the whole category of operads, since a morphism $\phi(0) : \mathsf{P}(0) \to \mathsf{Q}(0)$ is automatically the identity of the initial object 0 when $\mathsf{P}(0) = \mathsf{Q}(0) = 0$. In the case where the base category is equipped with the cartesian product as tensor structure, and $\mathbb{1}$ is the final object $*$, we may also use the notation $\mathcal{O}p_*$ (as in the foreword) instead of $\mathcal{O}p_1$.

In principle, our operads are supposed to be unital in the sense that they are equipped with a unit morphism $\eta : \mathbb{1} \to \mathsf{P}(1)$ (corresponding to a unit element $1 \in \mathsf{P}(1)$ in the point-set context), and this has not to be confused with the requirement that an operad is unitary $\mathsf{P}(0) = \mathbb{1}$. The notion of non-unitary operad, similarly, has not to be confused with the complement of the class of unitary operads. In the point-set context, the unitary operations $p \in \mathsf{P}(0)$ have not to be confused with the unit element $1 \in \mathsf{P}(1)$ too.

The adjectives unitary and unital are used interchangeably in the literature. In particular, the expression of unital operad is used in [72] for what we call the category of unitary operads. In what follows, we prefer to distinguish the application of these terminologies, and we will reserve the term of unital for references to operadic units. We therefore introduce the term unitary as a references to unitary operations acting on algebras and for the related structures.

The purpose of this section is to give an introduction to particular structures attached to unitary operads, mostly with the aim of explaining a reduced construction of unitary operads by generators and relations. The overall idea is that unitary operads P_+ can be produced by the addition of the unit object $\mathbb{1}$ to the arity 0 component of non-unitary operads P, which can also be put apart from composition structures.

In a first step, we distinguish a particular subpart of the composition structure of a unitary operad P_+ involving compositions with unitary operations.

I.4.1. *The deletion structure attached to unitary operads.* To be explicit, we consider composition patterns of the form

$$\begin{aligned}\mathsf{P}_+(n) &\xleftarrow{\simeq} \mathsf{P}_+(n)\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}\\ &\xrightarrow{id\otimes\eta_*}\mathsf{P}_+(n)\otimes\mathsf{P}_+(0)\otimes\cdots\otimes\mathsf{P}_+(1)\otimes\cdots\otimes\mathsf{P}_+(1)\otimes\cdots\otimes\mathsf{P}_+(0)\\ &\xrightarrow{\mu}\mathsf{P}_+(m),\end{aligned}$$

where the morphism η_* is given by the application of operadic units $\eta:\mathbb{1}\to\mathsf{P}_+(1)$ at places specified by an increasing sequence $1\leq k_1<\cdots<k_m\leq n$, and by the identity $id:\mathbb{1}\xrightarrow{=}\mathsf{P}_+(0)$ at the remaining places of the tensor product $\mathbb{1}^{\otimes n}=\mathbb{1}\otimes\cdots\otimes\mathbb{1}$. Since specifying an increasing sequence $1\leq k_1<\cdots<k_m\leq n$ amounts to giving an increasing injection $u:\{1<\cdots<m\}\to\{1<\cdots<n\}$ with $u(i)=k_i$, for $i=1,\ldots,m$, this construction returns a morphism $u^*:\mathsf{P}_+(m)\to\mathsf{P}_+(n)$ associated to any such map $u:\{1<\cdots<m\}\to\{1<\cdots<n\}$.

In the point-set context, these particular composition operations read

$$u^*(p)=p(*,\ldots,1,\ldots,1,\ldots,*),$$

where we use the notation $*$ to refer to the distinguished element of the operad P_+ in arity 0. Intuitively, the application of such composition operations amounts to removing (filling up) the inputs $k\neq i_1,\ldots,i_n$ of the operad element $p\in\mathsf{P}(n)$. Therefore, we coin the expression of deletion operation (or deletion morphism) to refer to morphisms of this form $u^*:\mathsf{P}(n)\to\mathsf{P}(m)$, and some generalizations of these morphisms which we consider in subsequent chapters. If we use the variable interpretation of operadic composites, then we obtain the following expression $u^*(p)=p(*,\ldots,x_1,\ldots,x_m,\ldots,*)$ for the outcome of the deletion process.

In practice, we mostly consider the particular deletion morphisms $\partial_i:\mathsf{P}_+(r)\to\mathsf{P}_+(r-1)$, $i=1,\ldots,r$, associated with the increasing injection $\partial^i:\{1<\cdots<r-1\}\to\{1<\cdots<r\}$ jumping over $i\in\{1<\cdots<r\}$, and corresponding to the partial composition operations $\partial_i(p)=p\circ_i *$. In §I.1.3, we mention that all operadic composites are composites of partial composition products. In the particular case of deletion morphisms, this result can also be deduced from the following proposition and the fact that all increasing injections are composites of elementary maps of the form ∂^i.

Proposition I.4.2. *Let P_+ be any unitary operad.*

(a) *The deletion operations $\mathsf{P}_+(r)\xrightarrow{u^*}\mathsf{P}_+(s)\xrightarrow{v^*}\mathsf{P}_+(t)$ associated to any sequence of increasing injections $\{1<\cdots<t\}\xrightarrow{v}\{1<\cdots<s\}\xrightarrow{u}\{1<\cdots<r\}$, satisfy the relation $v^*u^*=(uv)^*$ on $\mathsf{P}(r)$.*

(b) *The deletion morphisms $\mathsf{P}_+(n)\to u^*\mathsf{P}_+(m)$ associated to increasing injections $\{1<\cdots<m\}\xrightarrow{u}\{1<\cdots<n\}$, with $m>0$, also satisfy an*

equivariance relation, expressed by the commutativity of the diagram

$$\begin{tikzcd} \mathsf{P}(n) \arrow[r, "s"] \arrow[d, "u^*"'] & \mathsf{P}(n) \arrow[d, "s_*(u)^*"] \\ \mathsf{P}(m) \arrow[r, "u^*(s)"'] & \mathsf{P}(m) \end{tikzcd}$$

for all permutations $s \in \Sigma_n$, where $u^*(s) \in \Sigma_m$ is obtained by applying our deletion operation to $s \in \Sigma_n$ within the permutation operad, and $s_*(u)$ denotes the increasing injection such that $s \cdot u = s_*(u) \cdot u^*(s)$ in the set of maps from $\{1 < \cdots < m\}$ to $\{1 < \cdots < n\}$.

Proof. The first assertion follows from a straightforward application of the associativity relation of Figure I.3 to the unitary operad P_+. In the application this associativity relation, we put unitary factors $\mathsf{P}_+(0)$ at appropriate places, specified by the injections u and v, and we insert morphisms $\eta : \mathbb{1} \to \mathsf{P}_+(1)$ to fill up the remaining composition places. By the way, we also use the unit axiom to identify composites $\mathbb{1} \otimes \mathsf{P}_+(0) \xrightarrow{\eta \otimes id} \mathsf{P}_+(1) \otimes \mathsf{P}_+(0) \xrightarrow{\mu} \mathsf{P}_+(0)$, arising in this process, with the canonical isomorphism $\mathbb{1} \otimes \mathsf{P}_+(0) \xrightarrow{\simeq} \mathsf{P}_+(0)$. The second assertion of the proposition follows from the equivariance axiom of Figure I.4, where we take $n_i = 1$ when $i = u(1), \ldots, u(m)$ and $n_i = 0$ otherwise. □

The unit axiom of operads also implies:

Proposition I.4.3. *For any unitary operad P_+, the deletion operation $\mathsf{P}_+(1) \xrightarrow{\eta^*} \mathsf{P}_+(0) = \mathbb{1}$ associated to the initial map of the one-point set defines a retraction of the operadic unit $\eta : \mathbb{1} \to \mathsf{P}_+(1)$,*

Proof. Immediate. □

We also have associativity relations, which can be deduced from the associativity axioms of §I.1, but we do not need to make these relations explicit for the moment.

I.4.4. *Deletion morphisms on permutations.* In §4, we use the results of Proposition I.4.2 to integrate the action of the symmetric groups and the deletion operations on the underlying sequence of a unitary operad P_+ into the structure of a diagram over the category Λ_* formed by finite ordinals $\underline{\mathsf{n}} = \{1 < \cdots < n\}$ and injective maps $f : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ (not-necessarily increasing) between them. For this aim, we use that any injective map $f : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ admits a unique decomposition $f = v \cdot t$ such that t is a permutation of $\{1, \ldots, m\}$ and v is an increasing injection from $\{1 < \cdots < m\}$ to $\{1 < \cdots < n\}$.

By the way, in Proposition I.4.2, we implicitly use that the permutation $u^*(s) \in \Sigma_m$ produced by the application of a deletion operation $u^* : \Sigma_n \to \Sigma_m$ in the permutation operad can be identified with the unique permutation t occurring in a decomposition of this form $f = v \cdot t$ form the composite map $f = s \cdot u$. We suggest the reader to test this formula on examples.

I.4.5. *Unitary extensions and connected unitary operads.* We say that a non-unitary operad P, satisfying $\mathsf{P}(0) = 0$, admits a unitary extension when we have a unitary operad P_+ agreeing with P up to the arity 0 term $\mathsf{P}_+(0) = \mathbb{1}$, and of which composition operations extend the composition operations of P, so that the canonical embedding $i_+ : \mathsf{P} \to \mathsf{P}_+$ defines a morphism in the category of operads.

We often use an expression of the form $\mathsf{Q} = \mathsf{P}_+$ to assert that a given operad Q forms a unitary extension of another given (non-unitary) operad P. To give basic examples, the operad of unitary associative algebras As_+ in §I.1.15 is a unitary extension of the corresponding non-unitary operad As, and similarly as regards the operad of unitary commutative algebras Com_+.

In general, any unitary operad P_+ is obviously a unitary extension of the non-unitary operad P defined by the truncation

$$\mathsf{P}(r) = \begin{cases} 0, & \text{for } r = 0, \\ \mathsf{P}_+(r), & \text{for } r > 0. \end{cases}$$

Since we observe in §I.1.3 that the composition structure of an operad can be fully determined by giving the partial composition operations $\circ_i$, we can see that the composition structure of a unitary operad P_+ is determined by internal composition operations of the associated non-unitary operad $\circ_i : \mathsf{P}(m) \otimes \mathsf{P}(n) \to \mathsf{P}(m+n-1)$ together with the deletion operations $\partial_i : \mathsf{P}_+(r) \to \mathsf{P}_+(r-1)$, which represents the partial composition operations with the additional unit term on P_+. In the sequel, we only apply this reconstruction process to free operads, and in the connected context (which we consider soon), in order to define a reduced version of free objects in the category of unitary operads.

Nonetheless, we can easily identify the morphisms of non-unitary operads $\phi : \mathsf{P} \to \mathsf{Q}$ which admit a unitary extension $\phi_+ : \mathsf{P}_+ \to \mathsf{Q}_+$. Indeed, such a morphism ϕ_+ is necessarily given the identity of the unit object in arity 0 (by definition of the category of unitary operads) and is therefore entirely determined by the associated morphism of non-unitary operads ϕ. Since the internal partial composites of P and the deletion operations $\partial_i : \mathsf{P}_+(r) \to \mathsf{P}_+(r-1)$ exhaust all partial composites on the unitary operad, we immediately obtain:

Proposition I.4.6. *The mapping $\mathsf{P}_+ \mapsto \mathsf{P}$ defines a faithful functor from the category of unitary operads $\mathcal{O}p_1$ to the category of non-unitary operads $\mathcal{O}p_0$.*

Furthermore, a morphism of non-unitary operads $\phi : \mathsf{P} \to \mathsf{Q}$ lies in the image of this functor if and only if the extension of this morphism by the identity of $\mathbb{1}$ in arity 0 intertwines the additional deletion operations ∂_i associated with given unitary extensions P_+ and Q_+ of the operads P and Q. □

I.4.7. *Connected unitary operads.* We now focus on the setting where our unitary operads satisfy $\mathsf{P}_+(r) = \mathbb{1}$ for $r = 0$ and $r = 1$. We say that an operad satisfying these conditions is connected as a unitary operad. We use the notation $\mathcal{O}p_{11}$ for the category defined by these operads, where, as usual, we add lower-scripts to mark the required first terms of the objects. In the case where $\mathbb{1} = *$ (the final object of the base category), we may also use the notation $\mathcal{O}p_{**}$ instead of $\mathcal{O}p_{11}$.

We assume, as in the definition of $\mathcal{O}p_1$, that the morphisms of $\mathcal{O}p_{11}$ are given by the identity of the unit object $\mathbb{1}$ in arity 0. The category of connected unitary operads $\mathcal{O}p_{11}$ forms a full subcategory of the category of unitary operads $\mathcal{O}p_1$, but is not full in the whole category of operads $\mathcal{O}p$, at least in general (this is the case when, for instance, we have $\mathbb{1} = *$).

We aim to define a reduced version of free objects in the category of connected unitary operads. We start with the adjunction of Theorem I.2.14, between the free operad $\mathbb{O} : \mathsf{M} \mapsto \mathbb{O}(\mathsf{M})$ and the augmentation ideal functor $\overline{\omega} : \mathcal{O}p_0 \to \mathcal{S}eq_{00}$ on the

category of connected (non-unitary) operads. We essentially extend this adjunction to objects equipped with deletion operations in order to reach our result. We first introduce an appropriate category of symmetric sequences with deletion operations in order to define a suitable target for an extended version of the augmentation ideal functor $\overline{\omega}$, which we define as the composite mapping $\mathsf{P}_+ \mapsto \mathsf{P} \mapsto \overline{\mathsf{P}}$ in the context of unitary operads.

When we consider a unitary symmetric sequence M_+, satisfying $\mathsf{M}_+(0) = \mathbb{1}$ and which may form the underlying sequence of a unitary operad without any alteration, we simply assume that M_+ is equipped with a contravariant action of increasing injections $u^* : \mathsf{M}(n) \to \mathsf{M}(m)$, so that the functoriality and equivariance relations of Proposition I.4.2 hold. When we deal with a non-unitary symmetric sequence M, satisfying $\mathsf{M}(0) = 0$, we assume the existence of such a structure on a unitary symmetric sequence M_+, associated to M, such that $\mathsf{M}_+(0) = \mathbb{1}$ and $\mathsf{M}_+(r) = \mathsf{M}(r)$ for $r > 0$. But, for connected symmetric sequences, which we want to regard as the underlying structures of the augmentation ideals $\overline{\mathsf{P}}$ of unitary operads P_+, we need to provide a more elaborate definition, involving an additional unit condition.

I.4.8. *Connected symmetric sequences with deletion operations.* We precisely call connected symmetric sequence with deletion operations the structure formed by a symmetric sequence M such that $\mathsf{M}(0) = \mathsf{M}(1) = 0$, together with deletion morphisms $\mathsf{M}_{++}(n) \xrightarrow{u^*} \mathsf{M}_{++}(m)$, associated to all increasing injection $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$, and satisfying the relations of Proposition I.4.2-I.4.3, on the extended symmetric sequence

$$\mathsf{M}_{++}(r) = \begin{cases} \mathbb{1}, & \text{for } r = 0, 1, \\ \mathsf{M}(r), & \text{for } r > 1. \end{cases}$$

When we consider the relation of Proposition I.4.3, we also assume, to be precise, that the identity $\mathsf{M}_{++}(1) = \mathbb{1}$ provides a canonical unit morphism $\eta : \mathbb{1} \to \mathsf{M}_{++}(1)$ on this extended symmetric sequence M_{++}.

We adopt the notation $\mathcal{S}eq_{00}^{+}$ for the category formed by connected symmetric sequence with deletion operations, with as morphisms the morphisms of connected symmetric sequences preserving the extra structure.

We immediately see that the augmentation ideal functor induces a functor $\overline{\omega} : \mathcal{O}p_{11} \to \mathcal{S}eq_{00}^{+}$ from unitary operads to connected symmetric sequence with deletion operations. We have the following theorem:

THEOREM I.4.9. *Let M be a connected symmetric sequence with deletion operations. Let $\mathbb{O}(\mathsf{M})$ be the free operad associated to the symmetric sequence M, where we forget about the extra deletion structure.*

(a) *The free operad $\mathbb{O}(\mathsf{M})$ has a unique unitary extension $\mathbb{O}(\mathsf{M})_+$ such that the morphism $\iota : \mathsf{M} \to \overline{\mathbb{O}}(\mathsf{M})$ defines a morphisms of connected symmetric sequences with deletion operations.*

(b) *Furthermore, the mapping $\mathbb{O}_+ : \mathsf{M} \mapsto \mathbb{O}(\mathsf{M})_+$ defines a left adjoint of the functor $\overline{\omega} : \mathcal{O}p_{11} \to \mathcal{S}eq_{00}^{+}$, with the adjunction unit $\iota : \mathsf{M} \to \overline{\mathbb{O}}(\mathsf{M})$ inherited from the non-unitary free operad $\mathbb{O}(\mathsf{M})$. The augmentation of this adjunction $\lambda_+ : \mathbb{O}(\overline{\mathsf{P}})_+ \to \mathsf{P}_+$, which we associate to any connected unitary operad $\mathsf{P}_+ \in \mathcal{O}p_{11}$, is also identified with the extension of the augmentation morphism $\lambda : \mathbb{O}(\overline{\mathsf{P}}) \to \mathsf{P}$, inherited from the non-unitary free operad $\mathbb{O}(\mathsf{M})$, by the identity of the unit object $\mathbb{1}$ in arity 0.*

Explanations. We refer to the appendix part (§B.5) for the formal proof of this statement. At this stage, we can content ourselves with the following informal explanations. We explained in §I.2 that the free operad $\mathbb{O}(\mathsf{M})$ intuitively consists, in the point-set context, of formal composites of elements of the generating symmetric sequence M. We basically replace the application of deletion morphisms u^* in the free operad $\mathbb{O}(\mathsf{M})_+$ by equivalent composites with a unitary operation $*$. We then use the associativity of operadic composition products to decompose the operation on the free operad $\mathbb{O}(\mathsf{M})_+$ into the evaluation of internal deletion operations $v^*(\xi) = \xi(*, \ldots, 1, \ldots, 1, \ldots, *)$ on generating elements $\xi \in \mathsf{M}(n)$. We identify the result of these operations with

- an element of $\mathsf{M}(m)$ in the case where more than $m > 1$ inputs are left in the composite $v^*(\xi) = \xi(*, \ldots, 1, \ldots, 1, \ldots, *)$,
- a (multiple of the) unit operation $1 \in \mathbb{O}(\mathsf{M})(1)$, in the case where this composite leaves one input $v^*(\xi) = \xi(*, \ldots, 1, \ldots, *)$ only,
- a (multiple of the) unitary operation $*$ in the case where we get a full unitary composite $v^*(\xi) = \xi(*, \ldots, *)$, leaving no input at all in ξ.

For instance, let $u : \{1 < 2 < 3\} \to \{1 < 2 < 3 < 4 < 5\}$ be the injection such that $u(1) = 1$, $u(2) = 2$, $u(3) = 4$. For a composite $p = (1\ 5) \cdot ((a \circ_1 b) \circ_4 c)$, where $a \in \mathsf{M}(2)$, $b \in \mathsf{M}(3)$, $c \in \mathsf{M}(2)$, and on which we apply the transposition $(1\ 5) \in \Sigma_5$, we readily obtain a result of the form:

$$u^*((1\ 5) \cdot ((a \circ_1 b) \circ_4 c)) = v^*(b) \cdot (1\ 2\ 4) \cdot (a \circ_2 c).$$

In this expression, we consider the map $v : \{1\} \to \{1 < 2 < 3\}$ such that $v(1) = 2$, and we assume that we work in the category modules over a ring $\Bbbk$, so that the application of the morphism $v^* : \mathsf{M}(3) \to \Bbbk$ to $b \in \mathsf{M}(3)$ returns a scalar $v^*(b) \in \Bbbk$ which we put in front of our result. □

We now specialize our study to operads in modules over a ring $\mathcal{M} = \mathcal{M}od$. We explain in §I.2.9 that operads in module categories can be defined by generators and relations as quotients $\mathsf{P} = \mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} >$, where we consider an ideal $< z^\alpha, \alpha \in \mathcal{I} >$ in a free operad $\mathbb{O}(\mathsf{M})$. In the context of operads with deletion operations, we have the following simple consequence of Theorem I.4.9:

Proposition I.4.10. *Let M be a connected symmetric sequence with deletion operations (in $\Bbbk$-modules). We apply the construction of Theorem I.4.9 to obtain a unitary extension of the free operad associated to M. Let $\mathsf{S} =< z^\alpha, \alpha \in \mathcal{I} >$ be the ideal generated by a collection of elements $z^\alpha \in \mathsf{S}(n_\alpha)$ in the free operad $\mathbb{O}(\mathsf{M})$. If for each $i = 1, \ldots, n_\alpha$, we have $\partial_i(z^\alpha) \equiv 0$ in the unitary operad $\mathbb{O}(\mathsf{M})_+$, then:*

(a) *The operad $\mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} >$ admits a unitary extension $\mathbb{O}(\mathsf{M})_+ / < z^\alpha, \alpha \in \mathcal{I} >$ forming a quotient of the free unitary operad $\mathbb{O}(\mathsf{M})_+$.*

(b) *The morphisms of unitary operads $\bar{\phi}_f : \mathbb{O}(\mathsf{M})_+ / < z^\alpha, \alpha \in \mathcal{I} > \to \mathsf{Q}_+$, towards some $\mathsf{Q}_+ \in \mathcal{O}p_{11}$, are in bijection with the morphisms $\phi_f : \mathbb{O}(\mathsf{M})_+ \to \mathsf{Q}_+$ satisfying $\phi_f(z^\alpha) = 0$ for each generating element of the ideal $z^\alpha \in \mathsf{S}(n_\alpha)$.*

Proof. Straightforward. □

In the next paragraph, we explain applications of this proposition to the unitary operads considered in §I: the associative operad As_+, and the commutative operad Com_+. We also address the definition of a unitary version of the Poisson operad

Pois_+. We work within a category of modules over a fixed ground ring, as required by the construction of Proposition I.4.10.

We can actually adapt Proposition I.4.10 in the set-theoretic context, by using reflexive coequalizers instead of operad ideals (see §I.2.5), and description by generators and relations of the associative (respectively, commutative) operad in sets. We can moreover extend the result of Proposition I.2.7 in the unitary setting and gives a complete operadic counterpart of the results of Proposition I.1.16-I.1.17. We leave this review to interested readers. We focus on applications of these constructions in the module context for the moment.

I.4.11. *Examples of unitary operads constructed by generators and relations.* We consider the case of the associative operad first. Recall that $\mathsf{As} = \mathbb{O}(\Bbbk\,\mu(x_1,x_2)\oplus \Bbbk\,\mu(x_1,x_2)) \,/ < \mu(\mu(x_1,x_2),x_3)-\mu(x_1,\mu(x_2,x_3)) >$ for a generating symmetric sequence such that $\mathsf{M}_{\mathsf{As}}(2) = \Bbbk\,\mu(x_1,x_2)\oplus\Bbbk\,\mu(x_1,x_2)$ and $\mathsf{M}_{\mathsf{As}}(r) = 0$ for $r\neq 2$. Since M_{As} vanishes in arity $r>2$, we only have to specify a tail of deletion operations $\partial_1,\partial_2 : \mathsf{M}_{\mathsf{As}}(2)\to\Bbbk$. (The remaining deletion operations are either trivial or determined by the unit relation $\partial_1(1)=1$ in arity 1.) We take $\partial_1\mu=\partial_2\mu=1$, to reflect the unit relations $\mu(a,e)=a=\mu(e,a)$ in associative algebras. By applying the associativity of deletion morphisms with respect to operadic composition structures, we obtain $\partial_1(\mu(\mu,1)-\mu(1,\mu)) = \mu(\mu(*,1),1)-\mu(1(*),\mu) = \mu(1,1)-1(\mu)=0$ and similarly $\partial_2(\mu(\mu,1)-\mu(1,\mu))=0$. Hence, the assumptions of Proposition I.4.10 are fulfilled, so that the operad As inherits deletion operations, and as a consequence, admits a unitary extension As_+ satisfying $\mathsf{As}_+(0)=\Bbbk$ and $\mathsf{As}_+(r)=\mathsf{As}(r)=\Bbbk\{\Sigma_r\}$ for $r>0$. This operad As_+ is actually identified with the image of the permutation operad under the functor $\Bbbk\{-\}:\mathcal{S}et\mathcal{O}p\to\mathcal{M}od\mathcal{O}p$ (see §II.1.5 in the next chapter for explanations on this approach).

The case of the commutative operad is similar. We take the same expression as in the associative case for the value of the deletion morphisms $\partial_1,\partial_2:\mathsf{M}_{\mathsf{Com}}(2)\to\Bbbk$ on the generating operation $\mu\in\mathsf{M}_{\mathsf{Com}}(2)$, and the remaining verification are same. (Note that the deletion morphisms $\partial_1,\partial_2:\mathsf{M}_{\mathsf{P}}(2)\to\Bbbk$ considered in these constructions fulfil our equivariance requirement in both cases $\mathsf{P}=\mathsf{As},\mathsf{Com}$.) Hence, the assumptions of Proposition I.4.10 are also fulfilled for the commutative operad, which therefore inherits deletion operations, and admits a unitary extension Com_+ satisfying $\mathsf{Com}_+(0)=\Bbbk$ and $\mathsf{Com}_+(r)=\mathsf{Com}(r)=\Bbbk$ for $r>0$. This operad Com_+ is actually identified with the image of the one-point set operad under our functor $\Bbbk\{-\}:\mathcal{S}et\mathcal{O}p\to\mathcal{M}od\mathcal{O}p$ (see the concluding discussion of §II.1 again).

The unitary extension process can also be applied to the Poisson operad Pois. Recall that this operad has a generating symmetric sequence such that $\mathsf{M}_{\mathsf{Pois}}(2) = \Bbbk\,\mu(x_1,x_2)\oplus\Bbbk\,\lambda(x_1,x_2)$, where $\mu=\mu(x_1,x_2)$ represents a (symmetric) commutative product and $\lambda=\lambda(x_1,x_2)$ represents an (symmetric) Lie bracket. We take $\partial_1\mu=\partial_2\mu=1$ (as usual) and $\partial_1\lambda=\partial_2\lambda=0$ to reflect standard unit relations associated with the structure of Poisson algebra. Actually, the vanishing of $\partial_1=\partial_2$ on λ is forced by the antisymmetry relation $(1\;2)\cdot\lambda=-\lambda$ and the equivariance requirement associated with deletion morphisms. Again, we can check that the generating relations of the Poisson operad (see §I.2.12) are canceled by the deletion morphisms, so that the operad Pois inherits deletion operations, and as a consequence, admits a unitary extension Pois_+ satisfying $\mathsf{Pois}_+(0)=\Bbbk$ and $\mathsf{Pois}_+(r)=\mathsf{Pois}(r)$ for $r>0$.

I.5. Appendix: filtered colimits and reflexive coequalizers

The existence of colimits in the category of operads, and in categories of algebras over an operad similarly, relies on the existence of particular colimits (filtered colimits and reflexive coequalizers), which we create in the base category. The purpose of this appendix section is to recall the definition of these fundamental colimits in a general context. We assume that $\mathcal{C}$ is any category. In view towards applications to operads, we also study the image of filtered colimits and reflexive coequalizers under a multifunctor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ with the example of r-fold tensor products $T(X_1, \ldots, X_r) = X_1 \otimes \cdots \otimes X_r$ in mind.

I.5.1. *Filtered colimits.* Recall (see [65, §IX.1]) that a small category $\mathcal{I}$ is filtering when:

- for any pair of objects $\alpha, \beta \in \mathcal{I}$, we have morphisms
$$\begin{tikzcd} \alpha \arrow[dr, dotted, "u"] & \\ & \gamma \\ \beta \arrow[ur, dotted, "v"'] & \end{tikzcd}$$
meeting at the same target object γ in $\mathcal{I}$;
- for any pair of parallel morphisms $u, v : \alpha \rightrightarrows \beta$, we have a coequalizing morphism
$$\begin{tikzcd} \alpha \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \beta \arrow[r, dotted, "t"] & \gamma \end{tikzcd}$$
such that $tu = tv$ in $\mathcal{I}$.

We say that a colimit $\operatorname{colim}_{\alpha \in \mathcal{I}} X_\alpha$ is filtered when the indexing category $\mathcal{I}$ of the diagram X_α is filtering.

We have the following observation:

Proposition I.5.2. *Suppose that the multifunctor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves filtered colimits on each input in the sense that the natural morphism*
$$\operatorname*{colim}_{\alpha \in \mathcal{I}} T(X^1, \ldots, X^k_\alpha, \ldots, X^r) \to T(X^1, \ldots, \operatorname*{colim}_{\alpha \in \mathcal{I}} X^k_\alpha, \ldots, X^r)$$
is iso for any diagram $\{X^k_\alpha\}_\alpha$ over a filtering category $\mathcal{I}$ and all $X^i \in \mathcal{C}$, $i = 1, \ldots, \widehat{k}, \ldots, n$. Then the functor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves filtered colimits on the product category $\mathcal{C}^{\times r}$ in the sense that the natural morphism
$$\operatorname*{colim}_{\alpha \in \mathcal{I}} T(X^1_\alpha, \ldots, X^r_\alpha) \to T(\operatorname*{colim}_{\alpha \in \mathcal{I}} X^1_\alpha, \ldots, \operatorname*{colim}_{\alpha \in \mathcal{I}} X^r_\alpha)$$
is iso for any collection of diagrams $\{X^i_\alpha\}_\alpha$, $i = 1, \ldots, r$, over the same given filtering category $\mathcal{I}$.

Proof. Exercise, or see [38, Proposition 1.2.2] or [79, Lemma 2.3.2]. □

I.5.3. *Reflexive coequalizers.* Recall that a coequalizer is the colimit of a diagram formed by a parallel pair of morphisms $d_0, d_1 : X_1 \rightrightarrows X_0$. For a colimit of this particular shape, we use the notation $\operatorname{coker}\{d_0, d_1 : X_1 \rightrightarrows X_0\}$.

In many examples, a parallel pair of morphisms is given together with an extra morphism $s_0 : X_0 \to X_1$ satisfying $d_0 s_0 = id = d_1 s_0$. In this situation, we say that $\operatorname{coker}\{d_0, d_1 : X_1 \rightrightarrows X_0\}$ forms a reflexive coequalizer and we may also use the notation
$$\operatorname{coker}\{X_1 \overset{\curvearrowleft}{\rightrightarrows} X_0\}$$

in order to stress the existence of the reflection $s_0 : X_0 \to X_1$.

Note that the addition of the reflection $s_0 : X_0 \to X_1$ to the diagram $X_1 \rightrightarrows X_0$ does not change the colimit. The importance of reflexive coequalizers lies in the following stability assertion:

PROPOSITION I.5.4. *Suppose that the multifunctor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves reflexive coequalizers on each input in the sense that the natural morphism*

$$\begin{aligned}\operatorname{coker}\{T(X^1,\dots,X_1^k,\dots,X^r) \overset{\frown}{\rightrightarrows} T(X^1,\dots,X_0^k,\dots,X^r)\}\\ \to T(X^1,\dots,\operatorname{coker}\{X_1^k \overset{\frown}{\rightrightarrows} X_0^k\},\dots,X^r)\end{aligned}$$

is iso for any reflexive diagram $\{X_1^k \overset{\frown}{\rightrightarrows} X_0^k\}$ and all $X^i \in \mathcal{C}$, $i = 1,\dots,\widehat{k},\dots,n$. Then the functor $T : \mathcal{C}^{\times r} \to \mathcal{C}$ preserves reflexive coequalizers on the product category $\mathcal{C}^{\times r}$ in the sense that the natural morphism

$$\begin{aligned}\operatorname{coker}\{T(X_1^1,\dots,X_1^r) \overset{\frown}{\rightrightarrows} T(X_0^1,\dots,X_0^r)\}\\ \to T(\operatorname{coker}\{X_1^1 \overset{\frown}{\rightrightarrows} X_0^1\},\dots,\operatorname{coker}\{X_1^r \overset{\frown}{\rightrightarrows} X_0^r\})\end{aligned}$$

is iso for any collection of reflexive diagram $\{X_1^i \overset{\frown}{\rightrightarrows} X_0^i\}$, $i = 1,\dots,r$, in the base category.

PROOF. Exercise or see [38, Proposition 1.2.1] or [79, Lemma 2.3.2]. □

The fundamental role of reflexive coequalizers is also asserted by the following proposition:

PROPOSITION I.5.5. *If coproducts and reflexive coequalizers exist in a category $\mathcal{C}$, then so does any kind of small colimit in $\mathcal{C}$.*

PROOF. Exercise. Check [18, §2] and [19, §4.3]. □

This proposition is applied in §§I.2-I.3 in order to prove the existence of colimits (of any shape) in the category of operads and in categories of algebras over operads.

CHAPTER II

Operads in Monoidal Categories

In the previous chapter §I, we have worked in the setting of a base category $\mathcal{B}ase$ equipped with a tensor product $\otimes : \mathcal{B}ase \times \mathcal{B}ase \to \mathcal{B}ase$ preserving colimits on each side. The colimit assumption is required for the application of categorical constructions (like colimits, free objects) to operads (§§I.2-I.3), and is also implicitly used as soon as we deal with endomorphisms operads (see §I.1). The basic definition of an operad, on the other hand, makes sense in any symmetric monoidal category $\mathcal{M}$, without assuming that the tensor product of $\mathcal{M}$ satisfies any other requirement than the fundamental unit, associativity and symmetry axioms §0.6(a-c).

The overall purpose of the present chapter is to examine the application of general symmetric monoidal category concepts to operads (regardless of colimit issues). In §II.1, we study the definition of operads in general symmetric monoidal categories, and the applications of symmetric monoidal category changes to operads. In §II.2, we study operads in augmented cocommutative coalgebras (our main example of elaborate symmetric monoidal category). To be specific, we establish in §II.2 that operads in augmented cocommutative coalgebras are equivalent to augmented cocommutative coalgebra objects in the category of operads. In general, we rather use the term of Hopf operad to refer to these structures.

An appendix section §II.3 is devoted to recollections on functors between symmetric monoidal categories. To be more specific, we review the definition of functors preserving symmetric monoidal category structures. We briefly recall the standard definition of the notion of a (strict) symmetric monoidal functor, and we introduce classes of functors which are intermediate between these (strict) symmetric monoidal functors and the standard notion of lax/colax symmetric monoidal functor, classically considered in the literature (see [65]).

Throughout this chapter, we deal with a generalization of the notion of commutative algebra and of the notion of cocommutative coalgebra in the setting of symmetric monoidal categories. Before starting our main matter, we devote a preliminary section to a short review of this subject.

The ideas considered in this chapter are again not original. Our purpose is to give a comprehensive and detailed account of concepts and constructions scattered over the literature.

II.0. Commutative algebras and cocommutative coalgebras in symmetric monoidal categories

The main purpose of this zeroth section, as we just explained, is to make explicit the definition of the notion of commutative algebra, and of the dual notion of cocommutative coalgebra, in the context of symmetric monoidal categories. We address the case of commutative algebras first.

Let us mention that the definitions and constructions developed in the present section extend to associative (non-commutative) algebras, and to coassociative coalgebras dually. In the non-commutative context, we just lose the coproduct (respectively, product) interpretation of the tensor product of algebras (respectively, coalgebras) which we explain soon (see §§II.0.2-II.0.3), but we still have a symmetric monoidal structure on algebras (respectively, coalgebras) inherited from the underlying category (see §§II.0.2-II.0.3).

II.0.1. *Commutative algebras in symmetric monoidal categories.* For a symmetric monoidal category $\mathcal{M}$, we define a unitary commutative algebra in $\mathcal{M}$ as a structure formed by an object $A \in \mathcal{M}$, together with morphisms $\eta : \mathbb{1} \to A$ and $\mu : A \otimes A \to A$ that make the following diagrams commute:

$$\begin{tikzcd} A\otimes\mathbb{1} \arrow[r, "id\otimes\eta"] \arrow[dr, "\simeq"'] & A\otimes A \arrow[d, "\mu"] & \mathbb{1}\otimes A \arrow[l, "\eta\otimes id"'] \arrow[dl, "\simeq"] \\ & A & \end{tikzcd} , \quad \begin{tikzcd} A\otimes A\otimes A \arrow[r, "id\otimes\mu"] \arrow[d, "\mu\otimes id"'] & A\otimes A \arrow[d, "\mu"] \\ A\otimes A \arrow[r, "\mu"'] & A \end{tikzcd} , \quad \begin{tikzcd} A\otimes A \arrow[r, "(1\ 2)^*"] \arrow[dr, "\mu"'] & A\otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} .$$

The morphism η, respectively μ, represents the unit, respectively the product, attached to this commutative algebra A. The above diagrams naturally express the unit, associativity and commutativity relations of the algebra structure on A.

In the basic case, where $\mathcal{M}$ is the category of sets $\mathcal{M} = \mathcal{S}et$ (respectively, the category of modules $\mathcal{M} = \mathcal{M}od$ over a ground ring $\Bbbk$), we obviously retrieve the classical notion of a commutative monoid with unit (respectively, of a commutative $\Bbbk$-algebra with unit). More examples of unitary commutative algebras in symmetric monoidal categories are considered in this monograph later on.

In general, we refer to a commutative algebra by specifying the underlying object of this structure $A \in \mathcal{M}$, and we abusively assume that the unit morphism η and the product μ are part of the internal structure of this object A. In the same vein, we use the letter η (respectively, μ) as a generic notation for all unit (respectively, product) morphisms attached to a unitary commutative algebra structure. If we need to specify the algebra to which these morphisms is attached, then we simply set $\eta = \eta_A$ (respectively, $\mu = \mu_A$) to mark the object $A \in \mathcal{M}$ in the notation.

The unitary commutative algebras in $\mathcal{M}$ form a category, which we denote by $\mathcal{M}\,\mathcal{C}om_+$, or just by $\mathcal{C}om_+ = \mathcal{M}\,\mathcal{C}om_+$ when the monoidal category $\mathcal{M}$ is fixed by the context. Naturally, we define a morphism of unitary commutative algebras $f : A \to B$ as a morphism of $\mathcal{M}$ that makes the following diagrams commute:

$$\begin{tikzcd} \mathbb{1} \arrow[r, "="] \arrow[d, "\eta_A"'] & \mathbb{1} \arrow[d, "\eta_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} , \quad \begin{tikzcd} A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "\mu_A"'] & B\otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} .$$

Recall that we use the lower script $+$ to mark the consideration of unitary structures (as in §I.1.15). The category of non-unitary commutative algebras, which we denote by $\mathcal{M}\,\mathcal{C}om$, or just by $\mathcal{C}om = \mathcal{M}\,\mathcal{C}om$ when the monoidal category $\mathcal{M}$ is fixed by the context, is obviously defined by dropping the reference to unit morphisms in all definitions.

Note that the unit object of the underlying category $\mathbb{1}$ inherits a natural commutative algebra structure, and represents the initial object of the category of

unitary commutative algebras $\mathcal{M}\,\mathcal{C}om_+$. One can prove that the obvious forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{M}$ creates limits in unitary commutative algebras, whenever limits exist in $\mathcal{M}$. But the forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{M}$ does not preserve colimits in general. (To give the simplest example, we have already observed that the unit object $\mathbb{1}$, which generally differs from the initial object of $\mathcal{M}$, is the initial object of $\mathcal{M}\,\mathcal{C}om_+$.) In the case where the tensor product of $\mathcal{M}$ satisfies the colimit requirement of §0.6, one can prove that some particular colimits can be created in the ground category $\mathcal{M}$, and that $\mathcal{M}\,\mathcal{C}om_+$ inherits colimits of any shape. Indeed, this statement is a particular case of the general result which we establish in the framework of algebras over operads in §I.3. Now, we always have coproducts in $\mathcal{M}\,\mathcal{C}om_+$ (without assuming any colimit requirement on the tensor structure), and we explain this observation in the next paragraph.

II.0.2. *The symmetric monoidal category of commutative algebras.* The category of unitary commutative algebras in a symmetric monoidal category $\mathcal{M}\,\mathcal{C}om_+$ inherits a symmetric monoidal structure from the underlying category $\mathcal{M}$.

Indeed, a tensor product of commutative algebras $A \otimes B$ inherits a canonical unit morphism $\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\eta_A \otimes \eta_B} A \otimes B$, and a canonical product, defined by the composite $A \otimes B \otimes A \otimes B \xrightarrow{(2\ 3)^*} A \otimes A \otimes B \otimes B \xrightarrow{\mu \otimes \mu} A \otimes B$, so that $A \otimes B$ forms a commutative algebra itself.

For the unit object $\mathbb{1}$, which represents the initial object of the category of commutative algebras $\mathcal{M}\,\mathcal{C}om_+$, the isomorphisms $A \otimes \mathbb{1} \xrightarrow{\simeq} A \xleftarrow{\simeq} \mathbb{1} \otimes A$, formed in the underlying monoidal category $\mathcal{M}$, are isomorphisms of commutative algebras. Hence, the unit relations of the tensor product hold within the category of commutative algebras. The associativity and symmetry relations of the tensor product remain valid in the category of commutative algebras too. Thus, we have a whole symmetric monoidal structure on $\mathcal{M}\,\mathcal{C}om_+$, as claimed at the beginning of this paragraph.

One can actually observe that the tensor product $A \otimes B$ represents the coproduct of A and B in $\mathcal{C}om_+$ (and therefore coproducts exist in $\mathcal{C}om_+$ without any assumption on the tensor product). The universal morphisms $A \xrightarrow{i} A \otimes B \xleftarrow{j} B$ are given by the tensor products $i = id_A \otimes \eta_B$ and $j = \eta_A \otimes id_B$, where we consider the unit morphism $\eta_A : \mathbb{1} \to A$ (respectively, $\eta_B : \mathbb{1} \to B$) associated to A (respectively, B).

II.0.3. *Cocommutative coalgebras in symmetric monoidal categories.* The structure of an augmented cocommutative coalgebra in a symmetric monoidal category is defined by duality from the definition of a commutative algebra with unit.

In brief, an augmented cocommutative coalgebra in $\mathcal{M}$ consists of an object $C \in \mathcal{M}$, equipped with morphisms $\epsilon : C \to \mathbb{1}$ and $\Delta : C \to C \otimes C$ such that the following diagrams commute:

$$\begin{tikzcd}
 & C \arrow[dl, "\simeq"'] \arrow[d, "\Delta"] \arrow[dr, "\simeq"] & \\
C \otimes \mathbb{1} & C \otimes C \arrow[l, "id \otimes \epsilon"] \arrow[r, "\epsilon \otimes id"'] & \mathbb{1} \otimes C
\end{tikzcd}
,
\quad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\
C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C
\end{tikzcd}
,
\quad
\begin{tikzcd}
 & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[rr, "(1\ 2)^*"'] & & C \otimes C
\end{tikzcd}
.$$

The morphism ϵ (respectively, Δ) is called the counit or augmentation (respectively, the coproduct) of the cocommutative algebra C. The above diagrams respectively

express the counit, coassociativity and cocommutativity relations attached to a coalgebra structure.

Naturally, a morphism of augmented cocommutative coalgebras $f : C \to D$ is a morphism of $\mathcal{M}$ that makes the following diagrams commute

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\epsilon_C"'] & D \arrow[d, "\epsilon_D"] \\ \mathbb{1} \arrow[r, "="'] & \mathbb{1} \end{tikzcd} , \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D \end{tikzcd} ,$$

where we adopt the same conventions as in the context of algebras for the notation of the structure morphisms attached to coalgebras. The category of augmented cocommutative coalgebras will be denoted by $\mathcal{C}om_+^c$ with a superscript c simply added to notation in order to mark the consideration of coalgebras.

The basic notion of augmented cocommutative coalgebra, classically considered in the literature, corresponds to the case where $\mathcal{M} = \mathcal{M}od$ is a category of modules over a ground ring $\Bbbk$. In the case where $\mathcal{M}$ is the category of sets $\mathcal{M} = \mathcal{S}et$ (and more generally when the tensor structure is given by the cartesian structure of the category), any object $X \in \mathcal{S}et$ inherits an augmentation $\epsilon : X \to *$, because the unit object is the final object $*$ (the one-point set when $\mathcal{M} = \mathcal{S}et$), as well as a diagonal $\Delta : X \to X \times X$. Moreover, the above counit, coassociativity and cocommutativity relations obviously hold for this structure. Hence, any $X \in \mathcal{S}et$ inherits a tautological coalgebra structure. The definition of this diagonal is actually forced by the counit relation, and consequently, we have an isomorphism of categories $\mathcal{S}et\,\mathcal{C}om_+^c = \mathcal{S}et$.

The tensor unit $\mathbb{1}$ inherits a coalgebra structure, defined by inverting the orientation of the arrows in the definition of the algebra structure of §II.0.1, and represents the terminal object of the category of augmented cocommutative coalgebras. Besides, we can also dualize the construction of the tensor product of algebras in §II.0.2 to obtain that a tensor product of coalgebras $C \otimes D$ inherits a coalgebra structure, with the composite morphism $C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}$ as augmentation, and the morphism $C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{(2\ 3)^*} C \otimes D \otimes C \otimes D$ as coproduct.

This tensor product $C \otimes D$ also represents the cartesian product of C and D in the category of augmented cocommutative coalgebras. The universal morphisms $C \xleftarrow{p} C \otimes D \xrightarrow{q} D$ are given by the tensor products $p = id \otimes \epsilon_D$ and $q = \epsilon_C \otimes id$, where we consider the augmentation $\epsilon_C : C \to \mathbb{1}$ (respectively, $\epsilon_D : D \to \mathbb{1}$) of C (respectively, D).

The previous assertion is the exact dual of an observation of §II.0.2 on the tensor product of unitary commutative algebras. One can also check that the forgetful functor $\omega : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{M}$ creates colimits whenever colimits exist in $\mathcal{M}$, just like the dual forgetful functor on the category of commutative algebras creates limits. But, we can not dualize the construction of general colimits in the category of unitary commutative algebras to get limits in the category of augmented cocommutative coalgebras, because we should require that tensor products preserve limits (instead of colimits) then, and this assumption is not fulfilled in general.

We can also use the tensor product construction to provide the category of augmented cocommutative coalgebras with a symmetric monoidal structure, as in the dual context of unitary commutative algebras.

II.0.4. *Change of underlying symmetric monoidal categories.* To complete this preliminary section, we examine the application of a change of symmetric monoidal category to algebra and coalgebra structures.

First, we consider the case where we have a unit-pointed functor $S : \mathcal{M} \to \mathcal{N}$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{N}$ together with a symmetric monoidal transformation $\theta : S(A) \otimes S(B) \to S(A \otimes B)$ (see §II.3.1). Let A be a unitary commutative algebra in a symmetric monoidal category $\mathcal{M}$. Then the object $S(A) \in \mathcal{N}$ forms a commutative algebra in $\mathcal{N}$. Indeed, we have a unit morphism

$$\mathbb{1} \xrightarrow{=} S(\mathbb{1}) \xrightarrow{S(\eta)} S(A)$$

as well as a product

$$S(A) \otimes S(A) \xrightarrow{\theta} S(A \otimes A) \xrightarrow{S(\mu)} S(A),$$

inherited from A, and which satisfy the unit, associativity, and commutativity axioms of §II.0.1 as soon as the natural transformation θ fulfills the coherence constraints of §II.3.1 with respect to the internal symmetric monoidal structures of categories (easy verification).

This construction is obviously functorial with respect to the commutative algebra A. Hence, the mapping $S : A \mapsto S(A)$ induces a functor between algebra categories $S : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{N}\,\mathcal{C}om_+$. This functor is unit-pointed as well, and we readily see, moreover, that the symmetric monoidal transformation $\theta : S(A) \otimes S(B) \to S(A \otimes B)$, inherited from S, defines a morphism of unitary commutative algebras when $A, B \in \mathcal{M}\,\mathcal{C}om_+$. Thus, the functor $S : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{N}\,\mathcal{C}om_+$ induced by $S : \mathcal{M} \to \mathcal{N}$ is unit-pointed and comes equipped with a symmetric monoidal transformation in the category of unitary commutative algebras in $\mathcal{N}$.

In the dual case, when $S : \mathcal{M} \to \mathcal{N}$ is a unit-pointed functor equipped with a symmetric comonoidal transformation $\theta : S(X \otimes Y) \to S(X) \otimes S(Y)$, we readily see that the image of a unitary cocommutative coalgebra under S inherits a unitary cocommutative coalgebra structure so that S induces a functor between coalgebra categories $S : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{N}\,\mathcal{C}om_+^c$. Furthermore, this functor $S : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{N}\,\mathcal{C}om_+^c$, induced by $S : \mathcal{M} \to \mathcal{N}$, is unit-pointed and comes also equipped with a symmetric comonoidal transformation, inherited from the one of the underlying functor $S : \mathcal{M} \to \mathcal{N}$, in the category of augmented cocommutative algebras in $\mathcal{N}$.

In the optimal situation where our functor $S : \mathcal{M} \to \mathcal{N}$ is symmetric monoidal (see §II.3.1), we have an functor induced by S both on algebras $S : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{N}\,\mathcal{C}om_+$ and on coalgebras $S : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{N}\,\mathcal{C}om_+^c$. These functors are both symmetric monoidal too.

In the case where we have functors $S : \mathcal{M} \rightleftarrows \mathcal{N} : T$ forming a symmetric monoidal adjunction in the sense of §II.3.3, we have an induced a symmetric monoidal adjunction at the level of algebra categories $S : \mathcal{M}\,\mathcal{C}om_+ \to \mathcal{N}\,\mathcal{C}om_+ : T$, and at the level of coalgebra categories $S : \mathcal{M}\,\mathcal{C}om_+^c \to \mathcal{N}\,\mathcal{C}om_+^c : T$ as well. Indeed, we readily see that the unit $\eta : X \to T(S(X))$ and the augmentation

$\epsilon : T(S(A)) \to A$ of such an adjunction define morphisms of unitary commutative algebras (respectively, augmented cocommutative coalgebras) when X (respectively, A) has such a structure, and therefore, define the unit and the augmentation morphism of an adjunction at the algebra (respectively, coalgebra) level.

II.0.5. *Basic applications of symmetric monoidal category changes.* To give a simple example of symmetric monoidal category change, we consider the free $\Bbbk$-module functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$ from sets to $\Bbbk$-modules, where $\Bbbk$ is any fixed ground ring. This functor is symmetric monoidal (see §II.3.2). Thus, we have a symmetric monoidal functor $\Bbbk\{-\} : \mathcal{S}et\,\mathcal{C}om_+ \to \mathcal{M}od\,\mathcal{C}om_+$, from unitary commutative monoids to unitary commutative algebras, induced by $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$. We clearly retrieve the classical algebra of a monoid when we apply this construction.

On the other hand, since we have a category isomorphism $\mathcal{S}et = \mathcal{S}et\,\mathcal{C}om_+^c$, we immediately obtain that $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$ induces a functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od\,\mathcal{C}om_+^c$ from sets to augmented cocommutative coalgebras in $\Bbbk$-modules. We explicitly have $\epsilon(x) = 1$ and $\Delta(x) = x \otimes x$ for each generating element $x \in X$ in a free $\Bbbk$-modules $\Bbbk\{X\}$.

Let C be any augmented cocommutative coalgebra. We generally say that an element $c \in C$ is group-like when we have $\epsilon(c) = 1$ and $\Delta(c) = c \otimes c$ in C. We use the notation $\mathbb{G}(C)$ for the set of group-like elements of C. From now on, we set $\mathcal{C}om_+^c = \mathcal{M}od\,\mathcal{C}om_+^c$ to abbreviate notation. We can easily check that the mapping $\mathbb{G} : C \mapsto \mathbb{G}(C)$ defines a right-adjoint of the above functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{C}om_+^c$, from sets to augmented cocommutative coalgebras. The unit of this adjunction is the obvious set embedding $\iota : X \to \Bbbk\{X\}$, and the augmentation $\rho : \Bbbk\{\mathbb{G}(C)\} \to C$ is identified with the obvious $\Bbbk$-module morphism induced by the tautological set-theoretic inclusion $\mathbb{G}(C) \subset C$.

We have observed in §II.0.4 that the functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{C}om_+^c$ is symmetric monoidal, since our initial functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$, from sets to $\Bbbk$-modules, is already so. We can record that the group-like functor $\mathbb{G} : \mathcal{C}om_+^c \to \mathcal{S}et$ is symmetric monoidal too, because this functor, as a right-adjoint, preserves final objects and cartesian products, with which the unit and tensor product of the symmetric monoidal structure of coalgebras are identified (see §II.0.3). We can easily check that the unit morphism and the augmentation morphism of the adjunction $\Bbbk\{-\} : \mathcal{S}et \rightleftarrows \mathcal{C}om_+^c : \mathbb{G}$ are also symmetric monoidal transformations, so that our adjoint functors define a symmetric monoidal adjunction in the sense of §II.3.3.

II.1. Operads in symmetric monoidal categories

In §I.1, we formulate all definitions and constructions in the context of a base category $\mathcal{B}ase$, which we will take among the category of sets, topological spaces, simplicial sets, modules over a ground ring, or among variants of these categories. Nevertheless, the definition of an operad in §I.1 (and the definition of algebra over an operad similarly) makes sense any category $\mathcal{M}$ equipped with a tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, as long as the unit, associativity and symmetry axioms of symmetric monoidal categories are satisfied. Hence, we have a good definition for the notion of an operad (respectively, of an algebra over operads) in general symmetric monoidal categories. The precise purpose of this section is to explain general operad constructions in the context of symmetric monoidal categories.

To start with, we establish a generalization of the results of Proposition I.1.9 and Proposition I.1.17 in the context of symmetric monoidal categories:

PROPOSITION II.1.1. *In any symmetric monoidal category $\mathcal{M}$, we can form an operad Com_+ such that $\mathsf{Com}_+(r) = \mathbb{1}$ for every $r \in \mathbb{N}$, where $\mathbb{1}$ refers to the unit object of $\mathcal{M}$.*

The structure of this operad is precisely defined as follows: each component of the operad $\mathsf{Com}_+(r) = \mathbb{1}$ is equipped with a trivial action of the corresponding symmetric group Σ_r, the unit morphism $\eta : \mathbb{1} \to \mathsf{Com}_+(1)$ is the identity of $\mathbb{1}$, and the composition products $\mu : \mathsf{Com}_+(r) \otimes \mathsf{Com}_+(n_1) \otimes \cdots \otimes \mathsf{Com}_+(n_r) \to \mathsf{Com}_+(n_1 + \cdots + n_r)$ are given by the canonical isomorphisms $\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}$.

The collection Com such that $\mathsf{Com}(0) = 0$ and $\mathsf{Com}(r) = \mathbb{1}$ for $r > 0$ inherits an operad structure as well, and is actually a sub-object of Com_+ in the category of operads.

PROOF. The equivariance, unit, and associativity relations of the operadic composition structure of Com_+ follow from the internal coherence relations satisfied by the unit, associativity, and symmetry isomorphisms in symmetric monoidal categories.

We define the composition products of the operad Com by restriction from the composition structure of Com_+, and for this purpose, we essentially have to check that the composite $\mathbb{1} \otimes 0 \otimes \cdots \otimes 0 \to \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \to \mathbb{1}$ factors through $\mathsf{Com}(0) = 0$ when we deal with a composition product of the form $\mathsf{Com}(r) \otimes \mathsf{Com}(0) \otimes \cdots \otimes \mathsf{Com}(0) \to \mathsf{Com}(0)$. But this assertion is just a direct consequence of the functoriality of the unit isomorphism $\mathbb{1} \otimes X \simeq X$. □

PROPOSITION II.1.2. *Let $\mathcal{M}$ be any symmetric monoidal category. The category of unitary commutative algebras in $\mathcal{M}$, as defined in §II.0.1, is isomorphic to the category of algebras over the operad Com_+ of Proposition II.1.1. The category of non-unitary commutative algebras in $\mathcal{M}$ is isomorphic to the category of algebras over the non-unitary operad Com formed by dropping the term of arity 0 in Com_+.*

PROOF. The result of this proposition concerning the categories of commutative algebras in $\mathcal{M}$ follows from a formal extension, in the setting of monoidal categories, of the arguments of Proposition I.1.16-I.1.17. □

In the case $\mathcal{M} = \mathcal{S}et$, where we have $\mathbb{1} = pt$, we exactly retrieve the result of Proposition I.1.17, where the category of commutative monoids with unit is identified with the category of algebras over the one-point set operad. Indeed, the operad defined in the proposition is a generalization of the one-point set operad of Proposition I.1.9, and our construction gives a version of the unitary commutative operad Com_+ attached to any symmetric monoidal category $\mathcal{M}$.

The second basic example of application of Proposition I.1.17 is the category of $\Bbbk$-modules $\mathcal{M} = \mathcal{M}od$. In this case, we obtain that the usual category of unitary commutative algebras over $\Bbbk$ is isomorphic to the category of algebras over the operad Com_+ such that $\mathsf{Com}_+(r) = \Bbbk$ for every $r \in \mathbb{N}$, and similarly in the non-unitary setting.

In the situation where the ground symmetric monoidal category $\mathcal{M}$ has colimits and the tensor product preserves colimits, we can also use a presentation by generators and relations, as in §I.2.10, to define a commutative operad (unitary and non-unitary) in $\mathcal{M}$. One can check that the definition by generators and relations,

whenever this one makes sense, returns the same operad as the direct construction of Proposition II.1.1 (adapt the argument line of Proposition I.2.7).

To continue this study, we examine the application of a change of underlying symmetric monoidal category to operads. We consider a functor $S : \mathcal{M} \to \mathcal{N}$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{N}$. Recall that S is said to be unit-pointed when we have $S(\mathbb{1}) = \mathbb{1}$, and we say that a unit-pointed functor S is equipped with a symmetric monoidal transformation if we have a bifunctor morphism $\theta : S(A) \otimes S(B) \to S(A \otimes B)$, $A, B \in \mathcal{M}$, satisfying natural unit, associativity, and symmetry constraints (see §II.3.1 for details). We have the following result:

Lemma II.1.3. *Let P be an operad in $\mathcal{M}$. If $S : \mathcal{M} \to \mathcal{N}$ is a unit-pointed functor equipped with a symmetric monoidal transformation $\theta : S(A) \otimes S(B) \to S(A \otimes B)$, then the collection of objects $S(\mathsf{P}(r)) \in \mathcal{N}$, $r \in \mathbb{N}$, defined by applying S termwise to the underlying collection of P, forms an operad $S(\mathsf{P})$ in $\mathcal{N}$. Indeed:*

(a) *The functor S maps the morphisms $s : \mathsf{P}(r) \to \mathsf{P}(r)$ giving the action of permutations $s \in \Sigma_r$ on $\mathsf{P}(r)$ to morphisms of N, so that $S(\mathsf{P}(r)) \in \mathcal{N}$ inherits an action of the symmetric group Σ_r, and this for all $r \in \mathbb{N}$.*

(b) *The collection $S(\mathsf{P})(r) = S(\mathsf{P}(r))$ also inherits a unit morphism*

$$\mathbb{1} \xrightarrow{=} S(\mathbb{1}) \xrightarrow{S(\eta)} S(\mathsf{P}(1))$$

as well as composition products

$$\begin{aligned} S(\mathsf{P}(r)) \otimes S(\mathsf{P}(n_1)) \otimes \cdots \otimes S(\mathsf{P}(n_r)) \\ \xrightarrow{\theta} S(\mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r)) \\ \xrightarrow{S(\mu)} S(\mathsf{P}(n_1 + \cdots + n_r)), \end{aligned}$$

and the equivariance, unit and associativity relations of operads (§I.1.1) hold for this operadic composition structure.

Proof. The unit, associativity and symmetry constraints of symmetric monoidal transformations (§II.3.1) imply that the equivariance, unit and associativity relations of operads on $S(\mathsf{P})$ reduce to the corresponding relations on P, and hence hold at the level of that collection $S(\mathsf{P})$. □

The construction of the operad structure in this lemma is clearly functorial in $\mathsf{P} \in \mathcal{M}\mathcal{O}p$. Furthermore, for a unitary operad P_+ (in the sense of §I.4), we have $S(\mathsf{P}_+(0)) = S(\mathbb{1}) = \mathbb{1}$, so that $S(\mathsf{P}_+)$ is still unitary, with $S(\mathsf{P})$ as associated non-unitary operad (see §I.4.5). Finally, we have the following proposition:

Proposition II.1.4. *If $S : \mathcal{M} \to \mathcal{N}$ is a unit-pointed functor equipped with a symmetric monoidal transformation, then S induces a functor on operad categories $S : \mathcal{M}\mathcal{O}p \to \mathcal{N}\mathcal{O}p$. This functor is given by the construction of Lemma II.1.3 on objects $\mathsf{P} \in \mathcal{M}\mathcal{O}p$.*

This functor also preserves unitary extensions (in the sense of §I.4.5), since we have the identity $S(\mathsf{P}_+) = S(\mathsf{P})_+$, for any unitary operad $\mathsf{P}_+ \in \mathcal{M}\mathcal{O}p_1$. □

In our applications, we essentially need to transport operads from one symmetric monoidal category to another, and we base our constructions on the previous proposition. For the sake of completeness, we can also record that the functor $S : \mathcal{M} \to \mathcal{N}$ in Proposition II.1.4 induces a functor from the category of algebras

over $\mathsf{P} \in \mathcal{M}\mathcal{O}p$ to the category of algebras over the operad $S(\mathsf{P}) \in \mathcal{N}\mathcal{O}p$ associated to P in $\mathcal{N}$. To check this assertion, simply observe that the image of a P-algebra under S inherits evaluation morphisms $S(\mathsf{P}(r)) \otimes S(A)^{\otimes r} \to S(\mathsf{P}(r) \otimes A^{\otimes r}) \to S(A)$ providing $S(A)$ with a natural $S(\mathsf{P})$-algebra structure.

II.1.5. *Examples of functors between operad categories.* The functors considered in the appendix §II.3.2 give fundamental examples of applications of Proposition II.1.4.

(a) Let us begin with the simplest example, namely the functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$ mapping a set $X \in \mathcal{S}et$ to the associated free $\Bbbk$-module $\Bbbk\{X\} \in \mathcal{M}od$. Proposition II.1.4 implies that this functor induces a functor $\Bbbk\{-\} : \mathcal{S}et\mathcal{O}p \to \mathcal{M}od\mathcal{O}p$, from the category of operads in sets towards the category of operads in $\Bbbk$-modules, and similarly as regards the extension of this functor to simplicial sets $\Bbbk\{-\} : \mathcal{S}imp \to s\,\mathcal{M}od$.

(b) The geometric realization functor $|-| : \mathcal{S}imp \to \mathcal{T}op$ induces a functor $\Bbbk\{-\} : \mathcal{S}imp\mathcal{O}p \to \mathcal{T}op\mathcal{O}p$, from the category of operads in simplicial sets towards the category of topological operads. In the converse direction, the singular complex functor $\mathtt{Sing}_\bullet(-) : \mathcal{T}op \to \mathcal{S}imp$ induces a functor $\mathtt{Sing}_\bullet(-) : \mathcal{T}op\mathcal{O}p \to \mathcal{S}imp\mathcal{O}p$, from the category of topological operads towards the category of operads in simplicial sets.

Recall that the geometric realization and singular complex functors define an instance of symmetric monoidal adjunction. In such a situation, we have the following additional result:

Proposition II.1.6. *The functors on operad categories $S : \mathcal{M}\,\mathcal{O}p \leftrightarrows \mathcal{N}\,\mathcal{O}p : T$ induced by the functors of a symmetric monoidal adjunction $S : \mathcal{M} \leftrightarrows \mathcal{N} : T$ are still adjoint to each other. The augmentation $\epsilon : S(T(\mathsf{Q})) \to \mathsf{Q}$ and the unit $\eta : \mathsf{P} \to T(S(\mathsf{P}))$, of this adjunction (at the operad level) are given by the arity-wise application of the augmentation and unit of the underlying adjunction, between the categories $\mathcal{M}$ and $\mathcal{N}$.*

Proof. The augmentation $\epsilon : S(T(Y)) \to Y$ and the unit $\eta : X \to T(S(X))$, of the adjunction $S : \mathcal{M} \rightleftarrows \mathcal{N} : S$ are symmetric monoidal transformations by definition of the notion of a symmetric monoidal adjunction. This observation immediately implies that these morphisms can be applied arity-wise to operads in order to yield morphisms at the operad category level. The structure relations between adjunction augmentations and adjunction units remain obviously valid for these induced operad morphisms, and therefore, we still have an adjunction relation at the level of operad categories, with the unit and augmentation morphisms specified in the proposition. □

Let us record the application of this result to the geometric realization and singular complex functors into a proposition:

Proposition II.1.7. *The functors on operad categories $|-| : \mathcal{S}imp\mathcal{O}p \leftrightarrows \mathcal{T}op\mathcal{O}p : \mathtt{Sing}_\bullet(-)$ induced by the realization of simplicial sets and by the singular complex functor are adjoint to each other. The augmentation $\epsilon : |\mathtt{Sing}_\bullet(\mathsf{Q})| \to \mathsf{Q}$, respectively the unit $\eta : \mathsf{P} \to \mathtt{Sing}_\bullet(|\mathsf{P}|)$, of this adjunction is given by the arity-wise application of the augmentation, respectively unit, of the underlying adjunction between simplicial sets and topological spaces.*

Further examples of applications of Proposition II.1.4-II.1.6 are studied all through the book. For instance, the result of Proposition II.1.6 applies the adjunction $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{C}om_+^c : \mathbb{G}$, between sets and augmented cocommutative coalgebras, involving the extension of the functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$ to coalgebras as left adjoint (see §II.0.5).

In the sequel, we often face adjunction relations $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ such that the right adjoint functor G is symmetric monoidal, but not the left adjoint F (or conversely). In this situation, we still have a functor $G : \mathcal{N}\mathcal{O}p \to \mathcal{M}\mathcal{O}p$, but we can not apply Proposition II.1.4 to get a functor on operads from F. On the other hand, we will see that the functor $G : \mathcal{N}\mathcal{O}p \to \mathcal{M}\mathcal{O}p$, obtained by the arity-wise application of $G : \mathcal{N} \to \mathcal{M}$, preserves limits. In practice, we can apply adjoint functor theorems to retrieve an adjunction relation on operad categories from the single functor $G : \mathcal{N}\mathcal{O}p \to \mathcal{M}\mathcal{O}p$, and we obtain that way an operadic replacement $F_\sharp : \mathcal{M}\mathcal{O}p \to \mathcal{N}\mathcal{O}p$ of the functor $F : \mathcal{M} \to \mathcal{N}$. In §4, we follow this approach to produce a Sullivan's model functor from topological operads to cooperads in cosimplicial commutative algebras (the structures dual to operads in simplicial cocommutative coalgebras).

To prepare this subsequent study, we will study the definition of operads in cocommutative coalgebras in details in the next section.

To conclude this section, observe that we can apply the functor $\Bbbk\{-\} : \mathcal{S}et\mathcal{O}p \to \mathcal{M}od\mathcal{O}p$ to the permutation (respectively, one-point set) operad of §I.1 in order to obtain a model of the associative (respectively, commutative) operad in $\Bbbk$-modules. In the case of the permutation operad, we obtain an operad such that $\mathsf{As}_+(r) = \Bbbk\{\Sigma_r\}$ for $r \in \mathbb{N}$ (unitary case). In the case of the one-point set operad, we obtain an operad such that $\mathsf{Com}_+(r) = \Bbbk\{pt\} = \Bbbk$ for $r \in \mathbb{N}$. In the non-unitary setting, we simply replace the arity 0 component of these operads by the null module. In any case, we exactly retrieve the expansion of §§I.2.10-I.2.11 for the operads defined by generators and relations in §I.2.10. This identification gives an analogue of the results of Proposition I.2.7 in the context of $\Bbbk$-modules. Note that $\mathsf{Com}_+(r) = \Bbbk$ can also be identified with a particular instance of the commutative operad of Proposition II.1.1-II.1.2 since $\Bbbk$ represents the unit object of the category of $\Bbbk$-modules.

II.2. Hopf Operads

The purpose of this section, as we explained after Proposition II.1.7, is to unravel the internal structures of operads in augmented cocommutative coalgebras. To be more specific, one of our aims is to explain that operads in augmented cocommutative coalgebras are equivalent to augmented cocommutative coalgebra objects in the category of operads. The existence of these multiple equivalent definitions motivates us to adopt specific conventions for this category of operads. Basically, we will generally use the terminology of Hopf operad, rather than the expression of operad in augmented cocommutative coalgebras, to refer to the objects of this category, unless we want to specify a particular definition of our structures. Similarly, we generally use the notation $\mathcal{H}opf\mathcal{O}p$, rather than $\mathcal{C}om_+^c\mathcal{O}p$, to refer to the category of Hopf operads. In general, we assume by convention that the costructure of a Hopf object is augmented and cocommutative (or unitary and commutative after dualization) of our notions. Hence, when we use the expression

of Hopf operad, we implicitly assume that we deal with operads equipped with an augmented cocommutative coalgebra structure.

The constructions of the next paragraphs are still valid in an arbitrary ambient monoidal category $\mathcal{M}$, which we fix until the end of the section.

II.2.1. *The definition of Hopf operads as operads in augmented cocommutative coalgebras.* The symmetric monoidal structure of the category of augmented cocommutative coalgebras $\mathcal{C}om^c_+ = \mathcal{M}\mathcal{C}om^c_+$ is defined in §II.0.3. Recall simply that the tensor product of coalgebras $A, B \in \mathcal{C}om^c_+$ is obtained by providing the tensor product of A and B in the underlying symmetric monoidal category with a natural coalgebra structure. The unit, associativity, and symmetry isomorphisms of the tensor product of coalgebras are inherited from the ambient symmetric monoidal category, and the forgetful functor $\omega : \mathcal{M}\mathcal{C}om^c_+ \to \mathcal{M}$ is, as a consequence, symmetric monoidal in the sense of §II.3.1.

To define operads in augmented cocommutative coalgebras, we simply apply the general definition of §I.1.1 to the symmetric monoidal category of coalgebras $\mathcal{C}om^c_+$. Under this approach, an operad in augmented cocommutative coalgebras (a Hopf operad in our synonymous terminology) consists of a collection of augmented cocommutative coalgebras $\mathsf{P}(r)$, together with an action of the symmetric group Σ_r on $\mathsf{P}(r)$, for each $r \in \mathbb{N}$, a unit morphism $\eta : \mathbb{1} \to \mathsf{P}(1)$, and product morphisms $\mu : \mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r) \to \mathsf{P}(n_1 + \cdots + n_r)$, all formed within the category of augmented cocommutative coalgebras and satisfying the equivariance, unit, and associativity relations of §I.1.1 in that category $\mathcal{C}om^c_+$.

II.2.2. *The internal structure of (cocommutative) Hopf operads.* An operad in augmented cocommutative coalgebras forms an operad in the ground category since, as we just observed, the forgetful functor $\omega : \mathcal{M}\mathcal{C}om^c_+ \to \mathcal{M}$ is symmetric monoidal by construction. As such, an operad in augmented cocommutative coalgebras P can be identified with an operad in $\mathcal{M}$ so that the symmetric group Σ_r acts on $\mathsf{P}(r)$ by morphisms of cocommutative coalgebras, for each $r \in \mathbb{N}$, and the unit morphism $\eta : \mathbb{1} \to \mathsf{P}(1)$ preserves coalgebra structures, as well as the product morphisms $\mu : \mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r) \to \mathsf{P}(n_1 + \cdots + n_r)$. Accordingly, to completely unravel the definition, we simply need to go back to the definition of the coalgebra structure on the unit object $\mathbb{1}$, and on the tensor product $\mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r)$ in order to make explicit the conditions which η and μ have to satisfy as coalgebra morphisms. The result reads as follows: the preservation of coalgebra structures by the operadic unit $\eta : \mathbb{1} \to \mathsf{P}(1)$ is equivalent to the commutativity of the diagrams

(a)
$$\begin{tikzcd} \mathbb{1} \arrow[r, "\eta"] \arrow[dr, "="'] & \mathsf{P}(1) \arrow[d, "\epsilon"] \\ & \mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\eta"] \arrow[d, "\simeq"'] & \mathsf{P}(1) \arrow[d, "\Delta"] \\ \mathbb{1} \otimes \mathbb{1} \arrow[r, "\eta \otimes \eta"'] & \mathsf{P}(1) \otimes \mathsf{P}(1) \end{tikzcd} ,$$

where we use the notation ϵ (respectively, Δ) to refer to the counit (respectively, coproduct) of each $\mathsf{P}(n)$; the preservation of coalgebra structures by the composition product $\mu : \mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r) \to \mathsf{P}(n_1 + \cdots + n_r)$ is equivalent to the

commutativity of the diagrams

(b)
$$\begin{tikzcd}
\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[r, "\eta"] \arrow[d, "\epsilon\otimes\epsilon\otimes\cdots\otimes\epsilon"'] & \mathsf{P}(n_1+\cdots+n_r) \arrow[d, "\epsilon"] \\
\mathbb{1} \arrow[r, "="'] & \mathbb{1}
\end{tikzcd},$$

$$\begin{tikzcd}
\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[rr, "\mu"] \arrow[d, "\Delta\otimes\Delta\otimes\cdots\otimes\Delta"'] & & \mathsf{P}(n_1+\cdots+n_r) \arrow[dd, "\Delta"] \\
(\mathsf{P}(r)\otimes\mathsf{P}(r))\otimes(\mathsf{P}(n_1)\otimes\mathsf{P}(n_1))\otimes\cdots\otimes(\mathsf{P}(n_r)\otimes\mathsf{P}(n_r)) \arrow[d, "\simeq"'] & & \\
(\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r))\otimes(\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r)) \arrow[rr, "\mu\otimes\mu"'] & & \mathsf{P}(n_1+\cdots+n_r)\otimes\mathsf{P}(n_1+\cdots+n_r)
\end{tikzcd}.$$

In the case where $\mathcal{M}$ is the category of $\Bbbk$-modules and $\mathbb{1} = \Bbbk$ (and similarly in the context of graded, differential graded, simplicial and cosimplicial modules), the requirement that $\eta : \mathbb{1} \to \mathsf{P}(1)$ is a morphism of coalgebras amounts to the assumption that the operadic unit element $1 \in \mathsf{P}(1)$ (determining η) is group-like, because so is the unit 1 in the ground ring $\Bbbk$, regarded as a coalgebra. In point-wise terms, the commutation relation expressed by the diagrams in (b) read

$$\epsilon(p(q_1,\ldots,q_r)) = \epsilon(p)\cdot(\epsilon(q_1)\cdot\ldots\cdot\epsilon(q_r))$$
$$\text{and}\qquad \sum_{(p),(q_1),\ldots,(q_r)} p'(q_1',\ldots,q_r')\otimes p''(q_1'',\ldots,q_r''),$$

for any $p \in \mathsf{P}(r)$, $q_1 \in \mathsf{P}(n_1),\ldots,q_r \in \mathsf{P}(n_r)$, where we use the notation $\Delta(x) = \sum_{(x)} x'\otimes x''$ to represent the expansion of the coproduct of any element x in a coalgebra.

In general, the observations of this paragraph imply that we can define operads in augmented cocommutative coalgebras as operads in the ground category P, where each $\mathsf{P}(r)$ is equipped with a counit $\epsilon : \mathsf{P}(r) \to \mathbb{1}$ and a coproduct $\Delta : \mathsf{P}(r) \to \mathsf{P}(r)\otimes\mathsf{P}(r)$, defining an augmented cocommutative coalgebra structure on $\mathsf{P}(r)$, and so that the diagrams (a-b) commute, for all $r \geq 0$, $n_1,\ldots,n_r \geq 0$.

To give an abstract interpretation of the compatibility conditions expressed by these commutative diagrams, we will check that the category of operads inherits a tensor product from the ground category $\boxtimes : \mathcal{O}p\times\mathcal{O}p \to \mathcal{O}p$, so that the doubled factors in the tensor products of (a-b) can be interpreted as components of a tensor square $\mathsf{P}^{\boxtimes 2}$ in $\mathcal{O}p$. We devote the next paragraphs to this subject. This tensor product $\boxtimes : \mathcal{O}p\times\mathcal{O}p \to \mathcal{O}p$ will be called the arity-wise tensor product of operads.

II.2.3. *The arity-wise tensor product of operads.* Let $\mathsf{P},\mathsf{Q} \in \mathcal{O}p$. The components of the operad $\mathsf{P}\boxtimes\mathsf{Q}$ are given by the obvious formula $(\mathsf{P}\boxtimes\mathsf{Q})(r) = \mathsf{P}(r)\otimes\mathsf{Q}(r)$ in each arity $r \in \mathbb{N}$, where we form the tensor product of the objects $\mathsf{P}(r)$ and $\mathsf{Q}(r)$ in the ground symmetric monoidal category $\mathcal{M}$. The diagonal action of permutations $w \in \Sigma_r$ on the tensor product $\mathsf{P}(r)\otimes\mathsf{Q}(r)$ provides the object $(\mathsf{P}\boxtimes\mathsf{Q})(r) = \mathsf{P}(r)\otimes\mathsf{Q}(r)$ with an action of the symmetric group Σ_r, for each $r \in \mathbb{N}$. The unit of the operad $\mathsf{P}\boxtimes\mathsf{Q}$ is given by the composite morphism

$$\mathbb{1} \xrightarrow{\simeq} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\eta_{\mathsf{P}}\otimes\eta_{\mathsf{Q}}} \mathsf{P}(1)\otimes\mathsf{Q}(1)$$

involving the operadic units of P and Q on the different factors of the tensor product $(\mathsf{P}\boxtimes\mathsf{Q})(1) = \mathsf{P}(1)\otimes\mathsf{Q}(1)$. The composition products of $\mathsf{P}\boxtimes\mathsf{Q}$ are defined by the composite morphisms

$$\begin{aligned}(\mathsf{P}(r)\otimes\mathsf{Q}(r))\otimes(\mathsf{P}(n_1)\otimes\mathsf{Q}(n_1))\otimes\cdots\otimes(\mathsf{P}(n_r)\otimes\mathsf{Q}(n_r))&\\ \xrightarrow{\simeq}(\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r))\otimes(\mathsf{Q}(r)\otimes\mathsf{Q}(n_1)\otimes\cdots\otimes\mathsf{Q}(n_r))&\\ \xrightarrow{\mu_{\mathsf{P}}\otimes\mu_{\mathsf{Q}}}\mathsf{P}(n_1+\cdots+n_r)\otimes\mathsf{Q}(n_1+\cdots+n_r),&\end{aligned}$$

where we apply an appropriate tensor permutation to gather the factors attached to each operad P and Q before applying the composition products of these operads. We immediately check that these structure morphisms satisfy the equivariance, unit and associativity axioms of operads. Accordingly, our construction, which is also obviously natural with respect to $\mathsf{P},\mathsf{Q}\in\mathcal{O}p$, yields a bifunctor $\boxtimes:\mathcal{O}p\times\mathcal{O}p\to\mathcal{O}p$.

We can readily see that the commutative operad Com_+, defined in Proposition II.1.2 and consisting of the unit object $\mathbb{1}$ in all arities $\mathsf{Com}_+(r)=\mathbb{1}$, forms a unit for the arity-wise tensor product of operads. We also have a natural associativity (respectively, symmetry) isomorphism on $\boxtimes$ given by the arity-wise application of the associativity (respectively, symmetry) isomorphism of the tensor product $\otimes$ in the ambient category $\mathcal{M}$. We simply have to check that the structure isomorphisms obtained that way preserves the internal structure of operads, but this assertion follows from formal verifications. We conclude that the bifunctor $\boxtimes:\mathcal{O}p\times\mathcal{O}p\to\mathcal{O}p$ is the tensor product of a symmetric monoidal structure on $\mathcal{O}p$.

An augmented cocommutative coalgebra in $\mathcal{O}p$ formally consists of an operad $\mathsf{P}\in\mathcal{O}p$ equipped with a counit (an augmentation), defined by a morphism $\epsilon:\mathsf{P}\to\mathsf{Com}_+$, and a coproduct $\Delta:\mathsf{P}\to\mathsf{P}\boxtimes\mathsf{P}$, all formed in the category of operads, so that the counit, coassociativity, and cocommutativity relations of §II.0.3 hold. We immediately see, by definition of the arity-wise tensor product $\boxtimes$, that the given of these structure morphisms amounts to providing each $\mathsf{P}(r)$ with an augmented cocommutative coalgebra structure commuting with the action of symmetric groups. We also immediately see that, for the morphisms $\epsilon:\mathsf{P}\to\mathsf{Com}_+$ and $\Delta:\mathsf{P}\to\mathsf{P}\boxtimes\mathsf{P}$, the preservation of operad units and composition products amounts to the commutativity of the diagrams (a-b) in §II.2.2. Accordingly, we have the following result:

Proposition II.2.4. *The Hopf operads, initially defined as operads in augmented cocommutative coalgebras in §II.2.1, can equivalently be defined as augmented cocommutative coalgebras in operads, where we take the arity-wise tensor product of §II.2.3 to provide the category of operads with a symmetric monoidal structure.* □

We crucially need the equivalence established in this proposition for the definition Hopf operads by generators and relations (see Proposition II.2.10).

In §II.0.3, we mention that the tensor unit $\mathbb{1}$ represents the terminal object of the category of augmented cocommutative coalgebras, and the tensor product represents the cartesian product in that category. The same results hold in the operad context:

Proposition II.2.5.

(a) *The unitary commutative operad* Com_+ *inherits a natural Hopf operad structure (as unit object for the aritywise tensor product of operads) and represents the terminal object of the category of Hopf operads.*

(b) *The aritywise tensor product of Hopf operads inherits a natural Hopf operad structure, so that the aritywise tensor product induces a bifunctor* $\boxtimes : \mathcal{H}opf\mathcal{O}p \times \mathcal{H}opf\mathcal{O}p \to \mathcal{H}opf\mathcal{O}p$ *and gives a symmetric monoidal structure on the category of Hopf operads, with the unitary commutative operad* Com_+ *as unit object.*

(c) *The tensor product of Hopf operads* $\mathsf{P} \boxtimes \mathsf{Q} \in \mathcal{H}opf\mathcal{O}p$*, considered in (b), actually represents the cartesian product of* P *and* Q *in* $\mathcal{H}opf\mathcal{O}p$*. The structure projections* $\mathsf{P} \xleftarrow{p} \mathsf{P} \boxtimes \mathsf{Q} \xrightarrow{q} \mathsf{Q}$*, which characterize this cartesian product, are identified with the tensor products* $p = id \boxtimes \epsilon$ *and* $q = \epsilon \boxtimes id$*, where we consider the counit morphisms* $\epsilon : \mathsf{P} \to \mathsf{Com}_+$ *(respectively,* $\epsilon : \mathsf{Q} \to \mathsf{Com}_+$*) of the Hopf operad structure on* P *(respectively,* Q*).*

Proof. This result follows from the identity $\mathcal{H}opf\mathcal{O}p = \mathcal{O}p\mathcal{C}om_+^c$, established in Proposition II.2.4, and the observations of §II.0.3, concerning the categorical interpretation of the tensor product of coalgebras in a symmetric monoidal category, which we apply to the category of operads $\mathcal{M} = \mathcal{O}p$. □

The assertions of this proposition can also be deduced from the result of Proposition I.2.4, asserting that limits of operads are created in the underlying category.

II.2.6. *Hopf symmetric sequences and the definition of free Hopf operads.* We now examine the adjunction between symmetric sequences and operads in the context of Hopf operads. In parallel to the terminology of Hopf operad, we may use the expression of Hopf symmetric sequence to refer to a symmetric sequence in augmented cocommutative coalgebras. We may also use the notation $\mathcal{H}opf\mathcal{S}eq$, instead of $\mathcal{C}om_+^c\mathcal{S}eq$, to refer to that category of symmetric sequences.

We can also obviously extend the definition of the arity-wise tensor product to symmetric sequences. We then obtain a bifunctor $\boxtimes : \mathcal{S}eq \times \mathcal{S}eq \to \mathcal{S}eq$ providing $\mathcal{S}eq$ with a symmetric monoidal structure (we just retain the action of symmetric groups in the construction of §II.2.3), and Hopf symmetric sequences are identified with augmented cocommutative coalgebras with respect to this symmetric monoidal structure. We should mention that the tensor unit in $\mathcal{S}eq$ is still given by the unitary commutative operad Com_+, of which we forget the operadic composition structure. A Hopf symmetric sequence can clearly be identified with a symmetric sequence $\mathsf{M} \in \mathcal{S}eq$ equipped with an augmentation $\epsilon : \mathsf{M} \to \mathsf{Com}_+$ and a diagonal $\Delta : \mathsf{M} \to \mathsf{M} \boxtimes \mathsf{M}$, formed by the collection of augmentations $\epsilon : \mathsf{M}(r) \to \mathbb{1}$ and the diagonals $\Delta : \mathsf{M}(r) \to \mathsf{M}(r) \otimes \mathsf{M}(r)$ on the components of M, so that the counit, coassociativity, and cocommutative relations of coalgebras are satisfied in M. Accordingly, we also have an identity between the category of Hopf symmetric sequences and the category of augmented cocommutative coalgebras in $\mathcal{S}eq$. In our notation, we write $\mathcal{H}opf\mathcal{S}eq = \mathcal{C}om_+^c\mathcal{S}eq = \mathcal{S}eq\mathcal{C}om_+^c$.

We can apply the construction of the free operad to the symmetric monoidal category of augmented cocommutative coalgebras. We obtain in that context a Hopf operad $\mathbb{O}(\mathsf{M})$, naturally associated to any Hopf symmetric sequence M, and characterized by the universal property of Proposition I.2.2 in the category of Hopf operads (or by the equivalent adjunction relation of Theorem I.2.1).

We have already observed that the forgetful functor $\omega : \mathcal{C}om_+^c \to \mathcal{M}$, from augmented cocommutative coalgebras to the ground category, is symmetric monoidal by construction, and as a consequence, induces a functor $\omega : \mathcal{H}opf\mathcal{O}p \to \mathcal{O}p$ from Hopf operads to operads. According to the discussion of §§II.2.1-II.2.4, we can also identify this functor with a forgetful functor, which retains the operad structure in Hopf operads and forget about the coalgebra structure attached to each component. We also have an obvious forgetful functor $\omega : \mathcal{H}opf\mathcal{S}eq \to \mathcal{S}eq$ on Hopf symmetric sequences. We now study the interplay between these Hopf forgetful functors and the various free operad functors attached to each category.

The explicit construction of the free operad $\mathbb{O}(\mathsf{M})$ in Appendix B involves a combination of colimits and tensor products. On the other hand, we mention in §II.0.3 that the forgetful functor $\omega : \mathcal{C}om_+^c \to \mathcal{M}$ creates colimits (in addition to tensor products). From this observation, we may immediately deduce that the forgetful functor $\omega : \mathcal{H}opf\mathcal{O}p \to \mathcal{O}p$ preserves free operads. But we aim to establish this result by another approach, by relying on our interpretation of Hopf operads as coalgebras in operads. The argument is based on the following construction:

Lemma II.2.7. *Let M be a Hopf symmetric sequence. Let $\mathbb{O}(\mathsf{M})$ be the free operad associated to M, and formed in the ground category after forgetting the internal coalgebra structure of M.*

(a) *The augmentations $\epsilon : \mathsf{M}(r) \to \mathbb{1}$ and the diagonals $\Delta : \mathsf{M}(r) \to \mathsf{M}(r) \otimes \mathsf{M}(r)$, defining the augmented coalgebra structure of the object M, extend to operad morphisms $\epsilon : \mathbb{O}(\mathsf{M}) \to \mathsf{Com}_+$ and $\Delta : \mathbb{O}(\mathsf{M}) \to \mathbb{O}(\mathsf{M}) \boxtimes \mathbb{O}(\mathsf{M})$, providing $\mathbb{O}(\mathsf{M})$ with the structure of a Hopf operad.*

(b) *Let $f : \mathsf{M} \to \mathsf{P}$ be a morphism of Hopf symmetric sequences, where P is a Hopf operad. Let $\phi_f : \mathbb{O}(\mathsf{M}) \to \mathsf{P}$ be the unique morphism factorizing f in the category of operads. The free operad $\mathbb{O}(\mathsf{M})$ inherits a Hopf operad structure by assertion (a). The above morphism ϕ_f automatically preserves this additional coalgebra structure and as a consequence defines a factorization of f in the category of Hopf operads.*

(c) *In the construction of (a), the universal morphism attached to the free operad $\iota : \mathsf{M} \to \mathbb{O}(\mathsf{M})$ forms a morphism of Hopf symmetric sequences. In the construction of (b), if we form the morphism $\lambda : \mathbb{O}(\mathsf{P}) \to \mathsf{P}$, attached to the identity of P and defining the adjunction augmentation of the free operad, then we obtain a morphism of Hopf operads.*

Proof. Recall that the collection of augmentations $\epsilon : \mathsf{M}(r) \to \mathbb{1}$, attached to the coalgebra structure of each $\mathsf{M}(r)$, can be viewed as a morphism of symmetric sequences towards the unitary commutative operad Com_+. The existence of the operad morphism $\epsilon : \mathbb{O}(\mathsf{M}) \to \mathsf{Com}_+$ extending these augmentations immediately follows from the universal property of the free operad, as stated in Proposition I.2.2.

By composing the diagonals $\Delta : \mathsf{M}(r) \to \mathsf{M}(r) \otimes \mathsf{M}(r)$ with a tensor product of the universal morphisms $\iota : \mathsf{M}(r) \to \mathbb{O}(\mathsf{M})(r)$ in each arity, we also obtain a morphism $\Delta : \mathsf{M} \to \mathbb{O}(\mathsf{M}) \boxtimes \mathbb{O}(\mathsf{M})$. By applying the universal property of the free operad, we obtain again an operad morphism $\Delta : \mathbb{O}(\mathsf{M}) \to \mathbb{O}(\mathsf{M}) \boxtimes \mathbb{O}(\mathsf{M})$ extending this morphism of symmetric sequences.

By applying the uniqueness requirement in the universal property of free operads (see Proposition I.2.2 again), we immediately obtain that the counit, coassociativity and cocommutativity relations of coalgebras hold at the level of the free

operad $\mathbb{O}(\mathsf{M})$, for the just defined morphisms, as soon as they hold at the level of the symmetric sequence M.

The universal morphism $\iota : \mathsf{M} \to \mathbb{O}(\mathsf{M})$ forms a morphism of Hopf symmetric sequences by construction of the coalgebra structure on $\mathbb{O}(\mathsf{M})$. Thus, the first assertion of (c) is immediate. The uniqueness requirement in the universal property of free operads also implies that the morphism $\phi_f : \mathbb{O}(\mathsf{M}) \to \mathsf{P}$ associated to a morphism of Hopf symmetric sequences in (b) intertwines coalgebra structures and hence, forms a morphism of Hopf operads. The second assertion of (c), regarding the adjunction augmentation $\lambda : \mathbb{O}(\mathsf{P}) \to \mathsf{P}$, is also immediate from this result. □

Then we obtain:

Proposition II.2.8. *The free operad $\mathbb{O}(\mathsf{M})$, together with the Hopf structure constructed in the previous lemma, forms the free object associated to M in the category of Hopf operads.*

Proof. This proposition is a formal consequence of the results of assertions (b-c) in Lemma II.2.7. □

Lemma II.2.7 also implies the following result on the free operad adjunction:

Proposition II.2.9. *The functors defined by the forgetting of coalgebra structures in Hopf objects fit in a commutative diagram of functors*

$$\begin{tikzcd}
\mathcal{H}opf\mathcal{S}eq \arrow[r, shift left, "\mathbb{O}"] \arrow[d, dotted, "\omega"'] & \mathcal{H}opf\mathcal{O}p \arrow[l, shift left, "\omega"] \arrow[d, dotted, "\omega"] \\
\mathcal{S}eq \arrow[r, shift left, "\mathbb{O}"] & \mathcal{O}p \arrow[l, shift left, "\omega"]
\end{tikzcd},$$

where we consider the adjoint forgetful and free object functors between symmetric sequences and operads. These forgetful functors also induce mappings on morphism sets

$$\begin{tikzcd}
\mathtt{Mor}_{\mathcal{H}opf\mathcal{O}p}(\mathbb{O}(\mathsf{M}), \mathsf{P}) \arrow[r, "\simeq"] \arrow[d, dotted, "\omega"'] & \mathtt{Mor}_{\mathcal{H}opf\mathcal{S}eq}(\mathsf{M}, \mathsf{P}) \arrow[d, dotted, "\omega"] \\
\mathtt{Mor}_{\mathcal{O}p}(\mathbb{O}(\mathsf{M}), \mathsf{P}) \arrow[r, "\simeq"'] & \mathtt{Mor}_{\mathcal{S}eq}(\mathsf{M}, \mathsf{P})
\end{tikzcd}$$

that intertwine the correspondence (materialized by the horizontal arrows in the diagram) which arises from the definition of free operads as a left adjoint.

Proof. The assertion of Proposition II.2.8 implies that the forgetting of coalgebra structures preserves free objects in operads. In Lemma II.2.7, assertion (c) similarly implies that the forgetting of coalgebra structures preserves the unit morphism and the augmentation morphism of the free operad adjunction. From this observation, we immediately conclude that the forgetting of coalgebra structures also intertwines the adjunction correspondence on morphisms. □

In §I.2, we briefly explain that the free operad $\mathbb{O}(\mathsf{M})$ intuitively consists of formal operadic composites of elements $\xi \in \mathsf{M}(n)$ (whenever the notion of element makes sense). In this interpretation, the construction of Lemma II.2.7 amounts to extending the augmentation (respectively, diagonal) of M to such composites by using the point-wise commutation relations of §II.2.2. We use this idea soon, when

we explicitly determine the augmentation and diagonal of composition products in operads defined by generators and relations (see §II.2.11).

We now specialize our study to Hopf operads in modules over a ring $\mathcal{M} = \mathcal{M}od$. We explain in §I.2.9 that operads in module categories can be defined by generators and relations as quotients $\mathsf{P} = \mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} >$, where we consider an ideal $< z^\alpha, \alpha \in \mathcal{I} >$ in a free operad $\mathbb{O}(\mathsf{M})$. In the context of Hopf operads, we have the following result:

Proposition II.2.10. *Let M be a Hopf symmetric sequence (in $\Bbbk$-modules). We apply the construction of Lemma II.2.7 to obtain a Hopf structure on the free operad associated to M. Let $\mathsf{S} =< z^\alpha, \alpha \in \mathcal{I} >$ be the ideal generated by a collection of elements $z^\alpha \in \mathsf{S}(n_\alpha)$ in the free operad $\mathbb{O}(\mathsf{M})$. If $\epsilon(z^\alpha) = 0$ and $\Delta(z^\alpha) \in \mathsf{S}(n_\alpha) \otimes \mathbb{O}(\mathsf{M})(n_\alpha) + \mathbb{O}(\mathsf{M})(n_\alpha) \otimes \mathsf{S}(n_\alpha)$ for each $z^\alpha \in \mathsf{S}(n_\alpha)$, then:*

(a) *The operad $\mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} >$ inherits a quotient Hopf operad structure from the free operad $\mathbb{O}(\mathsf{M})$.*
(b) *The morphisms of Hopf operads $\bar{\phi}_f : \mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} > \to \mathsf{Q}$ defined on this quotient are in obvious bijection with the morphisms of Hopf operads $\phi_f : \mathbb{O}(\mathsf{M}) \to \mathsf{Q}$ such that $\phi_f(z^\alpha) = 0$ for each generating element of the ideal $z^\alpha \in \mathsf{S}(n_\alpha)$.*

In the situation of this proposition, we also say that the ideal $\mathsf{S} =< z^\alpha, \alpha \in \mathcal{I} >$ forms a Hopf ideal in the operad $\mathbb{O}(\mathsf{M})$.

Proof. The requirement $\epsilon(z^\alpha) = 0$ implies that ϵ induces a morphism on the quotient $\mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} >$, and hence provides this quotient operad with an augmentation $\epsilon : \mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} > \to \mathsf{Com}_+$. The requirement $\Delta(z^\alpha) \in \mathsf{S}(n_\alpha) \otimes \mathbb{O}(\mathsf{M})(n_\alpha) + \mathbb{O}(\mathsf{M})(n_\alpha) \otimes \mathsf{S}(n_\alpha)$ is equivalent to the vanishing of $\Delta(z^\alpha)$ in $(\mathbb{O}(\mathsf{M})/\mathsf{S} \boxtimes \mathbb{O}(\mathsf{M})/\mathsf{S})(n_\alpha) = \mathbb{O}(\mathsf{M})(n_\alpha)/\mathsf{S}(n_\alpha) \otimes \mathbb{O}(\mathsf{M})(n_\alpha)/\mathsf{S}(n_\alpha) = \mathbb{O}(\mathsf{M})(n_\alpha) \otimes \mathbb{O}(\mathsf{M})(n_\alpha)/\mathsf{S}(n_\alpha) \otimes \mathbb{O}(\mathsf{M})(n_\alpha) + \mathbb{O}(\mathsf{M})(n_\alpha) \otimes \mathsf{S}(n_\alpha)$, and implies that $\Delta : \mathbb{O}(\mathsf{M}) \to \mathbb{O}(\mathsf{M}) \boxtimes \mathbb{O}(\mathsf{M})$ induces a morphism $\Delta : \mathbb{O}(\mathsf{M})/\mathsf{S} \to \mathbb{O}(\mathsf{M})/\mathsf{S} \boxtimes \mathbb{O}(\mathsf{M})/\mathsf{S}$ on the quotient operad $\mathbb{O}(\mathsf{M})/\mathsf{S} = \mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} >$. These morphisms, obtained by a quotient process, naturally satisfy the counit, coassociativity, and cocommutativity relations of coalgebras and hence, provide the operad $\mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} >$ with a well-defined Hopf structure.

To check the second assertion of the proposition, simply observe that the morphism $\bar{\phi}_f : \mathbb{O}(\mathsf{M}) / < z^\alpha, \alpha \in \mathcal{I} > \to \mathsf{Q}$, induced by the morphism of Hopf operads $\phi_f : \mathbb{O}(\mathsf{M}) \to \mathsf{Q}$, naturally preserves coalgebra structures as well, and hence, defines a morphism of Hopf operads. □

II.2.11. *The basic examples of Hopf operads.* The assertions of Proposition II.2.5 include the statement that the unitary commutative operad Com_+ has a natural Hopf structure. The same result holds for the non-unitary version of this operad Com and can also be deduced from the identity between the components of this operad in arity $r > 0$ and the tensor unit $\mathbb{1}$. The augmentation $\epsilon : \mathsf{Com}(r) \to \mathbb{1}$ is given by the identity of $\mathbb{1}$ in arity $r > 0$, and by the initial morphism $0 \to \mathbb{1}$ in arity $r = 0$. The diagonal $\Delta : \mathsf{Com}(r) \to \mathsf{Com}(r) \otimes \mathsf{Com}(r)$ is given by the isomorphism $\mathbb{1} \xleftarrow{\simeq} \mathbb{1} \otimes \mathbb{1}$ in arity $r > 0$, and by the initial morphism $0 \to 0 \otimes 0$ in arity $r = 0$.

To illustrate our constructions, we check that this structure result can be retrieved from the statement of Proposition II.2.10 and from the presentation commutative operad in §I.2.10. We then assume that the ground symmetric monoidal category is a category of modules over a ring.

Recall that the generating symmetric sequence of the commutative operad is defined by $\mathsf{M}_{\mathsf{Com}}(2) = \Bbbk\{\mu(x_1, x_2)\} = \Bbbk$, where $\mu = \mu(x_1, x_2)$ denotes an operation on which Σ_2 acts trivially, and $\mathsf{M}_{\mathsf{Com}}(r) = 0$ for $r \neq 2$. We provide the module $\mathsf{M}_{\mathsf{Com}}(2) = \Bbbk\{\mu(x_1, x_2)\}$ with the coalgebra structure such that $\epsilon(\mu) = 1$ and $\Delta(\mu) = \mu \otimes \mu$ for this generating operation. The image of the generating relations of Com under the augmentation and the diagonal on the free operad is determined by using the preservation of operadic composition structures:

$$\begin{aligned}
\epsilon(\mu(\mu, 1) - \mu(1, \mu)) &= 1 - 1 = 0,\\
\Delta(\mu(\mu, 1) - \mu(1, \mu)) &= (\mu \otimes \mu)(\mu \otimes \mu, 1 \otimes 1) - (\mu \otimes \mu)(1 \otimes 1, \mu \otimes \mu)\\
&= \mu(\mu, 1) \otimes \mu(\mu, 1) - \mu(1, \mu) \otimes \mu(1, \mu)\\
&= (\mu(\mu, 1) - \mu(1, \mu)) \otimes \mu(\mu, 1) + \mu(1, \mu) \otimes (\mu(\mu, 1) - \mu(1, \mu)).
\end{aligned}$$

We see, from this computation, that the generating relations of the commutative operad generate a Hopf ideal. Hence, the assumptions of Proposition II.2.10 are satisfied, and we retrieve that Com inherits a well-defined Hopf operad structure, such that $\epsilon(\mu) = 1$ and $\Delta(\mu) = \mu \otimes \mu$ for the generating operation $\mu = \mu(x_1, x_2)$.

The unitary and the non-unitary version of the associative operad also inherits a Hopf structure. Let us see how to retrieve this structure result from the presentation again. The generating symmetric sequence of the associative operad is given by $\mathsf{M}_{\mathsf{As}}(2) = \Bbbk\{\mu(x_1, x_2), \mu(x_2, x_1)\} = \Bbbk\{\Sigma_2\}$, where $\mu = \mu(x_1, x_2)$ denotes an operation on which Σ_2 acts regularly, and $\mathsf{M}_{\mathsf{As}}(r) = 0$ for $r \neq 2$. We provide the module $\mathsf{M}_{\mathsf{As}}(2)$ with the coalgebra structure such that $\epsilon(\mu) = 1$ and $\Delta(\mu) = \mu \otimes \mu$. The definition of the augmentation and of the diagonal of the transposed operation $(1\ 2) \cdot \mu = \mu(x_2, x_1)$ is then forced by the equivariance requirement. We check, as in the case of the commutative operad, that $\mu(\mu, 1) - \mu(1, \mu)$ generates a Hopf ideal, from which we conclude again that the operad As inherits a well-defined Hopf structure.

In the case of the Lie operad, we have a generating symmetric sequence such that $\mathsf{M}_{\mathsf{Lie}}(2) = \Bbbk\{\lambda(x_1, x_2)\} = \Bbbk^{\pm}$ where $\Bbbk^{\pm}$ denotes the signature representation. We have in this case no possibility of fixing an augmentation $\epsilon(\lambda) \in \Bbbk$, and a diagonal $\Delta(\lambda) \in \mathsf{Lie}(2) \otimes \mathsf{Lie}(2)$, so that: the counit relations hold, the equivariance requirements of operad morphisms are satisfied and the Jacobi relation is canceled by the augmentation in $\Bbbk$, and by the diagonal in $\mathsf{Lie}(3) \otimes \mathsf{Lie}(3)$ as well. Hence, we have no Hopf structure on the Lie operad.

II.2.12. *The example of the Poisson operad.* Though we have no Hopf structure on the Lie operad, we can define an appropriate augmentation and diagonal for the corresponding generating operation λ in the Poisson operad. Recall that the Poisson operad Pois is defined by a presentation of the form

$$\begin{aligned}
\mathsf{Pois} = \ \mathbb{O}(\ \Bbbk\,\mu(x_1, x_2) \oplus \Bbbk\,\lambda(x_1, x_2)\ :\ & \mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3)),\\
& \lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) \equiv 0,\\
& \lambda(\mu(x_1, x_2), x_3) \equiv \mu(\lambda(x_1, x_3), x_2) + \mu(x_1, \lambda(x_2, x_3))\ ),
\end{aligned}$$

where the action of the symmetric group in arity 2 is determined by $(1\ 2)\cdot\mu=\mu$ and $(1\ 2)\cdot\lambda=-\lambda$. We extend the formula of the commutative operad to define the augmentation and the diagonal of the product operation $\mu=\mu(x_1,x_2)$. We define the augmentation and the diagonal of the Lie bracket operation $\lambda=\mu(x_1,x_2)$ by $\epsilon(\lambda)=0$ and $\Delta(\lambda)=\lambda\otimes\mu+\mu\otimes\lambda$. Again, we easily check (exercise, adapt the verifications performed in §II.2.11 for the commutative operad) that the generating relations of the Poisson operad form a Hopf ideal, and therefore we have a well-defined Hopf structure on the Poisson operad. We use a graded variant of this Hopf structure in our study of E_n-operads.

II.2.13. *Remark: tensor product of algebras over Hopf operads.* The existence of a Hopf structure on an operad P implies that the associated category of algebras $\mathcal{P}$ inherits a symmetric monoidal structure from the underlying symmetric monoidal category $\mathcal{M}$. Indeed, the tensor product of P-algebras $A,B\in\mathcal{P}$ inherits an action of P, given by the composite morphisms

$$\begin{aligned}\mathsf{P}(r)\otimes(A\otimes B)^{\otimes r}&\xrightarrow{\Delta}(\mathsf{P}(r)\otimes\mathsf{P}(r))\otimes(A\otimes B)^{\otimes r}\\&\xrightarrow{\simeq}(\mathsf{P}(r)\otimes A^{\otimes r})\otimes(\mathsf{P}(r)\otimes B^{\otimes r})\xrightarrow{\lambda_A\otimes\lambda_B}A\otimes B,\end{aligned}$$

for any $r\in\mathbb{N}$, where we consider the diagonal of P, followed by the obvious tensor permutation and the tensor product of the evaluation morphisms attached to the P-algebras. The tensor unit $\mathbb{1}$ also inherits an action of the operad P by restriction through the augmentation morphism $\epsilon:\mathsf{P}\to\mathsf{Com}_+$ (using the natural commutative algebra structure of $\mathbb{1}$). The counit, coassociativity, and cocommutativity relations, at the level of the coalgebra structure of the Hopf operad P, imply that the unit, associativity, and symmetry isomorphisms of the ground category define P-algebra morphisms when we deal with tensor products of P-algebras. Hence, we have a whole symmetric monoidal structure on the category of P-algebras.

In the case of the commutative operad, we retrieve with this observation the basic symmetric monoidal structure of §II.0.2. In the case of the associative operad, we retrieve the similarly defined symmetric monoidal structure alluded to in the introduction of §II.0.

II.2.14. *Changes in the context of connected operads.* In §I.1.18, we introduce the category of connected operads $\mathcal{O}p_{01}$, of which objects are the operads P satisfying $\mathsf{P}(0)=0$ and $\mathsf{P}(1)=\mathbb{1}$.

The constructions of §§II.2.3-II.2.5 can readily be adapted in the context of connected operads. If the tensor product of $\mathcal{M}$ preserves initial objects, then we immediately see that the tensor product of connected operads is connected. Otherwise, we adapt the definition of §II.2.3 by fixing $(\mathsf{P}\boxtimes\mathsf{Q})(0)=0$ for the arity 0 term of the tensor product in connected operads. In any case, we obtain a symmetric monoidal structure on connected operads by restriction, with, as unit object, the non-unitary version of the commutative operad Com (also defined in Proposition II.1.1).

The result of Proposition II.2.4 remains valid for connected operads, and so does the result of Proposition II.2.5, provided that we replace the unitary version of the commutative operad Com_+ by the non-unitary one Com.

II.2.15. *Unitary Hopf operads.* The results of §I.4, about the definition of (connected) unitary operads, makes sense in any base category equipped with a tensor product preserving colimits on each side (see §0.6), and as such can be applied

without change within the category of augmented cocommutative coalgebras, in order to give a description of (connected) unitary Hopf operads.

Now, we can also adapt the observations of §II.2 and regard (connected) unitary Hopf operads as augmented cocommutative coalgebras in the category of (connected) unitary operads. Indeed, the arity-wise tensor products $(\mathsf{P}\boxtimes\mathsf{Q})(r) = \mathsf{P}(r)\otimes\mathsf{Q}(r)$ of (connected) unitary operads is clearly a (connected) unitary operads. The action of a deletion morphism u^*, associated to some increasing injection u, on a tensor product of operads $\mathsf{P}\boxtimes\mathsf{Q}$ is simply given by the tensor product of the deletion morphisms determined by u on P and Q. Furthermore, the requirement that the deletion morphisms are morphisms in the base category of coalgebras amounts to the assertion that the augmentation $\Delta : \mathsf{P} \to \mathsf{Com}_+$ and the diagonal $\Delta : \mathsf{P} \to \mathsf{P}\boxtimes\mathsf{P}$ intertwine deletion morphisms.

Similarly, we can regard Hopf symmetric sequences with deletion operations either as symmetric sequences with deletion operations in the category augmented cocommutative coalgebras, or as augmented cocommutative coalgebras in the category of symmetric sequences with deletion operations.

The results of Proposition I.4.10 and Proposition II.2.10 can be combined to get a good definition of unitary Hopf operads by generators and relations. In this context, the input of our construction is a connected Hopf symmetric sequence with deletion operations, combining the deletion structures considered in §I.4.8 and the Hopf structures considered in Proposition II.2.10. The associative operad As_+, the commutative operad Com_+, and the Poisson operad Pois_+, give examples of connected unitary Hopf operads which we can define by a presentation by generators and relations. In fact, we simply have to check that the deletion morphisms defined in §I.4.11 preserve the coalgebra structure on the generating collection M_P of the operads $\mathsf{P} = \mathsf{As}, \mathsf{Com}, \mathsf{Pois}$ (see §§II.2.11-II.2.12) to conclude that each operad $\mathsf{P} = \mathsf{As}, \mathsf{Com}, \mathsf{Pois}$ has a unitary extension as Hopf operad.

II.3. Appendix: Functors between symmetric monoidal categories

In various constructions, we have to transport structures (like commutative algebras) from one symmetric monoidal category $\mathcal{M}$ to another $\mathcal{N}$ by using functors preserving the internal structures of symmetric monoidal categories. For this aim, we deal with functors preserving symmetric monoidal structures, in a strict or relaxed sense. The purpose of this appendix section is to make explicit extra structures, consisting of natural equivalences or natural transformations, which we use to govern the commutation of tensor products and functors $S : \mathcal{M} \to \mathcal{N}$.

II.3.1. *Symmetric monoidal transformations.* We often deal with functors $S : \mathcal{M} \to \mathcal{N}$ satisfying $S(\mathbb{1}) = \mathbb{1}$ for the unit object $\mathbb{1} \in \mathcal{M}$, and equipped with a natural transformation $\theta : S(A)\otimes S(B) \to S(A\otimes B)$, so that natural unit, associativity and symmetry constraints, expressed by the commutativity of the following diagrams, hold:

$$\begin{tikzcd} S(A)\otimes S(\mathbb{1}) \arrow[r, "\theta"] & S(A\otimes\mathbb{1}) \arrow[d, "\simeq"] \\ S(A)\otimes\mathbb{1} \arrow[u, "="] \arrow[r, "\simeq"'] & S(A) \end{tikzcd} , \qquad \begin{tikzcd} S(\mathbb{1})\otimes S(A) \arrow[r, "\theta"] & S(\mathbb{1}\otimes A) \arrow[d, "\simeq"] \\ \mathbb{1}\otimes S(A) \arrow[u, "="] \arrow[r, "\simeq"'] & S(A) \end{tikzcd} ,$$

$$\begin{tikzcd} S(A)\otimes S(B)\otimes S(C) \arrow[r, "\theta\otimes id"] \arrow[d, "id\otimes\theta"'] & S(A\otimes B)\otimes S(C) \arrow[d, "\theta"] \\ S(A)\otimes S(B\otimes C) \arrow[r, "\theta"'] & S(A\otimes B\otimes C) \end{tikzcd} , \qquad \begin{tikzcd} S(A)\otimes S(B) \arrow[r, "\theta"] \arrow[d, "\simeq"'] & S(A\otimes B) \arrow[d, "\simeq"] \\ S(B)\otimes S(A) \arrow[r, "\theta"'] & S(B\otimes A) \end{tikzcd} .$$

In this situation, we say that the functor S is unit-pointed (to refer to the identity $S(\mathbb{1}) = \mathbb{1}$) and that θ defines a symmetric monoidal transformation on S. We have a dual situation where our functor S is equipped with a natural transformation going in the converse direction $\theta : S(A \otimes B) \to S(A) \otimes S(B)$ and satisfying a dual of our unit, associativity and symmetry constraints. We then say that θ defines a symmetric comonoidal transformation associated to S.

We may deal with an optimal situation, where a unit-pointed functor S is equipped with a symmetric monoidal transformation θ that gives an isomorphism $\theta : S(A)\otimes S(B) \xrightarrow{\simeq} S(A\otimes B)$, for every $A, B \in \mathcal{M}$ (or dually in the case of a symmetric comonoidal transformation). We say in this case, following the conventions adopted by most authors, that θ forms a symmetric monoidal equivalence and that $S : \mathcal{M} \to \mathcal{N}$ is a symmetric monoidal functor from $\mathcal{M}$ to $\mathcal{N}$. (Some authors use the expression of strong symmetric monoidal functor to depict this situation.)

The functors which are unit pointed and equipped with a symmetric monoidal transformation in our sense form a subclass of the class of lax symmetric monoidal functors (simply called symmetric monoidal functors by certain authors). Indeed, to retrieve the general definition of a lax symmetric monoidal functor from our definition, we simply have to relax the identity requirement $S(\mathbb{1}) = \mathbb{1}$ and to assume the existence of a morphism $\eta : \mathbb{1} \to S(\mathbb{1})$ instead. Dually, the functors which are unit pointed and equipped with a symmetric comonoidal transformation in our sense form a subclass of the classical notion of colax symmetric monoidal functor.

Unit objects are preserved by all our examples of functors between symmetric monoidal categories. Therefore, we do not use the general notion of lax/colax functor in practice.

II.3.2. *Basic examples of symmetric monoidal functors.* The geometric realization functor $|-| : \mathcal{S}imp \to \mathcal{T}op$ (see §0.5) is a fundamental example of functor which carries a non trivial symmetric monoidal structure. Recall that the tensor product operation on simplicial sets and topological spaces is defined by the cartesian product of these categories. In this context, the canonical projections $K \xleftarrow{p} K \times L \xrightarrow{q} L$ induce morphisms $|K| \xleftarrow{p} |K \times L| \xrightarrow{q} |L|$ which we can put together to define a natural transformation $\theta : |K \times L| \xrightarrow{q} |K| \times |L|$. This natural transformation is actually a homeomorphism for all $K, L \in \mathcal{S}imp$ (see for instance [71, §III]), This result follows from a topological interpretation, in terms of simplicial decompositions of prisms, of the classical Eilenberg-Zilber equivalence (we refer to *loc. cit.* for details). For a point, we obviously have $|\,pt\,| = pt$, and the definition of the natural transformation $\theta : |K \times L| \xrightarrow{q} |K| \times |L|$ from universal categorical constructions automatically ensures that the unit, associativity and symmetry constraints of §II.3.1 are fulfilled.

The singular complex functor $\mathtt{Sing}_\bullet : \mathcal{T}op \to \mathcal{S}imp$, which defines the right adjoint of the geometric realization functor $| - | : \mathcal{S}imp \to \mathcal{T}op$ (see §0.5), is also symmetric monoidal. In this case, the identity $\mathtt{Sing}_\bullet(pt) = pt$ and the existence of an isomorphism $\mathtt{Sing}_\bullet(K \times L) \xrightarrow{\simeq} \mathtt{Sing}_\bullet(K) \times \mathtt{Sing}_\bullet(L)$ immediately follows from the definition of $\mathtt{Sing}_\bullet : \mathcal{T}op \to \mathcal{S}imp$ as a right adjoint.

To give another (even) simple(r) example: the functor $\Bbbk\{-\} : \mathcal{S}et \to \mathcal{M}od$, defined by assigning the free $\Bbbk$-module $\Bbbk\{X\}$ generated by X to any set $X \in \mathcal{S}et$ is symmetric monoidal since we have an obvious identity $\Bbbk\{pt\} = \Bbbk$ for the one point set $pt \in \mathcal{S}et$, a natural isomorphism $\Bbbk\{X\} \otimes \Bbbk\{Y\} \xleftarrow{\simeq} \Bbbk\{X \times Y\}$, for any cartesian product of sets $X, Y \in \mathcal{S}et$, and we can also easily check that this natural transformation fulfills our unit, associativity and symmetry constraints. We go back to this example in §II.0.5.

The simplicial extension of the free $\Bbbk$-module functor $\Bbbk\{-\} : \mathcal{S}imp \to s\,\mathcal{M}od$ (considered in §0.3) is also symmetric monoidal (the symmetric monoidal structure of simplicial modules will be studied in §4).

The normalized chain complex functor $\mathtt{N}_* : \mathcal{S}imp \to dg\,\mathcal{M}od$, of which we recall the definition later on, is an instance of functor which is not symmetric monoidal in the sense specified in §II.3.1. In the case of this functor, we have a natural transformation $\theta : \mathtt{N}_*(X) \times \mathtt{N}_*(Y) \to \mathtt{N}_*(X \times Y)$, called the Eilenberg-MacLane morphism, which satisfy our unit, associativity and symmetry constraints, but this morphism is only a weak-equivalence and not an isomorphism (see [64, §§VIII.6-8]).

II.3.3. *Symmetric monoidal adjunctions.* Suppose now we have a pair of adjoint functors $S : \mathcal{M} \rightleftarrows \mathcal{N} : T$ between symmetric monoidal categories such that both S and T are symmetric monoidal. We then say that the adjunction is symmetric monoidal if the adjunction augmentation $\epsilon : S(T(X)) \to X$ and the adjunction unit $\eta : A \to T(S(A))$ are identity morphisms on unit objects, and make commute the diagrams

$$\begin{tikzcd} S(T(X))\otimes S(T(Y)) \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\simeq"'] & X\otimes Y \\ S(T(X)\otimes T(Y)) \arrow[r, "\simeq"'] & S(T(X\otimes Y)) \arrow[u, "\epsilon"'] \end{tikzcd} , \qquad \begin{tikzcd} A\otimes B \arrow[r, "\eta"] \arrow[d, "\eta\otimes\eta"'] & S(T(A\otimes B)) \\ T(S(A))\otimes T(S(B)) \arrow[r, "\simeq"'] & T(S(A)\otimes S(B)) \arrow[u, "\simeq"'] \end{tikzcd} ,$$

involving the symmetric monoidal transformations attached to S and T.

One can check (exercise) that the augmentation $\epsilon : |\mathtt{Sing}_\bullet(X)| \to X$ and the unit $\eta : K \to \mathtt{Sing}_\bullet(|K|)$ of the adjunction between the geometric realization $|-| : \mathcal{S}imp \to \mathcal{T}op$ and the singular complex functor $\mathtt{Sing}_\bullet(-) : \mathcal{T}op \to \mathcal{S}imp$ satisfy these relations. Hence, this adjunction $|-| : \mathcal{S}imp \rightleftarrows \mathcal{T}op : \mathtt{Sing}_\bullet(-)$ is symmetric monoidal in the sense defined in the present paragraph.

Part 1

Models of $\mathbf{E_n}$-operads
&
Grothendieck-Teichmüller Groups

CHAPTER 1

Introduction to $\mathbf{E_n}$-operads

The first purpose of this chapter is to recall the basic definition of the operad of little n-discs D_n, and to explain the definition of the notion of E_n-operad. We devote the first section of the chapter (§1.1) to this objective. We give a survey of classical results on the homology of the little disc operads in the second section of the chapter (§1.2). The homology functor naturally goes from spaces to graded modules. In good cases, the homology of a space also inherits a coalgebra structure, which is dual to the standard commutative algebra structure of cohomology, and we will observe that the homology of the little disc operad naturally forms a graded Hopf operad, an operad in the symmetric monoidal category of augmented cocommutative coalgebras in graded modules. The ultimate aim of §1.2 is to determine this graded Hopf operad structure.

We have appended a section (§1.3) to the chapter in order to make explicit our conventions on the category of graded modules.

In this book, we deal with non-unitary operad structures as soon as we perform in-depth constructions on operads, and for technical reasons, we systematically regard unitary operads as unitary extensions of an underlying non-unitary operad. Therefore, in contrast with standard conventions, we assume $\mathsf{D}_n(0) = \emptyset$ for the version of the little n-discs operad which we denote by D_n. The usual unitary version of the operad little n-discs, which we denote by D_{n+}, is obtained by adding a base point in arity 0 to this non-unitary operad D_n. By convention, when we do not specify anything, we assume that our E_n-operads are non-unitary too.

Most results and concepts surveyed in this chapter come from [16, 17, 72], as regards the definition of the little discs operads and iterated loop spaces, and [2, 25, 26], as regards the homology computations.

1.1. Introduction to little discs operads

The purpose of this section, as we just explained, is to recall the definition of the little n-discs operad, for the sake of reference, and of the derived notion of an E_n-operad. To complete our account, we provide a short survey of the applications of operads to iterated loop spaces, because these original motivating applications yield some intuition on E_n-operads and on the associated algebra structures.

To begin with, we explain what the little discs are. We assume that n is a positive (finite) integer $n = 1, 2, \ldots$ for the moment.

1.1.1. *The little discs.* Let $\mathbb{D}^n$ denote the standard unit n-disc, defined as the subspace $\mathbb{D}^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \,|\, t_1^2 + \cdots + t_n^2 \leq 1\}$ in the euclidean space $\mathbb{R}^n$. The little n-discs, giving the name of the little n-discs operad, are affine embeddings $c : \mathbb{D}^n \to \mathbb{D}^n$ of the form

$$c(t_1, \ldots, t_n) = (a_1, \ldots, a_n) + r \cdot (t_1, \ldots, t_n),$$

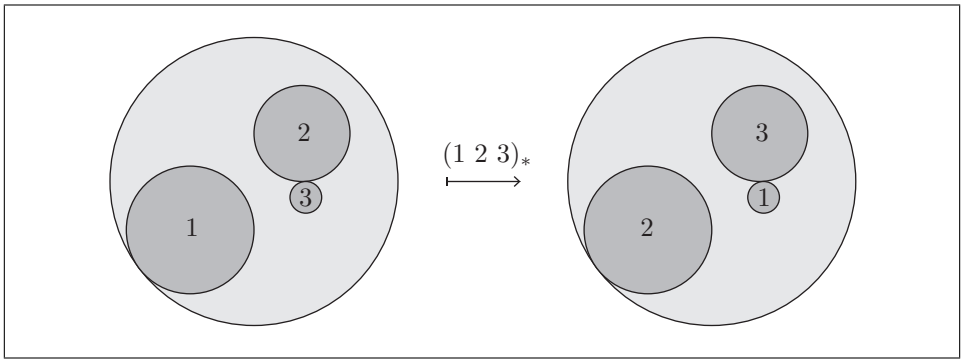


FIGURE 1.1. The representation of an element in the little 2-disc operad, and the action of the cyclic permutation (1 2 3) on this element.

for some translation vector $(a_1, \ldots, a_n) \in \mathbb{D}^n$ and multiplicative scalar $r > 0$ such that $r^2 < 1 - (a_1^2 + \cdots + a_n^2)$. To specify such an embedding, it is enough to give the subset $c(\mathbb{D}^n)$ since the definition of c amounts to assuming that $c(\mathbb{D}^n)$ forms an n-disc inside $\mathbb{D}^n$, with $(a_1, \ldots, a_n) = c(0, \ldots, 0) \in \mathbb{D}^n$ as center and $r > 0$ as radius. This observation implies that little 2-discs are determined by their graphical representations, which we can therefore safely use in order to illustrate our constructions. By abuse of notation, we set $c = c(\mathbb{D}^n)$ and we use the same letter c to define both the subspace in $\mathbb{D}^n$ and the corresponding mapping from $\mathbb{D}^n$ to $\mathbb{D}^n$.

The boundary of the unit n-disc $\mathbb{D}^n$, defined as the space of points $(t_1, \ldots, t_n) \in \mathbb{D}^n$ such that $t_1^2 + \cdots + t_n^2 = 1$, will be denoted by $\partial\,\mathbb{D}^n$. The interior of $\mathbb{D}^n$, defined as the complement of the subspace $\partial\,\mathbb{D}^n$ in $\mathbb{D}^n$, or equivalently, as the space of points $(t_1, \ldots, t_n) \in \mathbb{D}^n$ such that $t_1^2 + \cdots + t_n^2 < 1$, will be denoted by $\mathring{\mathbb{D}}^n$. We define the boundary of a little n-disc c as the subspace $\partial c = c(\partial\,\mathbb{D}^n)$ of $c = c(\mathbb{D}^n)$, and the interior as $\mathring{c} = c(\mathring{\mathbb{D}}^n)$.

1.1.2. *The definition of the little n-disc spaces.* The little n-discs space $\mathsf{D}_n(r)$ formally consists of r-tuples $\underline{c} = (c_1, \ldots, c_r)$ of affine embeddings $c_i : \mathbb{D}^n \to \mathbb{D}^n$, $i = 1, \ldots, r$, of the form considered in §1.1.1, and such that $\mathring{c}_i \cap \mathring{c}_j = \emptyset$ for all pairs $i \neq j$.

The space $\mathsf{D}_n(r)$ is equipped with the compact-open topology since the collection of affine maps $\underline{c} = (c_1, \ldots, c_r)$ is naturally identified with an element of the mapping space $\mathtt{Map}_{\mathcal{T}op}(\coprod^r \mathbb{D}^n, \mathbb{D}^n)$. Equivalently, we can use parameters associated with these maps, like the centers $(a_1, \ldots, a_n) = c_i(0, \ldots, 0) \in \mathbb{D}^n$ and the radius $r > 0$, to determine the topology of $\mathsf{D}_n(r)$. The first approach is more convenient when we deal with applications of little discs to iterated loop spaces. The second equivalent definition is more convenient when we examine the connections of little discs with configuration spaces (see §1.2.1).

Figure 1.1 gives the representation of an element $\underline{c} \in \mathsf{D}_n(3)$. In this picture, we use that the definition of $\underline{c}$ as an r-tuple $\underline{c} = (c_1, \ldots, c_r)$ amounts to assuming that the little n-discs $c_1, \ldots, c_r \subset \mathbb{D}^n$ are indexed by the elements $i = 1, \ldots, r$. We have

a natural mapping $s_* : \mathsf{D}_n(r) \to \mathsf{D}_n(r)$, associated to each permutation $s \in \Sigma_r$, formally defined by $s_*(c_1, \ldots, c_r) = (c_{s(1)}, \ldots, c_{s(r)})$, for any $\underline{c} = (c_1, \ldots, c_r) \in \mathsf{D}_n(r)$. Pictorially, the mapping $s_* : \mathsf{D}_n(r) \to \mathsf{D}_n(r)$ is given by an obvious reindexing operation: we apply the permutation $s \in \Sigma_*$ to the index $i = 1, \ldots, r$ associated with each little n-cube of $\underline{c} = (c_1, \ldots, c_r) \in \mathsf{D}_n(r)$ in order to get the picture of $s_*(\underline{c}) \in \mathsf{D}_n(r)$ from the picture of $\underline{c}$ (see Figure 1.1 for an example).

The collection $\mathsf{D}_n = \{\mathsf{D}_n(r), r \in \mathbb{N}\}$, where each space $\mathsf{D}_n(r)$ is equipped with this action of Σ_r, forms a symmetric sequence.

In certain applications, we may prefer to consider the symmetric collection associated to D_n, of which terms are indexed by arbitrary finite sets $\underline{\mathsf{r}}$, rather than this symmetric sequence. The elements of a term $\mathsf{D}_n(\underline{\mathsf{r}})$ in this symmetric collection are identified with collections of little cubes $\underline{c} = \{c_{i_1}, \ldots, c_{i_r}\}$ indexed by the elements of the given set $\underline{\mathsf{r}} = \{i_1, \ldots, i_r\}$ rather than by ordinal elements $i = 1, \ldots, r$. The action of finite set bijections $u \in \mathcal{B}ij(\underline{\mathsf{r}}, \underline{\mathsf{s}})$ on the symmetric collection $\mathsf{D}_n(\underline{\mathsf{r}})$ is the obvious extension of the reindexing process associated with permutations.

1.1.3. *The little n-disc operad.* We consider the symmetric sequence of little n-disc spaces defined in the previous paragraph. We have a natural unit element $1 \in \mathsf{D}_n(1)$ given by the 1-tuple $1 = (id)$, where we consider the identity mapping $id : \mathbb{D}^n \to \mathbb{D}^n$, with the full unit disc $\mathbb{D}^n = id(\mathbb{D}^n)$ as corresponding subspace $id(\mathbb{D}^n) \subset \mathbb{D}^n$.

We now define the partial composition operations $\circ_i : \mathsf{D}_n(r) \times \mathsf{D}_n(s) \to \mathsf{D}_n(r + s - 1)$ giving the operadic composition structure of D_n. To $\underline{a} = (a_1, \ldots, a_r) \in \mathsf{D}_n(r)$ and $\underline{b} = (b_1, \ldots, b_s) \in \mathsf{D}_n(s)$, we associate the $r + s - 1$-tuple of little discs

$$\underline{a} \circ_i \underline{b} = (a_1, \ldots, a_{i-1}, a_i \circ b_1, \ldots, a_i \circ b_s, a_{i+1}, \ldots, a_r) \in \mathsf{D}_n(r + s - 1),$$

where the expression $a_i \circ b_k$ refers to the composite of the maps $a_i : \mathbb{D}^n \to \mathbb{D}^n$ and $b_k : \mathbb{D}^n \to \mathbb{D}^n$. Note that such a composite $a_i \circ b_k$ is still an embedding of the form specified in §1.1.1. Intuitively, the little n-disc configuration $\underline{a} \circ_i \underline{b} \in \mathsf{D}_n(r + s - 1)$ is obtained by putting the configuration $\underline{b} = (b_1, \ldots, b_s)$ in the little disc of $\underline{a} = (a_1, \ldots, a_r)$ indexed by i, as depicted in Figure 1.2. In this process, we apply the affine mapping $a_i : \mathbb{D}^n \to \mathbb{D}^n$, equivalent to the given little n-disc $a_i = a_i(\mathbb{D}^n)$, in order to put the little n-disc configuration $\underline{b}$ at the appropriate position and scale.

The definition of the operad D_n, for $n = 1, 2, \ldots$, is now complete since we can immediately check, by a straightforward inspection of definitions, that the unit and associativity axioms of operads are satisfied by our composition operations.

1.1.4. *The deletion structure associated with the little n-disc operad.* In what follows, we take the convention that $\mathsf{D}_n(0) = \emptyset$ (as explained in the introduction of this chapter). On the other hand, we can formally extend the definitions of the previous paragraphs to include the case of an empty collection of little n-discs in arity 0. We then obtain a unitary version of the operad of little n-discs D_{n+}, with the empty collection as unique element in arity 0, so that $\mathsf{D}_{n+}(0) = *$. We use the notation $*$ for both this one-point set, and the empty collection, regarded as unique element of this component of the operad D_{n+}.

This operad D_{n+} forms a unitary extension of the non-unitary little n-cubes operad D_n (in the sense considered in §I.4.5), and the partial composites with the arity 0 element $* \in \mathsf{D}_{n+}(0)$ are equivalent to deletion operations $\partial_i : \mathsf{D}_{n+}(r) \to \mathsf{D}_{n+}(r - 1)$ so that $\partial_i(\underline{c}) = \underline{c} \circ_i *$ (see §I.4.1). The image of a little n-disc collection $\underline{c} = (c_1, \ldots, c_r)$ under the deletion map $\partial_i : \mathsf{D}_{n+}(r) \to \mathsf{D}_{n+}(r - 1)$ can readily be

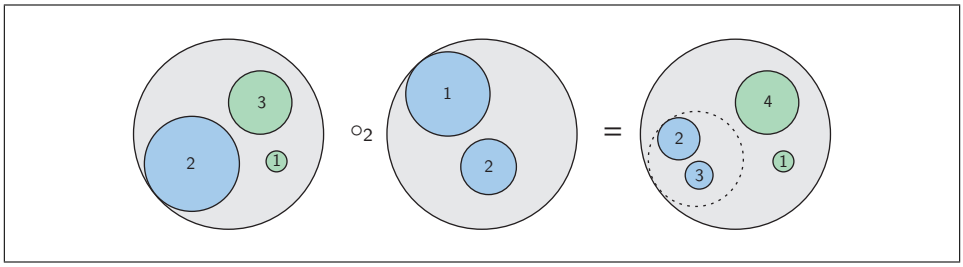


FIGURE 1.2. The composition of elements in the little 2-disc operad.

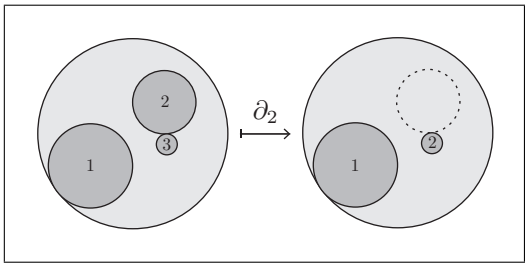


FIGURE 1.3. The representation of a deletion operation in the little 2-disc operad.

identified with the $r-1$-tuple $\partial_i(\underline{c}) = (c_1, \ldots, \widehat{c_i}, \ldots, c_r)$, where the ith term of $\underline{c}$ has been removed (see Figure 1.3 for an example).

The unitary operads D_{n+} naturally occur in applications to iterated loop spaces. The computation of the homology of the little n-discs operads (see the next section) involves the deletion morphisms associated to the operad D_{n+} too.

1.1.5. *The little disc operads as a nested sequence.* The operad of little n-cubes, as defined in the previous paragraphs for a finite integer $n = 1, 2, \ldots$, actually form a nested sequence of topological operads

$$\mathsf{D}_1 \hookrightarrow \mathsf{D}_2 \hookrightarrow \cdots \hookrightarrow \mathsf{D}_n \hookrightarrow \cdots .$$

We take the colimit $\mathsf{D}_\infty = \operatorname{colim}_n \mathsf{D}_n$ to add a terminal term to this sequence and to define the infinite dimensional version of the little disc operads. We have an extension of this construction in the unitary setting too.

We use the equatorial embedding of the n-disc $\mathbb{D}^n$ into the $n+1$-disc $\mathbb{D}^{n+1}$, formally defined by $e(t_1, \ldots, t_n) = (t_1, \ldots, t_n, 0)$, to regard $\mathbb{D}^n$ as a subspace of $\mathbb{D}^{n+1}$. To a little n-disc $c : \mathbb{D}^n \to \mathbb{D}^n$ we associate the little $n+1$-disc $e(c) : \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ with the same center $c(0, \ldots, 0)$ in the equatorial disc $\mathbb{D}^n \subset \mathbb{D}^{n+1}$ and the same radius $r > 0$. Thus, when $c(t_1, \ldots, t_n) = (a_1, \ldots, a_n) + r \cdot (t_1, \ldots, t_n)$, this little $n+1$-disc $e(c)$ is formally defined by $e(c)(t_1, \ldots, t_n, t_{n+1}) = (a_1, \ldots, a_n, 0) + r \cdot (t_1, \ldots, t_n, t_{n+1})$.

The operad embedding $e : \mathsf{D}_n \to \mathsf{D}_{n+1}$ is defined on each little n-discs space $\mathsf{D}_n(r)$ by the mapping such that $e(\underline{c}) = (e(c_1), \ldots, e(c_r))$ for any $\underline{c} = (c_1, \ldots, c_r) \in \mathsf{D}_n(r)$ (see Figure 1.4 for the graphical representation of this process). We readily see that the collection of these mappings preserve the internal structure of operads, and hence, do define operad morphisms, which moreover admit an obvious extension to the unitary version of the little cubes operads. We can check further that our

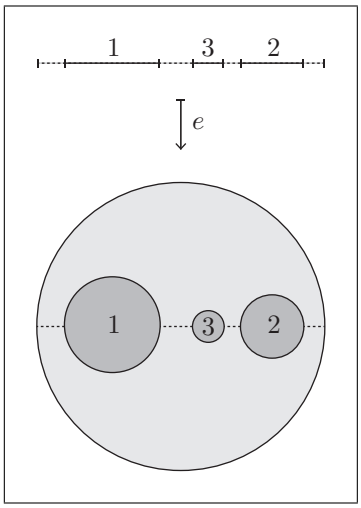


FIGURE 1.4. The image of a little 1-disc configuration in the little 2-disc operad.

mappings $e : \mathsf{D}_n(r) \to \mathsf{D}_{n+1}(r)$, are topological inclusions, for all $r \in \mathbb{N}$, and hence, the little n-disc space $\mathsf{D}_n(r)$ can really be identified with a subspace of $\mathsf{D}_{n+1}(r)$.

To complete our definitions, we record the following result (already mentioned in the chapter introduction) about the initial term of the sequence D_1 and the added terminal term D_∞:

PROPOSITION 1.1.6.

(a) *We have $\pi_0\,\mathsf{D}_1(r) = \Sigma_r$, for $r = 1, 2, \ldots,$ and the canonical maps $\mathsf{D}_1(r) \to \pi_0\,\mathsf{D}_1(r)$ define a weak-equivalence of topological operads $\mathsf{D}_1 \xrightarrow{\sim} \mathsf{As}$ from the little 1-disc operad D_1 to the associative operad As, formed in the category of sets and viewed as a discrete topological operad. In the unitary setting, we have similarly $\pi_0\,\mathsf{D}_{1+} \simeq \mathsf{As}_+$.*

(b) *We have $\pi_0\,\mathsf{D}_\infty(r) = *$, for $r = 1, 2, \ldots,$ and the canonical maps $\mathsf{D}_\infty(r) \to \pi_0\,\mathsf{D}_\infty(r)$ define a weak-equivalence of topological operads $\mathsf{D}_\infty \xrightarrow{\sim} \mathsf{Com}$ between D_∞ to the commutative operad Com, formed in the category of sets and viewed as a discrete topological operad. In the unitary setting, we have similarly $\pi_0\,\mathsf{D}_{\infty+} \simeq \mathsf{Com}_+$.*

PROOFS AND EXPLANATIONS. In the proposition, we consider the sets of path-connected components $\pi_0\,\mathsf{P}(r)$ associated to the topological spaces $\mathsf{P}(r)$ underlying an operad P. The collection of sets $\pi_0\,\mathsf{P}(r)$ inherits an operad structure from P. Moreover, the collection of maps $\mathsf{P}(r) \to \pi_0\,\mathsf{P}(r)$ defines a morphism of topological operads, where we regard the sets $\pi_0\,\mathsf{P}(r)$ as discrete topological spaces, as stated in the proposition. This assertions formally follows from the obvious observation that the mapping $\pi_0 : X \mapsto \pi_0 X$, from topological spaces to sets, defines a symmetric monoidal functor with the functors from sets to discrete spaces as adjoint (see §§II.1.3-II.1.7). The claim that $\mathsf{P} \to \pi_0\,\mathsf{P}$ defines a weak-equivalence of topological operads, as formulated in the proposition, amounts to the assertion that the path-connected components of the spaces $\mathsf{P}(r)$ are contractible.

In the case $\mathsf{P} = \mathsf{D}_1$, the embedding of a collection of little intervals (of little 1-discs) $\underline{c} = (c_1, \ldots, c_r) \in \mathsf{D}_1(r)$ in the one dimensional space $\mathbb{D}^1 = [-1, 1]$ determines

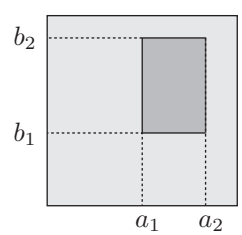
an order relation between the intervals. To be explicit, we set $c_i < c_j$ when we have $c_i(0) < c_j(0)$, or equivalently, when $c_i(s) \leq c_j(t)$ for all $s, t \in \mathbb{D}^1$. The obtained ordering $c_{i_1} < \cdots < c_{i_r}$ determines a permutation $(i_1, \ldots, i_r)$ of the indices $(1, \ldots, r)$ which we associate to our the little 1-disc configuration $(c_1, \ldots, c_r)$. For the little configuration of Figure 1.4, for instance, we obtain the permutation $(1, 3, 2)$.

This assignment gives a map $\mathsf{D}_1(r) \to \Sigma_r$, for any $r \in \mathbb{N}$, and we can easily check, by providing a map in the converse direction and a contracting homotopy, that this map is indeed a homotopy equivalence. From this verification, we conclude that $\pi_0 \mathsf{D}_1(r) = \Sigma_r$ and the path-connected components of $\mathsf{D}_1(r)$ are contractible, as asserted. Recall that the permutation groups Σ_r, $r > 0$, define the underlying collection of the associative operad in sets As. By inspection of definitions, we can also easily check that the relation $\pi_0 \mathsf{D}_1 = \mathsf{As}$ holds as an identity of operads. In the unitary context, where we simply consider an additional base point in arity 0 on both sides of the identity $\pi_0 \mathsf{D}_1 = \mathsf{As}$, and we similarly check that we have an identity $\pi_0 \mathsf{D}_{1+} = \mathsf{As}_+$ in the category of operads.

We refer to [17, Lemma 2.50] for a proof that each space $\mathsf{D}_\infty(r)$ is contractible. We therefore have $\pi_0 \mathsf{D}_\infty(r) = *$ for each $r > 0$, where we use the notation $*$ for the one-point set. Recall that the commutative operad in sets Com is also given by $\mathsf{Com}(r) = *$, for all $r > 0$. The existence of the relation $\pi_0 \mathsf{D}_\infty(r) = *$ for each $r > 0$ automatically implies the identity $\pi_0 \mathsf{D}_\infty = \mathsf{Com}$ in the category of operads in this case, and we similarly obtain the operad identity $\pi_0 \mathsf{D}_{\infty+} = \mathsf{Com}_+$ in the unitary setting. $\square$

We check soon (from homology computations) that the operads D_n, unlike D_1 and D_∞, are not weakly-equivalent to discrete operads when $1 < n < \infty$. We can readily see, nonetheless, that the spaces $\mathsf{D}_n(r)$ are path-connected for $n > 1$. Accordingly, the identity of the theorem $\pi_0 \mathsf{D}_n = \mathsf{Com}$ in assertion (b) holds as soon as $n > 1$, and similarly $\pi_0 \mathsf{D}_{n+} = \mathsf{Com}_+$ when we add a base point in arity 0.

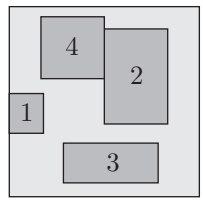
1.1.7. *Relationship with the little n-cubes operad.* The little n-cubes operad is a variant of the little n-discs operad D_n of which elements consist of collections of cube (rather than disc) embeddings. To be precise, we define a little cube c as a map $c_i : [0,1]^n \hookrightarrow [0,1]^n$, of the form $c_i(t_1, \ldots, t_n) = (a_1 + (b_1 - a_1)t_1, \ldots, a_r + (b_n - a_n)t_n)$, for $(t_1, \ldots, t_n) \in [0,1]^n$, for n-tuples of parameters $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0,1]^n$ such that $0 \leq a_k < b_k \leq 1$, for all k. Thus the space $c = c([0,1]^n)$ defines in this case an n-dimensional cube in $[0,1]^n$ with non-empty interior $\mathring{c}_i$ and faces parallel to the faces of the ambient unit cube, as in the following picture:



The n-tuples $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0,1]^n$ represent the extremal vertices of this little cube.

The spaces $\mathsf{C}_n(r)$, forming the little n-cubes operad C_n, consists of r-tuples of little n-cubes $\underline{c} = (c_1, \ldots, c_r)$ with disjoint interiors. Thus, a typical element of the

little n-cubes operad is represented by a picture of the following form:



The definition of the operad structure on little n-cubes is an obvious variation of the definition of the operad structure on little n-discs. One can prove that the operad of little n-discs is weakly-equivalent as an operad to the operad of little n-cubes. Some constructions possible with little n-cubes can not be performed with little n-discs, and conversely, but both operads suit equally well for the constructions considered in this book.

1.1.8. *Iterated loop spaces.* The little n-discs, as we explain soon, represent composition patterns for continuous maps $\alpha : \mathbb{D}^n \to X$ towards a space X equipped with a fixed base point x_0 and so that $\alpha \mid_{\partial \mathbb{D}^n} = x_0$. The space formed by these maps

$$\Omega^n X = \{\alpha \in \mathtt{Map}_{\mathcal{T}op}(\mathbb{D}^n, X) \mid \alpha \mid_{\partial \mathbb{D}^n} = x_0\},$$

together with the topology inherited from $\mathtt{Map}_{\mathcal{T}op}(\mathbb{D}^n, X)$, is one of the possible equivalent definitions for the n-fold loop space associated to X. In the case $n = 1$, we retrieve with this construction the basic definition of the space of loops $\alpha : \mathbb{D}^1 \to X$ based at x_0. This 1-fold loop space is more usually denoted by ΩX (with the dimension exponent dropped from the notation).

The pairs (X, x_0), consisting of a topological space X together with a distinguished base point $x_0 \in X$, form the objects of the category of pointed spaces. The morphisms of this category $\mathcal{T}op_*$ are the morphisms of topological spaces preserving the base point. In general, we use the expression of the underlying space X for the objects of $\mathcal{T}op_*$, and the notation $*$ to refer to the base point attached to any such space (except in particular cases where the base point has to be specified). Implicitly, we abusively consider that a space X, regarded as an object of $\mathcal{T}op_*$, comes together with a natural base point, which is part of its internal structure.

The loop space $\Omega^n X$ comes together with a natural base point, defined by the constant map towards the base point of X, and the assignment $\Omega^n : X \mapsto \Omega^n X$ defines a functor $\Omega^n : \mathcal{T}op_* \to \mathcal{T}op_*$ on the category of pointed spaces $\mathcal{T}op_*$. The n-fold loop space functor $\Omega^n : \mathcal{T}op_* \to \mathcal{T}op_*$ can formally be identified with the n-fold composite of the basic single loop space functor $\Omega : \mathcal{T}op_* \to \mathcal{T}op_*$. This observation motivates the terminology of iterated loop space for spaces of the form $Y = \Omega^n X$.

1.1.9. *Operations on iterated loop spaces associated to little discs.* Let $\underline{c} = (c_1, \ldots, c_r) \in \mathsf{D}_{n+}(r)$ be any element sequence of little discs c_i, $i = 1, \ldots, r$ (possibly empty, $r = 0$), defining an element in the (unitary) little n-discs operad (unitary structures are needed for applications to iterated loop spaces). The assumption that each little disc c_i has a radius $r > 0$ implies that the map $c_i : \mathbb{D}^n \to \mathbb{D}^n$ induces an affine isomorphism between $\mathbb{D}^n$ and $c_i = c_i(\mathbb{D}^n)$.

To a collection of n-fold loop space elements $\alpha_1, \ldots, \alpha_r \in \Omega^n X$, we associate the map $\alpha : \mathbb{D}^n \to X$ such that

$$\alpha(t_1, \ldots, t_n) = \begin{cases} \alpha_i(c_i^{-1}(t_1, \ldots, t_n)), & \text{when } (t_1, \ldots, t_n) \text{ belongs to the image} \\ & \text{of a small disc } c_i = c_i(\mathbb{D}^n), \\ * \quad (\text{the base point of } X), & \text{otherwise.} \end{cases}$$

The assumption $\alpha_i \mid_{\partial \mathbb{D}^n} = *$ for the elements of $\Omega^n X$ ensures that this map is well defined and continuous over $\mathbb{D}^n$. Moreover, we clearly have $\alpha \mid_{\partial \mathbb{D}^n} = *$. Thus, the map $\alpha : \mathbb{D}^n \to X$ defines an element of the n-fold loop space $\alpha = \underline{c}(\alpha_1, \ldots, \alpha_r) \in \Omega^n X$ naturally associated to $\alpha_1, \ldots, \alpha_r \in \Omega^n X$.

To reformulate the construction, the composite $\alpha = \underline{c}(\alpha_1, \ldots, \alpha_r) : \mathbb{D}^n \to \Omega^n X$ is obtained by applying the maps α_i to the little n-discs of the configuration $\underline{c}$. The composition with c_i^{-1} simply amounts to performing a suitable change of scale before applying α_i. The complement of the little n-discs inside $\mathbb{D}^n$ is sent to the base point by our map α.

Under the conventions of §I.1.13, the mapping $\underline{c} : (\alpha_1, \ldots, \alpha_r) \mapsto \underline{c}(\alpha_1, \ldots, \alpha_r)$ defines an operation $\underline{c} : \Omega^n X \times \cdots \times \Omega^n X \to \Omega^n X$ naturally associated to $\underline{c} \in \mathsf{D}_{n+}(r)$. We easily see that:

Proposition 1.1.10. *The above construction provides each n-fold loop space $\Omega^n X$ with an action of the (unitary version of the) little n-discs operad D_{n+} so that $\Omega^n X$ forms an algebra over this operad.* □

Basically, this proposition gives the construction of an algebraic structure (an algebra over D_{n+}) from a topological object (an n-fold loop space). The question is how far the algebraic structure provides a faithful picture of the topological objects. The answer is provided by the following recognition theorem, which gave the first motivation for the introduction of operads in topology:

Theorem 1.A (J. Boardman, R. Vogt [16, 17], P. May [72]). *For any space Y equipped with an action of the (unitary) operad of little n-discs D_{n+}, we have a pointed space $B_n Y$, naturally associated to Y, together with maps $\Omega^n B_n Y \leftarrow \cdot \xrightarrow{\sim} Y$ commuting with D_{n+}-actions, where the middle term is again equipped with a D_{n+}-action and the right-hand side map is a weak-equivalence.*

The left-hand side map is a weak-equivalence too when Y is path-connected (or, more generally, group-like). □

The cited references provide different approaches of this theorem. The arguments of [72] rely on an approximation theorem (see Theorem 2.7 in *loc. cit.*) asserting that free algebras over D_{n+} are weakly-equivalent to iterated loop spaces of suspensions $\Omega^n \Sigma^n X$ (see again *loc. cit.*) and returns the n-fold delooping $B_n Y$ in one step. The arguments of [16, 17] rely on an inductive delooping process.

The space $\Omega^n B_n Y$ is not weakly-equivalent to Y in general (when Y is not group-like), but forms a so-called group completion of Y (see [1] for an introduction to this notion and further references on this subject).

We will not go further into the applications of operads to iterated loop spaces. We refer to the literature, notably the already mentioned monographs [17, 72], for a comprehensive account of that subject. We simply want to explain, in order to complete the above survey, that the action of the little n-discs operad on n-fold loop spaces represents a fine homotopical structure underlying the classical definition of the homotopy groups of pointed spaces.

1.1.11. *Basic motivations: the definition of homotopy groups.* The nth homotopy group $\pi_n(X, x_0)$ of a space X equipped with a base point $x_0 \in X$ can be defined as the set of homotopy classes of maps $u : \mathbb{D}^n \to X$ which are identical to the base point $x_0 \in X$ on $\partial\,\mathbb{D}^n$. Simply recall that a homotopy between any such maps $u_0, u_1 : \mathbb{D}^n \to X$ consists of a map $h : [0,1] \times \mathbb{D}^n \to \mathbb{D}^n$ such that $h(0,\cdot) = u_0$, $h(1,\cdot) = u_1$ and $h(s,\cdot)\,{}_{|\partial\,\mathbb{D}^n} = x_0$, for all $s \in I$.

The group $\pi_1(X, x_0)$ is identified with the fundamental group of X because a based loop on the pointed space X is nothing but a map $u : [-1,1] \to X$ such that $u\,{}_{|\partial[-1,1]} = x_0$, and we have a similar identification for homotopies. We review the definition of the group structure on $\pi_n(X, x)$ soon. Simply recall for the moment that the fundamental group $\pi_1(X, x_0)$ is not abelian in general while all higher homotopy groups $\pi_n(X, x_0)$, $n > 1$, are. We give an operadic interpretation of this structure.

We have a formal identity between $\pi_n(X, x_0)$ and the set of path-connected components of the n-fold loop space $\Omega^n X$. The group multiplication of $\pi_n(X, x_0)$, as most usually defined (see [93, §IV]), can be identified with an operation $\mu : \Omega^n X \times \Omega^n X \to \Omega^n X$, formed at the loop space level, associated with the little n-cubes operad (after considering an n-cube model of $\Omega^n X$ rather than our n-disc model). When we chose to deal with discs instead of cubes, we can obtain an equivalent result by considering an operation $\mu : \Omega^n X \times \Omega^n X \to \Omega^n X$ associated with some little n-disc pair $\underline{c} = (c_1, c_2) \in \mathsf{D}_n(2)$.

To be more precise, when we assume $n > 1$, all operations $\underline{c} : \Omega^n X \times \Omega^n X \to \Omega^n X$ associated to a little n-disc configuration $\underline{c} = (c_1, c_2) \in \mathsf{D}_n(2)$ are the same up to homotopy: indeed, since $\mathsf{D}_n(2)$ is path-connected, any pair of little n-disc configurations $\underline{c}^0, \underline{c}^1 \in \mathsf{D}_n(2)$ are connected by a path $\underline{c}^s$ and the associated maps $\underline{c}^s : \Omega^n X \times \Omega^n X \to \Omega^n X$, $s \in [0,1]$, determine a homotopy between the operations associated to $\underline{c}^0$ and $\underline{c}^1$. We obtain in particular that the multiplication determined by any element $\underline{c} \in \mathsf{D}_n(2)$ is homotopy equivalent to the multiplication determined by the transposed operation $(1\ 2) \cdot \underline{c} \in \mathsf{D}_n(2)$. The commutativity of the multiplication on $\pi_n(X, x_0)$ actually follows from this observation. In the case of $n = 1$, we have two choices of multiplications in homotopy, corresponding to the two path-connected components of the space $\mathsf{D}_1(2)$, and these multiplications are transposed to each other. Thus we retrieve the non-commutativity of the fundamental group $\pi_1(X, x_0)$.

The homotopy, giving the associativity of the multiplication on homotopy groups, can also be defined by a one parameter family of triple operations $\mu_3^s(\cdot,\cdot,\cdot) : \Omega^n X \times \Omega^n X \times \Omega^n X \to \Omega^n X$, $s \in [0,1]$, associated with a path in the little n-discs space $\mathsf{D}_{n+}(3)$. The inversion operation is apart because the homotopies giving this operation are not included in the structure associated with the little n-discs operad (the connectedness assumption in the formulation of Theorem 1.A, the recognition theorem of iterated loop spaces, is related to this point).

By pushing our operadic analysis further, we can regard the associativity (respectively, commutativity) of the group structure on $\pi_n(X, x_0)$ as a consequence of the operad identity $\pi_0\,\mathsf{D}_{1+} = \mathsf{As}_+$ (respectively, $\pi_0\,\mathsf{D}_{n+} = \mathsf{Com}_+$ for $n > 1$). We mention after Proposition 1.1.6 that the operads D_{n+} are not componentwise contractible for $1 < n < \infty$. We actually prove (soon) that $\mathsf{D}_{n+}(2)$ is homotopy equivalent to a sphere $\mathbb{S}^{n-1}$ and that each space $\mathsf{D}_{n+}(r)$ has a non-trivial homology. Fine structures arising from the operad D_{n+} can be revealed by studying homology

groups $\mathtt{H}_*(\Omega^n X, \Bbbk)$ rather than restricting our consideration to the set of connected components $\pi_n(X, x_0) = \pi_0(\Omega^n X)$. The monograph [25] gives a complete description of these homological structures in the case where the coefficient ring of the homology is a field.

1.1.12. *The notion of E_n-operad.* To set the definition once and for all: a non-unitary (respectively, unitary) E_n-operad in topological spaces is an operad P, in the category of topological spaces, which is isomorphic to the operad of little n-discs D_n (respectively, D_{n+}) in the homotopy category of topological operads $\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)$.

By definition of the homotopy category $\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)$, this definition amounts to assuming that P is connected to D_n by a chain of morphisms of topological operads

$$\mathsf{P} \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \cdots \overset{\sim}{\rightarrow} \mathsf{D}_n$$

inducing isomorphisms on homotopy groups, and hence, defining weak-equivalences in the category of topological operads as specified by the marks $\sim$ in the expression of this chain. Since $\mathcal{T}op\mathcal{O}p$ forms a model category, we can assume that such a chain is reduced to two weak-equivalences

$$\mathsf{P} \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \mathsf{D}_n\,.$$

The same observations hold in the unitary context.

In many applications, authors take the additional assumption that E_n-operads are cofibrant as symmetric collections in order to ensure that the category of algebras associated with different models of E_n-operads are Quillen equivalent (see §III.4, and more particularly Theorem III.4.6, for recollections on this subject). The interesting reader can notice that all instances of E_n-operads considered in this book (including the reference model of little n-discs by the way) are cofibrant as symmetric collections. But we will not pay attention to this technical point. Furthermore, as soon as we consider homotopy automorphism groups, we need to deal with cofibrant models of E_n-operads, and this requirement is actually stronger than being cofibrant as a symmetric collection (see for instance [14]).

In the cofibrant case, the model category axioms implies that we can reduce our chain of a weak-equivalences, connecting P and D_n, to a single element $\mathsf{P} \overset{\sim}{\rightarrow} \mathsf{D}_n$, but we usually do not need to make this weak-equivalence explicit too.

In the case $n = 1, \infty$, the result of Proposition 1.1.6 immediately implies (see also the explanations after that statement):

Proposition 1.1.13.

(a) *A non-unitary operad P is E_1 if and only if we have $\pi_0\,\mathsf{P}(r) = \Sigma_r$, for $r = 1, 2, \ldots$, and the canonical maps $\mathsf{P}(r) \to \pi_0\,\mathsf{P}(r)$ define a weak-equivalence of topological operads $\mathsf{P} \overset{\sim}{\rightarrow} \mathsf{As}$, where we regard the associative operad As, formed in the category of sets, as a discrete topological operad. A similar result holds in the unitary context, with the non-unitary associative operad As replaced by the unitary one As_+.*

(b) *A non-unitary operad P is E_∞ if and only if we have $\pi_0\,\mathsf{P}(r) = *$, for $r = 1, 2, \ldots$, and the canonical maps $\mathsf{P}(r) \to \pi_0\,\mathsf{P}(r)$ define a weak-equivalence of topological operads $\mathsf{P} \overset{\sim}{\rightarrow} \mathsf{Com}$, where we regard the commutative operad Com, formed in the category of sets, as a discrete topological operad. A similar result holds in the unitary context, with the non-unitary commutative operad Com replaced by the unitary one Com_+.* □

Since the operads D_n are not equivalent to discrete operads for $1 < n < \infty$, we do not have such a simple characterization of E_n-operads in general. On the other hand, the existence of weak-equivalences $\mathsf{P} \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \mathsf{D}_n$ implies that E_n-operads have the same homology as the operad of little n-discs (and similarly in the unitary context). This already gives a simple criterion for the recognition of E_n-operads. But the study of the homology of E_n-operads gives the subject of the next section.

1.2. The homology (and cohomology) of E_n-operads

The goal of this section is to give a description of the homology of the little n-discs operads D_n, and as a byproduct of any E_n-operad. To simplify the exposition, we only consider the case of coefficients in a field $\Bbbk = \mathbb{Q}$ when we state the result of the homology computation for the operads of little discs. In short, we use coalgebra structures on the homology which are not defined for all operads when we deal with $\mathbb{Z}$ coefficients. Nonetheless, we can mention that these difficulties vanish in the particular case of the little n-disc operad and the given result extends to the case of $\mathbb{Z}$ coefficients.

Since the homology gives a functor towards the category of graded modules, we naturally deal with objects defined within this category $gr\,\mathcal{M}od$. The category of graded modules inherits a symmetric monoidal structure, of which definition is recalled in the appendix section §1.3. We soon recall that the homology of a space (with coefficients in a field) forms an augmented cocommutative coalgebra in graded modules (an augmented graded cocommutative coalgebras for short), and that the homology of a topological operad forms an operad in augmented graded cocommutative coalgebras (a graded Hopf operad). We precisely aim to determine the graded Hopf operad structure attached to the homology of the little discs operads.

To explain our conventions, when we deal with objects defined within the base category of graded modules $gr\,\mathcal{M}od$, we use the prefix graded as a reference for this underlying category, rather than the full expression of graded module. First examples include augmented graded cocommutative coalgebras, unital graded commutative algebras, and graded operads.

For objects defined in categories of augmented cocommutative coalgebras, we also follow the conventions of §II.2 and we generally use the prefix Hopf, rather than a full category designation, as a short reference to the underlying coalgebra structure. Thus, we use the terminology of graded Hopf operad to refer to the same structure as an operad in augmented graded cocommutative coalgebras. In mathematical expressions, we similarly use the notation $gr\,\mathcal{C}om^c_+$, rather than $gr\,\mathcal{M}od\,\mathcal{C}om^c_+$, to refer to the category of augmented graded cocommutative coalgebras, the notation $gr\,\mathcal{O}p$, rather than $gr\,\mathcal{M}od\,\mathcal{O}p$, for the category of graded operads, and the notation $gr\,\mathcal{H}opf\,\mathcal{O}p$ to refer to the category of graded Hopf operads. In Proposition II.2.4, we observed that Hopf operads can be identified with augmented cocommutative coalgebras in operads. Thus, in the graded context, we have categorical identities $gr\,\mathcal{H}opf\,\mathcal{O}p = gr\,\mathcal{C}om^c_+\,\mathcal{O}p = gr\,\mathcal{O}p\,\mathcal{C}om^c_+$. In this section, we also deal with the variant of this category formed by the connected graded Hopf operads equipped with deletion morphisms. In our conventions, we use the notation $gr\,\mathcal{H}opf\,\mathcal{O}p^+_{01}$ to refer to this category.

The homology of the little n-discs operads is essentially trivial when $n = 1, \infty$, since the topological spaces underlying these operads have contractible connected

components (and similarly in the unitary context). Therefore most of our efforts is devoted to the cases where $1 < n < \infty$

The previously alluded to cocommutative coalgebra structure on the homology of a space $\mathtt{H}_*(X)$ is dual to the standard algebra structure of the cohomology $\mathtt{H}^*(X)$. In a first stage, we forget operadic composition structures. We give a description of the cohomology of each space $\mathsf{D}_n(r)$ as a commutative algebra. In this context, we can replace the little n-discs spaces $\mathsf{D}_n(r)$ by homotopy equivalent configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$, which do not form an operad but are more suitable for the analysis of topological structures. To begin with, we recall the definition of these spaces:

1.2.1. *Configuration spaces.* The configuration spaces, which can be associated to any topological space $M \in \mathcal{T}op$, are simply defined by:

$$\mathsf{F}(M, r) = \{(a_1, \ldots, a_r) \in M^r | a_i \neq a_j \text{ for all pairs } i \neq j\},$$

for all $r \in \mathbb{N}$. In what follows, we only consider the configuration space associated to the open n-discs $M = \mathring{\mathbb{D}}^n$. The configuration space associated to the euclidean space $M = \mathbb{R}^n$ is more usually considered in the operadic litterature. But the standard homeomorphism between the euclidean space and the open n-disc induces a homeomorphism at the configuration space level. Therefore, we can deduce results involving one of these configuration spaces from results involving the other.

Proposition 1.2.2. *We have a homotopy equivalence $\omega : \mathsf{D}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ defined by mapping a collection of little n-discs $(c_1, \ldots, c_r) \in \mathsf{D}_n(r)$ to their corresponding centers $(c_1(0), \ldots, c_r(0)) \in \mathsf{F}(\mathring{\mathbb{D}}^n, r)$.*

Proof. Exercise or see [72, §4]. □

We refer to this homotopy equivalence $\omega : \mathsf{D}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ as the disc center mapping.

We have no operadic composition products on configuration spaces, nonetheless:

Proposition 1.2.3. *The collection of configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ inherits the structure of a symmetric sequence with deletion operations so that the collection of disc center mappings $\omega : \mathsf{D}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ defines a weak-equivalence of symmetric sequences with deletion operations.*

Explanations. To define the deletion structure of configuration spaces, we consider, as explained in §I.4.7, a unitary extension of the configuration space sequence $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$, $r \in \mathbb{N}$, defined by adding a one-point set term $\mathsf{F}(\mathring{\mathbb{D}}^n, 0)_+ = *$ in arity 0. The definition of this unitary extension parallels the definition of the unitary little n-cubes operad in §1.1.4, and we can actually regard the unique element of this one-point set as an empty configuration of points in $\mathring{\mathbb{D}}^n$.

The action of a permutation $s \in \Sigma_r$ on an element $\underline{a} = (a_1, \ldots, a_r) \in \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ is defined by the obvious component permutation $s_*(\underline{a}) = (a_{s(1)}, \ldots, a_{s(r)})$ associated to s, and the ith deletion morphism $\partial_i : \mathsf{F}(\mathring{\mathbb{D}}^n, r)_+ \to \mathsf{F}(\mathring{\mathbb{D}}^n, r-1)_+$ is defined by the removal operation $\partial_i(\underline{a}) = (a_1, \ldots, \widehat{a_i}, \ldots, a_r)$. The preservation of the symmetric sequence structure and of deletion operations by the disc center mapping follows from obvious verifications. □

We now examine the topological structure of the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ with the aim of determining the cohomology of these spaces. We begin with the following simple observation:

PROPOSITION 1.2.4. *We have a homotopy equivalence* $\mathsf{F}(\mathring{\mathbb{D}}^n, 2) \xrightarrow{\sim} \mathbb{S}^{n-1}$, *between the configuration space of two points* $\mathsf{F}(\mathring{\mathbb{D}}^n, 2)$ *and the* $n-1$*-sphere* $\mathbb{S}^{n-1}$, *explicitly defined as the map sending an element* $(a, b) \in \mathsf{F}(\mathring{\mathbb{D}}^n, 2)$ *to the normalized vector* $\overrightarrow{ab}/||\overrightarrow{ab}|| \in \mathbb{S}^{n-1}$.

PROOF. Exercise. □

1.2.5. *The definition of fundamental classes.* For $n > 1$, this result implies that

$$\mathtt{H}_*(\mathsf{F}(\mathring{\mathbb{D}}^n, 2)) = \mathtt{H}_*(\mathbb{S}^{n-1}) = \begin{cases} \mathbb{Q}, & \text{if } * = 0, n-1, \\ 0, & \text{otherwise,} \end{cases}$$

and similarly as regards the cohomology $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$. We use the notation $[\mathbb{S}^{n-1}]$ for the generator of $\mathtt{H}_{n-1}(\mathbb{S}^{n-1})$, defined by the fundamental class of the sphere (which we equip with an orientation), and for the corresponding element in the homology $\mathtt{H}_*(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$. We will also use the notation $[pt]$ for the canonical generator of the homology $H_*(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$ in degree 0. In the cohomological context, we consider the element $\omega \in \mathtt{H}^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$, dual to $[\mathbb{S}^{n-1}]$, in order to obtain a canonical generator of $\mathtt{H}^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$.

Let now $r \geq 2$. For each pair $1 \leq i < j \leq r$, we consider the mapping $\phi_{ij} : \mathsf{F}(\mathring{\mathbb{D}}^n, r) \to \mathsf{F}(\mathring{\mathbb{D}}^n, 2)$ such that $\phi_{ij}(a_1, \ldots, a_r) = (a_i, a_j)$, and we set $\omega_{ij} = \phi_{ij}^*(\omega)$ for the image of $\omega \in \mathtt{H}^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$ under the restriction map $\phi_{ij}^* : \mathtt{H}^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, 2)) \to \mathtt{H}^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$. Observe that ϕ_{ij} is the the deletion morphism associated with the injection $\rho_{ij} : \{1 < 2\} \to \{1 < \cdots < r\}$ such that $\rho_{ij}(1) = i$ and $\rho_{ij}(2) = j$.

Let $\mathbb{S}(\omega_{ij}, i < j)$ be the graded symmetric algebra generated by the classes ω_{ij} in degree $n-1$. We have the following result:

THEOREM 1.2.6 (See V. Arnold [2], F. Cohen [25]). *Let* $n > 1$. *Let* $r \geq 2$.

(a) *In* $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$, *we have the relation* $\omega_{ij}^2 = 0$ *for each pair* $i < j$, *and the relation* $\omega_{ij}\omega_{jk} - \omega_{ik}\omega_{jk} - \omega_{ij}\omega_{ik} = 0$ *for each triple* $i < j < k$.

(b) *The morphism* $\mathbb{S}(\omega_{ij}, i < j) \to \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$, *mapping the generator* ω_{ij} *to the corresponding cohomology class in* $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$, *induces an isomorphism*

$$\frac{\mathbb{S}(\omega_{ij}, i < j)}{(\omega_{ij}^2, \omega_{ij}\omega_{jk} - \omega_{ik}\omega_{jk} - \omega_{ij}\omega_{ik})} \xrightarrow{\simeq} \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)),$$

when we form the quotient of the symmetric algebra $\mathbb{S}(\omega_{ij}, i < j)$ *by the ideal generated by the relations of (a).* □

This theorem is established in the cited references, by using euclidean spaces $\mathbb{R}^n$ instead of open discs $\mathring{\mathbb{D}}^n$. This does not change the result since the homeorphism between the euclidean n-space $\mathbb{R}^n$ and the open n-disc $\mathring{\mathbb{D}}^n$ induces a homeorphism at the configuration space level. In the case $n = 2$, we can still use the complex plane $\mathbb{C}$ instead of $\mathbb{R}^2$. The reference [2] gives this case $n = 2$ of the theorem, by using the complex differential form $d(z_i - z_j)/(z_i - z_j)$ as a representative of the class ω_{ij} in the de Rham complex of the configuration space $\mathsf{F}(\mathbb{C}, r) = \{(z_1, \ldots, z_r) \in \mathbb{C}^r \,|\, z_i \neq z_j\}$. The reference [25] gives the general case $n \geq 1$ of the theorem. The computation involves the Leray-Serre spectral sequences associated to projection maps

$$f : \mathsf{F}(\mathbb{R}^n \setminus \{b_1, \ldots, b_m\}, r) \to \mathsf{F}(\mathbb{R}^n \setminus \{b_1, \ldots, b_m\}, r-1),$$

where $\{b_1, \ldots, b_m\}$ is an auxiliary set of punctures. The article [82] provides a comprehensive survey, with little background, of this homological computation.

The result of Theorem 1.2.6 is used in this form in §7, when we study the commutative algebra part of the deformation complex of E_2-operads. For our purpose, we also need to determine the morphisms $\partial_i^* : \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r-1)) \to \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ induced by the deletion operations on configuration spaces. Since the cohomology defines a functor from spaces to unitary commutative algebras, these deletion morphisms are fully determined by the following result:

Proposition 1.2.7. *Let $n > 1$ again. The morphisms $\partial_i^* : \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r-1)) \to \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$, induced by the deletion operations on little cube spaces, are determined by the expression*

$$\partial_i^*(\omega_{kl}) = \begin{cases} \omega_{kl}, & \text{if } i \neq k \neq l, \\ 0, & \text{otherwise}, \end{cases}$$

on the generating cohomology classes $\omega_{kl} \in H^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$.

Proof. Exercise. □

1.2.8. *Homology and monoidal structures.* We now examine the structure of the homology of the little n-discs operad. We can use the result of Theorem 1.2.6, implying the existence of a cohomology isomorphism $\omega^* : \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) \xrightarrow{\simeq} \mathtt{H}^*(\mathsf{D}_n(r))$ and the duality pairing

$$\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) \otimes \mathtt{H}_*(\mathsf{D}_n(r)) \xrightarrow{\simeq} \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) \otimes \mathtt{H}_*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) \xrightarrow{\langle -,- \rangle} \Bbbk$$

to determine $\mathtt{H}_*(\mathsf{D}_n(r))$ as a coalgebra (we recall the general definition of the coalgebra structure in homology soon): the homology $\mathtt{H}_*(\mathsf{D}_n(r))$, associated to each individual space $\mathsf{D}_n(r)$, is simply the dual object of the commutative algebra $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ determined by Theorem 1.2.6. But our new objective is to give a description of the operadic composition operations.

We have already used the classical result that the cohomology defines a functor from spaces to commutative algebras. We first carefully check the formulation of the dual statement, concerning the existence of a coalgebra structure in homology, and we explain the definition of operad structure on the homology of an operad. We use the formalism of symmetric monoidal functors, as set in §II.3.1.

We obviously have $\mathtt{H}_*(pt) = \Bbbk$, by definition of ordinary homology, so that the mapping $\mathtt{H}_* : X \mapsto \mathtt{H}_*(X)$ defines a unit pointed functor from topological spaces to graded modules. We consider the Künneth morphism $\kappa : \mathtt{H}_*(X) \otimes \mathtt{H}_*(Y) \to \mathtt{H}_*(X \times Y)$. We have the following classical statement:

Proposition 1.2.9 (See [64, §VIII] or [85, §5.3]).

(a) *The Künneth morphism defines a symmetric monoidal transformation on the homology mapping $\mathtt{H}_* : X \mapsto \mathtt{H}_*(X)$, regarded as a functor from the symmetric monoidal category of spaces $\mathcal{T}op$ towards the symmetric monoidal category of graded modules $gr\,\mathcal{M}od$.*

(b) *If the coefficient ring is a field, then the Künneth morphism is an iso, so that the homology defines a symmetric monoidal functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{M}od$.* □

We can therefore apply the general constructions of §II.0.4 to obtain:

PROPOSITION 1.2.10. *If the coefficient ring is a field, then the homology functor* $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{M}od$ *induces a functor from the category of topological spaces* $\mathcal{T}op$ *towards the category of augmented cocommutative coalgebras in graded modules* $gr\,\mathcal{C}om^c_+$*, and this functor* $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{C}om^c_+$ *is also symmetric monoidal.*

EXPLANATIONS. In §II.0.4, we deal with the general case of a functor between symmetric monoidal categories. In the context of Proposition 1.2.10, we consider the homology functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{M}od$ between topological spaces and graded modules. The first result of that proposition, the existence of an augmented cocommutative coalgebra structure on the homology, follows from Proposition 1.2.9 and the observation that any space X naturally forms an augmented cocommutative coalgebra in the category of spaces, with the constant map $\epsilon : X \to pt$ as augmentation, and the diagonal map $\Delta : X \to X \times X$ as counit. The second result of the proposition, the definition of the symmetric monoidal functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{C}om^c_+$, arises from the observations of §II.0.4.

To prepare our subsequent study of the homology of little cubes, we examine the applications of the general constructions with more details. First, the augmented graded cocommutative coalgebra structure on the homology of a space $\mathtt{H}_*(X)$ is formed as follows:

(a) to define the counit of this coalgebra, we simply consider the morphism $\mathtt{H}_*(X) \to \mathtt{H}_*(pt) = \Bbbk$, associated to the constant map $X \to pt$;
(b) to define the coproduct, we form the composite

$$\mathtt{H}_*(X) \xrightarrow{\Delta_*} \mathtt{H}_*(X \times X) \xleftarrow{\simeq} \mathtt{H}_*(X) \otimes \mathtt{H}_*(X),$$

where we consider the morphism induced by the diagonal of the space X, followed by the Künneth isomorphism.

The unit, associativity and symmetry constraints, fulfilled by the Künneth isomorphism, ensures that the obtained coalgebra structure satisfies the counit, coassociativity, and cocommutativity relations of augmented graded cocommutative coalgebras (see §II.0.4).

The above coproduct actually forms the dual morphism of the product $\mu : \mathtt{H}^*(X) \otimes \mathtt{H}^*(X) \to \mathtt{H}^*(X)$, defining the commutative algebra structure of the cohomology $\mathtt{H}^*(X)$, because this product can also be defined as a composite

$$\mathtt{H}^*(X) \otimes \mathtt{H}^*(X) \xrightarrow{\kappa} \mathtt{H}^*(X \times X) \xrightarrow{\Delta^*} \mathtt{H}^*(X),$$

where we consider a cohomological version of the Künneth morphism, followed by the morphism induced by the diagonal of X. Note that the commutative algebra structure of the cohomology is, unlike the coalgebra structure of the cohomology, still defined when the Künneth morphism is not iso. To give a more explicit formulation of this duality between product and coproduct, we consider the natural pairing $\langle -, - \rangle : \mathtt{H}^*(X) \otimes \mathtt{H}_*(X) \to \Bbbk$, between the cohomology and the homology of X. If we set $\Delta(c) = \sum_i a_i \otimes b_i$ for the coproduct of an element c in $\mathtt{H}_*(X)$, then we have the adjunction relation

$$\langle \alpha \cdot \beta, c \rangle = \sum_i \pm \langle \alpha, a_i \rangle \cdot \langle \beta, b_i \rangle,$$

for every $\alpha, \beta \in \mathtt{H}^*(X)$, where the sign $\pm$ is produced by the commutation of the factors α and a_i in this expression.

The tensor product $\otimes : gr\,\mathcal{C}om^c_+ \times gr\,\mathcal{C}om^c_+ \to gr\,\mathcal{C}om^c_+$ of the category of augmented graded cocommutative coalgebras is inherited from the category of graded modules by definition (see §II.0.3). The construction implies that the Künneth morphism $\mathtt{H}_*(X) \otimes \mathtt{H}_*(Y) \to \mathtt{H}_*(X \times Y)$ defines a morphism of augmented graded cocommutative coalgebras, and satisfies the unit, associativity, and symmetry constraints of §II.3.1 in that category $gr\,\mathcal{C}om^c_+$ (see §II.0.4). Thus, improving on the assertion of Proposition 1.2.9, we finally obtain that the homology functor defines a symmetric monoidal functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{C}om^c_+$, between spaces and augmented graded cocommutative coalgebras, as asserted in the proposition. □

From the general result of Proposition II.1.4, we then obtain:

Proposition 1.2.11. *Let* P *be any operad in topological spaces.*

(a) *In general, the collection of graded modules* $\mathtt{H}_*(\mathsf{P}) = \{\mathtt{H}_*(\mathsf{P}(r)), r \in \mathbb{N}\}$ *associated to the spaces* $\mathsf{P}(r)$ *forms a graded operad naturally associated to* P.

(b) *If the ground ring is a field, then this operad* $\mathtt{H}_*(\mathsf{P})$ *is actually an operad in augmented graded cocommutative coalgebras, where we use Proposition 1.2.10 to get the coalgebra structure on the homology modules* $\mathtt{H}_*(\mathsf{P}(r))$.

Explanations. In Proposition II.1.4, we deal again with the general case of a functor between symmetric monoidal categories. In the context of Proposition 1.2.10, we consider the homology functor $\mathtt{H}_* : X \mapsto \mathtt{H}_*(X)$ towards the category of graded modules (respectively, augmented graded cocommutative coalgebras). The definition of an operad structure on the homology $\mathtt{H}_*(\mathsf{P})$ is exactly the result of Lemma II.1.3 applied in this context. To prepare our subsequent study of the homology of little cubes, we check the application of the general construction with more details again:

(a) the morphisms $w_* : \mathtt{H}_*(\mathsf{P}(r)) \to \mathtt{H}_*(\mathsf{P}(r))$, induced by the action of permutations $w \in \Sigma_r$ at the topological level, give the action of permutations at the homology level;

(b) the morphism $\Bbbk = \mathtt{H}_*(pt) \xrightarrow{\eta_*} \mathtt{H}_*(\mathsf{P}(1))$, induced by the operadic unit of the topological operad P, gives the natural operadic unit of the homology;

(c) by composition with the Künneth morphism, the partial composition products of the topological operad P induce natural morphisms

$$\mathtt{H}_*(\mathsf{P}(m)) \otimes \mathtt{H}_*(\mathsf{P}(n)) \to \mathtt{H}_*(\mathsf{P}(m) \times \mathsf{P}(n)) \xrightarrow{(\circ_i)_*} \mathtt{H}_*(\mathsf{P}(m+n-1))$$

giving the partial composition products of the homology operad $\mathtt{H}_*(\mathsf{P})$.

The unit, associativity and symmetry constraints of symmetric monoidal functors ensures that the obtained structure fulfills the equivariance, unit and associativity axioms of operads (see §II.1). Depending on the context (a-b), we can form the morphisms giving this operad structure in the category of graded modules or in the category of augmented cocommutative coalgebras.

Recall that an operad mapping $\mathtt{H}_* : \mathsf{P} \mapsto \mathtt{H}_*(\mathsf{P})$ as defined in this proposition preserves unitary extensions: for any unitary operad, we have the identity $\mathtt{H}_*(\mathsf{P}_+) = \mathtt{H}_*(\mathsf{P})_+$. □

Recall that, following the conventions of §II.2, we may use the terminology of graded Hopf operad to refer to an operad in augmented graded commutative coalgebras, and the notation $gr\,\mathcal{H}opf\,\mathcal{O}p$ (instead of $gr\,\mathcal{C}om^c_+\,\mathcal{O}p$) for the category formed

by these operads. Similarly, we may use the terminology of graded Hopf symmetric sequence, and the notation $gr\,\mathcal{H}opf\mathcal{S}eq$, to refer to the category of symmetric sequence in augmented graded cocommutative coalgebras. The above proposition therefore asserts that the homology functor $\mathtt{H}_* : \mathcal{T}op \to gr\,\mathcal{C}om^c_+$ induces a functor $\mathtt{H}_* : \mathcal{T}op\mathcal{O}p \to gr\,\mathcal{H}opf\mathcal{O}p$.

For $\mathsf{P} = \mathsf{D}_1$ (respectively $\mathsf{P} = \mathsf{D}_\infty$), the existence of a weak-equivalences towards the discrete operad of associative (respectively, commutative) monoids implies:

Proposition 1.2.12.

(a) *We have an identity of graded Hopf operads* $\mathtt{H}_*(\mathsf{D}_1) = \mathsf{As}$*, where we consider the associative operads in* $\Bbbk$*-modules* As*, regarded as a graded operad concentrated in degree* 0*, together with the coproduct inherited from the corresponding set operad (see the concluding paragraph of* §*II.1). In the unitary setting, we have similarly* $\mathtt{H}_*(\mathsf{D}_{1+}) = \mathsf{As}_+$*.*

(b) *We have an identity of graded Hopf operads* $\mathtt{H}_*(\mathsf{D}_\infty) = \mathsf{Com}$*, where we consider the commutative operads in* $\Bbbk$*-modules* Com*, regarded as a graded operad concentrated in degree* 0*, together with the coproduct inherited from the corresponding set operad (see the concluding paragraph of* §*II.1 again). In the unitary setting, we have similarly* $\mathtt{H}_*(\mathsf{D}_{\infty+}) = \mathsf{Com}_+$*.*

Recall that our main objective is to give the description of $\mathtt{H}_*(\mathsf{D}_n)$ as a graded Hopf operad when $1 < n < \infty$. We give an abstract definition of this sequence of graded Hopf operads first and we explain the identity with the homology of little discs afterwards.

1.2.13. *The Gerstenhaber operad.* The graded Hopf operads, which we now consider, are graded versions, associated to any $1 < n < \infty$, of the Poisson operad of §I.2.12. We use the notation Gerst_n, and the terminology of n-Gerstenhaber operad, for the nth term of this operad sequence. Some authors use the terminology of Poisson operad of degree $n-1$. We actually define this operad Gerst_n by the same presentation as the Poisson operad

$$\begin{aligned}\mathsf{Gerst}_n = \ \mathbb{O}(\ \Bbbk\,\mu(x_1,x_2) \oplus \Bbbk\,\lambda(x_1,x_2)\ :\ & \\ & \mu(\mu(x_1,x_2),x_3) = \mu(x_1,\mu(x_2,x_3)),\\ & \lambda(\lambda(x_1,x_2),x_3) + \lambda(\lambda(x_2,x_3),x_1) + \lambda(\lambda(x_3,x_1),x_2) = 0,\\ & \lambda(\mu(x_1,x_2),x_3) = \mu(\lambda(x_1,x_3),x_2) + \mu(x_1,\lambda(x_2,x_3))\ ),\end{aligned}$$

with a generating operation $\mu = \mu(x_1,x_2)$ of degree 0 and such that $(1\ 2)\cdot\mu = \mu$ (as in the case of the Poisson operad), but where $\lambda = \lambda(x_1,x_2)$ now represents a generating operation of degree $n-1$, satisfying a symmetry relation $(1\ 2)\cdot\lambda = (-1)^n\lambda$ that depends on the degree, n, of the operad.

As in the Poisson case, we implement the associativity relation $\mu(\mu(x_1,x_2),x_3) = \mu(x_1,\mu(x_2,x_3))$ to make μ a representative of associative product in Gerst_n. One can check that the suboperad of Gerst_n generated by μ is isomorphic to the commutative operad Com (see [44, 66]). The Jacobi relation $\lambda(\lambda(x_1,x_2),x_3) + \lambda(\lambda(x_2,x_3),x_1) + \lambda(\lambda(x_3,x_1),x_2) = 0$ makes λ a graded version of Lie bracket. The suboperad of Gerst_n generated by λ is isomorphic to a suspension of the Lie operad Lie (we refer to [44] for the proof of this claim and the definition of the suspension of operads). The distribution relation $\lambda(\mu(x_1,x_2),x_3) = \mu(\lambda(x_1,x_3),x_2) + \mu(x_1,\lambda(x_2,x_3))$ in Gerst_n implies again that any composite of products and Lie bracket in the Gerstenhaber operad is equal to a product of Lie monomials.

To be more precise, one can prove, as in the Poisson case, that the components of the operad $\mathsf{Gerst}_n(r)$ are the $\Bbbk$-modules spanned by formal products

$$p(x_1,\dots,x_r) = p_1(x_{11},\dots,x_{1r_1})\cdot\dots\cdot p_m(x_{m1},\dots,x_{mr_m}),$$

where each $p_i = p_i(x_{i1},\dots,x_{ir_i})$ is a Lie monomial, formed by composites of the Lie bracket λ, of degree 1 with respect to each variable x_{ik}, $k=1,\dots,r_i$, and so that the variable subsets $\{x_{i1},\dots,x_{ir_i}\}$ form a partition of $\{x_1,\dots,x_r\}$. The description of the Lie operad in §I.2.10, remains also valid in the graded context. Thefore, in the above expansion, we can assume that the monomials $p_i = p_i(x_{i1},\dots,x_{ir_i})$ have a reduced form

$$p_i(x_{i1},\dots,x_{ir_i}) = \lambda(\cdots\lambda(\lambda(x_{i1},x_{i2}),x_{i3}\cdots),x_{ir_i}),$$

where we have $x_{i1} < x_{ik}$, for all $1<k$, with respect to the natural ordering inherited from the ambient set of variables $\{x_1<\dots<x_r\}$.

We provide the operad Gerst_n with a Hopf structure, extending the Hopf structure of the commutative operad $\mathsf{Com}\subset\mathsf{Gerst}_n$ (see §II.2.11), and such that $\epsilon(\lambda)=1$ and $\Delta(\lambda)=\lambda\otimes\mu+\mu\otimes\lambda$ for the Lie element $\lambda\in\mathsf{Gerst}_n(2)$. We can readily see, as in the Poisson case (see §II.2.12), that the ideal of generating relations forms a Hopf ideal, so that this Hopf structure is well defined.

1.2.14. *The unitary Gerstenhaber operad.* Naturally, we again refer to a non-unitary n-Gerstenhaber operad when we perform the construction of the previous paragraph. To define a unitary version of the n-Gerstenhaber operad, we observe, as in the Poisson case, that the generating operations of Gerst_n inherits deletion morphisms, determined by the expressions $\partial_1\mu=\partial_2\mu=1$ and $\partial_1\lambda=\partial_2\lambda=0$. We moreover check that the application of these deletion operations cancel the generating relations of Gerst_n. We can therefore apply the process of §I.4.10 to obtain a unitary extension Gerst_{n+} of the operad Gerst_n.

The deletion morphisms naturally preserve the Hopf structure considered in the previous paragraph so that Gerst_{n+} actually forms a unitary extension of Gerst_n in the category of Hopf operads.

In the computation of the homology of the operad of little discs, we use the deletion morphisms, associated with this unitary extension of the Gerstenhaber operad, as well as the Hopf structure. First, we have the following theorem:

THEOREM 1.B (V. Arnold [2], F. Cohen [25]). *Let $n>1$.*

(a) *The elements $\mu=[pt]\in\mathtt{H}_0(\mathsf{D}_n(2))$ and $\lambda=[\mathbb{S}^{n-1}]\in\mathtt{H}_{n-1}(\mathsf{D}_n(2))$ satisfy the symmetry relations $(1\ 2)\cdot\mu=\mu$ and $(1\ 2)\cdot\lambda=\lambda$ as well as the generating relations of the Poisson operad*

$$\begin{aligned}&\mu(\mu(x_1,x_2),x_3)=\mu(x_1,\mu(x_2,x_3)),\\&\lambda(\lambda(x_1,x_2),x_3)+\lambda(\lambda(x_2,x_3),x_1)+\lambda(\lambda(x_3,x_1),x_2)=0,\\&\lambda(\mu(x_1,x_2),x_3)=\mu(\lambda(x_1,x_3),x_2)+\mu(x_1,\lambda(x_2,x_3))\end{aligned}$$

in the homology of the little n-discs.

(b) *Besides, we have the formulas*

$$\Delta[pt]=[pt]\otimes[pt]\quad\text{and}\quad\Delta[\mathbb{S}^{n-1}]=[\mathbb{S}^{n-1}]\otimes[pt]+[pt]\otimes[\mathbb{S}^{n-1}]$$

for the coproduct of these elements in the homology of $\mathsf{D}_n(2)$, the obvious formulas $\epsilon[pt]=1$ and $\epsilon[\mathbb{S}^{n-1}]=0$ for the coalgebra augmentation, as well

as the formulas $\partial_i[pt] = 1$ *and* $\partial_i[\mathbb{S}^{n-1}] = 0$ *for the deletion morphisms inherited from* D_n.

(c) *The mapping* $\mu \mapsto [pt] \in \mathtt{H}_0(\mathsf{D}_n(2))$ *and* $\lambda \mapsto [\mathbb{S}^{n-1}] \in \mathtt{H}_{n-1}(\mathsf{D}_n(2))$ *induces an isomorphism of graded Hopf operads*

$$h : \mathsf{Gerst}_n \xrightarrow{\simeq} \mathtt{H}_*(\mathsf{D}_n),$$

which also admits a unitary extension $h_+ : \mathsf{Gerst}_{n+} \xrightarrow{\simeq} \mathtt{H}_*(\mathsf{D}_{n+})$.

Explanations and references for the proof. We refer to [25] for the proof of the identities of (a) in the homology of the little discs operad (see also [82] for another nice reference on this topic). The identities of (b) are obvious.

We deduce, after this preliminary verification, that we have a morphism of graded operads $h : \mathsf{G}_n \to \mathtt{H}_*(\mathsf{D}_n)$ mapping the generating operation $\mu \in \mathsf{Gerst}_n(2)$ (respectively, $\lambda \in \mathsf{Gerst}_n(2)$) to the element $[pt] \in \mathtt{H}_0(\mathsf{D}_n(2))$ (respectively $[\mathbb{S}^{n-1}] \in \mathtt{H}_{n-1}(\mathsf{D}_n(2))$), as specified in the theorem. As the coproduct of the homology classes $[pt]$ and $[\mathbb{S}^{n-1}]$ matches the definition of the coproduct of the corresponding generating operations in the Gerstenhaber operad, we immediately conclude that coproducts are preserved by our morphism, which therefore forms a morphism of graded Hopf operads.

We still have to check that this morphism is an iso. We can deduce this result from the computation of the cohomology of configuration spaces in Theorem 1.2.6, and from the following proposition, which involves the definition of the morphism (and no more). We refer to [82] for details.

The result of the theorem also follows from the computation of [25], giving the expression of the homology $\mathtt{H}_*(S_*(\mathsf{D}_n, X))$ as a functor in $\mathtt{H}_*(X)$, for any space X, where $S_*(\mathsf{D}_n, X)$ refers to the free D_n-algebra associated to X modulo base point (see *loc. cit.* for details). In the case of rational coefficients, the result of [25] asserts that this functor is precisely the free Gerst_n-algebra on $\mathtt{H}_*(X)$, and the identity between Gerst_n and $\mathtt{H}_*(\mathsf{D}_n)$ is actually equivalent to this functor identity.

The preservation of deletion morphisms implies that our morphism h extends to a morphism of unitary operads h_+ which is obviously an iso too as soon as h is. □

Proposition 1.2.15. *Let* $\omega_{ij} \in \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ *be any of the generating elements of the cohomology algebra* $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$, *as defined in §1.2.5. Let* $p = p(x_1, \ldots, x_r) \in \mathsf{Gerst}_n(r)$. *We apply the morphism of Theorem 1.B to regard* p *as an element of* $\mathtt{H}_*(\mathsf{D}_n(r))$. *Then we have the duality relation*

$$\langle \omega_{ij}, p\rangle = \begin{cases} 1, & \text{in the case } p = x_1 \cdot \ldots \cdot \lambda(x_i, x_j) \cdot \ldots \cdot \widehat{x_j} \cdot \ldots \cdot x_r, \\ 0, & \text{otherwise,} \end{cases}$$

with respect the pairing $\langle \cdot, \cdot \rangle : \mathtt{H}_*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) \otimes \mathtt{H}_*(\mathsf{D}_n(r)) \to \Bbbk$ *considered in §1.2.8.*

We will see that the definition of the morphism in Theorem 1.2.6 forces the adjunction relation of this proposition. On the other hand, we do not need that the morphism is an iso to check this proposition.

The expression of the pairing $\langle \pi, p\rangle$ associated to any monomial $\pi = \omega_{i_1 j_1} \cdot \ldots \cdot \omega_{i_r j_r}$ can be obtained from the result of this proposition, and the adjunction relation between the product of $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ and the coproduct of $\mathtt{H}_*(\mathsf{D}_n(r))$ (see §1.2.8). The combinatorial formula arising from this process is worked out in [82].

Proof. We use that the disc center map $\omega : \mathsf{D}_n(r) \to \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ defines a weak-equivalence of symmetric sequences with deletion morphisms. Recall that the map $\phi_{ij} : \mathsf{F}(\mathring{\mathbb{D}}^n, r) \to \mathsf{F}(\mathring{\mathbb{D}}^n, 2)$, considered in the definition of the element ω_{ij}, is identified with the composite deletion morphism $\phi_{ij} = \partial_r \cdot \ldots \cdot \widehat{\partial_j} \cdot \ldots \cdot \widehat{\partial_i} \cdot \ldots \cdot \partial_1$ on $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$. By functoriality of the pairing between cohomology and homology, and the preservation of deletion morphisms, we obtain:

$$\langle \omega_{ij}, p\rangle = \langle (\phi_{ij})^*(\omega), p\rangle = \langle \omega, (\phi_{ij})_*(p)\rangle$$

for any $p = p(x_1, \ldots, x_r) \in \mathsf{Gerst}_n(r)$. Then the result of the proposition follows from the expression of the deletion morphisms in §1.2.7, the expression of the elements of $\mathsf{Gerst}_n(r)$ in terms of products of Lie monomials in §1.2.13, and from the definition of $\omega \in \mathtt{H}^{n-1}(\mathsf{F}(\mathring{\mathbb{D}}^n, 2))$ as the dual element of $\lambda = [\mathbb{S}^{n-1}]$. □

1.3. Appendix: the symmetric monoidal category of graded modules

We fix a ground ring $\Bbbk$. In §0.1, we define the category of graded modules $gr\,\mathcal{M}od$ as the category formed by $\Bbbk$-modules K equipped with a fixed splitting $K = \bigoplus_{n\in\mathbb{Z}} K_n$. A morphism of graded modules is a morphism of $\Bbbk$-modules $f : K \to L$ such that $f(K_n) \subset L_n$, for all $n \in \mathbb{Z}$. We say that an element $x \in K$ is homogeneous when we have $x \in K_n$ for some integer $n \in \mathbb{Z}$, which defines the degree $\deg(x) = n$ of this element x.

The purpose of this appendix is to explain the definition of our symmetric monoidal structure on graded modules. By the way, we also check the existence of graded hom-objects $\mathtt{Hom}_{gr\,\mathcal{M}od}(-,-) : gr\,\mathcal{M}od^{op} \times gr\,\mathcal{M}od \to gr\,\mathcal{M}od$ defining an internal hom in this monoidal category $gr\,\mathcal{M}od$.

1.3.1. *The symmetric monoidal structure of graded modules.* The tensor product of $K, L \in gr\,\mathcal{M}od$ in the category of graded modules is the tensor product of K and L as $\Bbbk$-modules, which we equip with the decomposition $K\otimes L = \bigoplus_{n\in\mathbb{Z}}(K\otimes L)_n$ such that $(K \otimes L)_n = \bigoplus_{p+q=n} K_p \otimes L_q$. This construction obviously gives a bifunctor $\otimes : gr\,\mathcal{M}od \times gr\,\mathcal{M}od \to gr\,\mathcal{M}od$ with the ground ring $\Bbbk$ regarded as a graded module concentrated in degree 0 as unit object and the associativity isomorphism $(K \otimes L) \otimes M \simeq K \otimes (L \otimes M)$ inherited from $\Bbbk$-modules.

We also have an obvious symmetry isomorphism, inherited from $\Bbbk$-modules, but we actually modify this basic symmetry isomorphism in order to implement the signs of dg-algebra in the category of graded modules.

We precisely define our symmetry isomorphism $c : K \otimes L \to L \otimes K$ by the formula $c(x \otimes y) = (-1)^{pq} y \otimes x$, for any pair of homogeneous elements $x \in K_p$ and $y \in L_q$, where we consider the sign $(-1)^{pq}$ determined from the rules of §0.2. Following the general convention of §0.2, we generally simply add the symbol $\pm$ to mark the occurrence of such a sign arising from a permutation of homogeneous elements. In general, there is no need to make this sign explicit.

The whole construction of this paragraph gives the definition of the symmetric monoidal structure on graded modules.

We immediately see that the tensor product of graded modules satisfies the colimit requirement §0.6(d). We mention in §0.11 that this extra condition is related to the existence of an internal hom in the category of graded modules. We make this internal hom explicit in the next paragraph.

1.3.2. *The internal hom of graded modules.* We basically define the internal hom of graded modules $L, M \in gr\,\mathcal{M}od$ as the graded module $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M)$

spanned in degree n by the morphisms of $\Bbbk$-modules $f : L \to M$ such that $f(L_p) \subset L_{p+n}$. Thus, we set $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M)_n = \prod_p \mathtt{Hom}_{gr\,\mathcal{M}od}(L_p, M_{p+n})$, for each $n \in \mathbb{Z}$. The adjunction relation $\mathtt{Mor}_{gr\,\mathcal{M}od}(K \otimes L, M) \simeq \mathtt{Mor}_{gr\,\mathcal{M}od}(K, \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M))$ easily follows from the adjunction relation of $\Bbbk$-modules. Note that a morphism of graded modules is identified with a homomorphism of degree 0, where according to the conventions of §0.10, we use the term of homomorphism to refer to an element of the graded hom $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M)$.

In §0.10, we mention that, for general reasons, the internal hom-objects of a symmetric monoidal category inherit a composition product, an internal tensor product operation, and an evaluation action on the objects of the category. In the context of graded modules, the evaluation morphism is identified with the morphism of graded modules $\epsilon : \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M) \otimes L \to M$ mapping any tensor $f \otimes x$, where $f \in \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M)$, $x \in L$, to the element $f(x) \in M$ defined by applying the $\Bbbk$-module map $f : L \to M$ to $x \in L$. Note that $\mathtt{Hom}_{gr\,\mathcal{M}od}(L, M) \otimes L$ refers to the tensor product of graded modules in this construction. The composition product $\circ : \mathtt{Hom}_{gr\,\mathcal{M}od}(L, M) \otimes \mathtt{Hom}_{gr\,\mathcal{M}od}(K, L) \to \mathtt{Hom}_{gr\,\mathcal{M}od}(K, M)$ is induced by the obvious composition operation on $\Bbbk$-module morphisms. The tensor product operation $\otimes : \mathtt{Hom}_{gr\,\mathcal{M}od}(K, L) \otimes \mathtt{Hom}_{gr\,\mathcal{M}od}(M, N) \to \mathtt{Hom}_{gr\,\mathcal{M}od}(K \otimes M, L \otimes N)$ maps (homogeneous) homomorphisms $f : K \to L$ and $g : M \to N$ to the homomorphism $f \otimes g : K \otimes L \to M \otimes N$ such that $(f \otimes g)(x \otimes y) = \pm f(x) \otimes g(y)$, for any pair of (homogeneous) elements $x \in K$ and $y \in L$, where the sign $\pm$ is produced by the commutation of g and x.

CHAPTER 2

Braids and the Recognition of E_2-operads

Recall that an operad P is E_n when we have weak-equivalences of topological operads $\mathsf{P} \overset{\sim}{\leftarrow} \cdot \overset{\sim}{\rightarrow} \mathsf{D}_n$ connecting P to the operad of little n-discs D_n. In this situation, we also say that P is weakly-equivalent to D_n. In many applications the issue is to prove that a given operad P is E_n. The usual method is to apply an appropriate recognition criterion, building the required weak-equivalences from internal structures of P.

In the previous chapter, we observed that a topological operad P is E_1 if only if each space $\mathsf{P}(r)$ has contractible components which form an operad in sets $\pi_0\,\mathsf{P}$ isomorphic to the operad of associative monoids As. This criterion actually implies that P is weakly-equivalent to the set operad As, viewed as a discrete operad in topological spaces. The weak-equivalence with the little 1-discs operad follows from the observation that the operad D_1 is itself weakly-equivalent to As. Similarly, we observed that a topological operad P is E_∞ if only if each space $\mathsf{P}(r)$ is contractible. This criterion actually implies that P is weakly-equivalent to the discrete set operad of commutative monoids Com. The weak-equivalence with D_∞ follows, again, from the observation that D_∞ consists itself of contractible spaces and is itself weakly-equivalent to Com.

The main objective of this chapter is to explain a similar characterization of E_2-operads, given by the work of Fiedorowicz [34].

We consider the universal coverings $\widetilde{\mathsf{D}}_2(r)$ of the spaces of little 2-discs $\mathsf{D}_2(r)$. We start with the observation that each space $\mathsf{D}_2(r)$ is an Eilenberg-MacLane space such that $\pi_1\,\mathsf{D}_2(r) = P_r$, where P_r denotes the pure braid group on r strands. We deduce from this observation that $\widetilde{\mathsf{D}}_2(r)$ is contractible and comes equipped with an action of the braid group P_r so that $\widetilde{\mathsf{D}}_2(r)/P_r = \mathsf{D}_2(r)$. The idea is to characterize E_2-operads from structures defined at the level of these contractible spaces $\widetilde{\mathsf{D}}_2(r)$.

Following Fiedorowicz, we have to consider a braided version of the classical symmetric operads of §I, because we deal with objects equipped with braid group actions instead of symmetric group actions. Indeed, the already considered action of P_r on $\widetilde{\mathsf{D}}_2(r)$ extends to an action of the full braid group B_r which also lifts the action of the symmetric group Σ_r on the little 2-disc space $\mathsf{D}_2(r)$. The collection of covering spaces $\widetilde{\mathsf{D}}_2 = \{\widetilde{\mathsf{D}}_2(r)\}_r$ inherits, on the other hand, a plain operadic composition structure from the litte 2-discs spaces $\mathsf{D}_2(r)$. Thus, to get the full structure of the collection of covering spaces $\widetilde{\mathsf{D}}_2 = \{\widetilde{\mathsf{D}}_2(r)\}_r$, we only have to change the symmetric group actions in the definition of an operad into braid group actions. This observation gives the basis of Fiedorowicz's construction of E_2-operads.

We mainly apply Fiedorowicz's method to check that the classifying spaces of a certain operad in groupoids, the operad of colored braids, forms an instance of E_2-operad.

In a preliminary section §2.0, we give a survey of the definition of braid groups. In §2.1, we explain the definition of a braided operad and we state Fiedorowicz's recognition criterion. In §2.2, we introduce the operad of colored braids, and we explain our construction of a model of E_2-operad from the classifying spaces of this operad in groupoids. In §2.3, we explain that the operad of colored braids is equivalent to an operad in groupoids which we naturally obtain by applying the fundamental groupoid construction to the underlying spaces of the little 2-discs operad. The goal of this observation is essentially to give a complement on the results of the previous sections. In a concluding section §2.4, we give a brief introduction to more general recognition theorems, aiming to give similar characterizations of E_n-operads for all $n \geq 1$.

Throughout this chapter, we adopt the plan of §1 each time we deal with E_2-operads: we address constructions in the non-unitary setting first; and we observe afterwards that our definitions make sense in degenerate situations involving unitary operations, so that each construction of the present chapter has an obvious extension to unitary operads.

In the next chapter, we consider braids equipped with an extra structure, the parenthesization, already mentioned in the book introduction. The addition of this structure does not change the homotopy type of our classifying spaces, but is required for the definition of the Grothendieck-Teichmüller group. Therefore, we put off the introduction of these parenthesizations until the next chapter, when we tackle this subject.

The ideas and results of §§2.1-2.2 are, as we mentioned, mostly borrowed from [34]. The preprint [95] essentially provides a generalization of this approach for the recognition of operads made from Eilenberg-MacLane spaces. In §2.3, we sketch another (independent) approach of similar results, involving the adjunction between classifying spaces and fundamental groupoids, which we intend to use for the definition of Quillen's model of operads in subsequent works. In short, our approach provides an appropriate setting for an extension of constructions of [77] to operads (see also the Malcev completion process considered in the next chapter).

2.0. Braid groups

In the previous chapter, we introduce the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ as a suitable model of the little n-discs spaces $\mathsf{D}_n(r)$, which we use to perform cohomology and homology computations. In passing, we observed that the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^1, r)$, where $n = 1$, have contractible connected components indexed by the permutations of $(1, \ldots, r)$, just as the little 1-discs spaces $\mathsf{D}_1(r)$. Let us begin this chapter with the following preliminary observation about the homotopy of the spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ in the case $n > 1$:

Proposition 2.0.1. *The spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ are connected for all $n > 1$. If $n > 2$, then we have $\pi_1 \mathsf{F}(\mathring{\mathbb{D}}^n, r) = 0$ too. If $n = 2$, then we have in contrast $\pi_* \mathsf{F}(\mathring{\mathbb{D}}^2, r) = 0$, for all $* \neq 1$.*

Proof. In the previous chapter, we recall, by referring to [25], that the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n \setminus \{b_1, \ldots, b_m\}, r)$, where $\{b_1, \ldots, b_m\}$ is any set of punctures,

are connected by fibrations

$$\mathsf{F}(\mathring{\mathbb{D}}^n \setminus \{b_1, \ldots, b_m\}, r) \xrightarrow{f} \mathsf{F}(\mathring{\mathbb{D}}^n \setminus \{b_1, \ldots, b_m\}, r-1).$$

These maps are defined by the projections $f(a_1, \ldots, a_{r-1}, a_r) = (a_1, \ldots, a_{r-1})$, which forget the last element a_r of our configurations $(a_1, \ldots, a_{r-1}, a_r)$. The idea is to deduce the proposition from an inspection of the homotopy exact sequences associated to these fibrations (in the particular case $m = 0$).

Let $(a_1, \ldots, a_r) \in \mathsf{F}(\mathring{\mathbb{D}}^n, r)$. We use the notation $\underline{a} = (a_1, \ldots, a_m)$ for any $m \leq r$ in order to refer to the base points of the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, m)$ which we extract from our sequence.

The homotopy exact sequence associated to our fibration $f : \mathsf{F}(\mathring{\mathbb{D}}^n, r) \to \mathsf{F}(\mathring{\mathbb{D}}^n, r-1)$ reads

$$\begin{aligned}
&\cdots \to \pi_*(f^{-1}(\underline{a}), a_r) \to \pi_*(\mathsf{F}(\mathring{\mathbb{D}}^n, r), \underline{a}) \xrightarrow{f_*} \pi_*(\mathsf{F}(\mathring{\mathbb{D}}^n, r-1), \underline{a}) \to \cdots \\
&\cdots \to \pi_1(f^{-1}(\underline{a}), a_r) \to \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^n, r), \underline{a}) \xrightarrow{f_*} \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^n, r-1), \underline{a}) \to \underbrace{\pi_0(f^{-1}(\underline{a}), a_r)}_{=*},
\end{aligned}$$

where $f^{-1}(\underline{a})$ refers to the fiber of f at $\underline{a} = (a_1, \ldots, a_{r-1}) \in \mathsf{F}(\mathring{\mathbb{D}}^n, r-1)$. This fiber is identified with the punctured space $\{(a_1, \ldots, a_{r-1}, a) \in \mathring{\mathbb{D}}^n | a \neq a_1, \ldots, a_{r-1}\} = \mathring{\mathbb{D}}^n \setminus \{a_1, \ldots, a_{r-1}\}$, which is connected as long as $n > 1$. Hence, we have the identity $\pi_0(f^{-1}(\underline{a}), a_r) = *$ as noted in the above sequence.

The connectedness of the fiber $f^{-1}(\underline{a})$ implies, by an easy induction on r, that the spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ are connected for all $n > 1$. In the case $n > 2$, we have besides $\pi_1(f^{-1}(a_1, \ldots, a_{r-1}), a_r) = \pi_1(\mathring{\mathbb{D}}^n \setminus \{a_1, \ldots, a_{r-1}\}, a_r) = *$, and by an immediate induction, we deduce from the degree 1 terms of the homotopy exact sequence that the spaces $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ are simply connected too, for all $r > 0$. In the case $n = 2$, we have $\pi_*(f^{-1}(a_1, \ldots, a_{r-1}), a_r) = \pi_*(\mathring{\mathbb{D}}^2 \setminus \{a_1, \ldots, a_{r-1}\}, a_r) = *$ for $* > 1$, and we use the higher terms of the homotopy exact sequence to conclude that $\pi_*(\mathsf{F}(\mathring{\mathbb{D}}^2, r), \underline{a})$ vanishes for all $* > 1$. $\square$

The result of the proposition obviously holds for the little disc spaces $\mathsf{D}_n(r)$ since we have a homotopy equivalence $\omega : \mathsf{D}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^n, r)$ (see Proposition 1.2.2) which induces an isomorphism on homotopy groups. Briefly recall that this homotopy equivalence, which we call the disc center mapping, sends an r-tuple of little n-discs $\underline{c} = (c_1, \ldots, c_n)$, defining an element of $\mathsf{D}_n(r)$, to the configuration of the disc centers $c_i(0, \ldots, 0) \in \mathring{\mathbb{D}}^n$. In this chapter, we heavily use this process to deduce results on the little 2-discs spaces $\mathsf{D}_2(r)$ from structure results on the fundamental group of the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$.

First of all, the previous proposition implies that the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$, and hence the little 2-disc spaces $\mathsf{D}_2(r)$, are Eilenberg-MacLane spaces $K(P_r, 1)$, where we set $P_r = \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r), *)$. This group P_r is the pure braid group on r strands.

The purpose of this preliminary section is to recall the definition of braid groups and the usual representation of their elements in terms of braid diagrams. We will go back to the subject of Eilenberg-MacLane spaces in subsequent sections, where we explain the application of braid groups to little 2-discs operads.

2.0.2. *Braid groups.* The fundamental group of $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ is the pure braid group P_r. In what follows, we rather deal with the full braid group B_r, which includes P_r as a distinguished subgroup. This group B_r can be defined as follows.

The space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ is equipped with an action of the symmetric group Σ_r given by the standard formula

$$w_*(a_1, \ldots, a_r) = (a_{w^{-1}(1)}, \ldots, a_{w^{-1}(r)}),$$

on any element $(a_1, \ldots, a_r) \in \mathsf{F}(\mathring{\mathbb{D}}^2, r)$, and for all permutations $w \in \Sigma_r$ (see Proposition 1.2.3). The standard braid group on r strands B_r is precisely defined as the fundamental group of the quotient of the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ under this action:

$$B_r = \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r, *).$$

The quotient map $q : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ induces a morphism $q_* : P_r \to B_r$.

To understand the connection between these groups, we use the following observation, whose proof reduces to a straightforward verification:

Lemma 2.0.3. *The symmetric group Σ_r acts freely and properly on $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ so that the quotient map $q : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ defines a covering map.* □

Then we apply standard results of covering theory to obtain:

Proposition 2.0.4. *The morphism $q_* : P_r \to B_r$ fits in an exact sequence of groups $1 \to P_r \xrightarrow{q_*} B_r \xrightarrow{p_*} \Sigma_r \to 1$, where $p_* : B_r \to \Sigma_r$ is deduced from the action of $B_r = \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r, *)$ on the fiber of the covering $q : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ at any base point $* \in \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$.* □

2.0.5. *Braids and braid diagrams.* The braids, giving the name of braid groups, come from a representation of paths on the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ and from a representation of the corresponding homotopy classes defining the elements of our fundamental groups. In our context, we more naturally consider braids defined in the cylinder $\mathring{\mathbb{D}}^2 \times [0, 1]$, and in a first stage, we fix equidistant contact points

$$(x_k^0, 0, 0), (x_k^0, 0, 1), \quad \text{so that } x_k^0 = -1 + (2k-1)/(r+1),\ k = 1, \ldots, r,$$

on the axis $y = 0$ in the boundary discs $\mathring{\mathbb{D}}^2 \times \{0\}$, $\mathring{\mathbb{D}}^2 \times \{1\}$, of that cylinder $\mathring{\mathbb{D}}^2 \times [0, 1]$. In the literature, authors more usually deal with braids in the euclidean plan $\mathbb{R}^2$ rather than in the open 2-disc $\mathring{\mathbb{D}}^2$, but we can use the standard homeomorphism between these spaces to transport any usual construction or result to our setting.

By definition [4], a braid with r strands is defined as a collection of r disjoint arcs $\alpha_i : [0, 1] \to \mathring{\mathbb{D}}^2 \times [0, 1]$, $i = 1, \ldots, r$, of the form

$$\alpha_i(t) = (x_i(t), y_i(t), t), \quad t \in [0, 1],$$

and so that $\alpha_i(t^0) \in \{(x_k^0, 0, t^0) | k = 1, \ldots, r\}$ when we take the origin $t^0 = 0$ and end-point $t^0 = 1$.

The requirement that the arcs α_i are disjoint amounts to the relation $(x_i(t), y_i(t)) \neq (x_j(t), y_j(t))$ for all $i \neq j$ and every $t \in [0, 1]$. In the case $t^0 = 0, 1$, this assumption implies that the r-tuple $(\alpha_1(t^0), \ldots, \alpha_r(t^0)) = ((x_1(t^0), 0, t^0), \ldots, (x_r(t^0), 0, t^0))$ forms a permutation of $((x_1^0, 0, t^0), \ldots, (x_r^0, 0, t^0))$. The mapping $s : k \mapsto s(k)$ which we read from the relations

$$x_i(0) = x_k^0, \quad x_i(1) = x_{s(k)}^0, \quad \text{for } i = 1, \ldots, r,$$

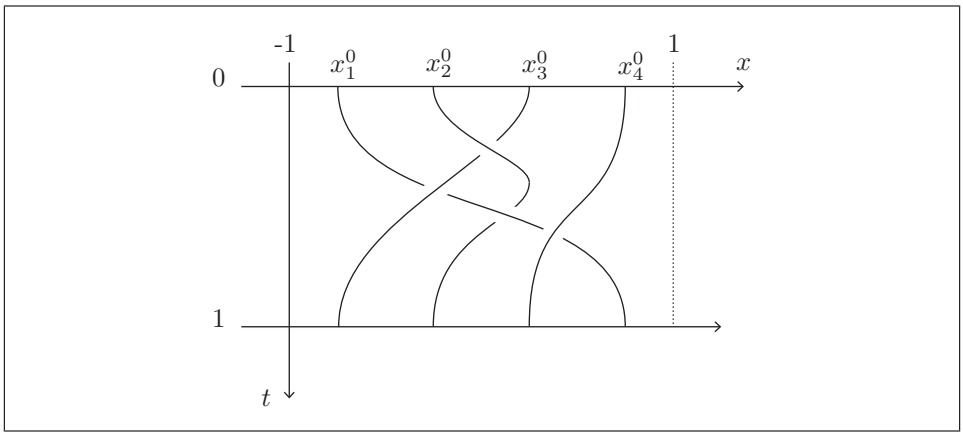


FIGURE 2.1. An instance of braid diagram. In the next pictures, we generally do not specify the abscissa x_i^0 of the contact points. We simply mark contact points by the corresponding indices i when we have to give this information.

defines a permutation $s \in \Sigma_r$ naturally associated to our braid. Following a standard convention, we also refer to this permutation s as the underlying permutation of the braid α. The braids associated to the identity permutation define the set of pure braids.

For the moment, we take the convention that the strand collection of a braid $\alpha = (\alpha_1, \ldots, \alpha_r)$ is ordered so that $\alpha_i(0) = (x_i^0, 0, 0)$, for all $i = 1, \ldots, r$, and we refer to the arc α_i as the ith strand of α. In this setting, we have $(\alpha_1(1), \ldots, \alpha_1(1)) = ((x_{s(1)}^0, 0, 1), \ldots, (x_{s(r)}^0, 0, 1))$, where $s \in \Sigma_r$ is the permutation associated to the braid α. In §2.2, we consider braids equipped with additional structures, for which the above ordering $\alpha = (\alpha_1, \ldots, \alpha_r)$ is not natural.

In most situations, we perform a projection onto the plan (x, t) in order to obtain a convenient representation of our braids. Figure 2.1 gives an instance of such a representation for a braid on 4 strands which has

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

as underlying permutation. The projection picture works for braids such that the intersection between the projected arcs $(x_i(t), t)$ reduce to isolated points, and so that each intersection $(x_i(t), t) = (x_j(t), t)$ involves no more than two arcs $(x_i(t), t)$, $(x_j(t), t)$. In this context, the habitual practice is to insert a gap at each intersection point $(x_i(t), t) = (x_j(t), t)$, as in the example of Figure 2.1, in order to mark the strand going under the other with respect to the y-coordinate. Such a figure is called a braid diagram.

In the next paragraph, we recall the definition of the isotopy relation between braids. The notion of isotopy can be formalized in terms of braid diagrams, and one can prove that braid diagrams are enough to give a faithful picture of braids up to isotopy. This observation is originally due to E. Artin, and we refer to his article [4], or to the subsequent textbook [55] by C. Kassel and V. Turaev,

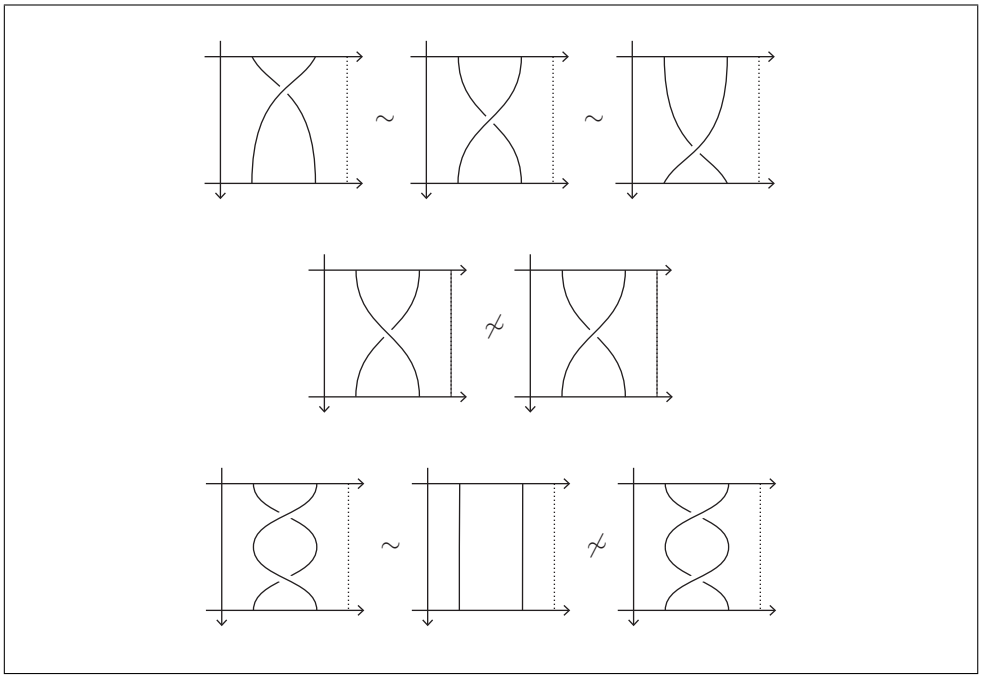


FIGURE 2.2. Basic examples of isotopic and non-isotopic braids

for more explanations about the relationship between braids and braid diagrams. In what follows, we just use braid diagrams informally, in order to illustrate our constructions.

2.0.6. *Braid isotopies.* By definition, an isotopy from a braid α to another one β is a continuous family of braids h_s such that $h_0 = \alpha$ and $h_1 = \beta$. Two braids are isotopic if we have an isotopy between them, and in this case we write $\alpha \sim \beta$. The isotopy relation is clearly an equivalence relation on the set of braids. Figure 2.2 gives simple instances of braid isotopies and fundamental examples of non-isotopic braids.

Let us regard a braid as a single map $\alpha(t) = (\alpha_1(t), \ldots, \alpha_r(t))$ rather than a collection. The assumption that the underlying braids of an isotopy h_s form a continuous family amounts to the requirement that the two parameter map $h : (s,t) \mapsto h_s(t)$ is continuous over $[0,1] \times [0,1]$. By continuity, the requirement that $h_s(1)$ belongs to the discrete space $\{((x^0_{w(1)}, 0, 1), \ldots, (x^0_{w(k)}, 0, 1)) | w \in \Sigma_r\}$ implies that the map $s \mapsto h_s(1)$, given by the endpoints of the isotopy, is constant. Hence, we see that isotopic braids have the same underlying permutation.

By a standard abuse of language, we may use the term of braid to refer to an isotopy class of braids as soon as the context is sufficient to avoid confusion.

2.0.7. *Relationship with the fundamental groups.* We immediately see that a pure braid on r-strands $\alpha_i(t) = (x_i(t), y_i(t), t)$ is equivalent to a based loop $\gamma(t) = ((x_1(t), y_1(t)), \ldots, (x_r(t), y_r(t)))$ in the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$, where we take $\underline{a}^0 = ((x^0_1, 0), \ldots, (x^0_r, 0))$, with $x_i = -1 + (2i-1)/(r+1)$, as base point. Similarly, an isotopy of pure braids is equivalent to a homotopy of based loops in $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$.

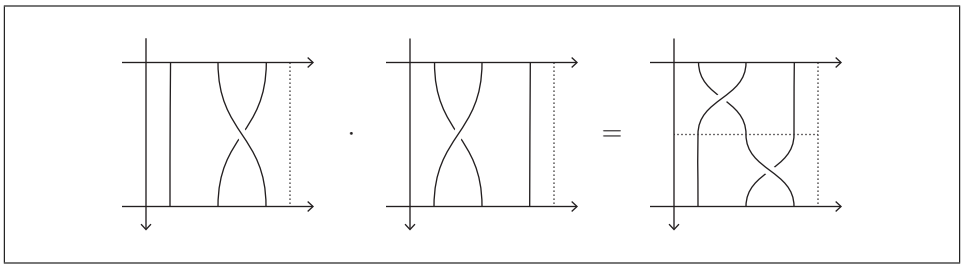


FIGURE 2.3. A braid concatenation

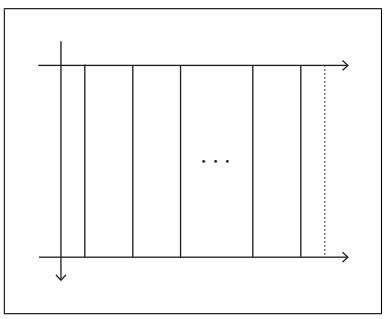


FIGURE 2.4. The identity braid

Thus, the pure braid group P_r, which we define as the fundamental group of the space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$, is identified with the set of isotopy classes of pure braids.

Let $\underline{b}^0 = q(\underline{a}^0)$ be the image of the element $\underline{a}^0 = ((1,0),\dots,(r,0))$ in the quotient space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$. The fiber of this point $\underline{b}^0$ under the covering map $q : \mathsf{F}(\mathring{\mathbb{D}}^2, r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ is $q^{-1}(\underline{b}^0) = \{((x^0_{w(1)}, 0),\dots,(x^0_{w(1)}, 0)),\ w \in \Sigma_r\}$. The set of all braids on r strands is identified with the set of paths connecting $\underline{a}^0$ to another point $w\underline{a}^0 = ((x^0_{w(1)}, 0),\dots,(x^0_{w(1)}, 0))$ in this fiber. Braid isotopies are also equivalent to path homotopies. By standard results of covering theory, any loop γ based at $\underline{b}^0$ in the quotient space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$ lifts to a path of this form $\tilde{\gamma}$, with $\tilde{\gamma}(0) = \underline{a}^0$ and $\tilde{\gamma}(1) = w\underline{a}^0$, for some $w \in \Sigma_r$. Moreover, such a lifting is unique once we fix the starting point $\tilde{\gamma}(0) = \underline{a}^0$, and any homotopy of based loops lifts to a path homotopy. Hence, we obtain that the full braid group B_r, which we define as the fundamental group of the quotient space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r$, is identified with the set of isotopy classes of all braids.

In both cases P_r and B_r, the group multiplication can readily be identified with a natural concatenation operation on braids, of which the Figure 2.3 gives an example. The unit element with respect to this group multiplication is given by the identity braid, represented in Figure 2.4. (In what follows, we also use the notation *id* to refer to this braid, because we soon identify braids with the morphisms of a category, in which the unit braid represents an identity morphism.) Note that we perform compositions downwards, in the increasing direction of the t coordinates, in contrast with conventions adopted by other authors. Our choice is more natural

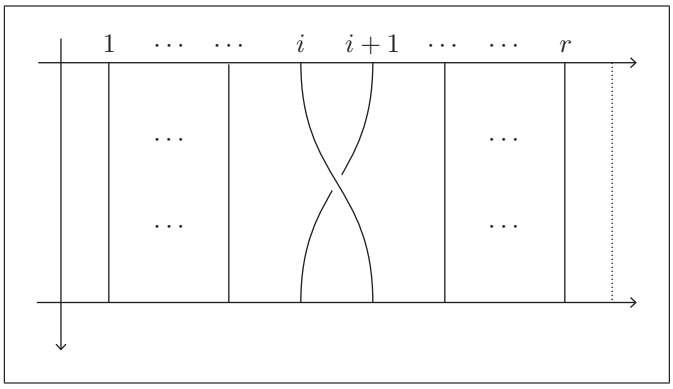


FIGURE 2.5. The generating braids

when we regard braids as morphisms oriented from a source to a target object, and we heavily use this interpretation next.

In Proposition 2.0.3, we refer to a general result of covering theory in order to define the morphism $p_* : B_r \to \Sigma_r$. By going back to the proof of this result, we immediately see that the morphism $p_* : B_r \to \Sigma_r$ is identified with the map sending the isotopy class of a braid α to its underlying permutation s. The natural embedding of the subset of pure braids into the set of all braids gives the morphism $q_* : P_r \to B_r$. Thus we have a full interpretation of the exact sequence of groups $1 \to P_r \to B_r \to \Sigma_r \to 1$ in terms of isotopy classes of braids.

2.0.8. *Generating elements.* For $i = 1, \ldots, r-1$, we consider the element $\tau_i \in B_r$ represented by the diagram of Figure 2.5

The mapping $q_* : B_r \to \Sigma_r$ assigns the elementary transposition $t_i = (i\ i+1) \in \Sigma_r$ to this braid $\tau_i \in B_r$. In §0.7, we recall that the symmetric group has a simple presentation by generators and relations involving these transpositions t_i, $i = 1, \ldots, r-1$, as generating elements. For the braid group, we have the following classical result:

THEOREM 2.0.9 (see [3]). *The braid group B_r is generated by the elements τ_i, $i = 1, \ldots, r-1$, and has the commutation relations*

$$\tau_i \tau_j = \tau_j \tau_i, \quad \textit{for } i, j = 1, \ldots, r-1 \textit{ such that } |i-j| \geq 2,$$

together with the braid relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \textit{for } i = 1, \ldots, r-2,$$

as generating relations (see also the representation of these relations in Figure 2.6). □

In other words, the braid group B_r is given by the same presentation as the symmetric group Σ_r, except that we drop the involution relation $t_i^2 = 1$ associated to transpositions. The idea of this result goes back to the work of E. Artin [3] cited in reference. We refer to [15], [36], and [55] for various proofs of the theorem.

The inverse of a generator τ_i in the braid group can actually be obtained by switching the disposition of the strands in the representation of Figure 2.5 (the $i+1$th strand comes in the foreground and ith strand goes in the background). The case $r = 2$, where the braid is reduced to these overlapping strands, has already been represented in Figure 2.2, to give an example of non-isotopic braids.

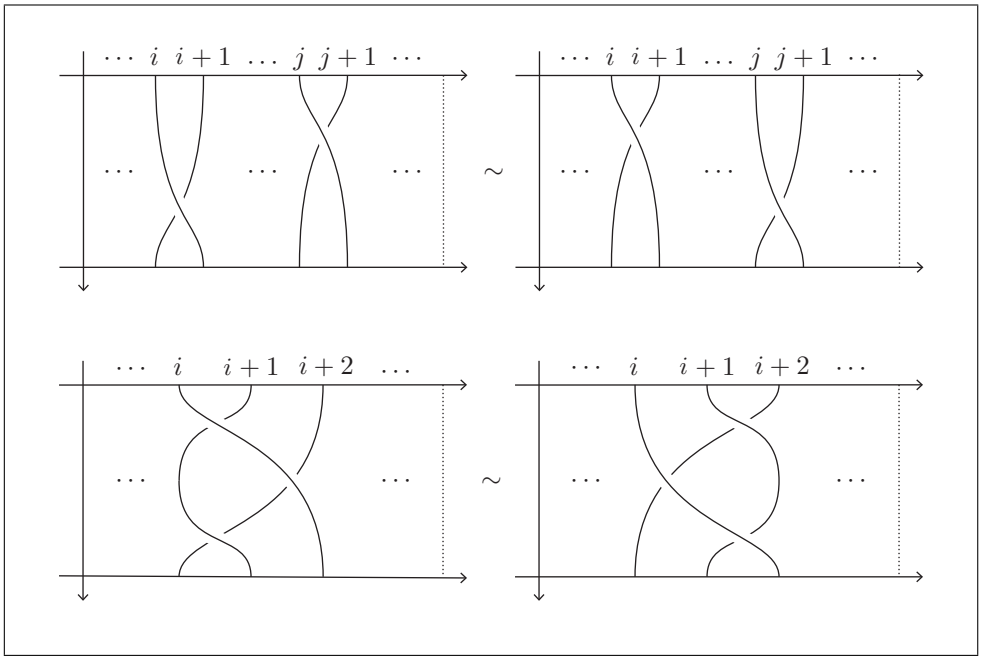


FIGURE 2.6. The commutation and braid relations in braid groups.

2.0.10. *Change of contact points.* In the definition of §2.0.5, we assume that the origin points of a braid belong to the subset $\{(x_k^0, 0, 0)|k = 1, \ldots, r\}$, where $x_k^0 = -1 + (2k-1)/(r+1)$, and the end points belong to the subset $\{(x_k^0, 0, 1)|k = 1, \ldots, r\}$. Equivalently, our braids correspond to paths in the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ starting at the element $((x_1^0, 0), \ldots, (x_r^0, 0))$ and ending at a permutation $((x_{w(1)}^0, 0), \ldots, (x_{w(r)}^0, 0))$ of this base configuration $((x_1^0, 0), \ldots, (x_r^0, 0))$.

In principle, we obtain isomorphic groups if we replace the chosen configuration $((x_1^0, 0), \ldots, (x_r^0, 0))$ by any other base point $((a_1, b_1), \ldots, (a_r, b_r))$ of the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ in the definition. Equivalently, we may assume that the contact points of a braid are given by any fixed subset $\{(a_k, b_k, t^0)|k = 1, \ldots, r\}$ in the plans $t^0 = 0, 1$. The definition of such the isomorphism comparing such contact points to the canonical ones involves the choice of a path $\gamma(t) = (\gamma_1(t), \ldots, \gamma_r(t))$ going from $\gamma(0) = ((x_1^0, 0), \ldots, (x_r^0, 0))$ to $\gamma(1) = ((a_1, b_1), \ldots, (a_r, b_r))$ in $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$. Formally, we use the arc γ and the corresponding permutations of this path to go from one base point configuration to the other one.

Note however that this just defined isomorphism is not canonical and depends on the homotopy class of the path γ. In subsequent constructions, we will implicitly use such changes of base points, but we also need a strict control of the isomorphism involved in the operation. For this aim, we restrict ourselves to base configurations of the form $((a_1, 0), \ldots, (a_r, 0))$, where all points lie on the line $y = 0$, and so that $a_1 < \cdots < a_r$. Equivalently, we only consider base configurations arising from a configuration $(a_1, \ldots, a_r)$ in the equatorial 1-disc $\mathring{\mathbb{D}}^1 \subset \mathring{\mathbb{D}}^2$, and belonging to the connected component of $(1, \ldots, r)$ within the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^1, r)$. Since

$\mathsf{F}(\mathring{\mathbb{D}}^1, r)$ has contractible connected components, all paths $\gamma(t) = (\gamma_1(t), \ldots, \gamma_r(t))$ going from one such configuration to another in the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^1, r)$ are homotopic and hence, induce the same isomorphism at the fundamental group level. Thus, all choices of contact points on the the line $y = 0$ yields the same braid group up to a canonical and well determined isomorphism.

2.0.11. *Degenerate cases.* We should note that the definition of the braid group B_r makes sense for $r = 0$. We then deal with a degenerate situation of braid with an empty set of strands. We therefore have $B_0 = *$ for formal reasons. We can also identify this group B_0 with the fundamental group of the one-point set $*$ considered in the unitary extension of the sequence of configurations spaces.

The braid group B_1 is also trivial (like the symmetric group Σ_1), with the isotopy class of a one-strand vertical braid as single element.

2.1. Braided operads and $\mathbf{E_2}$-operads

Let $\widetilde{\mathsf{D}}_2(r)$ be the universal coverings of the spaces of little 2-discs $\mathsf{D}_2(r)$. The main purpose of this section is to prove, following [34], that the collection of spaces $\widetilde{\mathsf{D}}_2(r)$ inherits a braided operad structure from the little 2-discs. The main application of this construction, as we explained in the chapter introduction, is a simple characterization of E_2-operads from associated contractible braided operads. We conclude the section by the proof of this recognition theorem.

In a preliminary stage, we give a general definition of the notion of braided operad. The plan of this definition parallels the definition of a symmetric operad in §I.1.1. To summarize, the idea consists in replacing the symmetric group actions in that definition by braids group actions.

2.1.1. *Braided operads.* Explicitly, a braided operad P in a base category $\mathcal{B}ase$ consists of a sequence of objects $\mathsf{P}(r) \in \mathcal{B}ase$, $r \in \mathbb{N}$, where $\mathsf{P}(r)$ is now equipped with an action of the braid group B_r, together with

(a) a unit morphism $\eta : \mathbb{1} \to \mathsf{P}(1)$
(b) and composition products $\mu : \mathsf{P}(r) \otimes \mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_r) \to \mathsf{P}(n_1 + \cdots + n_r)$, given for every $r \geq 0$, and all $n_1, \ldots, n_r \geq 0$,

which, all together, satisfy equivariance, unit and associativity relations. These relations are shaped on the same diagrams as in the case of symmetric operad. We therefore refer to Figure I.1-I.3 in §I for the form of these diagrams. In the case of braided operads, we simply have to consider the action of braids $\alpha \in B_r$ (respectively, $\beta_1 \in B_{n_1}, \ldots, \beta_r \in B_{n_r}$) instead of permutations $s \in \Sigma_r$ (respectively, $t_1 \in \Sigma_{n_1}, \ldots, t_r \in \Sigma_{n_r}$) in the equivariance diagram of Figure I.1, and similarly in the variant of Figure I.4. In the braided context, the definition of the composite elements $\alpha_*(n_1, \ldots, n_r)$ (respectively, $\beta_1 \oplus \cdots \oplus \beta_r$) occurring in these equivariance relations arises from the following statement:

Proposition 2.1.2. *Let $r \in \mathbb{N}$. Let $n_1, \ldots, n_r \in \mathbb{N}$.*

(a) *The direct sum of permutations, regarded as a mapping $\Sigma_{n_1} \times \cdots \times \Sigma_{n_r} \to \Sigma_{n_1 + \cdots + n_r}$, has a unique lifting to braid groups*

$$B_{n_1} \times \cdots \times B_{n_r} \to B_{n_1 + \cdots + n_r},$$

given by the picture of Figure 2.7 for direct sums $id \oplus \cdots \oplus \tau_k \oplus \cdots \oplus id$ involving a single generating element $\tau_k \in B_{n_i}$, and so that the following

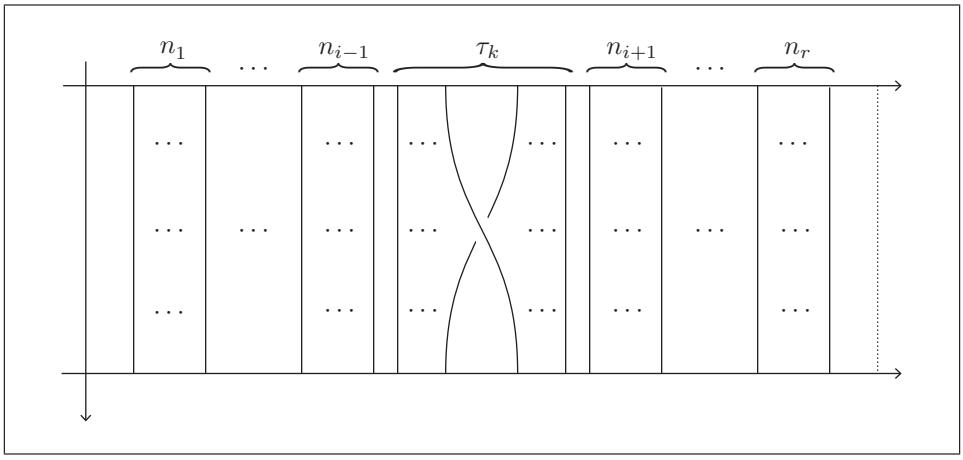


FIGURE 2.7. The direct sum $id \oplus \cdots \oplus \tau_k \oplus \cdots \oplus id$ in the braid group.

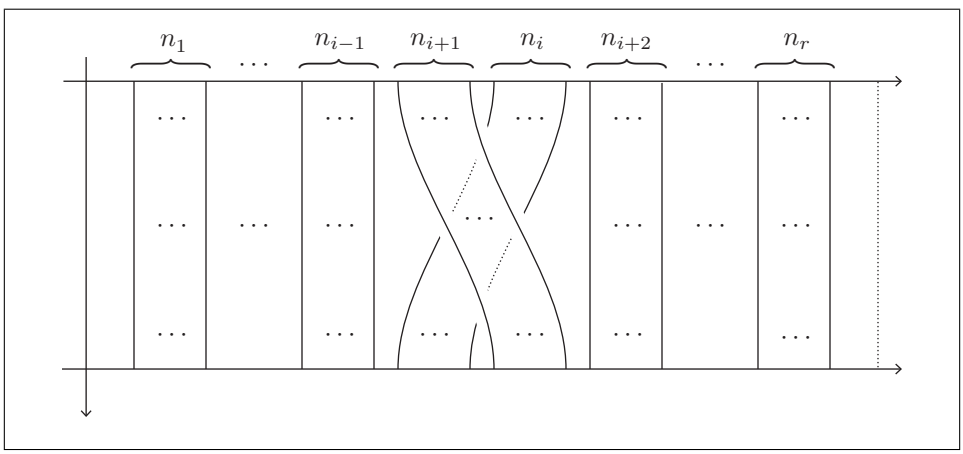


FIGURE 2.8. The block braid $(\tau_i)_*(n_1, \ldots, n_r)$

multiplication relation holds

$$(\alpha_1 \cdot \beta_1 \oplus \cdots \oplus \alpha_r \cdot \beta_r) = (\alpha_1 \oplus \cdots \oplus \alpha_r) \cdot (\beta_1 \oplus \cdots \oplus \beta_r),$$

for all $(\alpha_1, \ldots, \alpha_r), (\beta_1, \ldots, \beta_r) \in B_{n_1} \times \cdots \times B_{n_r}$.

(b) *The block permutation construction, regarded as a mapping* $\Sigma_r \to \Sigma_{n_1+\cdots+n_r}$, *has a unique lifting to braid groups*

$$B_r \to B_{n_1+\cdots+n_r},$$

given by the picture of Figure 2.8 for the generating elements $\tau_i \in B_r$, *and so that the following multiplication relation holds*

$$(\alpha \cdot \beta)_*(n_1, \ldots, n_r) = \alpha_*(n_1, \ldots, n_r) \cdot \beta_*(n_{s(1)}, \ldots, n_{s(r)}),$$

for all $\alpha, \beta \in B_r$, *and where* s *denotes the underlying permutation of* α.

(c) *Besides, we have the commutation relation*

$$\beta_1 \oplus \cdots \oplus \beta_r \cdot \alpha_*(n_1, \ldots, n_r) = \alpha_*(n_1, \ldots, n_r) \cdot \beta_{s(1)} \oplus \cdots \oplus \beta_{s(r)}$$

for all $\alpha \in B_r$, every $(\beta_1, \ldots, \beta_r) \in B_{n_1} \times \cdots \times B_{n_r}$, and where s denotes again the underlying permutation of α.

Proof. The multiplication relation implies that the liftings of (a-b) are uniquely determined by the expression of the image of generating elements. In both cases, we simply have to check that our mapping preserves generating relations in order to prove the coherence of our definition.

In (a), we have to deal with the internal generating relations of braid groups, within each factor B_{n_i}, and with the commutation relation

$$\begin{aligned}(id \oplus \cdots \oplus \tau_k \oplus \cdots \oplus id) \cdot (id \oplus \cdots \oplus \tau_l \oplus \cdots \oplus id)& \\ = (id \oplus \cdots \oplus \tau_l \oplus \cdots \oplus id) \cdot (id \oplus \cdots \oplus \tau_k \oplus \cdots \oplus id)&\end{aligned}$$

when we take generating elements in disjoint factors B_{n_i} and B_{n_j}, $i \neq j$, of the cartesian product $B_{n_1} \times \cdots \times B_{n_r}$. Our mapping visibly preserves all these identities. The case of construction (b) is addressed by a similar straightforward inspection.

The multiplication relations again implies that we are reduced to check the identity of (c) in the case where one element α, or $\beta_1, \ldots, \beta_r$ is a generating braid τ_k, and all the others are units. The relation is visibly satisfied in this case. □

The braids of Figure 2.7 and Figure 2.8 can also be defined purely algebraically, in terms of the generating elements of $B_{n_1+\cdots+n_r}$. Let $k_i = n_1 + \cdots + n_{i-1}$, $i = 1, \ldots, r$. In the case of Figure 2.7, we have:

$$id \oplus \cdots \oplus \tau_k \oplus \cdots \oplus id = \tau_{k_i+k},$$

for all $\tau_k \in B_{n_i}$. In the case of Figure 2.8, we obtain:

$$\begin{aligned}(\tau_k)_*(n_1, \ldots, n_r) = & \\ & (\tau_{k_i+n_i} \cdot \tau_{k_i+n_i-1} \cdot \ldots \cdot \tau_{k_i+1}) \cdot (\tau_{k_i+n_i+1} \cdot \tau_{k_i+n_i} \cdot \ldots \cdot \tau_{k_i+2}) \cdot \ldots \\ & \ldots \cdot (\tau_{k_i+n_i+n_{i+1}} \cdot \tau_{k_i+n_i+n_{i+1}-1} \cdot \ldots \cdot \tau_{k_i+n_{i+1}}).\end{aligned}$$

The definition of the permutation operad in Proposition I.1.8 has the following braided analogue:

Proposition 2.1.3. *The collection of braid groups B_n, $n \in \mathbb{N}$, forms a braided operad in sets so that:*

(a) *the action of the braid group on each B_n is given by left translations;*
(b) *the single element of $B_1 = \{id\}$ gives the operadic unit,*
(c) *and the composition product $\mu : B_r \times (B_{n_1} \times \cdots \times B_{n_r}) \to B_{n_1+\cdots+n_r}$ maps a collection $\alpha \in B_r$, $(\beta_1, \ldots, \beta_r) \in B_{n_1} \times \cdots \times B_{n_r}$, to the product element*

$$\alpha(\beta_1, \ldots, \beta_r) = \beta_1 \oplus \cdots \oplus \beta_r \cdot \alpha_*(n_1, \ldots, n_r)$$

in $B_{n_1+\cdots+n_r}$.

Proof. This proposition follows easily from the relations of Proposition 2.1.2. □

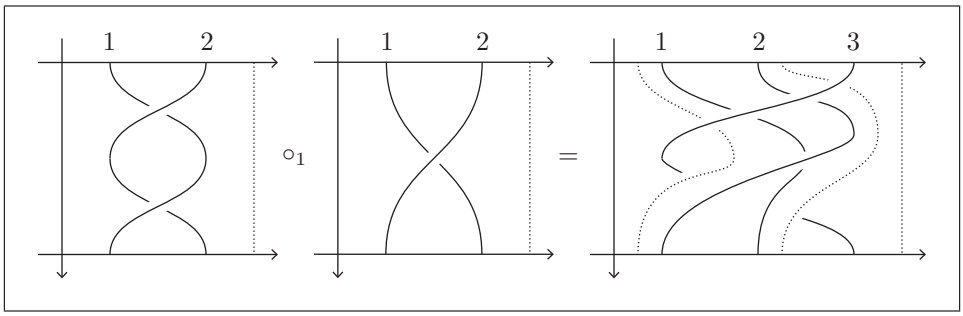


FIGURE 2.9. An operadic composite of braids

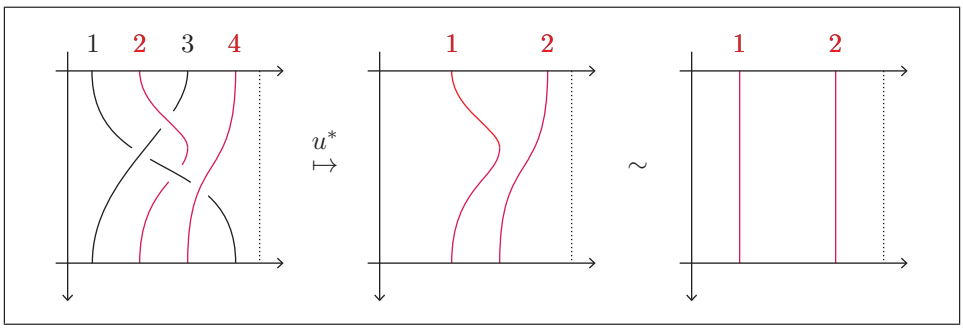


FIGURE 2.10. An instance of deletion operation in the braid operad, where the considered injection $u : \{1 < 2\} \to \{1 < 2 < 3 < 4\}$ is such that $u(1) = 2$ and $u(2) = 4$.

By convention, we include the degenerate arity 0 component B_0 in the primitive braid operad defined by this proposition (just as we do in the case of the permutation operad).

The result of §A.1, the equivalence between the plain definition of an operad and the definition in terms of partial composition operations has an obvious extension to braided operads. In the sequel, we use the definition in terms of partial composites rather than the plain definition of §2.1.3.

Let $\alpha \in B_m$, $\beta \in B_n$. To illustrate the definition, we give an instance of operadic composition of braids $\alpha \circ_i \beta = \alpha(id, \ldots, \beta, \ldots, id) \in B_{m+n-1}$ in Figure 2.9. Intuitively, the operadic composite $\alpha \circ_i \beta$ is obtained by inserting the braid β on the ith strand of the braid α. In order to ease the understanding of the picture, we have added dotted lines marking the array in which the braid β is inserted.

In the definition of the colored braid operad (§§2.2.8-2.2.11), we will implicitly use that the strands defining the composite $\alpha \circ_i \beta$ are canonically in bijection with the strands of the braid α, minus the ith one α_i, plus the strands of the braid β.

2.1.4. *Unitary braided operads and deletion operations.* The notion of a non-unitary and of a unitary operad have obvious analogues in the braided setting, and similarly as regards the notion of unitary extension of non-unitary. Furthermore, any unitary braided operad P_+ inherits deletion morphisms $u^* : \mathsf{P}_+(n) \to \mathsf{P}_+(m)$

associated to all increasing injections $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$, defined like the deletion morphisms of symmetric operads in §I.4.1, and satisfying the same relations (see Proposition I.4.2). In the formulation of the equivariance relation, we have to consider the action of braid elements $\sigma \in B_n$ instead of the action of permutations $s \in \Sigma_n$, and the braid $u^*(\sigma) \in B_m$ obtained by performing a deletion operation within the braid operad instead of a permutation $u^*(s) \in \Sigma_n$. The injection $s^*(u)$ occurring in the equivariance relation is again determined by a decomposition $s \cdot u = s_*(u) \cdot u^*(s)$, but s now refers to the underlying permutation of the considered braid σ.

The result $u^*(\alpha)$ of a deletion operation $u^* : B_n \to B_m$ in the braid groups B_n is identified with the withdrawal of strands α_k, indexed by elements $k \neq u(1), \ldots, u(m)$, which do not lie in the image of u. Figure 2.10 gives an instance of application of this deletion process for the injection $u : \{1 < 2\} \to \{1 < 2 < 3 < 4\}$ such that $u(1) = 2$ and $u(2) = 4$.

In the sequel, we mostly consider, again, the deletion operations $\partial_i : \mathsf{P}_+(r) \to \mathsf{P}_+(r-1)$ corresponding to the partial composites $\partial_i(p) = p \circ_i *$ and associated with the increasing injection $\partial^i : \{1 < \cdots < r-1\} \to \{1 < \cdots < r\}$ that jumps over i in $\{1 < \cdots < r\}$.

The components of a symmetric operad naturally inherit an action of braid groups (by restriction through the canonical morphism $p_* : B_r \to \Sigma_r$) so that any symmetric operad naturally forms a braided operad. The next proposition gives a functor in the converse direction:

Proposition 2.1.5.

(a) *Let* P *be any braided operad. Let* $\mathtt{Sym}\,\mathsf{P}(r) = \mathsf{P}(r)/P_r$. *The collection of these objects* $\mathtt{Sym}\,\mathsf{P}(r)$ *inherits, by quotient, a symmetric structure and an operadic composition structure from the braided operad* P. *Hence the collection of quotient objects* $\mathtt{Sym}\,\mathsf{P}(r) = \mathsf{P}(r)/P_r$ *forms a symmetric operad* $\mathtt{Sym}\,\mathsf{P}$ *naturally associated to* P.

(b) *The mapping* $\mathtt{Sym} : \mathsf{P} \mapsto \mathtt{Sym}\,\mathsf{P}$ *provides a left adjoint of the obvious restriction functor from symmetric operads to braided operads (the functor defined by the componentwise restriction of group actions). The collection of quotient morphisms* $\mathsf{P}(r) \to \mathsf{P}(r)/P_r$ *forms a morphism of braided operads* $\mathsf{P} \to \mathtt{Sym}\,\mathsf{P}$ *which represents the augmentation of this adjunction.*

(c) *In the case of the braid operad* $B(r) = B_r$, *we have* $\mathtt{Sym}\,B(r) = B_r/P_r = \Sigma_r$ *and the symmetric operad* $\mathtt{Sym}\,B$ *is identified with the permutation operad, as defined in Proposition I.1.8.*

(d) *The mapping* $\mathtt{Sym} : \mathsf{P} \mapsto \mathtt{Sym}\,\mathsf{P}$ *preserves unitary extensions. To be more explicit, under the conventions of §I.4.5, we have an identity* $\mathtt{Sym}(\mathsf{P}_+) = \mathtt{Sym}(\mathsf{P})_+$, *for any unitary braided operad* P_+,

Proof. Since $\Sigma_r = B_r/P_r$, we immediately obtain that the action of B_r on $\mathsf{P}(r)$ induces an action of the symmetric group Σ_r on the quotient object $\mathsf{P}(r)/P_r$.

The operadic unit of P obviously defines a unit morphism $\mathbb{1} \xrightarrow{\eta} \mathtt{Sym}\,\mathsf{P}(1)$ at the level of the collection $\mathtt{Sym}\,\mathsf{P}$ since $\mathtt{Sym}\,\mathsf{P}(1) = \mathsf{P}(1)/P_1 = \mathsf{P}(1)$. Recall that the direct sums $\beta_1 \oplus \cdots \oplus \beta_r$ as well as the block braids $\alpha_*(n_1, \ldots, n_r)$ of Proposition 2.1.2 are lifting of corresponding operations on permutations. If $\beta_1, \ldots, \beta_r$ are pure braids, then so is the direct sum $\beta_1 \oplus \cdots \oplus \beta_r$ because we have the identity $id_{n_1} \oplus \cdots \oplus id_{n_r} = id_{n_1 + \cdots + n_r}$ at the level of permutations, and similarly in the case of the block braid

$\alpha_*(n_1,\ldots,n_r)$. Thus, the permutations $\beta_1\oplus\cdots\oplus\beta_r$ and $\alpha_*(n_1,\ldots,n_r)$ occurring in the equivariance relations of braided operads are pure whenever α and $\beta_1,\ldots,\beta_r$ are pure braids. From this observation, we immediately deduce that the composition products of P induce composition products on $\mathtt{Sym}\,\mathsf{P}$ by the quotient process

$$\begin{tikzcd}
\mathsf{P}(r)\otimes\mathsf{P}(n_1)\otimes\cdots\otimes\mathsf{P}(n_r) \arrow[r, "\mu"] \arrow[d] & \mathsf{P}(n_1+\cdots+n_r) \arrow[d] \\
\mathsf{P}(r)/P_r\otimes\mathsf{P}(n_1)/P_{n_1}\otimes\cdots\otimes\mathsf{P}(n_r)/P_{n_r} \arrow[r, dotted, "\exists\mu"'] & \mathsf{P}(n_1+\cdots+n_r)/P_{n_1+\cdots+n_r}
\end{tikzcd},$$

and the equivariance, unit and associativity relations of Figure I.1-I.3 remain obviously satisfied in the quotient $\mathtt{Sym}\,\mathsf{P}$. This completes the construction of the symmetric operad $\mathtt{Sym}\,\mathsf{P}$ associated to P. The assertion about the adjunction relation follows from a straightforward inspection of our construction.

The identity between the symmetrization of the braided operad and the permutation operad follows from the observation that the composition operation on braids $\alpha(\beta_1,\ldots,\beta_r)=\beta_1\oplus\cdots\oplus\beta_r\cdot\alpha_*(n_1,\ldots,n_r)$ lifts a corresponding operation on permutations.

The last assertion of the proposition is immediate. □

Our main objective is to prove that the topological operad of little 2-discs is the symmetrization of a contractible braided operad in topological spaces. For this purpose, we consider the universal coverings $\widetilde{\mathsf{D}}_2(r)$ of the little 2-discs spaces $\mathsf{D}_2(r)$. Our result precisely reads:

THEOREM 2.1.6. *The universal coverings $\widetilde{\mathsf{D}}_2(r)$ of the little 2-disc spaces $\mathsf{D}_2(r)$ form a braided operad in topological spaces $\widetilde{\mathsf{D}}_2$ with the operad of little 2-discs D_2 as associated symmetric operad.*

The construction of this operad structure works for the unitary extension of the operad of little 2-discs D_{2+} as well, and gives in this case a unitary extension of the non-unitary operad $\widetilde{\mathsf{D}}_2$.

The proof of this theorem is deferred to a series of constructions and lemmas. We focus on the non-unitary part of the theorem, and we skip the examination of degenerate cases involved in the unitary extension of our statement, because this additional verification reduces from a straightforward inspection of our constructions.

Recall that the definition of a universal covering depends on the choice of a base point in the base space. To be precise, the universal coverings associated to different base points are isomorphic, but the isomorphisms connecting them is not canonical. We really need a rigid construction in order to check the operad relations at the level of universal covers. Therefore, we explain how we fix base points in little 2-disc spaces first.

2.1.7. *The choice of base points.* Recall that the operad of little 1-discs embeds into the little 2-discs operad by a topological inclusion $\mathsf{D}_1\hookrightarrow\mathsf{D}_2$. In Proposition 1.1.6, we prove that each space $\mathsf{D}_1(r)$ has contractible connected components $\mathsf{D}_1(r)_w$ indexed by permutations $w\in\Sigma_r$. Moreover, we observe that $\pi_0\,\mathsf{D}_1$ is isomorphic to the permutation operad as an operad. Equivalently, the partial composition product $\mu:\mathsf{D}_1(m)\times\mathsf{D}_1(n)\to\mathsf{D}_1(m+n-1)$ maps each cartesian product of connected components $\mathsf{D}_1(m)_s\times\mathsf{D}_1(n)_t$ into the connected component

$\mathsf{D}_1(m+n-1)_{s\circ_i t}$, associated with the composition product $s\circ_i t$ of the permutations $s \in \Sigma_m$ and $t \in \Sigma_n$, formed within the permutation operad.

We consider the contractible space $\mathsf{D}_1(r)_{id}$ associated to the identity permutation $id \in \Sigma_r$, and the corresponding subspace in $\mathsf{D}_2(r)$, which, according to definitions (check §1.1.5), consist of little disc configurations of the form represented in Figure 2.11. We fix such a disc configuration $\underline{c}^0$, coming from $\mathsf{D}_1(r)_{id}$, as base point for the little 2-disc space $\mathsf{D}_2(r)$, and from now on, we use the notation $\widetilde{\mathsf{D}}_2(r)$ to refer to the universal covering of $\mathsf{D}_2(r)$ formed at that base point. We recall an explicit description of such covering spaces in the next paragraph.

Any disc configuration $\underline{c}$ coming from the subspace $\mathsf{D}_1(r)_{id} \hookrightarrow \mathsf{D}_2(r)$ can be connected to our base point $\underline{c}^0$ by a path γ^0 in that subspace $\mathsf{D}_1(r)_{id} \hookrightarrow \mathsf{D}_2(r)$. All paths of this form belong to the same homotopy class since $\mathsf{D}_1(r)_{id}$ is contractible. Such a path gives a canonical isomorphism between the universal covering of $\mathsf{D}_2(r)$ determined at the base point $\underline{c}$ and the universal covering $\widetilde{\mathsf{D}}_2(r)$ determined at our chosen base point $\underline{c}^0$. We explain this process in the next paragraph, where we apply a standard construction of covering theory to give an explicit definition of our covering spaces.

2.1.8. *The construction of the universal coverings.* The covering spaces $\widetilde{\mathsf{D}}_2(r)$ can be defined as quotient sets

$$\widetilde{\mathsf{D}}_2(r) = \{\gamma : [0,1] \to \mathsf{D}_2(r) | \gamma(0) = \underline{c}^0\}/\simeq,$$

formed by all homotopy classes of paths $\gamma : [0,1] \to \mathsf{D}_2(r)$ with our base point $\underline{c}^0$ as origin, and which we equip with a suitable topology (see for instance [70, §10]). In what follows, we omit to check the continuity of our constructions on covering spaces. These verifications reduce to straightforward inspections, at all steps of argument line, by using the explicit definition of the topology of universal coverings.

The covering map $q : \widetilde{\mathsf{D}}_2(r) \to \mathsf{D}_2(r)$ assigns the endpoint $\gamma(1) \in \mathsf{D}_2(r)$ to the element of $\widetilde{\mathsf{D}}_2(r)$ defined by the homotopy class of the path γ. The quotient set $\widetilde{\mathsf{D}}_2(r)$ inherits an appropriate topology so that this map $p : \widetilde{\mathsf{D}}_2(r) \to \mathsf{D}_2(r)$ is indeed a covering because the little 2-disc space $\mathsf{D}_2(r)$ is locally contractible (we refer to [70], for instance, for the general argument).

In this representation, the isomorphism connecting $\widetilde{\mathsf{D}}_2(r)$ with the universal covering taken at another base point $\underline{c}$ is given by the concatenation of the paths $\gamma : [0,1] \to \mathsf{D}_2(r)$, defining the elements of $\widetilde{\mathsf{D}}_2(r)$, with a path $\gamma^0 : [0,1] \to \mathsf{D}_2(r)$, connecting our disc configurations $\underline{c}^0$ and $\underline{c}$. From this construction, we immediately see that this isomorphism is canonical as soon as the homotopy class of the path γ^0 connecting the base points is uniquely determined, and this is so when, as set in §2.1.7, we restrict ourselves to base points and connecting paths γ^0 lying within the component $\mathsf{D}_1(r)_{id}$ of the little 1-disc space $\mathsf{D}_1(r)$ inside $\mathsf{D}_2(r)$.

2.1.9. *The action of braid groups.* The pure braid group P_r can immediately be identified with the group of automorphisms of the covering $\widetilde{\mathsf{D}}_2(r) \to \mathsf{D}_2(r)$ because:

- the automorphism group of a universal covering is identified with the fundamental group of its base space,
- and the homotopy equivalence $\omega : \mathsf{D}_2(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^2, r)$, defined by the disc center mapping, gives an isomorphism of fundamental groups $\pi_1(\mathsf{D}_2(r), *) \xrightarrow{\simeq} \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r), *) = P_r$.

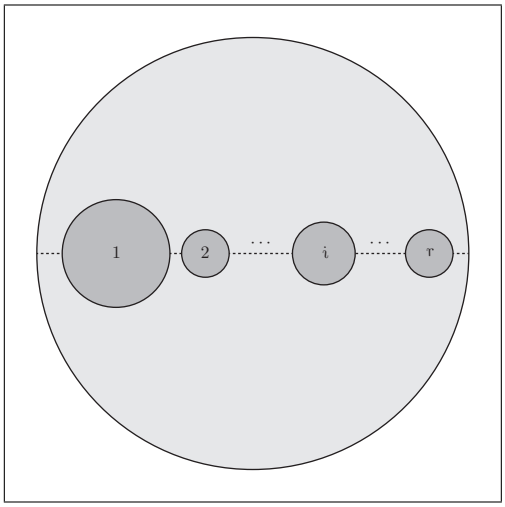


FIGURE 2.11. The form of a chosen base disc configuration, lying in the image of the contractible subspace $\mathsf{D}_1(r)_{id} \hookrightarrow \mathsf{D}_2(r)$.

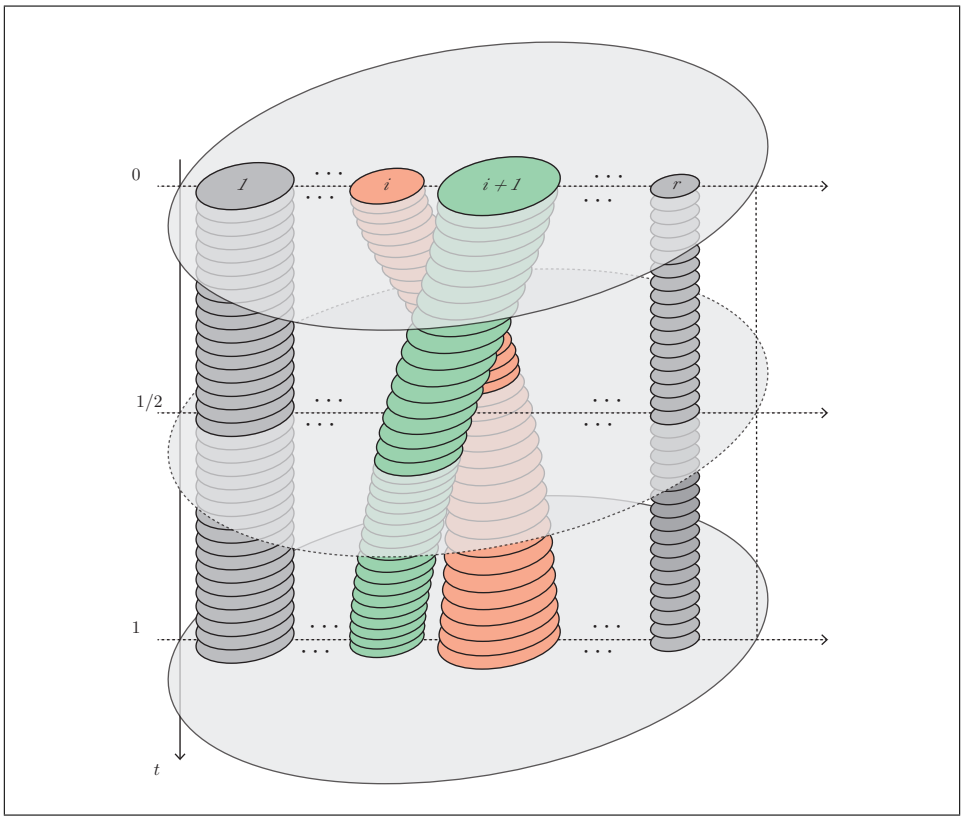


FIGURE 2.12. The path defining a representative of the generating braid τ_i in the little 2-disc space $\mathsf{D}_2(r)$.

One can adapt this approach in order to prove that the action of P_r on $\widetilde{\mathsf{D}}_2(r)$ extends to an action of the full braid group B_r. Indeed, we can also identify our covering space $\widetilde{\mathsf{D}}_2(r)$ with the universal covering of the quotient space $\mathsf{D}_2(r)/\Sigma_r$, for which we have $\pi_1(\mathsf{D}_2(r)/\Sigma_r, *) \xrightarrow{\simeq} \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r)/\Sigma_r, *) = B_r$.

In order to ease the proof of the operad equivariance relation, we prefer to give an explicit construction of this action by relying on the explicit definition of the universal covering $\widetilde{\mathsf{D}}_2(r)$, as given in §2.1.8. For this aim, we consider a path in the little 2-disc space $\tau_i : [0,1] \to \mathsf{D}_2(r)$ of the form represented in Figure 2.12. From this perspective representation, we immediately see that the image of this path under the disc center mapping $\omega : \mathsf{D}_2(r) \to \mathsf{F}(\mathring{\mathbb{D}}^2, r)$ gives a representative of the generating braid of Figure 2.5.

Note that the endpoint of this path $\tau_i(1)$ is identified with the image of our base disc configuration $\underline{c}^0$ under the action of the transposition $t_i = (i\ i+1)$.

Let now $\gamma : [0,1] \to \mathsf{D}_2(r)$ be a path in $\mathsf{D}_2(r)$, with $\gamma(0) = \underline{c}^0$ as origin, so that the homotopy class of this path $[\gamma]$ defines an element of the covering space $\widetilde{\mathsf{D}}_2(r)$. We apply the transposition t_i to this path in order to obtain a path $t_i\gamma$ with $t_i\gamma(0) = t_i \cdot \underline{c}^0$ as origin. We can then concatenate $t_i\gamma$ with the path represented in Figure 2.12 to obtain a new path $(t_i\gamma) \cdot \tau_i : [0,1] \to \mathsf{D}_2(r)$ with $\underline{c}^0$ as origin, and of which homotopy class $[(t_i\gamma) \cdot \tau_i]$ determines an element of $\widetilde{\mathsf{D}}_2(r)$ associated to $[\gamma]$.

The following lemma follows from an immediate visual inspection:

Lemma 2.1.10.

(a) *The mapping $\tau_i : [\gamma] \mapsto [(t_i\gamma)\cdot\tau_i]$ defines a lifting to the covering space $\widetilde{\mathsf{D}}_2(r)$ of the map $t_i : \mathsf{D}_2(r) \to \mathsf{D}_2(r)$ given by the action of the transposition $t_i = (i\ i+1)$ on the little 2-disc space $\mathsf{D}_2(r)$.*

(b) *The maps $\tau_i : \widetilde{\mathsf{D}}_2(r) \to \widetilde{\mathsf{D}}_2(r)$, obtained by this construction for $i = 1, \ldots, r-1$, satisfy the generating relations of braids groups, and hence, determine an action of the braid group B_r on the covering space $\widetilde{\mathsf{D}}_2(r)$.* □

This result completes the construction of the braided structure on the collection $\widetilde{\mathsf{D}}_2 = \{\widetilde{\mathsf{D}}_2(r)\}_{r\in\mathbb{N}}$.

We can use a similar composition process $[\gamma] \mapsto [\gamma \cdot \alpha]$ when $\omega : [0,1] \to \mathsf{D}_2(r)$ is any loop based at $\omega(0) = \omega(1) = \underline{c}^0$ in order to determine the action of the fundamental group $\pi_1(\mathsf{D}_2(r), \underline{c}^0)$ on the universal covering $\widetilde{\mathsf{D}}_2(r)$. We immediately see that this action corresponds to a restriction of the action considered in Lemma 2.1.10 when we apply the isomorphism $\pi_1(\mathsf{D}_2(r), *) \xrightarrow{\simeq} \pi_1(\mathsf{F}(\mathring{\mathbb{D}}^2, r), *)$ to identify $\pi_1(\mathsf{D}_2(r), *)$ with the pure braid group P_r.

The following statement follows from this identification and standard results of covering theory:

Lemma 2.1.11. *The covering map $q : \widetilde{\mathsf{D}}_2(r) \to \mathsf{D}_2(r)$, as defined in §2.1.8, induces a homeomorphism $q_* : \widetilde{\mathsf{D}}_2(r)/P_r \xrightarrow{\simeq} \mathsf{D}_2(r)$, where the quotient space $\widetilde{\mathsf{D}}_2(r)/P_r$ is formed by considering the restriction of the action of Lemma 2.1.10 to the pure braid group P_r.* □

2.1.12. *The operadic composition structure.* We now aim at providing the collection $\widetilde{\mathsf{D}}_2$ with an operadic composition structure.

We can assume that our base point in arity $r = 1$ is given by the operadic unit of the little 2-disc operad $1 \in \mathsf{D}_2(1)$. We take the homotopy class of the constant path $1(t) = 1$ associated to this element $1 \in \mathsf{D}_2(1)$ as operadic unit for $\widetilde{\mathsf{D}}_2$.

We proceed as follows to define the composition products of $\widetilde{\mathsf{D}}_2$. Let $\alpha : [0,1] \to \mathsf{D}_2(m)$ and $\beta : [0,1] \to \mathsf{D}_2(n)$ be paths giving elements in our universal covering spaces. Let $\alpha(0) = \underline{a}^0$ and $\beta(0) = \underline{b}^0$ be the corresponding base points. We fix a composition index $i \in \{1, \ldots, m\}$. By performing the operadic composition of little 2-discs point-wise, we obtain a path $\alpha \circ_i \beta : [0,1] \to \mathsf{D}_2(m+n-1)$ with $\alpha \circ_i \beta(0) = \underline{a}^0 \circ_i \underline{b}^0$ as origin. This composite disc configuration $\underline{a}^0 \circ_i \underline{b}^0$ is not necessarily equal to the chosen base point $\underline{c}^0$ of the little 2-disc space $\mathsf{D}_2(m+n-1)$. But, the assumption that $\underline{a}^0$ lies in the contractible space $\mathsf{D}_1(m)_{id} \hookrightarrow \mathsf{D}_2(m)$, and that $\underline{b}^0$ similarly arises from $\mathsf{D}_1(n)_{id} \hookrightarrow \mathsf{D}_2(n)$ implies that $\underline{a}^0 \circ_i \underline{b}^0$ lies in our distinguished subspace $\mathsf{D}_1(m+n-1)_{id}$ too, because the composition of these connected components is reflected by the composition structure of the permutation operad, in which we have $id \circ_i id = id$ (see Proposition I.1.8 and the explanations hereafter). Thus, as explained in §2.1.7, we have a path $\gamma^0 : [0,1] \to \mathsf{D}_2(m+n-1)$, going from $\gamma^0(0) = \underline{c}^0$ to $\gamma^0(1) = \underline{a}^0 \circ_i \underline{b}^0$, and of which homotopy class is canonically determined. We concatenate our composite $\alpha \circ_i \beta$ with such a path $\gamma^0 : [0,1] \to \mathsf{D}_2(m+n-1)$ with the prescribed base point $\underline{c}^0 \in \mathsf{D}_2(m+n-1)$ as origin and of which homotopy class $[\alpha \circ_i \beta \cdot \gamma^0]$ determines an element of $\widetilde{\mathsf{D}}_2(m+n-1)$ canonically associated to $[\alpha] \in \widetilde{\mathsf{D}}_2(m+n-1)$ and $[\beta] \in \widetilde{\mathsf{D}}_2(m+n-1)$. We obtain by this process a composition product on our universal coverings

$$\circ_i : \widetilde{\mathsf{D}}_2(m) \times \widetilde{\mathsf{D}}_2(n) \to \widetilde{\mathsf{D}}_2(m+n-1)$$

which obviously lifts the composition product $\circ_i$ of the little 2-discs operad. We prove that:

Lemma 2.1.13. *The operadic unit and composition products defined on the covering spaces $\widetilde{\mathsf{D}}_2(r)$ in the previous paragraphs fulfill the unit and associativity requirements of operadic composition structures, as well as the equivariance relation of braided operad.*

Proof. The proof of the unit and associativity relations follows from a quick visual inspection, by using the explicit definition of the composition structure given in the previous paragraph, of the composites occurring in these relations. The equivariance relation is checked similarly in the case of a generating braid τ_i, by using the explicit definition of §§2.1.9-2.1.10 for the action of these braids on our covering spaces, and we immediately conclude from this verification that the equivariance relation holds in full generality since the braids τ_i generate the whole braid group B_r. □

As the braided operad structure of $\widetilde{\mathsf{D}}_2$ is essentially defined by lifting the symmetric operad structure of the little 2-discs operad, we immediately obtain that the covering maps $q : \widetilde{\mathsf{D}}_2(r) \to \mathsf{D}_2(r)$ define a morphism of braided operads $q : \widetilde{\mathsf{D}}_2 \to \mathsf{D}_2$, and the assertion of Lemma 2.1.11 implies that this morphism induces an isomorphism by between the symmetrized operad $\mathtt{Sym}\,\widetilde{\mathsf{D}}_2$ and D_2. This verification finishes the proof of Theorem 2.1.6. □

Theorem 2.1.6 has the following consequence:

Theorem 2.1.14. *Let P be a braided operad in topological spaces. Suppose that the action of B_r on $\mathsf{P}(r)$ is free and proper, for all $r \in \mathbb{N}$. If the spaces $\mathsf{P}(r)$ are contractible for all $r \in \mathbb{N}$, then the symmetric operad naturally associated to P, and*

formed by the collection of quotient spaces $\mathtt{Sym}\,\mathsf{P}(r) = \mathsf{P}(r)/P_r$*, is an* E_2*-operad, and similarly in the unitary setting.*

Proof. We again focus on the non-unitary setting because the generalization of our statement to unitary operads follows from a straightforward extension of our arguments.

We form the arity-wise product $\mathsf{Q}(r) = \mathsf{P}(r) \times \widetilde{\mathsf{D}}_2(r)$ in the category of braided operads. The braid group B_r operates diagonally on $\mathsf{Q}(r)$, for each $r \in \mathbb{N}$, and the collection $\mathsf{Q} = \{\mathsf{Q}(r)\}_{r\in\mathbb{N}}$ comes also equipped with a natural operadic composition structure which is formed factor-wise as in the context of symmetric operads. We immediately check that the canonical projections

$$\mathsf{P}(r) \leftarrow \mathsf{P}(r) \times \widetilde{\mathsf{D}}_2(r) \rightarrow \widetilde{\mathsf{D}}_2(r)$$

define morphisms of braided operads $\mathsf{P} \leftarrow \mathsf{Q} \rightarrow \widetilde{\mathsf{D}}_2$.

Recall that the spaces $\mathsf{P}(r)$ are contractible by assumption, and we have already observed that the spaces $\widetilde{\mathsf{D}}_2(r)$ are contractible too. Thus, the considered projections are weak-equivalences between contractible spaces.

In general, weak-equivalences are preserved by the quotients over group actions which are free and proper. The braid group B_r operates freely and properly on $\mathsf{P}(r)$ by assumption, and on $\widetilde{\mathsf{D}}_2(r)$ as well by definition of this space as a universal covering. The diagonal action of B_r on $\mathsf{P}(r) \times \widetilde{\mathsf{D}}_2(r)$ is free and proper too. Thus, by performing the quotient over the action of $P_r \subset B_r$, we obtain weak-equivalences of spaces

$$\mathsf{P}(r)/P_r \xleftarrow{\sim} (\mathsf{P}(r) \times \widetilde{\mathsf{D}}_2(r))/P_r \xrightarrow{\sim} \widetilde{\mathsf{D}}_2(r)/P_r = \mathsf{D}_2(r),$$

yielding weak-equivalences of operads $\mathtt{Sym}\,\mathsf{P} \xleftarrow{\sim} \mathtt{Sym}\,\mathsf{Q} \xrightarrow{\sim} \mathsf{D}_2$, from which we conclude that $\mathtt{Sym}\,\mathsf{P}$ is E_2, as claimed in the statement of the theorem. □

2.2. The classifying spaces of the colored braid operad

Recall that an Eilenberg-MacLane space of type $K(G,1)$, where G is any group, is a connected space X such that $\pi_1(X) = G$ and $\pi_*(X) = 0$ for $* \neq 1$. These conditions actually determine the homotopy type of the space X (in plain terms, one can prove that all Eilenberg-MacLane spaces of a given type $K(G,1)$ are weakly-equivalent).

In the preliminary section §2.0, we mentioned that the underlying spaces of the little 2-discs operad D_2 are Eilenberg-MacLane spaces $K(P_r,1)$ associated to the pure braid groups P_r. This result follows from the existence of the homotopy equivalence $\mathsf{D}_2(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^2, r)$, established in Proposition 1.2.2, and the computation of the homotopy groups of the configuration spaces $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ in Proposition 2.0.1.

In topology, we have a standard simplicial model BG for the Eilenberg-MacLane space $K(G,1)$. This simplicial set BG also represents the base space of a universal G-principal bundle, and for that reason, is usually referred to as the classifying space of G. The objective of this section is to adapt the classifying space construction in order to define a simplicial model of the little 2-disc operad D_2.

For this aim, we need to consider classifying spaces of small categories, which include classifying spaces of groups as particular examples. The operads which we introduce soon are defined in the category of groupoids, which lie between small categories and groups, but classifying spaces are more naturally defined in the general setting of small categories.

To explain the problem, we need structures on G reflecting the operadic structures which we aim to define at the level of the classifying spaces BG. The pure braid groups $G = P_r$ lack the symmetric structure underlying an operad, and hence, do not fit our requirements. The idea is to introduce groupoids of colored braids $CoB(r)$, which include the pure braid groups P_r as particular morphism sets, and come equipped with the required symmetric structure, as well as a full operadic composition structure. To summarize: we establish that the groupoids of colored braids $CoB(r)$ form an operad, which we may call the colored braid operad, and we prove that the collection of classifying spaces $B\,CoB(r)$ associated with this operad in groupoids defines a simplicial model of E_2-operad.

To begin with, we briefly explain the definition of an operad in the category of small categories and in the category of groupoids. Then we recall the definition of the classifying space of a category, and we examine the application of this classifying space construction to operads in categories. The colored braid operad will be defined afterwards as an instance of operad in groupoids.

2.2.1. *The category of small categories and groupoids.* We use the notation $\mathcal{C}at$ for the category of small categories. The cartesian product of categories defines the underlying product $\times : \mathcal{C}at \times \mathcal{C}at \to \mathcal{C}at$ of a symmetric monoidal structure on $\mathcal{C}at$. The singleton pt, which is identified with the final object of the category of small categories, defines the unit object associated with this symmetric monoidal structure.

Groupoids will be more heavily used later on. Simply recall for the moment that a groupoid is a small category in which all morphisms are invertible. Groups can identified with the groupoids of which underlying object set is reduced to a point. In contrast, we can identify sets with discrete groupoids, which have no morphism outside the identity attached to each object. We use the notation $\mathcal{G}rd$ for the category of groupoids, which we regard as a full subcategory of the category of small categories $\mathcal{C}at$. We immediately see that the embedding $\mathcal{G}rd \hookrightarrow \mathcal{C}at$ creates products and final objects. Accordingly, the category of groupoids $\mathcal{G}rd$ is identified with a symmetric monoidal subcategory of $\mathcal{C}at$ with respect to the cartesian monoidal structure.

2.2.2. *Operads in small categories and groupoids.* We consider operads in the category of categories (and in the category of groupoids) which we define by applying the general definition of §I to this instance of symmetric monoidal categories $\mathcal{B}ase = \mathcal{C}at$ (respectively, $\mathcal{B}ase = \mathcal{G}rd$). According to our conventions, we will use the notation $\mathcal{C}at\mathcal{O}p$ (respectively, $\mathcal{G}rd\mathcal{O}p$) to refer to this category of operads.

To unravel definitions: an operad in the category of small categories $\mathsf{P} \in \mathcal{C}at\mathcal{O}p$ (an operad in categories for short) consists of a sequence of small categories $\mathsf{P}(r)$, $r \in \mathbb{N}$, each of which equipped with a symmetric group action, together with a unit morphism $\eta : pt \to \mathsf{P}(1)$, and composition products $\mu : \mathsf{P}(r) \times \mathsf{P}(n_1) \times \cdots \times \mathsf{P}(n_r) \to \mathsf{P}(n_1 + \cdots + n_r)$, all formed in the category of categories, so that the identities expressed by the diagrams of Figure I.1-I.3 hold; a morphism of operads in categories $f : \mathsf{P} \to \mathsf{Q}$ is a sequence of functors $f : \mathsf{P}(r) \to \mathsf{Q}(r)$ preserving the internal structures attached to operads. Since we define the category of groupoids as a subcategory of $\mathcal{C}at$, an operad in groupoids $\mathsf{P} \in \mathcal{G}rd\mathcal{O}p$ can be defined as an operad in categories of which underlying sequence $\mathsf{P}(r)$ consists of groupoids. The equivalence between the plain definition of an operad (§I.1) and the definition in terms of partial composition operations (§A.1) naturally holds in the context of categories

$\mathcal{B}ase = \mathcal{C}at$ (respectively, groupoids $\mathcal{B}ase = \mathcal{G}rd$), which fulfills all conditions that we require for our base categories (see §0.6). Hence, the composition structure of an operad in categories (respectively, groupoids) is equivalently determined by functors $\circ_i : \mathsf{P}(m) \times \mathsf{P}(n) \to \mathsf{P}(m+n-1)$, $i = 1, \ldots, m$, satisfying the equivariance, unit and associativity axioms of §A.1.

In the category of small categories, one distinguishes the class of category equivalences (functors which are invertible up to natural equivalence) besides the ordinary isomorphisms of the category (the functors which are strictly invertible). For operads in categories, we will naturally consider operad morphisms $f : \mathsf{P} \to \mathsf{Q}$ of which all underlying functors $f : \mathsf{P}(r) \to \mathsf{Q}(r)$ are equivalences of categories. In this situation, we will say that the operad morphism f is a categorical equivalence, and we will use the distinguishing mark $\sim$ in the notation of f. Note that the inverse equivalences of the functors $f : \mathsf{P}(r) \xrightarrow{\sim} \mathsf{Q}(r)$ do not necessarily define an operad morphism, and we do not set this requirement in our definition of a categorical equivalence of operads.

2.2.3. *Recollections on classifying spaces.* The classifying space of a category $\mathcal{C}$ is the simplicial set $B\,\mathcal{C}$ defined in dimension n by the n-fold sequences of composable morphisms of $\mathcal{C}$

$$\underline{\alpha} = \{x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n\}$$

together with the face operators such that

$$\begin{aligned}
d_0(\underline{\alpha}) &= x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n,\\
d_i(\underline{\alpha}) &= x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} x_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} x_{i+1} \xrightarrow{\alpha_{i+2}} \cdots \xrightarrow{\alpha_n} x_n, \quad \text{for } 0 < i < n,\\
d_n(\underline{\alpha}) &= x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} x_{n-1},
\end{aligned}$$

and the degeneracy operators given by the insertion of identity morphisms

$$s_j(\underline{\alpha}) = x_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_j} x_j \xrightarrow{id} x_j \xrightarrow{\alpha_{j+1}} \cdots \xrightarrow{\alpha_n} x_n,$$

for all $j = 0, \ldots, n$. One can prove that the simplicial set $B\,\mathcal{C}$ forms a Kan complex if and only if the category $\mathcal{C}$ is a groupoid (see for instance [48, §I.3]). In the case of a group G, this result can be used to check, by a direct and simple computation, that the geometric realization of BG is an Eilenberg-MacLane space (use the combinatorial definition of simplicial homotopy groups in [27, §2] or in [71, §1]).

The mapping $B : \mathcal{C} \mapsto B\,\mathcal{C}$ defines a functor from the category of small categories to the category of simplicial sets. To study the image of operads in categories under the classifying space construction, we use the following result:

Proposition 2.2.4. *The functor $B : \mathcal{C}at \to \mathcal{S}imp$ is symmetric monoidal in the sense of §II.3.1:*

(a) *In the case of a point pt, viewed as the unit object of the category of small categories, we have an obvious identity $B(pt) = pt$.*
(b) *In the case of a cartesian product of categories $\mathcal{C} \times \mathcal{D}$, the maps $B\,\mathcal{C} \xleftarrow{p_*} B(\mathcal{C} \times \mathcal{D}) \xrightarrow{q_*} B\,\mathcal{D}$, associated to the canonical projections $\mathcal{C} \xleftarrow{p} \mathcal{C} \times \mathcal{D} \xrightarrow{q} \mathcal{D}$ give rise to an isomorphism $B(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} B\,\mathcal{C} \times B\,\mathcal{D}$.*
(c) *The comparison isomorphisms of (a-b) fulfill the unit, associativity and symmetry constraints of §II.3.1 too.*

Proof. The proof of assertions (a-b) reduces to a straightforward inspection of definitions. The definition of the isomorphism $B(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} B\,\mathcal{C} \times B\,\mathcal{D}$ from

universal categorical constructions automatically ensures that the unit, associativity and symmetry constraints are fulfilled. □

From this statement, the result of Proposition II.1.4 gives:

PROPOSITION 2.2.5. *Let* P *be an operad in small categories. The collection of classifying spaces* $B\,\mathsf{P}(r)$ *associated to the categories* $\mathsf{P}(r)$ *forms an operad in simplicial sets naturally associated to* P. □

Recall that in the situation of Proposition 2.2.4, the mapping $B : \mathsf{P} \mapsto B\,\mathsf{P}$ also preserves unitary extensions, and in the formalism of §I.4.5, we have an identity $B(\mathsf{P}_+) = (B\,\mathsf{P})_+$ for any unitary operad in categories P_+ (see Proposition II.1.4).

In §II.3.2, we mention that the geometric realization functor $|-| : \mathcal{S}imp \to \mathcal{T}op$ is symmetric monoidal as well. We can apply this functor to the simplicial operad $B\,\mathsf{P}$ in order to form a classifying space operad in topological spaces naturally associated to P. In general, we abusively use the notation of the underlying simplicial operad $B\,\mathsf{P}$ for this associated operad in topological spaces too. But, if we need to distinguish the topological object from its underlying simplicial model, then we mark the application of the realization functor $|-|$ in the notation of this topological operad.

The functoriality claim in Proposition 2.2.5 implies that the mapping $B : \mathsf{P} \mapsto B\,\mathsf{P}$ defines a functor assigning a morphism of simplicial operads $Bf : B\,\mathsf{P} \to B\,\mathsf{Q}$ to any morphism of operads in the category of small categories $f : \mathsf{P} \to \mathsf{Q}$. The following proposition, which is an immediate corollary of a standard result on classifying spaces, is worth recording:

PROPOSITION 2.2.6. *The morphism* $Bf : B\,\mathsf{P} \to B\,\mathsf{Q}$ *associated to a categorical equivalence of operads* $f : \mathsf{P} \xrightarrow{\sim} \mathsf{Q}$ *is a weak-equivalence of simplicial operads.* □

The remainder of this section is devoted to the definition of the colored braid operad CoB and to the proof that the associated classifying space operad $B\,CoB$ defines an instance of E_2-operad. We also establish a unitary extension of this result.

In a first step, we define the underlying groupoids of the operad $CoB(r)$, $r \in \mathbb{N}$. Most usually, we regard the small categories, which include groupoids as particular instances, as formed of an object set $\mathtt{Ob}\,\mathcal{C}$ together with morphism sets $\mathtt{Mor}_{\mathcal{C}}(x, y)$ attached to all pairs of objects $x, y \in \mathtt{Ob}\,\mathcal{C}$. But for the definition of the groupoid of colored braid $CoB(r)$, we may adopt another point of view, motivated by the fact that the whole information is supported by the morphisms. Therefore, we revisit the general definition of a groupoid before tackling the definition of the colored braid groupoids.

2.2.7. *Groupoids revisited.* To summarize the idea: we can regard a groupoid $\mathcal{G}$ as formed of an object set $\mathtt{Ob}\,\mathcal{G}$ together with a single morphism set $\mathtt{Mor}\,\mathcal{G}$ collecting all morphisms of $\mathcal{G}$.

In this approach, a source $s : \mathtt{Mor}\,\mathcal{G} \to \mathtt{Ob}\,\mathcal{G}$ and target map $t : \mathtt{Mor}\,\mathcal{G} \to \mathtt{Ob}\,\mathcal{G}$ are given as part of the groupoid structure in order to identify the source and target of morphisms, and we similarly provide a map $e : \mathtt{Ob}\,\mathcal{G} \to \mathtt{Mor}\,\mathcal{G}$ such that $se = te = id$ in order to specify the identity morphism associated to any object. The morphism sets $\mathtt{Mor}_{\mathcal{G}}(x, y)$, which we have considered so far, are identified with the subsets of morphisms $\alpha \in \mathtt{Mor}\,\mathcal{G}$ associated to a given source $s(\alpha) = x$ and target $t(\alpha) = y$.

The fiber product

$$\begin{tikzcd} \mathtt{Mor}\,\mathcal{G} \times_{st} \mathtt{Mor}\,\mathcal{G} \arrow[r, dotted, "q"] \arrow[d, dotted, "p"'] & \mathtt{Mor}\,\mathcal{G} \arrow[d, "t"] \\ \mathtt{Mor}\,\mathcal{G} \arrow[r, "s"'] & \mathtt{Ob}\,\mathcal{G} \end{tikzcd},$$

more explicitly defined as the set $\mathtt{Mor}\,\mathcal{G} \times_{st} \mathtt{Mor}\,\mathcal{G} = \{(\alpha, \beta) | s(\alpha) = t(\beta)\}$, collects all pairs of composable morphisms of the groupoid. The composition operation of $\mathcal{G}$ is given by a product operation $\mu : \mathtt{Mor}\,\mathcal{G} \times_{st} \mathtt{Mor}\,\mathcal{G} \to \mathtt{Mor}\,\mathcal{G}$, defined on this fiber product, and such that $s\mu = sq$, $t\mu = tp$. In point-wise terms, these latter requirements read $s(\alpha\beta) = s(\beta)$ and $t(\alpha\beta) = t(\alpha)$ for all composable morphisms $(\alpha, \beta) \in \mathtt{Mor}\,\mathcal{G} \times_{st} \mathtt{Mor}\,\mathcal{G}$, where we set $\alpha\beta = \mu(\alpha, \beta)$. Thus, these requirements are simply added to fix the target and source of composites.

To define the inverse of morphisms in a groupoid, we similarly consider a map $\iota : \mathtt{Mor}\,\mathcal{G} \to \mathtt{Mor}\,\mathcal{G}$ such that $s\iota = t$ and $t\iota = s$. The unit, associativity, and inverse relations of the composition structure of groupoids can be written in terms of commutative diagrams, involving the fiber product $\mathtt{Mor}\,\mathcal{G} \times_{st} \mathtt{Mor}\,\mathcal{G}$, but, for the moment, we define the product and the inversion as point-set mappings, and therefore, we will still use the basic point-set interpretation of these relations.

2.2.8. *The groupoids of colored braids.* In the case of the colored braid groupoid $\mathcal{G} = CoB(r)$, the object set $\mathtt{Ob}\,CoB(r)$ is defined to be the set of permutations $w \in \Sigma_r$ which we regard as ordered sequences $(w(1), \ldots, w(r))$ of integers $w(i) \in \{1, \ldots, r\}$.

The morphisms of $\mathtt{Mor}\,CoB(r)$ are isotopy classes of braids α given together with a bijection between $\{1, \ldots, r\}$ and the collection of strands $\{\alpha_1, \ldots, \alpha_r\}$ defining α. Intuitively, the extra bijection assigns a color $i \in \{1, \ldots, r\}$ to each strand α_i, and this interpretation motivates the name of colored braid given to our groupoid.

The given of the bijection amounts to assuming that the strand collection is arranged on an r-tuple $\alpha = (\alpha_1, \ldots, \alpha_r)$. Note that we take the ordering determined by the bijection $i \mapsto \alpha_i$, given together with our braid α, in order to form this r-tuple, and not the ordering of the points $(\alpha_1(0), \ldots, \alpha_r(0))$ on the axis $(y = 0, t = 0)$, as in §2.0.5. Thus, we may have $(\alpha_1(0), \ldots, \alpha_r(0)) = ((x^0_{u(1)}, 0, 0), \ldots, (x^0_{u(r)}, 0, 0)$ for some permutation $u \in \Sigma_r$, where we again use the notation $x^0_k = -1 + (2k - 1)/(r + 1)$, $k = 1, \ldots, r$, for the abscissa of the contact points of braids on the axis $y = 0$ (see §2.0.5). This permutation precisely defines the source of our braid $u = s(\alpha)$ in the groupoid CoB. The target of the braid $v = t(\alpha)$ is the permutation v such that $(\alpha_1(1), \ldots, \alpha_r(1)) = ((x^0_{v(1)}, 0, 1), \ldots, (x^0_{v(r)}, 0, 1)$. Intuitively, we simply take the ordering of the origin points of the strands on the axis $(y = 0, t = 0)$ to determine a color ordering yielding the source permutation u of the colored braid α, and we the take the ordering of the strand end points on the axis $(y = 0, t = 1)$ to determine another color ordering yielding the target permutation v.

To illustrate these definitions, we give an instance of a colored braid in Figure 2.13. The source and target permutations associated to this colored braid are given by the ordered sequences $u = (2, 4, 3, 1)$ and $v = (3, 4, 1, 2)$.

The identity element, associated to any permutation $w \in \Sigma_r$, is given by the isotopy class of the identity braid $e(t) = ((x^0_{w(1)}, 0, t), \ldots, (x^0_{w(r)}, 0, t))$, where the given permutation w determines the coloring attached to this braid (see Figure 2.14

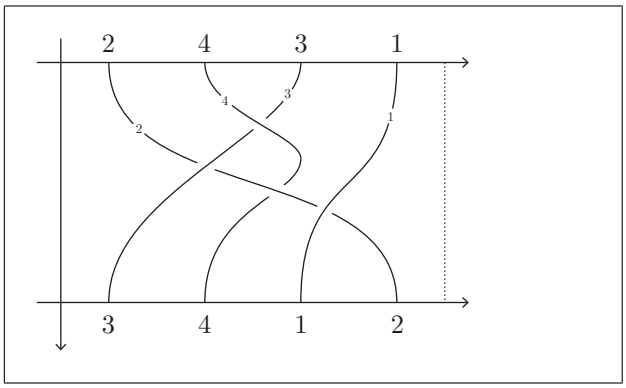


FIGURE 2.13. An instance of colored braid.

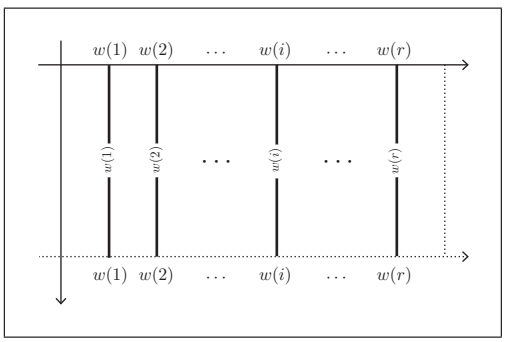


FIGURE 2.14. The representation of identity elements in the groupoid of colored braids.

for the representation), or equivalently, the ordering of the vertical strands $(x_k^0, 0, t)$, $k = 1, \ldots, r$. The composition of the groupoid is given by the standard concatenation operation on braids, inherited from the braid group, and represented in Figure 2.3. In our new context, we simply note that the colors assigned to strands agree on contact points precisely when our braids (α, β) satisfy the relation $s(\alpha) = t(\beta)$ and hence are composable in the sense of §2.2.7. In this situation, each composite strand inherits a single color from its components, which we use to define the coloring of the composite braid $\alpha \cdot \beta$.

The inversion of colored braids can also be deduced from the inversion operation in the braid groups, with the appropriate coloring switching the source and target permutations.

2.2.9. *Braid cosets and morphisms in the colored braid groupoids.* In the previous paragraph, we chose an approach which provides an intuitive definition of the colored braid groupoid. On the other hand, we immediately see, from this first definition, that the color indexing of an element $\alpha \in CoB(r)$ is determined by the given of the permutation $u = s(\alpha)$, which represents the source of α in the colored braid groupoid. Indeed, the ordered sequence $u = (u(1), \ldots, u(r))$ corresponds to the color indexing of the origin points $((x_1^0, 0, 0), \ldots, (x_r^0, 0, 0))$, which in turn determines the coloring of the braid strands. By using this observation, we can readily identify the morphism set $\mathtt{Mor}_{CoB(r)}(u, v)$ associated to fixed permutations $u, v \in \Sigma_r$ with the coset $q_*^{-1}(v^{-1}u) \subset B_r$, where we consider the natural group morphism

$q_* : B_r \to \Sigma_r$ from braids to permutations. The composition operation of $CoB(r)$ is also identified with the operation $q_*^{-1}(w^{-1}v) \times q_*^{-1}(v^{-1}u) \to q_*^{-1}(w^{-1}u)$ obtained by restriction of the natural multiplication of the braid group B_r. For a single permutation $w \in \Sigma_r$, we have an identity $\mathtt{Mor}_{CoB(r)}(w,w) = q_*^{-1}(w^{-1}w) = P_r$, and the identity morphism associated to w in the groupoid corresponds to the neutral element of the pure braid group P_r.

2.2.10. *The symmetric structure of the colored braid groupoids.* Each groupoid of colored braids $CoB(r)$ inherit a natural action of permutations. Therefore the collection $CoB = \{CoB(r), r \in \mathbb{N}\}$ forms a symmetric sequence of groupoids. To be explicit, the groupoid morphism $s_* : CoB(r) \to CoB(r)$ associated to any $s \in \Sigma_r$ is defined by the following process: for a permutation $w \in \Sigma_r$, representing an object of $CoB(r)$, we set $s_*(w) = sw$; for a braid α equipped with a strand coloring $i \mapsto \alpha_i$, we define $s_*(\alpha)$ by the same underlying braid as α, but we equip $s_*(\alpha)$ with the modified coloring $s(i) \mapsto \alpha_i$ which assigns the value $s(i) \in \{1,\ldots,r\}$ to the strand α_i which was previously colored by the index $i \in \{1,\ldots,r\}$. The mappings $s_* : \mathtt{Mor}\, CoB(r) \to \mathtt{Mor}\, CoB(r)$ and $s_* : \mathtt{Ob}\, CoB(r) \to \mathtt{Ob}\, CoB(r)$ clearly preserve the structure morphisms attached to our groupoid.

In the representation of §2.2.9, the morphism mapping $s_* : \mathtt{Mor}_{CoB(r)}(u,v) \to \mathtt{Mor}_{CoB(r)}(s_*(u), s_*(v))$ changes the source and target objects according to the rule $s_*(w) = sw$, but is given by the identity on the coset $q_*^{-1}(s_*(v)^{-1}s_*(u)) = q_*^{-1}(v^{-1}u) \subset B_r$ associated with both the source and the target of our mapping s_*.

2.2.11. *The operadic composition operations on colored braids.* We have an obvious identity $CoB(1) = pt$, giving a canonical operadic unit in the colored braid groupoids. We also have operadic composition operations, deduced from the operadic composition of permutations and braids, so that CoB inherits a full operad structure. We proceed as follows to define these operations.

On object sets $\mathtt{Ob}\, CoB(r) = \Sigma_r$, we simply use the operadic composition of permutations. (Accordingly, the collection $\mathtt{Ob}\, CoB$ is identified with the permutation operad in the category of sets.) On morphism sets $\mathtt{Mor}\, CoB(r)$, we use the operadic composition of braids, defined in §§2.1.2-2.1.3, together with an operadic composition of the braid colorings which we define as follows.

Let $\alpha \in \mathtt{Mor}\, CoB(m)$ and $\beta \in \mathtt{Mor}\, CoB(n)$ be colored braids. Intuitively, to define the composite $\alpha \circ_i \beta \in \mathtt{Mor}\, CoB(m+n-1)$, we insert the ith input braid β in the strand of α colored by $i \in \{1,\ldots,m\}$. We simply apply the standard operadic shift $k \mapsto k+i-1$ to the index of the strands of β in the composite braid, the shift $k \mapsto k+n-1$ to the index of the strands of α when $i < k$, and this gives the coloring of $\alpha \circ_i \beta$. In comparison with the process of §§2.1.2-2.1.3, we simply use an ordering defined by the color indexing of the strands of α instead of the natural ordering of the source points on the line $y = t = 0$. Thus, the composition of braids in the colored braid groupoid is formally defined by the composition operation of §§2.1.2-2.1.3 up to an input reordering, which we determine from the source permutation of the braid α. To illustrate this process, we give an instance of partial composition operation $\alpha \circ_1 \beta = \alpha(\beta, 1)$ in Figure 2.15. In order to ease the understanding of this picture, we have added dotted lines marking the array in which the braid β is inserted.

In the coset representation of morphism sets (§2.2.9), the partial composite $\mathtt{Mor}_{CoB(m)}(s,t) \times \mathtt{Mor}_{CoB(n)}(u,v) \xrightarrow{\circ_i} \mathtt{Mor}_{CoB(m+n-1)}(s \circ_i u, t \circ_i v)$ maps elements $\alpha \in q_*^{-1}(t^{-1}s)$ and $\beta \in q_*^{-1}(v^{-1}u)$ to the composite braid $\alpha \circ_{s^{-1}(i)} \beta$ which

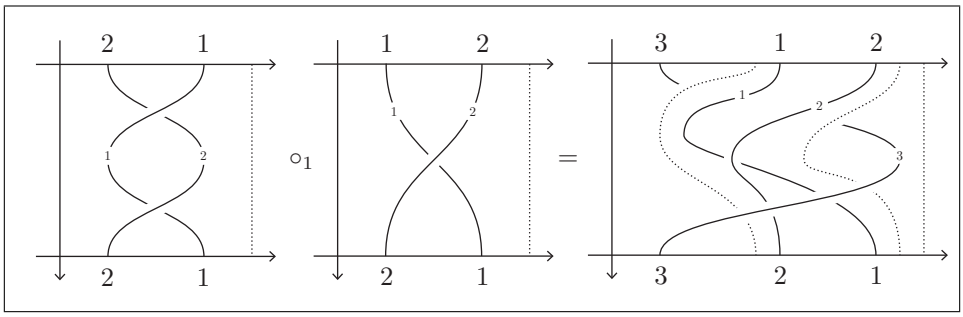


FIGURE 2.15. An operadic composition of colored braids

has $q_*(\alpha \circ_{s^{-1}(i)} \beta) = (t \circ_i v)^{-1} \cdot (s \circ_i u)$ as associated permutation. This operation obviously preserves the groupoid structure, and hence, gives a morphism $\circ_i : CoB(m) \times CoB(n) \to CoB(m+n-1)$ in the category of groupoids.

The verification of the operad axioms is straightforward from the results already obtained for the braid operad in §2.1.

The construction of this composition clearly extends to the degenerate case of a colored braid β with an empty set of strands, and we readily deduce from this observation that the operad CoB has a unitary extension CoB_+. The deletion operation $u^* : CoB_+(n) \to CoB_+(m)$ can actually be identified with a natural generalization to colored braids of the removal operations on braid groups, as described in §2.1.4, just like the operadic composition of colored braids define a generalization of the operadic composition of braids.

The definition of the colored braid operad is now complete and we aim to prove:

THEOREM 2.A (Z. Fiedorowicz [34]). *The classifying space operad $B(CoB)$ associated to the operad of colored braids CoB is an E_2-operad, and the operad $B(CoB)_+$, associated to the unitary extension of CoB, forms an instance of unitary E_2-operad similarly.*

We focus on the non-unitary context. The unitary extension of our result follows, again, from a straightforward adaptation of the arguments.

The idea is to identify $B(CoB)$ with the symmetrization of a contractible braided operad in order to deduce this result from the recognition theorem of §2.1. This contractible braided operad is formed by a collection of contractible classifying spaces EB_r naturally associated to the braid groups B_r. In a preliminary stage, we review the general definition of these contractible classifying spaces EG, which can be associated to any group G.

2.2.12. *Translation categories and their classifying spaces.* First, to a group G, we associate a translation category E_G which has $\mathtt{Ob}\, E_G = G$ as object set, and of which morphism sets are reduced to a single element $\mathtt{Mor}_{E_G}(\alpha, \beta) = \{\beta^{-1}\alpha\}$, for all $\alpha, \beta \in G$. This element $\beta^{-1}\alpha$ represents the right translation connecting β and α in G. This interpretation motivates the name of translation category assigned to E_G. The translation category E_G obviously forms a groupoid, for any group G.

The translation category E_G is naturally equipped with a left G-action, which assigns a functor $g_* : E_G \to E_G$ to each $g \in G$. This functor is given by the left

translation operation $g_*(\alpha) = g\alpha$ at the object level, and by the identity of the translation elements $(g\beta)^{-1}(g\alpha) = \beta^{-1}\alpha$ at the morphism level.

The classifying space associated to the translation category E_G is usually denoted by $EG = B(E_G)$. By definition of E_G, the n-simplices of this classifying space have a representation of the form

$$\underline{\alpha} = \{\alpha_0 \xrightarrow{\alpha_1^{-1}\alpha_0} \alpha_1 \xrightarrow{\alpha_2^{-1}\alpha_1} \cdots \xrightarrow{\alpha_n^{-1}\alpha_{n-1}} \alpha_n\},$$

where $(\alpha_0, \ldots, \alpha_n)$ runs over G^{n+1}. The morphisms occurring in this simplex are determined by the sequence of vertices $(\alpha_0, \ldots, \alpha_n)$, as we see in the above expression. Faces d_i (respectively, degeneracies s_j) are given by the omission (respectively, repetition) of a vertex α_i (respectively, α_j). By functoriality of the classifying space construction, the simplicial set EG inherits a left G-action from the translation category E_G. This group action maps a simplex $\underline{\alpha}$ as above to the simplex such that:

$$g_*(\underline{\alpha}) = \{g\alpha_0 \xrightarrow{(g\alpha_1)^{-1}(g\alpha_0)} g\alpha_1 \xrightarrow{(g\alpha_2)^{-1}(g\alpha_1)} \cdots \xrightarrow{(g\alpha_n)^{-1}(g\alpha_{n-1})} g\alpha_n\}.$$

Together with EG, we have a natural map $p_* : EG \to BG$ towards the classifying space of the group G. If we regard the group G as a category with a single object $*$, then this classifying space map is induced by the functor $p : E_G \to G$ defined by $p(\alpha) = *$ on objects and by $p(\beta^{-1}\alpha) = \beta^{-1}\alpha$ on morphisms. The following standard observations provide the usual motivations for the introduction of this space EG together with the map $p_* : EG \to BG$:

Observation 2.2.13.

(a) *The groupoid E_G is equivalent to a point and hence, the associated classifying space $EG = B(E_G)$ is contractible.*
(b) *The action of G on EG is free.*
(c) *The mapping $p_* : EG \to BG$ goes down to an isomorphism $EG/G \xrightarrow{\simeq} BG$ on the quotient space EG/G.*

These observations follow from immediate inspections. In the topological context, the contractibility of the simplicial set EG implies that the space $|EG|$ is contractible. The free action of G on EG gives rise to a free and proper action at the topological level. Furthermore, the mapping $p_* : EG \to BG$ induces a homeomorphism $|EG|/G = |EG/G| \xrightarrow{\simeq} |BG|$ since we have $|X/G| = |X|/G$ for any space X equipped with a G-action.

We now consider the translation categories E_{B_r} associated to the braid groups B_r. We immediately see that the collection E, formed by this sequence of groupoids $\mathsf{E}(r) = E_{B_r}$, inherits a natural braided operad structure from the braid groups: the natural action of the group B_r on each E_{B_r} gives the braided structure of the collection E; the identity $E_{B_1} = pt$ provides the operadic unit $\eta : pt \to \mathsf{E}(1)$; and the composition morphism is the functor $\circ_i : E_{B_m} \times E_{B_n} \to E_{B_{m+n-1}}$ determined by the operadic composition of braids at the level of the object sets $\mathtt{Ob}\, E_{B_n} = B_n$. We also easily check, by using the result obtained at the level of braid groups, that these morphisms fulfill the structure axioms of braided operads.

We apply the symmetrization functor of Proposition 2.1.5 in the context of the category of categories $\mathcal{B}ase = \mathcal{C}at$ to form a symmetric operad $\mathtt{Sym}\,\mathsf{E}$ naturally associated to E. We have the following identification:

LEMMA 2.2.14. *The colored braid operad CoB is identified, as a symmetric operad in groupoids, with the symmetrization of this braided operad E, formed by the translation categories $\mathsf{E}(r) = E_{B_r}$ of the braid groups B_r.*

PROOF. We use the definition of §2.2.9, where we identify the morphism sets of the groupoid $CoB(r)$ with the cosets $\mathtt{Mor}_{CoB(r)}(u,v) = q_*^{-1}(v^{-1}u)$ naturally associated to the morphism $q_* : B_r \to \Sigma_r$. We have an obvious functor $q_* : E_{B_r} \to CoB(r)$, given by the map $q_* : B_r \to \Sigma_r$ on the object set $\mathtt{Ob}\, E_{B_r} = B_r$, and by the embedding $\{\beta^{-1}\alpha\} \hookrightarrow q_*^{-1}(q_*(\beta)^{-1}q_*(\alpha))$ on each morphism set $\mathtt{Mor}_{E_{B_r}}(\alpha,\beta) = \{\beta^{-1}\alpha\}$, for $\alpha,\beta \in B_r$. We immediately see that this functor carries the action of B_r on E_{B_r} to the natural action of Σ_r on $CoB(r)$, the action of $P_r \subset B_r$ to a trivial action. We can also readily check, by unraveling the definition of a quotient object in the category of categories, that $q_* : E_{B_r} \to CoB(r)$ identifies $CoB(r)$ with the quotient category E_{B_r}/P_r.

We have already observed that $q_* : E_{B_r} \to CoB(r)$ transports the B_r-action on E_{B_r} to the natural Σ_r-action attached to $CoB(r)$. We readily obtain that q_* preserves the operadic composition structures too by using the coset definition of this structure in §2.2.11. Accordingly, the collection of functors $q_* : E_{B_r} \to CoB(r)$ defines a morphism $q_* : \mathsf{E} \to CoB$ in the category of braided operads, and the relation $E_{B_r}/P_r = CoB(r)$ immediately implies that this morphism identifies CoB with the symmetric operad naturally associated to E. □

The conclusion of Proposition 2.2.5 remains obviously valid in the context of braided operads. In the particular case of the translation categories associated to braid groups $\mathsf{E}(r) = E_{B_r}$, we deduce from this assertion:

FACT 2.2.15. *The collection of classifying spaces $B\,\mathsf{E}(r) = B(E_{B_r}) = EB_r$ inherits a braided operad structure.*

The geometric realization and classifying space functors naturally commute with quotients under group actions. In the case of the symmetrization functor $\mathtt{Sym}$, which is essentially given by such a quotient process, this observation implies:

OBSERVATION 2.2.16. *We have operad identities $\mathtt{Sym}\,|B\,\mathsf{E}| = |\mathtt{Sym}\,B\,\mathsf{E}| = |B(\mathtt{Sym}\,\mathsf{E})|$.*

Thus, from the identity $\mathtt{Sym}\,\mathsf{E} = CoB$ established in Lemma 2.2.14, we conclude that $|B(CoB)|$ is identified with the symmetrization of the contractible braided operad $|B\,\mathsf{E}|$. The braided operad $B\,\mathsf{E}$ is also contractible by observation 2.2.13 and the braid group B_r operates freely and properly at the level of the topological space $|B\,\mathsf{E}(r)| = EB_r$. By Theorem 2.1.14, these assertions imply that $|B(CoB)| = \mathtt{Sym}\,|B\,\mathsf{E}|$ forms an E_2-operad, as claimed in Theorem 2.A. □

2.2.17. *Remark.* The category of algebras associated with the unitary operad CoB_+ consists of braided categories equipped with a tensor product which is unitary and associative in the strict sense. This statement is an operadic version of a result of Joyal and Street [53] on free braided categories. In the next chapter, we establish a similar result for the operad of parenthesized braids PaB_+, which we associate to braided categories of which tensor product is unitary in the strict sense, but associative up to coherent isomorphisms (not necessarily given by the identity of objects).

2.3. Fundamental groupoids and operads

In the previous section, we recalled that the definition an Eilenberg-MacLane space gives a well determined homotopy type in the category of topological spaces. Hence, the identification of the spaces $\mathsf{D}_2(r)$ with Eilenberg-MacLane spaces $K(P_r, 1)$, obtained in §2.0, fully determines the homotopy type of the spaces underlying the little discs operad D_2. However, we have needed to replace the pure braid groups P_r by groupoids of colored braids $CoB(r)$ in order to retrieve a whole operad structure at the level of our classifying space model of the little 2-discs operad.

The purpose of this section is to explain the source of our problems and to give an a posteriori explanation for the introduction of colored braids in §2.2.

In short, the pure braid group P_r represents the fundamental group of the little 2-discs space $\mathsf{D}_2(r)$, and involves, by definition of the fundamental group, the choice of a base point in $\mathsf{D}_2(r)$. The problem comes from this choice: base points can not be chosen coherently with respect to all structure operations attached to an operad. The natural idea is to replace fundamental groups by fundamental groupoids in order to obtain the right objects, extending fundamental groups, on which the structures attached to topological operads can be transported. In the case of the little 2-discs operad D_2, we prove precisely that the fundamental groupoids of the spaces $\mathsf{D}_2(r)$ form an operad in groupoids which is equivalent to the colored braid operad of §2.2. The ultimate purpose of this section is to establish this result. Before, we quickly recall the definition of the fundamental groupoid, as a functor from spaces to groupoids, and we establish that the collection of fundamental groupoids attached to a topological operad inherit a full operad structure in general.

2.3.1. *Fundamental groupoids.* The fundamental groupoid of a topological space is denoted by πX. The object set of this groupoid πX is the underlying point-set of the space X. Let $x, y \in X$. The morphisms from x to y in πX are the homotopy classes of paths $\alpha : [0,1] \to X$ with $\alpha(0) = x$ as prescribed origin and $\alpha(1) = y$ as prescribed endpoint. The composition of morphisms in πX is given by the usual composition operation on paths, and extends the composition of based loops considered in the definition of the fundamental group. The unit relation, associativity and the existence of inverses in πX is proved by a straightforward extension of the arguments classically considered in the context of fundamental groups.

The fundamental group of X at a base point $x_0 \in X$ is clearly identified with the automorphism set of x_0 in the fundamental groupoid

$$\pi_1(X, x_0) = \mathtt{Mor}_{\pi X}(x_0, x_0)$$

and we have an isomorphism connecting $x_0 \in X$ to another point $x \in X$ in πX if an only if x_0 and x belongs to the same path connected component of X.

Thus, if we regard a group as a groupoid with one object, then we can also identify the fundamental group $\pi_1(X, x_0)$ at a base point x_0 with the full subcategory of πX generated by the single object $\{x_0\} \subset X = \mathtt{Ob}\,\pi X$, and, when X is path connected, the embedding $\pi_1(X, x_0) \hookrightarrow \pi X$, which arises from this categorical interpretation of the fundamental group, defines an equivalence of categories. In general, the fundamental groupoid is equivalent (as a groupoid) to the coproduct $\coprod_{[x_0] \in \pi_0(X)} \pi_1(X, x_0)$ formed by picking a representative $x_0 \in C$ in each path component $[x_0] = C \in \pi_0(X)$.

Even in the path connected case, we usually have no canonical choice for a single base point x_0 in X. In subsequent applications, we rather consider subsets

$A \subset X$ and the full subcategories, denoted by $\pi X_{\restriction A}$, which such subsets generate. The embedding $\pi X_{\restriction A} \hookrightarrow \pi X$ defines an equivalence of groupoids as soon as A includes a representative of each path connected of X.

The mapping $\pi : X \mapsto \pi X$ clearly gives a functor from spaces to groupoids, and usual results on fundamental groups extend to fundamental groupoids. But, in the groupoid context, we need to take care of the difference between the notion of isomorphism and the notion of equivalence. For instance, a homeomorphism induces an isomorphism on fundamental groupoids, but a homotopy equivalence $f : X \xrightarrow{\sim} Y$ induces a groupoid equivalence $f_* : \pi X \xrightarrow{\sim} \pi Y$, and no more, unless f is a bijection at the point set level.

In order to study the image of topological operads under the fundamental groupoid functor $\pi : \mathcal{T}op \to \mathcal{G}rd$, we establish as usual that:

Proposition 2.3.2. *The functor $\pi : \mathcal{T}op \to \mathcal{G}rd$ is symmetric monoidal:*

(a) *In the case of a point pt, viewed as the unit object of the category of spaces, we have an obvious identity $\pi\, pt = pt$.*
(b) *In the case of a cartesian product of spaces $X \times Y$, the maps $\pi X \xleftarrow{p_*} \pi(X \times Y) \xrightarrow{q_*} \pi Y$, associated to the canonical projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ give rise to an isomorphism $\pi(X \times Y) \xrightarrow{\simeq} \pi X \times \pi Y$.*
(c) *The comparison isomorphisms obtained in (a-b) fulfill the unit, associativity and symmetry constraints of §II.3.1 too.*

Proof. The proof of assertion (a) is immediate. The proof of assertion (b) reduces to a straightforward extension of arguments classically used in the case of fundamental groups. The definition of the isomorphism $\pi(X \times Y) \xrightarrow{\simeq} \pi X \times \pi Y$ from universal categorical constructions automatically ensures again that the unit, associativity and symmetry constraints of symmetric monoidal functors are fulfilled. □

From this statement, the result of Proposition II.1.4 gives:

Proposition 2.3.3. *Let P be an operad in topological spaces. The collection of groupoids $\pi\,\mathsf{P}(r)$ associated to the spaces $\mathsf{P}(r)$ forms an operad in groupoids naturally associated to P.* □

From Proposition II.1.4, we also deduce (as usual) that the mapping $\pi : \mathsf{P} \mapsto \pi\,\mathsf{P}$ preserves unitary extensions: in the formalism of §I.4.5, we have an identity $\pi(\mathsf{P}_+) = (\pi\,\mathsf{P})_+$ for any unitary operad in topological spaces P_+.

We use the expression of fundamental groupoid attached to the operad P in order to refer to the operad in groupoids $\pi\,\mathsf{P}$ defined in that proposition.

We now aim to prove the following result:

Theorem 2.3.4. *The fundamental groupoid of the little 2-discs operad $\pi\,\mathsf{D}_2$ is related to the colored braid operad CoB of §2.2 by categorical equivalences of operads in groupoids, and similarly in the unitary context.*

Proof. We focus on the verification of the result in the non-unitary context. The unitary case follows from a straightforward extension of our verifications (as usual).

We use the embedding $\mathsf{D}_1 \hookrightarrow \mathsf{D}_2$, defined in §1.1.5, to identify the operad of little 1-discs D_1 with a suboperad of D_2. For each r, we consider the groupoid

$\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)}$ which we formally define as the full subcategory of $\pi\,\mathsf{D}_2(r)$ generated by the image of $\mathsf{D}_1(r)$ in $\mathsf{D}_2(r)$ as object set. The collection of these groupoids $\pi\,\mathsf{D}_2\,\vert_{\mathsf{D}_1} = \{\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)}\}_{r\in\mathbb{N}}$ forms a suboperad of $\pi\,\mathsf{D}_2$ since the associated object sets $\mathsf{D}_1(r)$ are themselves identified with a suboperad of D_2.

We use this suboperad $\pi\,\mathsf{D}_2\,\vert_{\mathsf{D}_1}$ as an intermediate object between $\pi\,\mathsf{D}_2$ and the colored braid operad CoB. The embeddings $\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)} \hookrightarrow \pi\,\mathsf{D}_2(r)$ are equivalences of categories since each space $\mathsf{D}_2(r)$ is connected. Hence, at the operad level, we obtain that the embedding $\pi\,\mathsf{D}_2\,\vert_{\mathsf{D}_1} \hookrightarrow \pi\,\mathsf{D}_2$ is a categorical equivalence of operads. To complete the proof of the theorem, we define a new categorical equivalence of operads $\pi\,\mathsf{D}_2\,\vert_{\mathsf{D}_1} \xrightarrow{\sim} CoB$ connecting $\pi\,\mathsf{D}_2\,\vert_{\mathsf{D}_1}$ with the colored braid operad CoB. In a preliminary stage, we construct the collection of groupoid equivalences $\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)} \xrightarrow{\sim} CoB(r)$, which will form our operad morphism.

Let $\Pi(r)$ be the subset of the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ formed by the elements of the form $\underline{a}^0_w = ((w(1), 0), \ldots, (w(r), 0))$, where $w \in \Sigma_r$. If we go back to the construction of §2.2.8, where we define the groupoids of colored braids, then we immediately see that the isotopy classes of braids defining the morphisms of the colored operad are nothing but homotopy classes of paths between elements of $\Pi(r)$. In other words, we have a formal identity $CoB(r) = \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r)\,\vert_{\Pi(r)}$, for each $r \in \mathbb{N}$.

The homotopy equivalence $\omega : \mathsf{D}_2(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^2, r)$, given by the disc center mapping (see §§1.2.1-1.2.2), induces an equivalence of fundamental groupoids $\omega_* : \pi\,\mathsf{D}_2(r) \xrightarrow{\sim} \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r)$. In order to connect $\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)} \subset \pi\,\mathsf{D}_2(r)$ with $CoB(r) = \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r)\,\vert_{\Pi(r)}$, we pick a collection of little 2-discs $\underline{c}^0$ in the image of our embedding $\mathsf{D}_1(r) \to \mathsf{D}_2(r)$ so that $\omega(\underline{c}^0) = \underline{a}^0_{id} = ((1, 0), \ldots, (r, 0))$. Then we consider the subset $\Xi(r)$ formed by the elements $\underline{c}^0_w = w_*(\underline{c}^0)$, $w \in \Sigma_r$, in $\mathsf{D}_1(r) \hookrightarrow \mathsf{D}_2(r)$. The disc center mapping is clearly equivariant, so that $\omega(\underline{c}^0_w) = \underline{a}^0_w$, for all $w \in \Sigma_r$, and the equivalence $\pi\omega : \pi\,\mathsf{D}_2(r) \xrightarrow{\sim} \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ induces, by restriction to $\Xi(r) \subset \mathsf{D}_2(r)$, a groupoid isomorphism $\pi\,\mathsf{D}_2(r)\,\vert_{\Xi(r)} \xrightarrow{\simeq} \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r)\,\vert_{\Pi(r)}$.

To recap, we now have a groupoid diagram

$$\begin{tikzcd}
\pi\,\mathsf{D}_2(r)\,\vert_{\Xi(r)} \arrow[r, dotted, "\simeq"] \arrow[d, hook] & \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r)\,\vert_{\Pi(r)} \arrow[r, "="] \arrow[dd, hook] & CoB(r) \\
\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)} \arrow[d, hook] & & \\
\pi\,\mathsf{D}_2(r) \arrow[r, "\sim"'] & \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r) &
\end{tikzcd}$$

where vertical morphisms are embeddings of full subgroupoids, the bottom horizontal morphism is a groupoid equivalence, and the upper horizontal morphism is a groupoid isomorphism. The connectedness of $\mathsf{D}_2(r)$ implies that the vertical embedding $\pi\,\mathsf{D}_2(r)\,\vert_{\Xi(r)} \hookrightarrow \pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)}$ defines an equivalence of groupoids, just like $\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)} \hookrightarrow \pi\,\mathsf{D}_2(r)$. The groupoid equivalence $\pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)} \xrightarrow{\sim} \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r)\,\vert_{\Pi(r)} = CoB(r)$, which we aim to define, is obtained by picking an inverse equivalence of this embedding $\pi\,\mathsf{D}_2(r)\,\vert_{\Xi(r)} \hookrightarrow \pi\,\mathsf{D}_2(r)\,\vert_{\mathsf{D}_1(r)}$.

For that purpose, we essentially have to specify a path connecting any element $\underline{c}$ in the image of the embedding $\mathsf{D}_1(r) \hookrightarrow \mathsf{D}_2(r)$ to a little 2-disc configuration $\underline{c}^0_w \in$

$\Xi(r)$. Indeed, any such path γ represents an isomorphism between $\underline{c}$ and $\underline{c}^0_w$ in the fundamental groupoid $\pi\,\mathsf{D}_2(r)$. The required equivalence maps each object $\underline{c}$ to the corresponding element $\underline{c}^0_w$ in $\Xi(r)$ and is given, at the morphism level, by the composition with the isomorphism $[\gamma] \in \mathtt{Mor}_{\pi\,\mathsf{D}_2(r)}(\underline{c}, \underline{c}^0_w)$ determined by the homotopy class of the path connecting $\underline{c}^0_w$ and $\underline{c}$.

In our context, we perform choices as follows. Recall that the embedding of a configuration of little 1-discs $\underline{c} = (c_1, \ldots, c_r)$ in the interval $\mathring{\mathbb{D}}^1 = [-1, 1]$ determines a linear ordering $i_1 < \cdots < i_r$ of the indices of these 1-discs c_i. In proposition 1.1.6, we use this observation to assign a permutation $w = (i_1, \ldots, i_r)$ to each element $\underline{c} \in \mathsf{D}_1(r)$, and to establish the identity $\pi_0\,\mathsf{D}_1(r) = \Sigma_r$. To an element $\underline{c}$ in the image of $\mathsf{D}_1(r) \hookrightarrow \mathsf{D}_2(r)$, we associate the element $\underline{c}^0_w \in \Xi(r)$ formed by applying the permutation w associated with $\underline{c}$ to the initially chosen configuration of little 2-discs $\underline{c}^0$, which is also in the image of $\mathsf{D}_1(r) \hookrightarrow \mathsf{D}_2(r)$. Essentially, this construction amounts to considering the element $\underline{c}^0_w$ which comes from the same connected component $\mathsf{D}_1(r)_w$ of the 1-disc space $\mathsf{D}_1(r)$ as $\underline{c}$. To define our isomorphism between $\underline{c}$ and $\underline{c}^0_w$, we take a path γ connecting $\underline{c}$ and $\underline{c}^0_w$ in the subspace $\mathsf{D}_1(r)_w \hookrightarrow \mathsf{D}_2(r)$. Since $\mathsf{D}_1(r)_w$ is contractible, the paths of this form have the same homotopy class, and hence define the same isomorphism in the fundamental groupoid. Thus, we finally have an isomorphism $[\gamma] \in \mathtt{Mor}_{\pi\,\mathsf{D}_2(r)}(\underline{c}, \underline{c}^0_w)$ canonically associated to each little configuration $\underline{c}$ of $\mathsf{D}_1(r) \hookrightarrow \mathsf{D}_2(r)$.

The above construction is clearly equivariant with respect to the action of the symmetric group Σ_r. Hence, our choices provide an equivalence of groupoids $\pi\,\mathsf{D}_2(r) \mid_{\mathsf{D}_1(r)} \to \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, r) \mid_{\Pi(r)}$ commuting with the action of permutations. In the case $r = 1$, this morphism trivially maps the unit element of the operad $\pi\,\mathsf{D}_2 \mid_{\mathsf{D}_1}$ to the unit element of CoB since $CoB(1) = \pi\,\mathsf{F}(\mathring{\mathbb{D}}^2, 1) \mid_{\Pi(1)} = pt$. Hence, we essentially have to check that our morphism preserves the operadic composition structures to conclude that our construction provides a categorical equivalence of operad in groupoids from $\pi\,\mathsf{D}_2 \mid_{\mathsf{D}_1}$ to CoB.

The existence of a groupoid equivalence between $\pi\,\mathsf{D}_2(r) \mid_{\mathsf{D}_1(r)}$ and $CoB(r)$ implies that the morphisms of $\pi\,\mathsf{D}_2(r) \mid_{\mathsf{D}_1(r)}$ are composites of paths representing the generating braids τ_i in the little 2-discs space $\mathsf{D}_2(r)$. Since our morphism $\pi\,\mathsf{D}_2 \mid_{\mathsf{D}_1} \xrightarrow{\sim} CoB$ preserves the internal composition structure of groupoids, we are reduced to check the commutation with operadic composites for these generating morphisms. But we can easily see, by going back to our figures, that we retrieve the definition of the operadic composites of generating and identity braids in Proposition 2.1.2 when we form the operadic composites of paths in the little 2-discs operads (see Figure 2.12) corresponding to the generating elements of the braid group.

This inspection completes our verifications and the proof of Theorem 2.3.4. $\square$

2.3.5. *Addenda.* The results of Theorem 2.A and Theorem 2.3.4 are actually not independent though we give a direct proof of each statement. To explain the precise relationship between our results, we move from topological spaces to simplicial objects.

The classifying space construction of §2.2.3 is naturally given as a functor from categories to simplicial sets. The fundamental groupoid construction, considered in this section, has a combinatorial analogue, defined on the category of simplicial sets, and yielding a functor $\pi : \mathcal{S}imp \to \mathcal{G}rd$. This functor represents the left

adjoint of the restriction of the classifying space functor $B : \mathcal{C}at \to \mathcal{S}imp$ to the category of groupoids $\mathcal{G}rd \subset \mathcal{C}at$. The fundamental groupoid functor $\pi : \mathcal{S}imp \to \mathcal{G}rd$ from simplicial sets to groupoids, is also equal to the composite of the topological fundamental groupoid functor $\pi : \mathcal{T}op \to \mathcal{G}rd$ with the realization functor $|-| : \mathcal{S}imp \to \mathcal{T}op$. In one direction, one can prove that the adjunction augmentation $\pi B\,\mathcal{G} \to \mathcal{G}$ is an isomorphism of groupoids, for all $\mathcal{G} \in \mathcal{G}rd$. In the other direction, the adjunction unit $X \to B\pi X$ defines a weak-equivalence of simplicial sets precisely when X is a Kan complex with a trivial homotopy in degree $* > 1$. This approach can be used to prove the homotopy uniqueness of Eilenberg-MacLane spaces of a given type (the assertion recalled in the introduction of this section).

The simplicial version of the fundamental groupoid $\pi : \mathcal{S}imp \to \mathcal{G}rd$ is a symmetric monoidal functor, like the topological one (this result is a variation on the Eilenberg-Zilber correspondence). Therefore, the fundamental groupoid induces a functor $\pi : \mathcal{S}imp\mathcal{O}p \to \mathcal{G}rd\mathcal{O}p$ from simplicial operads to operads in groupoids, which is still left adjoint of the functor $B : \mathcal{G}rd\mathcal{O}p \to \mathcal{S}imp\mathcal{O}p$ defined by the aritywise application of the classifying space functor from groupoids to simplicial sets. By combining these adjoint functors with the singular complex and realization functors, we obtain a chain of adjunctions

$$\begin{tikzcd} \mathcal{T}op\mathcal{O}p \arrow[r, shift right, "\mathtt{Map}(\Delta^\bullet,-)"'] \arrow[r, phantom, "(1)"] & \mathcal{S}imp\mathcal{O}p \arrow[l, shift right, "|-|"'] \arrow[r, shift left, "\pi"] \arrow[r, phantom, "(2)"] & \mathcal{G}rd\mathcal{O}p \arrow[l, shift left, "B"] \end{tikzcd}$$

connecting the category of topological operads and the category of operads in groupoids.

Recall that (1) is a Quillen equivalence of model categories. In (2), we deduce from the corresponding results, obtained at the simplicial set level, that the adjunction augmentation $\pi B\,\mathsf{Q} \to \mathsf{Q}$ defines an isomorphism of operads in groupoids, for all $\mathsf{Q} \in \mathcal{G}rd\mathcal{O}p$, and the adjunction unit $\mathsf{P} \to B\pi\,\mathsf{P}$ defines a weak-equivalence of operads as soon as the underlying spaces of the operad P have a trivial homotopy in degree $* > 1$. From these observations, we see that the existence of weak-equivalences of operads $\mathsf{D}_2 \xleftarrow{\sim} \cdot \xrightarrow{\sim} B\,CoB$, asserted by Theorem 2.A, implies the existence of categorical equivalences of groupoids connecting $\pi\,\mathsf{D}_2$ and $\pi B\,CoB \xrightarrow{\simeq} CoB$. On the other hand, since we observed that the underlying spaces of the little 2-discs operad D_2 are Eilenberg-MacLane spaces, we have weak-equivalences connecting D_2 with the operad $B\pi\,\mathsf{D}_2$. Consequently, the existence of equivalences of operads in groupoids between $\pi\,\mathsf{D}_2$ and CoB, asserted by Theorem 2.3.4, implies the existence of weak-equivalences of simplicial operads connecting D_2 and $B\,CoB$, the claim of Theorem 2.A.

Our adjunctions (1-2) can also be used to give a necessary and sufficient recognition criterion of E_2-operads. Namely, an operad P is E_2 if and only if each space $\mathsf{P}(r)$ has a trivial homotopy in degree $* > 1$ and $\pi\,\mathsf{P}$ is equivalent to the colored braid operad CoB as an operad in groupoids.

2.4. The recognition of $\mathbf{E_n}$-operads for $\mathbf{n > 2}$

The recognition of E_n-operads is more difficult in the case $n > 2$ than in the case $n = 2$, because the underlying spaces of the little n-discs operads are no longer Eilenberg-MacLane spaces when $n > 2$. On the other hand, we do have sufficient

conditions asserting, as in Theorem 2.1.14, that certain operads $\mathtt{Sym}_n\,\mathsf{P}$ obtained by a quotient process from an appropriate contractible object P are E_n.

In the context of Theorem 2.1.14, we consider the category of braided operads, the obvious restriction functor from symmetric operads to braided operads, and the symmetrization functor which represents a left adjoint of this one. Nice analogues of these notions have been introduced by Michael Batanin's with the aim of defining higher dimensional generalizations of fundamental groupoids (see [8] for this part of the program). In Batanin's approach [10, 9, 11], the category of braided operads is replaced by a category of n-operads, which have an underlying collection $\mathsf{P}(\tau)$ indexed by n-level trees, representing certain composition patterns that can be formed from the structure of an n-category. We again have an obvious functor $\mathcal{O}p \to {}_n\,\mathcal{O}p$, from the category of ordinary operads to the category of n-operads, and we consider an n-symmetrization functor in the converse direction $\mathtt{Sym}_n : {}_n\,\mathcal{O}p \to \mathcal{O}p$. In [9], Batanin establishes that the symmetrization of a contractible n-operad forms an E_n-operad. In [10], He proves further that many usual models of E_n-operads, like the Fulton-MacPherson operads (see [44]), can be obtained as such.

These recognition criterions are used to define models of E_n-operads, for each n independently. In [12], Clemens Berger explains that models of the little n-discs operads, regarded as a nested sequence of operads, can be obtained from contractible (symmetric) operads equipped with an appropriate cell structure. The first application of this recognition method, given by Berger himself in [12], is the construction of simplicial models of E_n-operads from a basic simplicial operad, first considered by Barratt-Eccles in [7], and given by an application of the translation category construction of §§2.2.12-2.2.15 to the symmetric groups Σ_n. The obtained E_n-operads arising from the Barratt-Eccles operad are related to simplicial models of n-fold spaces of suspensions $\Omega^n\Sigma^n X$ (defined by Jeff Smith in [84]). Berger's method has also been applied successfully by Jim McClure and Jeff Smith in [74] to prove that a certain operad, defined by natural operations acting on Hochschild cochain complexes, is E_2. This result has lead to a new conceptual proof of the Deligne conjecture claiming the existence of a natural E_2-structure on the Hochschild cochain complex (see the preface of the book).

Other models of E_n-operads, related to the topics studied in the present chapter, arise from the iterated monoidal categories of [5], which generalize the classical braided monoidal categories of quantum algebra ($n = 2$) and yield higher intermediate structures between the standard (noncommutative) monoidal categories ($n = 1$) and symmetric monoidal categories ($n = \infty$).

CHAPTER 3

Malcev Completion of $\mathbf{E_2}$-operads & Grothendieck-Teichmüller Groups

The goal of this chapter is to explain the definition of the isomorphism between the Grothendieck-Teichmüller group and the group of homotopy automorphisms of E_2-operads over the rationals.

In a first step, addressed in §3.1, we give the definition of rational models E_2-operads. First, we quickly review the Malcev completion process (the rationalization) of groups. We observe that this process has an obvious extension to groupoids, which, in turn, gives rise to a suitable rationalization functor on operads in groupoids. In the previous chapter, we proved that the classifying space operad of the colored braid groupoids forms a model of E_2-operad. To define an instance of rational E_2-operad in topological spaces, we will precisely take the classifying space operad associated with the Malcev completion of this operad in groupoids CoB.

We go back to the rationalization of operads in the next chapters, where we introduce an operadic version of Sullivan's models to develop a complete rational homotopy theory of operads.

In a second step, addressed in §3.2, we explain that the pro-unipotent Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$, as defined by Drinfeld in [28], is identified with the automorphism group of an operad in groupoids. This operad is the (rational completion of the) operad of parenthesized braids PaB, mentioned in the book introduction, which we actually define by pulling back the operad of colored braids CoB to objects sets forming a free operad. This pullback operation does not change the homotopy type of classifying spaces but provides an operad in groupoids satisfying better invariance properties than the operad of colored braids.

To complete our account, we explain the definition of a graded version of the Grothendieck-Teichmüller group $GRT^1(\mathbb{Q})$ as a group of operad automorphisms associated to another categorical operad, the operad of parenthesized chord diagrams PaC, and we give a survey of the definition of Drinfeld associators. To be more precise, we explain that Drinfeld associators determine isomorphisms from the operad of parenthesized braids PaB towards the operad of parenthezised chord diagrams PaC. These topics will be included in the final version of this chapter.

The interpretation of the Grothendieck-Teichmüller groups in terms of operad automorphisms, and the similar interpretation of Drinfeld's associators in terms of operad isomorphisms, are reformulations of ideas of [6, 28, 91]. The purpose of §3.2 is to provide detailed arguments for this relationship between Drinfeld's initial definitions and the operadic interpretations. By the way, we prove that the operad of parenthesized braids PaB has the category of braided monoidal categories

with strict unit as associated category of algebras. This observation can be used to give an explicit realization, in terms of free algebras over the operad PaB, of the free braided monoidal categories considered by Joyal and Street in [53].

The statement of our main result, on homotopy automorphisms of E_2-operads, is the purpose of the concluding section of the chapter §3.3. In this chapter, we really start to explain our original results. The relationship of our statements with the existing literature is explained in detail alongside our account, and we give explicit references for the results, borrowed from other authors, which we use in our own work.

Throughout this chapter, we consider both non-unitary and unitary operads, in the sense of §I.4, and we again use the formalism of unitary extensions introduced in §I.4.

3.1. The Malcev completion of operads in groupoids and of E_2-operads

In the appendix chapter §E, we briefly recall that, according to Quillen [77, §A], the adjunction between groups and Hopf algebras can be used to process the Malcev completion of groups.

To be more explicit, we have already explained, in a previous chapter §II, that the free $\Bbbk$-module functor $\Bbbk\{-\} : X \mapsto \Bbbk\{X\}$ induces a functor from sets $\mathcal{S}et$ to augmented cocommutative coalgebras $\mathcal{C}om^c_+$, which has a functor of group-like elements $\mathbb{G} : \mathcal{C}om^c_+ \to \mathcal{S}et$ as right adjoint (see §II.0.5). Recall that the set of group-like elements $\mathbb{G}(C)$ of an (augmented cocommutative) coalgebra C consists of the elements $c \in C$ such that $\epsilon(c) = 1$ and $\Delta(c) = c \otimes c$.

The coalgebra $\Bbbk\{G\}$ associated to a group G inherits an extra structure, consisting of a unit element, an associative multiplication, and an antipode, so that the mapping $\Bbbk\{-\} : G \mapsto \Bbbk\{G\}$ actually gives a functor from groups to Hopf algebras. Conversely, the set of group-like elements of a Hopf algebra $\mathbb{G}(H)$ inherits a group structure, and the mapping $\mathbb{G}(-) : H \mapsto \mathbb{G}(H)$ also induces a functor from Hopf algebras to groups. These functors between the category of groups $\mathcal{G}rp$ and the category of Hopf algebras $\mathcal{H}opf\mathcal{G}rp$ are adjoint too.

From now on, we take the field of rationals as ground ring $\Bbbk = \mathbb{Q}$. To obtain the Malcev completion of groups, we consider an extension of this adjunction relation

$$\widehat{\mathbb{Q}}\{-\} : \mathcal{G}rp \rightleftarrows \widehat{\mathcal{H}opf\mathcal{G}rp} : \widehat{\mathbb{G}}$$

where the category of plain Hopf algebras $\mathcal{H}opf\mathcal{G}rp$ is replaced by a category of complete Hopf algebras $\widehat{\mathcal{H}opf\mathcal{G}rp}$. The complete Hopf algebra $\widehat{\mathbb{Q}}\{G\}$ associated to a group G can be defined explicitly as the completion $\widehat{\mathbb{Q}}\{G\} = \lim_n \mathbb{Q}\{G\}/\,\mathtt{I}^n\,\mathbb{Q}\{G\}$ of the Hopf algebra associated to G with respect to the powers of the augmentation ideal $\mathtt{I}\,\mathbb{Q}\{G\} = \ker(\epsilon : \mathbb{Q}\{G\} \to \mathbb{Q})$. The Malcev completion functor on groups can formally be defined as the composite functor $\widehat{Id} = \widehat{\mathbb{G}}(\widehat{\mathbb{Q}}\{-\})$ arising from this adjunction relation with complete Hopf algebras.

The first purpose of this section is to check that this completion process for groups formally extends to groupoids. Then we prove that our completion functor preserves symmetric monoidal structures, so that the Malcev completion of groupoids can be applied to operads arity-wise in order to yield a Malcev completion functor on operads in groupoids. Some care is necessary when we deal with groupoids, and not all arguments are generalizable from groups to groupoids, since

the morphism sets of groupoids, as opposed to the underlying set of a group, are not naturally pointed.

To begin with, we define a notion of Hopf groupoid, extending the classical notion of Hopf algebra, which we need to form the Hopf side of our completion process in the groupoid context. In summary, our Hopf groupoids are groupoids enriched in coalgebras. The main purpose of the next paragraphs is to unravel this definition.

3.1.1. *Hopf categories.* Before introducing Hopf groupoids, we make explicit the definition of the notion of a Hopf category, which we obtain by specializing the general definition of an enriched category (see §§0.9-0.10) to the coalgebra context. We formally define a Hopf category as a small category $\mathcal{C}$, together with a collection of hom-coalgebras $\mathtt{Hom}_{\mathcal{C}}(X,Y)$, $X,Y \in \mathtt{Ob}\,\mathcal{C}$, equipped with morphisms

$$\text{(a)} \qquad \Bbbk \xrightarrow{id_X} \mathtt{Hom}_{\mathcal{C}}(X,X),$$

$$\text{(b)} \qquad \mathtt{Hom}_{\mathcal{C}}(Y,Z) \otimes \mathtt{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\circ} \mathtt{Hom}_{\mathcal{C}}(X,Z),$$

given for any $X,Y,Z \in \mathtt{Ob}\,\mathcal{C}$, and satisfying the usual unit and associativity axioms of categories within the symmetric monoidal category of coalgebras (see §0.9). We also assume that coalgebra morphisms $\iota : \Bbbk\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\} \to \mathtt{Hom}_{\mathcal{C}}(X,Y)$ are given for any $X,Y \in \mathtt{Ob}\,\mathcal{C}$, so that the unit and composition structure of the hom-objects $\mathtt{Hom}_{\mathcal{C}}(X,Y)$ extends the unit and composition structure of the plain set-theoretic category given with $\mathcal{C}$ (see §0.10).

The morphism $id_X : \Bbbk \to \mathtt{Hom}_{\mathcal{C}}(X,X)$, which models the identity morphism associated to any object $X \in \mathtt{Ob}\,\mathcal{C}$, is formally determined by a group-like element $id_X \in \mathtt{Hom}_{\mathcal{C}}(X,X)$ such that $id_X(1) = id_X$. Furthermore, the object $\Bbbk\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}$ arising from the general construction of §0.10 can be identified with the coalgebra $\Bbbk\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\}$, associated to the set $\mathtt{Mor}_{\mathcal{C}}(X,Y)$, and of which each generating element $f \in \mathtt{Mor}_{\mathcal{C}}(X,Y)$ is group-like. Consequently, our morphism $\iota : \Bbbk\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\} \to \mathtt{Hom}_{\mathcal{C}}(X,Y)$ assigns a group like element $\iota(f) \in \mathtt{Hom}_{\mathcal{C}}(X,Y)$ to any $f \in \mathtt{Mor}_{\mathcal{C}}(X,Y)$. Recall that $\iota(id_X) = id_X$, by preservation of unit structures, and that the composition with the image of morphisms in hom-objects makes the collection of hom-objects into a bifunctor $\mathtt{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}om_c^+$ (see §0.10 for details).

In practice, one can define Hopf categories by giving the object set $\mathtt{Ob}\,\mathcal{C}$, the coalgebras $\mathtt{Hom}_{\mathcal{C}}(X,Y)$, the associated unit and composition structure, and define the morphism sets $\mathtt{Mor}_{\mathcal{C}}(X,Y)$ as particular subsets of group-like elements in the hom-objects $\mathtt{Hom}_{\mathcal{C}}(X,Y)$. In this context, the coalgebra morphism

$$\iota : \Bbbk\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\} \to \mathtt{Hom}_{\mathcal{C}}(X,Y),$$

associated to any pair of objects (X,Y) is defined by adjunction from the set embedding $\mathtt{Mor}_{\mathcal{C}}(X,Y) \subset \mathbb{G}(\mathtt{Hom}_{\mathcal{C}}(X,Y))$. On the other hand, we soon deal with Hopf categories with a prescribed underlying category in sets. Our initial definition is more natural in this setting, and therefore, we use this approach rather than the above identification. Nonetheless, we often assume that the natural coalgebra morphisms $\iota : \Bbbk\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\} \to \mathtt{Hom}_{\mathcal{C}}(X,Y)$ associated to a Hopf category $\mathcal{C}$ are adjoint to embeddings $\iota_\flat : \mathtt{Mor}_{\mathcal{C}}(X,Y) \hookrightarrow \mathbb{G}(\mathtt{Hom}_{\mathcal{C}}(X,Y))$. Thus, any Hopf category which we consider can be put in the above form, with morphisms identified with particular group-like elements of the hom-objects.

3.1.2. *Hopf groupoids.* We define a Hopf groupoid as a Hopf category $\mathcal{G}$ equipped with an extra inversion operation $\sigma : \mathtt{Hom}_{\mathcal{G}}(X,Y) \to \mathtt{Hom}_{\mathcal{G}}(Y,X)$, defined for any $X, Y \in \mathtt{Ob}\,\mathcal{G}$, and so that the following diagram commutes

$$\begin{tikzcd}
\mathtt{Hom}_{\mathcal{G}}(X,X) & & \mathtt{Hom}_{\mathcal{G}}(X,Y) \arrow[ll, "\eta\epsilon"'] \arrow[rr, "\eta\epsilon"] \arrow[d, "\Delta"] & & \mathtt{Hom}_{\mathcal{G}}(Y,Y) \\
\mathtt{Hom}_{\mathcal{G}}(Y,X)\otimes\mathtt{Hom}_{\mathcal{G}}(X,Y) \arrow[u, "\mu"] & & \mathtt{Hom}_{\mathcal{G}}(X,Y)\otimes\mathtt{Hom}_{\mathcal{G}}(X,Y) \arrow[ll, "\sigma\otimes id"] \arrow[rr, "id\otimes\sigma"'] & & \mathtt{Hom}_{\mathcal{G}}(X,Y)\otimes\mathtt{Hom}_{\mathcal{G}}(Y,X) \arrow[u, "\mu"']
\end{tikzcd} ,$$

for any $X, Y \in \mathtt{Ob}\,\mathcal{C}$. This diagram is naturally a coalgebra analogue of the inversion relation in groupoids. For a group-like element f, representing a morphism in $\mathcal{G}$, the relations read $f \cdot \sigma(f) = id$, $\sigma(f) \cdot f = id$, and hence amounts to the requirement that f is invertible with $\sigma(f) = f^{-1}$ as inverse. From this observation, we also deduce that the underlying category of a Hopf groupoid is a groupoid (assuming, as stated, that morphisms form subsets of the set of group-like elements in hom-objects). One can also prove, by an easy extension of the standard argument in the Hopf algebra context, that the inversion operation in a Hopf groupoid is unique and satisfies the relation $\sigma(id_X) = id_X$,for any $X \in \mathtt{Ob}\,\mathcal{G}$, as well as $\sigma(u \cdot v) = \sigma(v) \cdot \sigma(u)$, for any pair of composable homomorphisms $u \in \mathtt{Hom}_{\mathcal{G}}(Y,Z)$, $v \in \mathtt{Hom}_{\mathcal{G}}(X,Y)$.

We immediately see that a Hopf algebra is identified with a Hopf groupoid with one object.

3.1.3. *Morphisms of Hopf categories and of Hopf groupoids.* We have a natural notion of morphism associated to Hopf categories, which we use to form a category $\mathcal{H}opf\mathcal{C}at$ with Hopf categories as objects. To be explicit, a morphism of Hopf categories $\phi : \mathcal{C} \to \mathcal{D}$ consists of a functor between the underlying set-theoretic categories of $\mathcal{C}$ and $\mathcal{D}$, together with a collection of coalgebra morphisms

$$\text{(a)} \qquad \mathtt{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\phi} \mathtt{Hom}_{\mathcal{D}}(\phi X, \phi Y),$$

given for all $X, Y \in \mathcal{C}$, preserving the enriched category unit and products on hom-coalgebras, and making the following diagram commute

$$\text{(b)} \qquad \begin{tikzcd}
\Bbbk\{\mathtt{Mor}_{\mathcal{C}}(X,Y)\} \arrow[r, "\phi"] \arrow[d] & \Bbbk\{\mathtt{Mor}_{\mathcal{D}}(\phi X, \phi Y)\} \arrow[d] \\
\mathtt{Hom}_{\mathcal{C}}(X,Y) \arrow[r, dotted, "\phi"'] & \mathtt{Hom}_{\mathcal{D}}(\phi X, \phi Y)
\end{tikzcd} ,$$

for all $X, Y \in \mathtt{Ob}\,\mathcal{C}$, where we consider the natural morphisms, attached to $\mathcal{C}$ and $\mathcal{D}$, identifying the (set-theoretic morphisms) of these categories with group-like homomorphisms in the Hopf category structure.

We have a category of Hopf groupoids $\mathcal{H}opf\mathcal{G}rd$ as well, which we simply define as the full subcategory of the category of Hopf categories generated by Hopf groupoids. We should note that a morphism of Hopf groupoids automatically preserves the extra structure given by the inversion operations (this assertion is a variation of the uniqueness of inversion morphisms in Hopf algebras).

In subsequent constructions, we also consider subcategories of the category of Hopf categories $\mathcal{H}opf\mathcal{C}at$ formed by Hopf categories with a prescribed object set $\mathcal{X}$. We use the notation $\mathcal{H}opf\mathcal{C}at_{\mathcal{X}}$ to refer to such a category, of which morphisms are the morphisms of Hopf categories that reduced to the identity of $\mathcal{X}$ at the object level. We similarly consider subcategories $\mathcal{H}opf\mathcal{G}rd_{\mathcal{X}}$ of the category of

Hopf groupoids $\mathcal{H}opf\mathcal{G}rd$ formed by Hopf groupoids with a prescribed object set $\mathcal{X}$.

We have the following result, extending the classical adjunction relation between groups and Hopf algebras:

PROPOSITION 3.1.4.

(a) *The coalgebras* $\Bbbk\{\mathtt{Mor}_{\mathcal{G}}(X,Y)\}$ *associated to the morphism sets of a groupoid* $\mathcal{G}$ *define the hom-objects of a Hopf groupoid* $\Bbbk\{\mathcal{G}\}$ *with* $\mathcal{G}$ *as underlying groupoid in sets.*

(b) *The set of group-like elements* $\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(X,Y))$ *associated to the hom-coalgebras of a Hopf groupoid* $\mathcal{H}$ *form the hom-objects of a groupoid* $\mathbb{G}(\mathcal{H})$*, naturally associated to* $\mathcal{H}$*, and so that the mapping* $\mathbb{G}:\mathcal{H}\mapsto\mathbb{G}(\mathcal{H})$ *gives a right adjoint of the functor* $\Bbbk\{-\}:\mathcal{G}rd\to\mathcal{H}opf\mathcal{G}rd$ *defined in assertion (a).*

PROOF. The definition of the Hopf groupoid $\Bbbk\{\mathcal{G}\}$ in assertion (a) is a straightforward extension of the definition of the Hopf algebra $\Bbbk\{G\}$ in the group context, and similarly as regards the definition of a groupoid structure on the group-like element sets $\mathbb{G}(\mathtt{Hom}_{\mathcal{H}}(X,Y))$ associated to a Hopf groupoid $\mathcal{H}$ in assertion (b). One can also readily check that the adjunction relations between coalgebra morphisms $f:\Bbbk\{X\}\to C$ and set maps $g:X\to\mathbb{G}(C)$ make morphisms of Hopf groupoids $f:\Bbbk\{\mathcal{G}\}\to\mathcal{H}$ correspond to groupoid morphisms $g:\mathcal{G}\to\mathbb{G}(\mathcal{H})$ so that the mappings $\Bbbk\{-\}:\mathcal{G}\mapsto\Bbbk\{\mathcal{G}\}$ and $\mathbb{G}:\mathcal{H}\mapsto\mathbb{G}(\mathcal{H})$ define adjoint functors between the category of groupoids $\mathcal{G}rd$ and the category of Hopf groupoids $\mathcal{H}opf\mathcal{G}rd$. $\square$

To obtain the Malcev completion of groupoids, we consider, as in the group context, an extension of this adjunction relation involving a natural completion functor for Hopf groupoids. We now give the definition of this completion functor, and by the way, we make explicit the structure carried by the completion of a Hopf groupoid.

3.1.5. *The completion of Hopf categories and of Hopf groupoids.* To form the completion of a Hopf category $\mathcal{H}$, we assume that the hom-objects of $\mathcal{H}$ are equipped with decreasing filtrations

$$\mathtt{Hom}_{\mathcal{H}}(X,Y)=\mathtt{F}^0\,\mathtt{Hom}_{\mathcal{H}}(X,Y)\supset\mathtt{F}^1\,\mathtt{Hom}_{\mathcal{H}}(X,Y)\supset\cdots\supset\mathtt{F}^n\,\mathtt{Hom}_{\mathcal{H}}(X,Y)\supset\cdots$$

such that the relation

$$\Delta(\mathtt{F}^n\,\mathtt{Hom}_{\mathcal{H}}(X,Y))\subset\sum_{p=0}^{n}\mathtt{F}^p\,\mathtt{Hom}_{\mathcal{H}}(X,Y)\otimes\mathtt{F}^{n-p}\,\mathtt{Hom}_{\mathcal{H}}(X,Y)$$

holds with respect to the coproduct of each $\mathtt{Hom}_{\mathcal{H}}(X,Y)$, as well as the relation

$$\mathtt{F}^p\,\mathtt{Hom}_{\mathcal{H}}(Y,Z)\cdot\mathtt{F}^q\,\mathtt{Hom}_{\mathcal{H}}(X,Y)\subset\mathtt{F}^{p+q}\,\mathtt{Hom}_{\mathcal{H}}(X,Z)$$

with respect to the composition structure of the category, and so that the augmentation $\epsilon:\mathtt{Hom}_{\mathcal{H}}(X,Y)\to\Bbbk$ vanishes over $\mathtt{F}^n\,\mathtt{Hom}_{\mathcal{H}}(X,Y)$ for all $n>0$.

The completion

$$\widehat{\mathtt{Hom}}_{\mathcal{H}}(X,Y)=\lim_n\mathtt{Hom}_{\mathcal{H}}(X,Y)/\,\mathtt{F}^n\,\mathtt{Hom}_{\mathcal{H}}(X,Y)$$

of each hom-object $\mathtt{Hom}_{\mathcal{H}}(X,Y)$ forms an augmented cocommutative coalgebra in the category of complete $\Bbbk$-modules $\widehat{\mathcal{M}od}$ (a complete augmented cocommutative

coalgebra). Identity morphisms $\Bbbk \xrightarrow{id_X} \widehat{\mathtt{Hom}}_{\mathcal{H}}(X,X)$, simply defined by extending the identity morphism of $\mathtt{Hom}_{\mathcal{H}}(X,X)$ to $\widehat{\mathtt{Hom}}_{\mathcal{H}}(X,X)$, are associated to these completed hom-objects. The composition products of $\mathcal{H}$ also induce composition products

$$\cdot : \widehat{\mathtt{Hom}}_{\mathcal{H}}(Y,Z)\widehat{\otimes}\widehat{\mathtt{Hom}}_{\mathcal{H}}(X,Y) \to \widehat{\mathtt{Hom}}_{\mathcal{H}}(X,Z)$$

so that the completed hom-objects $\widehat{\mathtt{Hom}}_{\mathcal{H}}(X,Y)$ define the hom-objects of an enriched category in the category of complete augmented cocommutative coalgebras (a complete Hopf category) naturally associated to $\mathcal{H}$. We denote this category by $\widehat{\mathcal{H}}$. We still have coalgebra morphisms $\Bbbk\{\mathtt{Mor}_{\mathcal{H}}(X,Y)\} \xrightarrow{\iota} \widehat{\mathtt{Hom}}_{\mathcal{H}}(X,Y)$, extending the morphisms towards $\mathtt{Hom}_{\mathcal{H}}(X,Y)$, and $\widehat{\mathcal{H}}$ has the same underlying set-theoretic category as $\mathcal{H}$.

In the context of Hopf groupoids, we assume that the inversion morphism $\sigma : \mathtt{Hom}_{\mathcal{H}}(X,Y) \to \mathtt{Hom}_{\mathcal{H}}(Y,X)$ preserves the filtration on hom-objects. This requirement implies that σ induces a morphisms $\widehat{\mathtt{Hom}}_{\mathcal{H}}(X,Y) \xrightarrow{\sigma} \widehat{\mathtt{Hom}}_{\mathcal{H}}(Y,X)$, on the completed hom-object, so that a version of the inversion relation of §3.1.2 in the symmetric monoidal category of complete $\Bbbk$-modules holds. Hence, the complete Hopf category associated to a Hopf groupoid forms an analogue, in the category of complete $\Bbbk$-modules, of a Hopf groupoid. We will use the terminology of complete Hopf groupoid to refer to a general structure of this form.

3.1.6. *The category of complete Hopf categories and of complete Hopf groupoids.* We formally define a complete Hopf category as an analogue, in the category of complete $\Bbbk$-modules, of the Hopf categories of §3.1.1, and a complete Hopf groupoid as an analogue of the Hopf groupoids of §3.1.2. In summary, when we deal with complete structures, we simply assume that hom-objects are complete coalgebras, instead of plain coalgebras. Thus, to make our definitions fully explicit, we just have to replace the plain tensor product in the axioms of §§3.1.1-3.1.2 by the completed one. Similarly, we define morphisms of complete Hopf categories by considering morphisms of complete coalgebras (preserving the structure filtration) at the hom-object level instead of morphisms of plain augmented cocommutative coalgebras in the definition of §3.1.3.

We use the notation $\widehat{\mathcal{H}opf\mathcal{C}at}$ (respectively, $\widehat{\mathcal{H}opf\mathcal{G}rd}$) for the category of complete Hopf categories (respectively, groupoids). We also consider the full subcategory of $\widehat{\mathcal{H}opf\mathcal{C}at}$ (respectively, $\widehat{\mathcal{H}opf\mathcal{G}rd}$) formed by complete Hopf categories (respectively, groupoids), with a prescribed object set $\mathcal{X}$. We then simply add this object set $\mathcal{X}$ as a lower script to our notation, as usual.

3.1.7. *The complete Hopf groupoid associated to a groupoid.* The natural filtration of Hopf algebras, arising from the tensor powers of the augmentation ideal, has also a natural generalization in the context of a Hopf groupoid $\mathcal{H}$. We then consider the submodules $\mathtt{I}^n\,\mathtt{Hom}_{\mathcal{H}}(X,Y) \subset \mathtt{Hom}_{\mathcal{H}}(X,Y)$ spanned by all n-fold composites of composable homomorphisms $f_1 \cdot \ldots \cdot f_n$ such that $\epsilon(f_i) = 0$, for each $i = 1,\ldots,n$. We equivalently assume that each f_i lies in the kernel of the augmentation on hom-objects $\mathtt{I}\,\mathtt{Hom}_{\mathcal{H}}(-,-) = \ker\{\mathtt{Hom}_{\mathcal{H}}(-,-) \xrightarrow{\epsilon} \Bbbk\}$ giving the $n = 1$ layer of this submodule sequence. We readily check that the nested sequence of submodules $\mathtt{I}^n\,\mathtt{Hom}_{\mathcal{H}}(X,Y)$ fulfill the requirements of §3.1.5. We can therefore form a complete Hopf groupoid $\widehat{\mathcal{H}}$, naturally associated to $\mathcal{H}$, with $\widehat{\mathtt{Hom}}_{\mathcal{H}}(X,Y) = \lim_n \mathtt{Hom}_{\mathcal{H}}(X,Y)/\,\mathtt{I}^n\,\mathtt{Hom}_{\mathcal{H}}(X,Y)$ as hom-objects.

The mapping $\widehat{Id} : \mathcal{H} \mapsto \widehat{\mathcal{H}}$, defined by processing the completion with respect to this natural filtration, yields a functor $\widehat{Id} : \mathcal{H}opf\mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd}$ from Hopf groupoids to complete Hopf groupoids. On the category of groupoids, now, we consider the functor $\widehat{\Bbbk}\{-\} : \mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd}$ defined as the composite of the Hopf groupoid functor of Proposition 3.1.4 with this completion functor on Hopf groupoids:

$$\mathcal{G}rd \xrightarrow{\Bbbk\{-\}} \mathcal{H}opf\mathcal{G}rd \xrightarrow{\widehat{Id}} \widehat{\mathcal{H}opf\mathcal{G}rd}.$$

Recall that the set of group-like element $\mathbb{G}(C)$ associated to a coalgebra C consists of the elements $c \in C$ such that $\epsilon(c) = 1$ and $\Delta(c) = c \otimes c$. For a complete coalgebra $\widehat{C}$, we similarly set

$$\widehat{\mathbb{G}}(\widehat{C}) = \{c \in \widehat{C} | \epsilon(c) = 1 \text{ and } \Delta(c) = c \widehat{\otimes} c\}.$$

Proposition 3.1.4(b) has the following analogue in the context of complete Hopf groupoids:

Proposition 3.1.8. *The set of group-like elements $\widehat{\mathbb{G}}(\widehat{\mathtt{Hom}}_{\mathcal{H}}(X, Y))$ associated to the (complete) hom-coalgebras of a (complete) Hopf groupoid $\widehat{\mathcal{H}}$ form the hom-objects of a groupoid $\widehat{\mathbb{G}}(\widehat{\mathcal{H}}) \in \mathcal{G}rd$ naturally associated to $\widehat{\mathcal{H}} \in \widehat{\mathcal{H}opf\mathcal{G}rd}$, and the mapping $\widehat{\mathbb{G}} : \widehat{\mathcal{H}} \mapsto \widehat{\mathbb{G}}(\widehat{\mathcal{H}})$ defines a right-adjoint of the functor $\widehat{\Bbbk}\{-\} : \mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd}$.*

Proof. The definition of the groupoid $\widehat{\mathbb{G}}(\widehat{\mathcal{H}})$ is a straightforward generalization of the construction of Proposition 3.1.4(b).

Let $\widehat{C}$ be any complete coalgebra, satisfying $\widehat{C} = \lim_n \widehat{C}/\mathtt{F}^n\widehat{C}$ for some structure filtration $\widehat{C} = \mathtt{F}^0\widehat{C} \supset \cdots \supset \mathtt{F}^n\widehat{C} \supset \ldots$. To prove our adjunction relation, we first observe that we have $\widehat{\mathbb{G}}(\widehat{C}) = \lim_n \mathbb{G}(\widehat{C}/\mathtt{F}^n\widehat{C})$ for any such $\widehat{C}$, where we consider the set of group-like elements of the (discrete) coalgebras $\widehat{C}/\mathtt{F}^n\widehat{C}$ on the right-hand side. This identity follows from a straightforward verification. For a complete Hopf groupoid $\widehat{\mathcal{H}}$, we deduce from this preliminary statement on coalgebras that we have an identity (more properly, an isomorphism) of groupoids $\widehat{\mathbb{G}}(\widehat{\mathcal{H}}) = \lim_n \mathbb{G}(\widehat{\mathcal{H}}/\mathtt{F}^n\widehat{\mathcal{H}})$, induced by the canonical projections $\widehat{\mathbb{G}}(\widehat{\mathcal{H}}) \to \mathbb{G}(\widehat{\mathcal{H}}/\mathtt{F}^n\widehat{\mathcal{H}})$, where we now consider the groupoid of (complete) group-like elements in $\widehat{\mathcal{H}}$, and the groupoids of (discrete) group-like elements in the quotients Hopf groupoids $\widehat{\mathcal{H}}/\mathtt{F}^n\widehat{\mathcal{H}}$.

The definition of a morphism of complete Hopf groupoids implies that any such $f : \widehat{\Bbbk}\{\mathcal{G}\} \to \widehat{\mathcal{H}}$ arises from the limit of a tower of Hopf groupoid morphisms $f_n : \widehat{\Bbbk}\{\mathcal{G}\}/\mathtt{I}^n\widehat{\Bbbk}\{\mathcal{G}\} \to \widehat{\mathcal{H}}/\mathtt{F}^n\widehat{\mathcal{H}}$, where we consider the (discrete) Hopf groupoids $\widehat{\Bbbk}\{\mathcal{G}\}/\mathtt{I}^n\widehat{\Bbbk}\{\mathcal{G}\} = \Bbbk\{\mathcal{G}\}/\mathtt{I}^n\Bbbk\{\mathcal{G}\}$ associated to $\Bbbk\{\mathcal{G}\}$. The adjunction relation of Proposition 3.1.4 implies that the composites of the morphisms f_n with the projections $\Bbbk\{\mathcal{G}\} \to \Bbbk\{\mathcal{G}\}/\mathtt{I}^n\Bbbk\{\mathcal{G}\}$ are associated with a sequence of groupoid morphisms $g_n : \mathcal{G} \to \mathbb{G}(\widehat{\mathcal{H}}/\mathtt{F}^n\widehat{\mathcal{H}})$ which we can lift to the limit $\widehat{\mathbb{G}}(\widehat{\mathcal{H}}) = \lim_n \mathbb{G}(\widehat{\mathcal{H}}/\mathtt{F}^n\widehat{\mathcal{H}})$ in order to obtain a morphism $g : \mathcal{G} \to \widehat{\mathbb{G}}(\widehat{\mathcal{H}})$ naturally associated to $f : \widehat{\Bbbk}\{\mathcal{G}\} \to \widehat{\mathcal{H}}$. We easily check that this mapping defines a one-to-one correspondence so that the functors $\widehat{\Bbbk}\{-\} : \mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd}$ and $\widehat{\mathbb{G}} : \widehat{\mathcal{H}opf\mathcal{G}rd} \to \mathcal{G}rd$ are adjoint to each other as asserted in the proposition. □

The definition of a Hopf groupoid in §3.1.2 implies that the endomorphism coalgebra $\mathtt{Hom}_{\mathcal{H}}(X, X)$ of any object $X \in \mathtt{Ob}\,\mathcal{H}$ in a Hopf groupoid $\mathcal{H}$ forms a Hopf

algebra in the classical sense, just as the endomorphism set $\mathtt{Mor}_{\mathcal{G}}(X,X)$ of any object $X \in \mathtt{Ob}\,\mathcal{G}$ in a groupoid $\mathcal{G}$ forms a group, and similarly in the context of complete Hopf groupoids. We easily check that:

Lemma 3.1.9.

(a) *Let $\mathcal{H}$ be a Hopf groupoid. Suppose that any pair of objects $(X,Y) \in \mathtt{Ob}\,\mathcal{H}$ are connected by a group-like element in $\mathcal{H}$ (equivalently, the underlying set-theoretic groupoid of $\mathcal{H}$ is connected). The endomorphism coalgebras $\widehat{\mathtt{Hom}}_{\mathcal{H}}(X,X)$ of the objects $X \in \mathtt{Ob}\,\mathcal{H}$ in the completion of $\mathcal{H}$ are isomorphic to the completion of the Hopf algebras $\mathtt{Hom}_{\mathcal{H}}(X,X)$ associated to each object $X \in \mathtt{Ob}\,\mathcal{H}$ individually.*

(b) *For a connected groupoid $\mathcal{G} \in \mathcal{G}rd$, the endomorphism coalgebras $\widehat{\mathtt{Hom}}_{\Bbbk\{\mathcal{G}\}}(X,X)$ of the objects $X \in \mathtt{Ob}\,\mathcal{G}$ in the complete Hopf groupoid $\widehat{\Bbbk}\{\mathcal{G}\}$ associated to $\mathcal{G}$ are isomorphic to the complete group algebras $\widehat{\Bbbk}\{\mathtt{Mor}_{\mathcal{G}}(X,X)$ associated to each group $\mathtt{Mor}_{\mathcal{G}}(X,X)$, $X \in \mathtt{Ob}\,\mathcal{G}$, individually.*

Proof. To check the first assertion (a), we just observe that the filtration of §3.1.7, where we consider all composites of composable homomorphisms in $\mathcal{H}$, agrees with the filtration of the Hopf algebra $\mathtt{Hom}_{\mathcal{H}}(X,X)$ by the powers of the augmentation ideal of $\mathtt{Hom}_{\mathcal{H}}(X,X)$, where we only consider composites of endomorphisms of X in $\mathcal{H}$. The latter is obviously included in the former. The converse inclusion immediately follows from our assumption ensuring that we can insert appropriate invertible elements to convert any sequence of composable homomorphisms $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n$ going from $X_0 = X$ to $X_n = X$ into a sequence of endomorphisms of X.

The second assertion of the lemma is a corollary of the first one. □

3.1.10. *The Malcev completion of groupoids.* From now on, we take $\Bbbk = \mathbb{Q}$ as ring of coefficients. We define the Malcev completion $\widehat{\mathcal{G}}$ of a groupoid $\mathcal{G}$ as the image of $\mathcal{G}$ under the composite functor $\widehat{Id} = \widehat{\mathbb{G}}(\widehat{\mathbb{Q}}\{-\})$ on groupoids. We have a natural morphism $\eta : \mathcal{G} \to \widehat{\mathcal{G}}$ given by the unit of the adjunction $\widehat{\mathbb{Q}} : \mathcal{G}rd \rightleftarrows \widehat{\mathcal{H}opf\mathcal{G}}rd : \mathbb{G}$. We say that a groupoid is Malcev complete when this morphism is iso.

Lemma 3.1.9, together with the results obtained for Hopf algebras in [77, §A], readily imply:

Lemma 3.1.11. *The augmentation of the adjunction $\widehat{\mathbb{Q}}\{-\} : \mathcal{G}rd \rightleftarrows \widehat{\mathcal{H}opf\mathcal{G}}rd : \widehat{\mathbb{G}}$ defines an iso $\epsilon : \widehat{\mathbb{Q}}\{\widehat{\mathbb{G}}(\widehat{\mathcal{H}})\} \xrightarrow{\simeq} \widehat{\mathcal{H}}$, for any complete Hopf groupoid $\widehat{\mathcal{H}} \in \widehat{\mathcal{H}opf\mathcal{G}}rd$ of which underlying set-theoretic groupoid is connected.*

Proof. Let $\widehat{\mathcal{H}}$ be any complete Hopf groupoid. The endomorphism group $\mathtt{Mor}_{\widehat{\mathbb{G}}(\widehat{\mathcal{H}})}(X,X)$ in the groupoid $\widehat{\mathbb{G}}(\widehat{\mathcal{H}})$ agrees by definition with the group of group-like elements $\mathbb{G}(\mathtt{Hom}_{\widehat{\mathcal{H}}}(X,X))$ associated to the Hopf algebra $\mathtt{Hom}_{\widehat{\mathcal{H}}}(X,X)$ in the classical Malcev completion process.

The morphism $\epsilon : \widehat{\mathbb{Q}}\{\widehat{\mathbb{G}}(\mathtt{Hom}_{\widehat{\mathcal{H}}}(X,X))\} \to \mathtt{Hom}_{\widehat{\mathcal{H}}}(X,X)$ is iso for each endomorphism Hopf algebra $\mathtt{Hom}_{\widehat{\mathcal{H}}}(X,X)$ by [77, §A, Theorem 3.3]. To address the case of the hom-coalgebra $\mathtt{Hom}_{\widehat{\mathcal{H}}}(X,Y)$ associated to a pair of objects $(X,Y) \in \mathtt{Ob}\,\mathcal{H}$, we just use that the composition with an (invertible) morphism $\alpha \in \mathtt{Mor}_{\mathcal{H}}(X,Y)$ induced an iso between the endomorphism coalgebras associated to X and the homomorphism coalgebras associated to that pair (X,Y). □

From this lemma we immediately conclude:

PROPOSITION 3.1.12. *If $\mathcal{G}$ is a connected groupoid, then the natural morphism $\eta : \widehat{\mathcal{G}} \to \widehat{\widehat{\mathcal{G}}}$, associated to the completion of $\mathcal{G}$, is iso. Furthermore, any groupoid morphism $f : \mathcal{G} \to \widehat{\mathcal{H}}$, towards the Malcev completion of some groupoid $\mathcal{H} \in \mathcal{G}rd$, admits a unique factorization through $\widehat{\mathcal{G}}$.* □

Thus, the Malcev completion functor is idempotent on connected groupoids.

Lemma 3.1.9 also readily implies:

PROPOSITION 3.1.13. *The endomorphism group of an object $X \in \mathtt{Ob}\,\mathcal{G}$ in the Malcev completion $\widehat{\mathcal{G}}$ of a connected groupoid $\mathcal{G}$ is identified with the Malcev completion (in the classical sense) of the group of endomorphisms of X in $\mathcal{G}$.* □

3.1.14. *Symmetric monoidal structures on Hopf categories and Hopf groupoids.* In §2.2.1, we equip the category of categories with the symmetric monoidal structure defined by the cartesian product of categories. In §II.0.3, we observe that the tensor product defines the cartesian product in the category of augmented cocommutative coalgebras.

To Hopf categories $\mathcal{C}$ and $\mathcal{D}$, we now associate the Hopf category $\mathcal{C} \otimes \mathcal{D}$ with the cartesian product $\mathtt{Ob}(\mathcal{C} \otimes \mathcal{D}) = \mathtt{Ob}\,\mathcal{C} \times \mathtt{Ob}\,\mathcal{D}$ as object set, and the coalgebra tensor products $\mathtt{Hom}_{\mathcal{C} \otimes \mathcal{D}}((X,Y),(Z,T)) = \mathtt{Hom}_{\mathcal{C}}(X,Z) \otimes \mathtt{Hom}_{\mathcal{D}}(Y,T)$ as hom-coalgebras. These tensor products inherit identity morphisms and composition products from the hom-coalgebras of $\mathcal{C}$ and $\mathcal{D}$ so that $\mathcal{C} \otimes \mathcal{D}$ forms a Hopf category. We moreover have natural functors $\mathcal{C} \xleftarrow{p} \mathcal{C} \otimes \mathcal{D} \xrightarrow{q} \mathcal{D}$ given by the natural projections $\mathtt{Ob}\,\mathcal{C} \xleftarrow{p} \mathtt{Ob}\,\mathcal{C} \times \mathtt{Ob}\,\mathcal{D} \xrightarrow{q} \mathtt{Ob}\,\mathcal{D}$ on object sets, and yielded by the tensor products with augmentation morphisms $\mathtt{Hom}_{\mathcal{C}}(X,Z) \xleftarrow{\epsilon \otimes id} \mathtt{Hom}_{\mathcal{C}}(X,Z) \otimes \mathtt{Hom}_{\mathcal{D}}(Y,T) \xrightarrow{id \otimes \epsilon} \mathtt{Hom}_{\mathcal{D}}(Y,T)$ on hom-coalgebras (in §II.0.3, we precisely use these morphisms to identify the tensor product with the cartesian product in the category of coalgebras). This Hopf category $\mathcal{C} \otimes \mathcal{D}$ actually represents the cartesian product of $\mathcal{C}$ and $\mathcal{D}$ in the category of Hopf categories.

We can replace the plain tensor product by the completed one to define an analogous tensor product construction $\widehat{\mathcal{C}} \widehat{\otimes} \widehat{\mathcal{D}}$ in the context of complete Hopf categories. We also readily see that the complete Hopf category $\widehat{\mathcal{C}} \widehat{\otimes} \widehat{\mathcal{D}}$ obtained by this operation represents the cartesian product of $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ in the category of complete Hopf categories.

In §2.2.1, we observe that the cartesian product of groupoids $\mathcal{G} \times \mathcal{H}$, formed in the category of small categories, defines a groupoid and represents the cartesian product of $\mathcal{G}$ and $\mathcal{H}$ in the category of groupoids as well. In the context of Hopf categories, we can similarly prove that the tensor product of Hopf groupoids $\mathcal{G} \otimes \mathcal{H}$ forms a Hopf groupoid and defines the cartesian product of $\mathcal{G}$ and $\mathcal{H}$ in the category of Hopf groupoids too. Similar results hold in the context of complete Hopf groupoids.

We now consider operads in Hopf groupoids and in complete Hopf groupoids. We aim to define a Malcev completion process on operads by using the Malcev completion of groupoids. We actually check that each step of the process of §3.1.10 works well with operads, and for that purpose, we prove:

PROPOSITION 3.1.15.

(a) *The functors* $\widehat{\mathbb{Q}}\{-\} : \mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd}$ *and* $\widehat{\mathbb{G}} : \widehat{\mathcal{H}opf\mathcal{G}rd} \to \mathcal{G}rd$*, between groupoids and complete Hopf groupoids, are symmetric monoidal, as well as the adjunction relation between them.*

(b) *The above functors can be applied arity-wise to operads in order to yield functors* $\widehat{\mathbb{Q}}\{-\} : \mathcal{G}rd\,\mathcal{O}p \to \widehat{\mathcal{H}opf\mathcal{G}rd}\,\mathcal{O}p$ *and* $\widehat{\mathbb{G}} : \widehat{\mathcal{H}opf\mathcal{G}rd}\,\mathcal{O}p \to \mathcal{G}rd\,\mathcal{O}p$ *on operad categories, and we still have an adjunction relation* $\widehat{\mathbb{Q}}\{-\} : \mathcal{G}rd\,\mathcal{O}p \rightleftarrows \widehat{\mathcal{H}opf\mathcal{G}rd}\,\mathcal{O}p : \widehat{\mathbb{G}}$ *at this level.*

This result also holds for the plain version of our functors $\Bbbk\{-\} : \mathcal{G}rd \to \mathcal{H}opf\mathcal{G}rd$ and $\mathbb{G} : \mathcal{H}opf\mathcal{G}rd \to \mathcal{G}rd$ and for any choice of coefficient ring $\Bbbk$.

Proof. The functor $\widehat{\mathbb{G}} : \widehat{\mathcal{H}opf\mathcal{G}rd} \to \mathcal{G}rd$, defining a right-adjoint of $\widehat{\mathbb{Q}}\{-\} : \mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd}$, preserves terminal objects and cartesian products and is therefore symmetric monoidal since we observed that the (complete) tensor product of (complete) Hopf groupoids represent the cartesian product (as well as the cartesian product of groupoids).

For the trivial one-point set groupoid pt, we obviously have $\widehat{\mathbb{Q}}\{pt\} = \mathbb{Q}$. For a cartesian product of groupoids $\mathcal{G} \times \mathcal{H}$, we can easily check that the filtration of §3.1.7 satisfies

$$\mathtt{I}^n\,\mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((X,Y),(Z,T))\} = \sum_{p=0}^{n} \mathtt{I}^p\,\mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(X,Z)\} \otimes \mathtt{I}^{n-p}\,\mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(Y,T)\}$$

in the coalgebra tensor product

$$\begin{aligned}\mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((X,Y),(Z,T))\} &= \mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(X,Z) \times \mathtt{Mor}_{\mathcal{G}}(Y,T)\}\\ &= \mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(X,Z)\} \otimes \mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(Y,T)\}.\end{aligned}$$

Therefore, we have a limit identity

$$\begin{aligned}&\lim_n \mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((X,Y),(Z,T))\}/\,\mathtt{I}^n\,\mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}\times\mathcal{H}}((X,Y),(Z,T))\}\\ &= \lim_{p,q} \mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(X,Z)\}/\,\mathtt{I}^p\,\mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(X,Z)\} \otimes \mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(Y,T)\}/\,\mathtt{I}^q\,\mathbb{Q}\{\mathtt{Mor}_{\mathcal{G}}(Y,T)\}\end{aligned}$$

from which we deduce that the natural morphism $\widehat{\mathbb{Q}}\{\mathcal{G}\times\mathcal{H}\} \to \widehat{\mathbb{Q}}\{\mathcal{G}\}\widehat{\otimes}\widehat{\mathbb{Q}}\{\mathcal{H}\}$, deduced from the canonical projections $\mathcal{G} \xleftarrow{p} \mathcal{G}\times\mathcal{H} \xrightarrow{q} \mathcal{H}$ by using the interpretation of the complete tensor product as a categorical product, is an iso. As usual, the definition of this comparison isomorphism from categorical constructions immediately implies the verification of the unit, associativity and symmetry constraints of §II.3.1. The proof that the adjunction relation is symmetric monoidal is straightfoward as well. □

This proposition implies:

Proposition 3.1.16. *The Malcev completion functor on groupoids* $\widehat{Id} : \mathcal{G}rd \to \widehat{\mathcal{G}rd}$ *is symmetric monoidal (as a composite of symmetric monoidal functors) and can be applied arity-wise to operads in groupoids in order to yield a Malcev completion functor at the level of operad categories* $\widehat{Id} : \mathcal{G}rd\,\mathcal{O}p \to \widehat{\mathcal{G}rd}\,\mathcal{O}p$.

To recap the construction, the Malcev completion of an operad in groupoids $\mathsf{P} \in \mathcal{G}rd\,\mathcal{O}p$ is the operad $\widehat{\mathsf{P}}$ formed by the collection $\widehat{\mathsf{P}}(r)$, where we consider the Malcev completion of each groupoid $\mathsf{P}(r)$. We also have $\mathsf{P} = \widehat{\mathbb{G}}(\widehat{\mathbb{Q}}\{\mathsf{P}\})$, where $\widehat{\mathbb{Q}}\{\mathsf{P}\}$

is the operad in complete Hopf groupoids defined by the completion of the Hopf groupoid $\mathbb{Q}\{\mathsf{P}(r)\}$ associated to each $\mathsf{P}(r) \in \mathcal{G}rd$, and $\widehat{\mathbb{G}}(-)$ refers to the arity-wise application of the group-like functor on complete Hopf groupoids $\widehat{\mathbb{G}} : \widehat{\mathcal{H}opf\mathcal{G}rd} \to \widehat{\mathcal{G}rd}$.

Recall that in the situation of Proposition 3.1.15(a), the functors $\widehat{\mathbb{Q}}\{-\} : \mathcal{G}rd\,\mathcal{O}p \to \widehat{\mathcal{H}opf\mathcal{G}rd}\,\mathcal{O}p$ and $\widehat{\mathbb{G}} : \widehat{\mathcal{H}opf\mathcal{G}rd}\,\mathcal{O}p \to \mathcal{G}rd\,\mathcal{O}p$, preserves unitary extensions of operads (see Proposition II.1.4), and as a byproduct, so does the composite functors $\widehat{Id} = \widehat{\mathbb{G}}(\widehat{\mathbb{Q}}\{-\})$. In the notation of §I.4.5, we have the identity $\widehat{(\mathsf{P}_+)} = (\widehat{\mathsf{P}})_+$ for any unitary operad in simplicial sets P_+.

We apply the operadic Malcev completion functor, defined in this proposition, to the operad of colored braids CoB, as defined in §2.2 (and to the associated unitary operad CoB_+). We obtain the following result:

THEOREM 3.A. *The operad $B(\widehat{CoB})$ is a rationalization of the operad of little 2-discs D_2 in the sense that we have a chain of operad morphisms*

$$\mathsf{D}_2 \xleftarrow{\sim} \cdot \xrightarrow{\sim} B(CoB) \to B(\widehat{CoB})$$

inducing the Malcev completion at the level of fundamental groups, and we obviously still have

$$\pi_n(B(\widehat{CoB}(r))) = \pi_n(\mathsf{D}_2(r)) = *$$

for $n \neq 1$. The same result holds in the unitary setting, for the unitary extension $B(\widehat{CoB})_+ = B(\widehat{CoB}_+)$ of the operad $B(\widehat{CoB})$. □

3.2. The operad of parenthesized braids and the pro-unipotent Grothendieck-Teichmüller group

The pro-unipotent Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$, which we consider all through this monograph, has formally been defined by Drinfeld in [28] as a group of power series satisfying certain equations in the Malcev completion of the pure braid groups. In this initial approach, the elements $\phi \in GT^1(\mathbb{Q})$ are regarded as universal transformations acting on (the completion of) braided monoidal categories and the equations of [28] reflect coherence constraints associated with these monoidal structures. The goal of this section is to explain that the Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$ can be interpreted as a group of automorphisms associated to the Malcev completion of an operad in groupoids. This operad, the operad of parenthesized braid PaB, is a variant of the operad of colored braids CoB such that the object sets $\mathsf{Ob}\,PaB(r)$ form a free operad with a generating element in arity 2.

In §2.2.2, we distinguish the class formed by the morphisms of operads in groupoids $\phi : \mathsf{P} \to \mathsf{Q}$ which are equivalence of categories $\phi : \mathsf{P}(r) \to \mathsf{Q}(r)$ in each arity r. We use the expression of categorical equivalence of operads to refer to a morphism in this class. Since a weak-equivalence of topological spaces induces an equivalence of categories at the fundamental groupoid level, we immediately see that a weak-equivalence of topological operads induces a categorical equivalence at the fundamental groupoid level. Since an equivalence of categories induces a homotopy equivalence at the classifying space level, we also obtain that a categorical equivalence of operads in groupoids $\phi : \mathsf{P} \to \mathsf{Q}$ induces a weak-equivalence of topological operads $B\phi : B\,\mathsf{P} \to B\,\mathsf{Q}$ when we apply the classifying space construction.

On the other hand, we mentioned in §2.2.2 that categorical equivalences of operads can be inverted arity-wise, but not globally in general (as morphisms of operads in groupoids). Similarly, the weak-equivalence of topological operads $B\phi : B\,\mathsf{P} \to B\,\mathsf{Q}$ induced by a categorical equivalence is a homotopy equivalence of spaces arity-wise, but does not define a homotopy equivalence in the category of operads in general.

A first motivation for the introduction of the parenthesized braid operad lies in the following general proposition:

Proposition 3.A. *Let $\phi : \mathsf{P} \to \mathsf{Q}$ be a categorical equivalence of operads in groupoids. If the object sets of the operad Q form a free operad in sets, then we have a morphism of operads in groupoids $\psi : \mathsf{Q} \to \mathsf{P}$, going in the converse direction as ϕ, and of which components $\psi(r) : \mathsf{Q}(r) \to \mathsf{P}(r)$ define inverse equivalences of the groupoid morphisms $\phi(r) : \mathsf{P}(r) \to \mathsf{Q}(r)$, for all $r \in \mathbb{N}$.*

Furthermore, the equivalences $\theta(X)$ connecting the composite functors $\psi(r) \cdot \phi(r)$ (respectively, $\phi(r) \cdot \psi(r)$) to the identity functors on the groupoids $\mathsf{P}(r)$ (respectively, $\mathsf{Q}(r)$) are operadic in the sense that we have $\theta(1) = id_1$, for the operadic unit $1 \in \mathtt{Ob}\,\mathsf{P}(1)$ (respectively, $1 \in \mathtt{Ob}\,\mathsf{Q}(1)$), the equivariance relation $\theta(sX) = s\theta(X)$ for each $s \in \Sigma_r$ and $X \in \mathtt{Ob}\,\mathsf{P}(r)$ (respectively, $X \in \mathtt{Ob}\,\mathsf{Q}(r)$), as well as the multiplication relation $\theta(X \circ_i Y) = \theta(X) \circ_i \theta(Y)$ for each composite object $X \circ_i Y$ in P (respectively, Q).

Proof. This proposition is stated as a remark. Therefore the proof is left as an exercise for interested readers. □

For the operad of parenthesized braids PaB (which we define soon), this proposition implies that in any diagram

$$\begin{tikzcd} & \mathsf{Q} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ PaB \arrow[rr, dotted, "\phi"'] & & PaB \end{tikzcd}$$

such that the diagonal arrows are categorical equivalences of operads we have a fill-in morphism ϕ making the diagram commute up to an equivalence which is operadic (in the sense specified in the proposition). In subsequent applications, we consider morphisms which reduce to the identity in arity 2. This requirement implies that the equivalence arising from the construction of the proposition reduces to the identity (because the operad of objects $\mathtt{Ob}\,PaB$ is generated by an element in arity 2), and in this situation, the morphism ϕ, returned by the fill-in process, is an actual isomorphism (not a categorical equivalence) of operads in groupoids.

We now explain the definition of the operad PaB. We will give the precise definition of the Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$ afterwards. We formally define the operad PaB by applying a general pullback process to the operad of colored braids CoB, which we explain first.

3.2.1. *Object-wise pullbacks of groupoids.* Recall that $\mathcal{G}rd_{\mathcal{X}}$ denotes the category of groupoids with a prescribed object set $\mathcal{X}$. Suppose we have a map $f : \mathcal{Y} \to \mathcal{X}$ from one set $\mathcal{Y}$ to another $\mathcal{X}$. Then, to any $\mathcal{G} \in \mathcal{G}rd_{\mathcal{X}}$, we can associate a groupoid $f^*\,\mathcal{G} \in \mathcal{G}rd_{\mathcal{Y}}$ with $\mathtt{Ob}\,f^*\,\mathcal{G} = \mathcal{Y}$ as object set (as required in $\mathcal{G}rd_{\mathcal{Y}}$) and $\mathtt{Mor}_{f^*\,\mathcal{G}}(X, Y) = \mathtt{Mor}_{\mathcal{G}}(f(X), f(Y))$ as morphism sets, for all $X, Y \in \mathcal{Y}$. We take the structure unit of the groupoid $\mathcal{G}$ to define identity morphisms $id_X = id_{f(X)} \in$

$\mathtt{Mor}_{\mathcal{G}}(f(X), f(X))$ and the composition operation of $\mathcal{G}$ to define composition operations on these morphism sets $\mathtt{Mor}_{\mathcal{G}}(f(X), f(Y))$. We have a natural groupoid morphism $f : f^*\,\mathcal{G} \to \mathcal{G}$ defined by the mapping $f : \mathcal{X} \to \mathcal{Y}$ on object sets and the structure identity $id : \mathtt{Mor}_{f^*\,\mathcal{G}}(X, Y) \xrightarrow{=} \mathtt{Mor}_{\mathcal{G}}(f(X), f(Y))$ on morphism sets. This morphism is automatically fully faithful by definition and forms an equivalence of groupoids as soon as the given map $f : \mathcal{Y} \to \mathcal{X}$ is surjective.

The above construction gives a pullback functor $f^* : \mathcal{G}rd_{\mathcal{X}} \to \mathcal{G}rd_{\mathcal{Y}}$, and we have an obvious analogue of this functor in the context of Hopf groupoids $f^* : \mathcal{H}opf\mathcal{G}rd_{\mathcal{X}} \to \mathcal{H}opf\mathcal{G}rd_{\mathcal{Y}}$, as well as in the pro-nilpotent variant of the category of groupoids $f^* : \widehat{\mathcal{G}rd}_{\mathcal{X}} \to \widehat{\mathcal{G}rd}_{\mathcal{Y}}$ and in the complete variant of the category of Hopf groupoids $f^* : \widehat{\mathcal{H}opf\mathcal{G}rd}_{\mathcal{X}} \to \widehat{\mathcal{H}opf\mathcal{G}rd}_{\mathcal{Y}}$ too. Moreover, we easily see that these pullback functors f^* commute with all functors considered in the previous section, including the adjunction functors $\widehat{\Bbbk}\{-\} : \mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd} : \widehat{\mathbb{G}}(-)$ which we use to define the Malcev completion of groupoids.

3.2.2. *Object-wise pullbacks of operads in groupoids.* Suppose now that a morphism of set operads $f : \mathsf{B} \to \mathsf{A}$ is given. Let P be an operad in groupoids with $\mathtt{Ob}\,\mathsf{P} = \mathsf{A}$ as underlying object operad. The collection of groupoids $f^*\,\mathsf{P}(r)$ obtained by arity-wise pullbacks from the collection $\mathsf{P}(r)$ forms an operad $f^*\,\mathsf{P}$ with B as underlying object operad:

- the operadic unit of this operad $1 \in \mathtt{Ob}\,f^*\,\mathsf{P}(1)$ is inherited from the set operad B;
- the composition products $\circ_i : f^*\,\mathsf{P}(m) \otimes f^*\,\mathsf{P}(n) \to f^*\,\mathsf{P}(m+n-1)$. are given by the composition products of B on object sets and by the composition products

$$\begin{aligned}\mathtt{Mor}_{\mathsf{P}(m)}(f(X), f(Y)) \times \mathtt{Mor}_{\mathsf{P}(n)}(f(Z), f(T))& \\ \xrightarrow{\circ_i} \mathtt{Mor}_{\mathsf{P}(m+n-1)}(f(X \circ_i Z), f(Y \circ_i T))&,\end{aligned}$$

  inherited from P (using that f preserves operadic composites), on morphism sets.

The functors $f : f^*\,\mathsf{P}(r) \to \mathsf{P}(r)$ naturally define a morphism of operads in groupoids $f : f^*\,\mathsf{P} \to \mathsf{P}$.

We can again define an analogous construction for operads in Hopf groupoids, operads in pro-nilpotent groupoids, and operads in complete Hopf groupoids. We moreover see that these pullback constructions commute with our adjoint functors $\widehat{\Bbbk}\{-\} : \mathcal{G}rd \to \widehat{\mathcal{H}opf\mathcal{G}rd} : \widehat{\mathbb{G}}(-)$ which we apply to operads arity-wise.

Recall that the operad of colored braids has the permutation operad Π, with $\Pi(r) = \Sigma_r$, as underlying object operad. To define the operad of parenthesized braids PaB, we consider a free set-theoretic operad $\Omega = \mathbb{O}(\mu(x_1, x_2), \mu(x_2, x_1))$ with a generating operation $\mu = \mu(x_1, x_2)$ in arity 2 on which the symmetric group Σ_2 acts freely. Formally, we define PaB by pulling-back CoB to this object-set operad Ω. We also consider a unitary extension of this construction. Before performing this process, we review the definition of this particular free operad Ω, which we call the Magma operad, in order to relate our definition with other representations occurring in the literature.

3.2.3. *The Magma operad.* Intuitively, the elements of the operad Ω are formal composite of the operation $\mu(x_1, x_2)$, and of the associated transpose operation

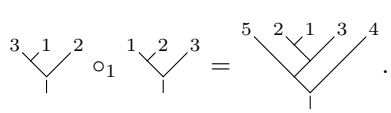
$\mu(x_2, x_1)$. If we use the short notation $x_1x_2 = \mu(x_1, x_2)$ for this generating operation, then the monomials defining the elements of Ω have the form of parenthesized words

$$\begin{aligned}
\Omega(2) &= \{(x_ix_j)|(i,j) \in \Sigma_2\},\\
\Omega(3) &= \{((x_ix_j)x_k), (x_i(x_jx_k))|(i,j,k) \in \Sigma_3\},\\
\Omega(4) &= \{(((x_ix_j)x_k)x_l),((x_i(x_jx_k))x_l), ((x_ix_j)(x_kx_l)),\\
&\qquad\qquad (x_i((x_jx_k)x_l)), (x_i(x_j(x_kx_l)))|(i,j,k,l) \in \Sigma_4\},\\
\Omega(5) &= \cdots
\end{aligned}$$

defined by providing any permutation of the variables $(x_1, \ldots, x_r)$ with a full binary bracketing (the parenthesization). Certain authors use the term of magma, borrowed from [20, §I.1], to refer to this structure. We actually use this expression as a proper noun, Magma, for the operad Ω. In this algebraic representation, the symmetric groups act by permuting variable indices, the operadic unit is defined by the one-variable word $1 = x_1$, and the operadic composition operations $\circ_i : \Omega(m) \times \Omega(n) \to \Omega(m+n-1)$ are defined by the insertion of parenthesized words on variables (after performing the usual index shift), as in the following example

$$((x_3x_1)x_2) \circ_1 ((x_2x_1)x_3) = ((x_5((x_2x_1)x_3))x_4).$$

The above monomials have a convenient graphical representation, in terms of planar binary trees, which is also used in the literature. The correspondence between the algebraic representation and the tree representation is given in Figure 3.1. The indexing of ingoing edges in this tree representation correspond to the variable indexing in the algebraic interpretation. The symmetric action, the operadic unit and the operadic composition operations are given on these trees by planar variants of the operations considered in §B.1. For instance, the previous example of composite parenthesized words (given to illustrate the algebraic definition of Ω) is equivalent to the following composition operation on trees:



In our general construction of the free operad, the elements $\xi \in \Omega(r)$ are represented by trees with r ingoing edges, indexed by $\{1, \ldots, r\}$, and of which vertices are labeled by the generating operation $\mu = \mu(x_1, x_2)$, and the associated transposed operation $t \cdot \mu = \mu(x_2, x_1)$, where we set $t = (1\ 2)$. In the general construction of free operads, we assume that the ingoing edges of vertices are in bijection with the inputs of the corresponding operation. In the case of the Magma operad Ω, this requirement implies that we have to restrict ourselves to binary trees, of which vertices have two ingoing edges, no more no less. The equivariance relations of §I.1.4 (see also §B.2) moreover implies that any tree-wise element of $\Omega(r)$ has a reduced form, defined by a planar binary tree of which all vertices are labeled by $\mu = \mu(x_1, x_2)$.

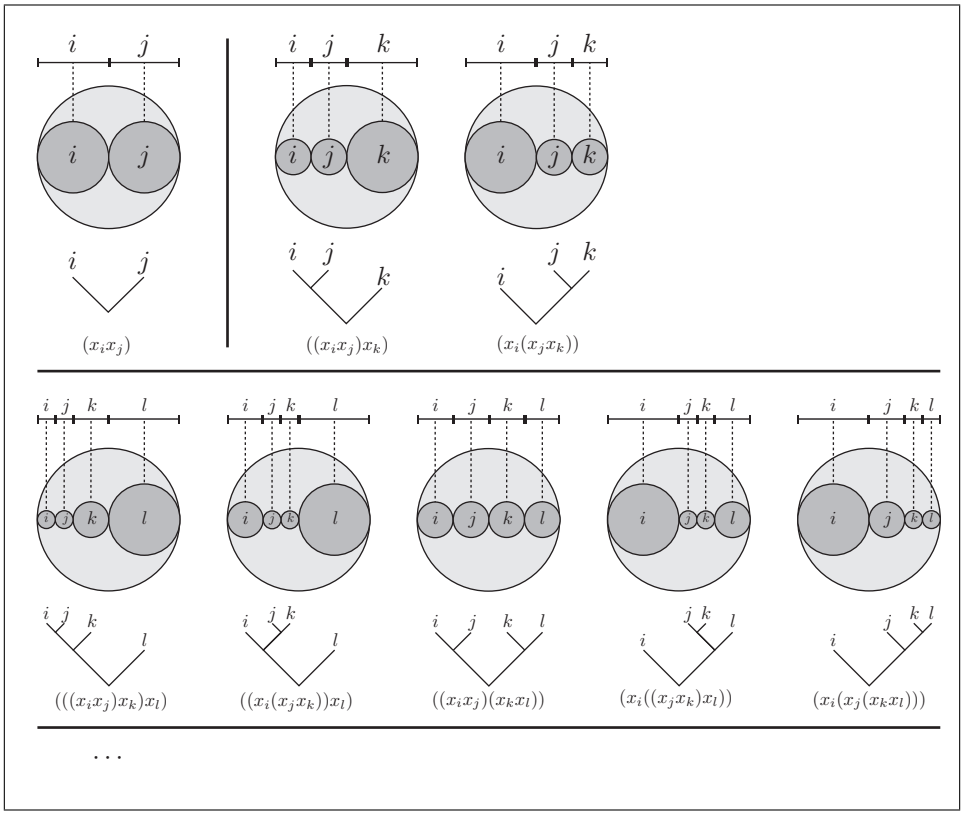
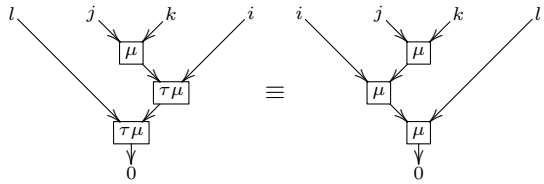


FIGURE 3.1. The correspondence between parenthesized words, binary trees, operadic composites of a generating little 2-disc configurations and diadic decompositions of the interval in arity $r = 2, 3, 4, \ldots$. The indices (i, j), (i, j, k), (i, j, k, l), ... run over permutations of $(1, 2)$, $(1, 2, 3)$, $(1, 2, 3, 4)$, ... The diadic decompositions of the interval are counterparts, in the little 1-disc operad, of the little 2-disc composites considered in this picture (see proof of Proposition 3.2.6 for detailed explanations).

To give an example, we have the relation



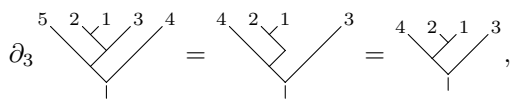
in $\Omega(4)$. Recall that the choice of a planar embedding is equivalent to the choice of an ordering between the ingoing edges of each vertex v (see again §I.1.4). This ordering determines the correspondence between the ingoing edges of the vertex v and the inputs of the operation $\mu = \mu(x_1, x_2)$ attached to this vertex v.

Thus, we finally retrieve, our planar binary representation of the Magma operad (we simply omit some information, given by the labeling of vertices, the edge orientation and the output mark 0, which become unnecessary in the reduced representation of tree-wise elements).

The composition process of the free operad, as defined in §B.3, preserves elements in reduced forms too, and so do symmetric group actions. This explains that the composition structure of the operad Ω can be given by the process introduced earlier in this paragraph, in terms of the single structure of planar binary trees.

3.2.4. *The unitary extension of the Magma operad.* The Magma operad has a natural unitary extension. We use the process of §I.4.9, and the construction of Ω as a free operad, to formalize the definition of this unitary extension. In short, as in the case of the associative operad (see §I.4.11) we provide the generating sequence of Ω with the deletion operations such that $\partial_1\mu = \partial_2\mu = 1$ for the generating operation $\mu = \mu(x_1, x_2)$. We complete the definition for the transposed element $t\mu = \mu(x_2, x_1)$ by using the equivariance of deletion morphisms.

Recall that the operations ∂_i represent partial composites $\partial_i(p) = p \circ_i *$ with a unitary element $*$, and we use the associativity of partial composites of operads in order to extend these operations from generating elements to the whole free operad. In fact, we can readily identify the action of the deletion morphisms $\partial_i : \Omega_+(r) \to \Omega_+(r-1)$ on the unitary Magma operad $\Omega_+(r)$ with the removal of ingoing edges in planar binary trees, as in the following example:



where we perform the appropriate shift on the input indices (as usual).

3.2.5. *The operad of parenthesized braids.* We now form the operad morphism $\rho : \Omega \to \Pi$ mapping the generating operation μ of Ω to the identity permutation $id \in \Sigma_2$. This morphism ρ is identified with the quotient morphism

$$\mathbb{O}(\mu(x_1, x_2), \mu(x_2, x_1)) \xrightarrow{\rho} \mathbb{O}(\mu(x_1, x_2), \mu(x_2, x_1) : \mu(\mu, 1) = \mu(1, \mu)))$$

arising from the definition of the associative operad by generators and relations (see §I.2.6) and the identity between the permutation operad and the associative operad (see §I.2.7). In the algebraic description of §3.2.3, the map ρ is defined on parenthesized words by forgetting the bracketing and retaining the variable permutation as single information. This map ρ has also an obvious unitary extension $\rho : \Omega_+ \to \Pi_+$.

We define the operad of parenthesized braids PaB as the pullback of the operad of colored braids CoB along this morphism of operads in sets $\Omega \xrightarrow{\rho} \Pi = \mathtt{Ob}\, CoB$. Accordingly, the operad $PaB = \rho^*\, CoB$ has $\mathtt{Ob}\, PaB = \Omega$ as underlying operad of objects, and the sets of colored braids $\mathtt{Mor}_{PaB(r)}(u, v) = \mathtt{Mor}_{CoB(r)}(\rho(u), \rho(v))$ as morphism sets, for $u, v \in \Omega(r)$. We have a categorical equivalence of operads in groupoids $\rho : PaB \to CoB$ by construction of PaB. We also consider a unitary extension of PaB, which we can obtain by a pullback $PaB_+ = \rho_+^*\, CoB_+$ along the unitary extension of ρ.

We can actually give a direct topological interpretation of the operad PaB (and of the associated unitary operad PaB_+), improving on the result of Theorem 2.A which connects the operad of colored braids CoB to the fundamental groupoid of the operad of little 2-discs $\pi\, \mathsf{D}_2$.

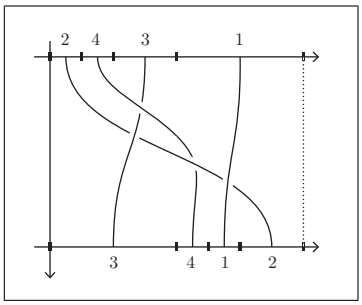


FIGURE 3.2. An instance of parenthesized braids.

To explain the idea, recall that in the definition of the operad of colored braids CoB we make a choice of contact points $\underline{a}$ on the medium axis $y = 0$ of the open disc $\mathring{\mathbb{D}}^2$. To give a topological interpretation of the composition structure of this operad, we have to use that these contact points lie in a contractible subspace of the configuration space $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$. This need clearly appears when we address the proof of Theorem 2.A. Indeed, to define the operad equivalences

$$\pi\,\mathsf{D}_2 \xleftarrow{\sim} \pi\,\mathsf{D}_2 \mid_{\mathsf{D}_1} \xrightarrow{\sim} CoB$$

occurring in the statement of that theorem, we picked a little 2-disc configuration $\underline{c} = (c_1, \ldots, c_r)$ with the chosen contact points $\underline{a} = (a_1, \ldots, a_r)$ as associated centers, for each $r \in \mathbb{N}$. But these little 2-disc configurations are not preserved by the operadic composition products in general. Therefore, in order to retrieve our chosen contact points, we have needed to introduce extra path concatenations in each operadic composition operation (see the proof of Theorem 2.A for details).

The planar binary trees, defining the objects of the groupoids $PaB(r)$, have a topological interpretation in terms of configurations of contact points: instead of the equidistant contact points of §2.0, we consider the centers of diadic decompositions of the axis $y = 0$ of the open disc $\mathring{\mathbb{D}}^2$, which actually correspond to particular composite elements in the little 2-disc operad. We precisely claim that we have a bijection between such particular configurations and planar binary trees. The formal definition of this correspondence arises from the next proposition. For the moment, simply look at the picture of Figure 3.1 to see the little 2-disc composites and the point configurations corresponding to the elements of the first terms of the Magma operad $\Omega(r)$, $r = 2, 3, 4, \ldots$.

The morphisms of the parenthesized braid groupoid $PaB(r)$ are, in this setting, identified with colored braids going from one configuration of contact points of the considered form to another one. The source (respectively, target) of the morphism is the binary tree corresponding to the configuration of contact points at the origin (respectively, end-point) of our braid. Figure 3.2 gives an example of application of this convention. In subsequent identifications, we deal with fundamental morphisms, the associator $\Phi \in \mathtt{Mor}_{PaB}(((x_1x_2)x_3), (x_1(x_2x_3)))$ and the braiding $\Theta \in \mathtt{Mor}_{PaB}((x_1x_2), (x_2x_1))$, of which representation is given in Figure 3.3.

The following proposition, giving the announced improvement of Theorem 2.A, motivates our interpretation of parenthesized braids in terms of strands connecting contact points which are associated with a diadic decomposition of the axis $y = 0$ of the open disc $\mathring{\mathbb{D}}^2$. In short, these configurations of contact points correspond to

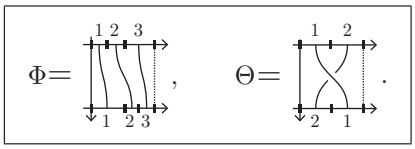
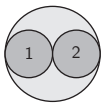


FIGURE 3.3. The representation of the associator Φ and of the braiding Θ in morphism sets of the parenthesized braid operad.

the centers of the little 2-disc configurations forming the free operad considered in the proposition.

PROPOSITION 3.2.6.

(a) *The little* 2*-disc configuration*

$\underline{\mu} :=$  $\in \mathsf{D}_2(2)$

generates a free operad, isomorphic to Ω*, within the little* 2*-disc operad* D_2*.*

(b) *The disc center mapping of §1.2.2, applied point-wise to paths in little* 2*-disc spaces, induces an isomorphism*

$$\omega_* : \pi\, \mathsf{D}_2 \mid_{\Omega} \xrightarrow{\simeq} PaB$$

from the restriction to Ω *of the fundamental groupoid operad of* D_2 *towards the operad of parenthesized braids* PaB*.*

(c) *The morphism* ω_* *of (b) has a unitary extension, yielding an isomorphism of unitary operads* $\omega_* : \pi\, \mathsf{D}_{2+} \mid_{\Omega_+} \xrightarrow{\simeq} PaB_+$ *too.*

PROOF. Let $\phi : \Omega \to \mathsf{D}_2$ be the morphism sending the generating element $\mu \in \Omega(2)$ of the free operad Ω to the little 2-disc configuration $\underline{\mu} \in \mathsf{D}_2(2)$ of assertion (a). The claim of assertion (a) is that this morphism defines an embedding.

In our verification, we use the symmetric collection representation of the operad Ω, and we consider Magma operations $p = p(x_{i_1}, \ldots, x_{i_r}) \in \Omega(\underline{\mathsf{r}})$ of which variables may be indexed by an arbitrary finite set $\underline{\mathsf{r}} = \{i_1, \ldots, i_r\}$ (not necessarily a standard ordinal). Since $\underline{\mu}$ visibly comes from the operad of little 1-discs D_1, regarded as a suboperad of D_2 (see §1.1.5 and §2.1.7), our morphism ϕ admits a factorization through D_1, and we are therefore reduced to prove that this factorization $\phi : \Omega \to \mathsf{D}_1$ is an injection. Equivalently, we look at the trace of little 2-disc configurations $\underline{c} \in \mathsf{D}_2(r)$ on the axis $y = 0$ in the ambient disc $\mathbb{D}^2$ to determine the counter-image of elements $\underline{c}$ in Ω. (Recall that the image of D_1 in D_2 consists of configurations of little disc centered on this axis $y = 0$, and the trace, considered in our process, can be used to determine the counter-image in D_1 of an element of D_2.)

The little interval configurations lying in the image of our map ϕ are associated with diadic decomposition of the interval $[-1, 1]$ (see Figure 3.1 for examples). To retrieve an element of Ω from the corresponding little interval configuration $\underline{c}$, just observe that we have $\underline{c} = \underline{\mu}(\underline{a}, \underline{b})$ where $\underline{a} \in \mathsf{D}_1(\{i_1, \ldots, i_m\})$ (respectively, $\underline{b} \in \mathsf{D}_1(\{j_1, \ldots, j_n\})$) is produced by applying the affine transformation $t \mapsto 2t + 1$ (respectively, $t \mapsto 2t-1$) the configuration of little intervals lying in $[-1, 0] \subset [-1, 1]$ (respectively, $[0, 1] \subset [-1, 1]$) in the collection $\underline{c}$. We continue by induction to obtain the full decomposition of $\underline{c}$ and to determine the counter-image of $\underline{c}$ in Ω.

$$\begin{tikzcd}
 & m(m(m(x_1,x_2),x_3),x_4) \arrow[dl, "m(a,1)"'] \arrow[dr, "a(m,1,1)"] & \\
m(m(x_1,m(x_2,x_3)),x_4) \arrow[d, "a(1,m,1)"'] & & m(m(x_1,x_2),m(x_3,x_4)) \arrow[dl, "a(1,1,m)"] \\
m(x_1,m(m(x_2,x_3),x_4)) \arrow[dr, "m(1,a)"'] & & \\
 & m(x_1,m(x_2,m(x_3,x_4))) &
\end{tikzcd}$$

FIGURE 3.4. The pentagon constraint

$$\begin{tikzcd}
 & m(m(x_1,x_2),x_3) \arrow[dl, "m(c,1)"'] \arrow[dr, "a"] & \\
m(m(x_2,x_1),x_3) \arrow[d, "a"'] & & m(x_1,m(x_2,x_3)) \arrow[d, "c(1,m)"] \\
m(x_2,m(x_1,x_3)) \arrow[dr, "m(1,c)"'] & & m(m(x_2,x_3),x_1) \arrow[dl, "a"] \\
 & m(x_3,m(x_2,x_1)) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & m(x_1,m(x_2,x_3)) \arrow[dl, "m(1,c)"'] \arrow[dr, "a^{-1}"] & \\
m(x_1,m(x_3,x_2)) \arrow[d, "a^{-1}"'] & & m(m(x_1,x_2),x_3) \arrow[d, "c(m,1)"] \\
m(m(x_1,x_3),x_2) \arrow[dr, "m(c,1)"'] & & m(x_3,m(x_1,x_2)) \arrow[dl, "a^{-1}"] \\
 & m(m(x_3,x_1),x_2) &
\end{tikzcd}$$

FIGURE 3.5. The hexagon constraints

The second claim of the proposition is a variation of the result of Theorem 2.A. Simply note that we now have a direct isomorphism $\omega_* : \pi\,\mathsf{D}_2 \upharpoonright_\Omega \xrightarrow{\simeq} PaB$ which actually lift the chain of category equivalences

$$\begin{tikzcd}
\pi\,\mathsf{D}_2(r) \upharpoonright_{\Omega(r)} \arrow[r, dotted, "\omega_*"] \arrow[d, hook] & PaB(r) \arrow[d] \\
\pi\,\mathsf{D}_2(r) \upharpoonright_{\mathsf{D}_1(r)} \arrow[r] \arrow[d, hook] & CoB(r) \\
\pi\,\mathsf{D}_2(r) &
\end{tikzcd}$$

considered in the proof of Theorem 2.A.

The proof that ω_* has a unitary extension, asserted in (b), reduces to the straightforward verification that ω_* preserves deletion operations (see Proposition I.4.6). $\square$

In §2.2.17, we briefly mention that the category of algebras associated with the operad CoB_+ consists of braided categories of which tensor product is unitary and associative in the strict sense. The operad PaB_+ governs the category formed by braided categories of which tensor product is unitary in the strict sense, but associative up to coherent isomorphism, not necessarily given by identity morphisms. This claim is a consequence of the following statement:

THEOREM 3.2.7.

(a) *Any morphism $\phi : PaB \to \mathsf{Q}$ towards an operad in the category of categories $\mathsf{Q} \in \mathcal{C}at\mathcal{O}p$ is uniquely determined by an object*

$$m = m(x_1, x_2) \in \mathtt{Ob}\,\mathsf{Q}(2)$$

and invertible morphisms

$$a = a(x_1, x_2, x_3) \in \mathtt{Mor}_{\mathsf{Q}(3)}(m(m(x_1, x_2), x_3), m(x_1, m(x_2, x_3)))$$
$$\text{and } c = c(x_1, x_2) \in \mathtt{Mor}_{\mathsf{Q}(2)}(m(x_1, x_2), m(x_2, x_1))$$

such that the diagrams of Figure 3.4-3.5 commute. The object m represents the image of the generating element of the magma operad $\mu = (x_1 x_2) \in \Omega(2)$ under the mapping $\Omega(2) = \mathtt{Ob}\,PaB(2) \xrightarrow{\phi} \mathtt{Ob}\,\mathsf{Q}(2)$. The morphism $a(x_1, x_2, x_3)$ represents the image of the associator Φ and $c(x_1, x_2)$ represents the image of the braiding Θ under the mapping defined by ϕ on morphisms.

(b) *When Q is equipped with a deletion structure, we have a morphism of unitary operads $\phi : PaB_+ \to \mathsf{Q}_+$ extending ϕ if and only if the object $m = m(x_1, x_2)$ satisfies $\partial_1 m = \partial_2 m = 1$ in $\mathtt{Ob}\,\mathsf{Q}(1)$, the morphism $a = a(x_1, x_2, x_3)$ satisfies $\partial_1 a = \partial_2 a = \partial_3 a = id_m$ in $\mathtt{Mor}_{\mathsf{Q}(2)}(m, m)$, and $c = c(x_1, x_2)$ satisfies $\partial_1 c = \partial_2 c = id_1$ in $\mathtt{Mor}_{\mathsf{Q}(1)}(1, 1)$.*

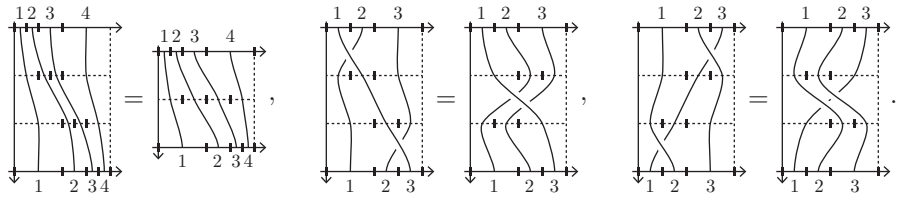
We explain how to retrieve our claim about the structures governed by the operad PaB before tackling the proof of this theorem. We consider the endomorphism operad $\mathtt{End}_{\mathcal{C}}$, associated to any category $\mathcal{C}$, of which arity r component $\mathtt{End}_{\mathcal{C}}(r)$ is the category formed by the multi-functors $F : \mathcal{C}^{\times r} \to \mathcal{C}$ as objects together with the natural transformations between them as morphisms. This operad is equipped with deletion morphisms, corresponding to a unitary structure, whenever the category $\mathcal{C}$ is provided with a unit object $\mathbb{1} \in \mathcal{C}$. Theorem 3.2.7 implies that a morphism $\phi : PaB \to \mathtt{End}_{\mathcal{C}}$, defining an action of the operad PaB on $\mathcal{C}$, is uniquely determined by a bifunctor $m(X_1, X_2) = X_1 \otimes X_2$ on $\mathcal{C}$ together with an associativity isomorphisms $a(X_1, X_2, X_3) : (X_1 \otimes X_2) \otimes X_3 \xrightarrow{\simeq} X_1 \otimes (X_2 \otimes X_3)$ and a symmetry isomorphism $c(X_1, X_2) : X_1 \otimes X_2 \xrightarrow{\simeq} X_2 \otimes X_1$ so the relations of Figure 3.4-3.5, which are actually nothing but the classical coherence constraints of braided monoidal categories, are fulfilled. The preservation of deletion structures is equivalent to the strict unit constraint $X \otimes \mathbb{1} = X = \mathbb{1} \otimes X$ at the tensor product level, together with the identities $a(\mathbb{1}, X_1, X_2) = a(X_1, \mathbb{1}, X_2) = a(X_1, x_2, \mathbb{1}) = id_{X_1 \otimes X_2}$ and $c(X, \mathbb{1}) = id_X = c(\mathbb{1}, X)$ on the corresponding associativity and symmetry isomorphisms.

We defer the proof of Theorem 3.2.7 to a series of lemma.

We immediately see that the morphisms $a = a(x_1, x_2, x_3)$ and $c = c(x_1, x_2)$, given as the image of the associativity iso Φ and of the braiding Θ in Theorem 3.2.7, have to fulfill the coherence constraints of Figure 3.4-3.5, because these relations involve structure operations of operads in groupoids (which are preserved by morphisms) and:

LEMMA 3.2.8. *The associativity iso Φ and the braiding Θ satisfy the relations of Figure 3.4-3.5 within the operad of parenthesized braids.*

PROOF. The verification of this lemma is entirely given by the following picture:



The pentagonal equation, which we identify as an identity of parenthesized braids in this picture, is also a formal consequence of our definition of the morphism sets of the parenthesized braid groupoids as object-wise pullbacks. □

The next lemma is a standard statement of the theory of braided monoidal categories (see [53]):

LEMMA 3.2.9. *If the morphisms $a(x_1, x_2, x_3)$ and $c(x_1, x_2)$ make the hexagon diagrams of Figure 3.5 commute, then the duodecagon*

$$\begin{tikzcd}
m(m(x_1,x_2),x_3) \arrow[r, "a"] \arrow[d, "m(c,1)"'] & m(x_1,m(x_2,x_3)) \arrow[r, "m(1,c)"] \arrow[dd, dotted, "c(1,m)"] & m(x_1,m(x_3,x_2)) \arrow[r, "a^{-1}"] \arrow[dd, dotted, "c(1,m)"] & m(m(x_1,x_3),x_2) \arrow[d, "m(c,1)"] \\
m(m(x_2,x_1),x_3) \arrow[d, "a"'] & & & m(m(x_3,x_1),x_2) \arrow[d, "a"] \\
m(x_2,m(x_1,x_3)) \arrow[d, "m(1,c)"'] & & & m(x_3,m(x_1,x_2)) \arrow[d, "m(1,c)"] \\
m(x_2,m(x_3,x_1)) \arrow[r, "a^{-1}"'] & m(m(x_2,x_3),x_1) \arrow[r, "m(c,1)"'] & m(m(x_3,x_2),x_1) \arrow[r, "a"'] & m(x_3,m(x_2,x_1))
\end{tikzcd},$$

tiled with two hexagons and one square, commutes as well.

We suggest the reader to make these relations explicit for the associator Φ and the braiding Θ of the parenthesized braid operad PaB.

PROOF. The left hand side and right hand side hexagons in the duodecagon tiling of the lemma are identified with the hexagons of Figure 3.5 (with a factor $a^{\pm 1}$ inverted) and therefore, these hexagons commute. The medium square commutes as well. Indeed, for the morphism $c = c(x_1, x_2)$, going from $m = m(x_1, x_2)$ to $(1\ 2) \cdot m = m(x_2, x_1)$, the functoriality of the composition product $\circ_2 : PaB(2) \times PaB(2) \to PaB(3)$ gives $c \circ_2 ((1\ 2) \cdot m) \cdot m \circ_2 c = c \circ_2 c = ((1\ 2) \cdot m) \circ_2 c \cdot c \circ_2 m$, which is the identity asserted by the commutation of that square. □

We now check that:

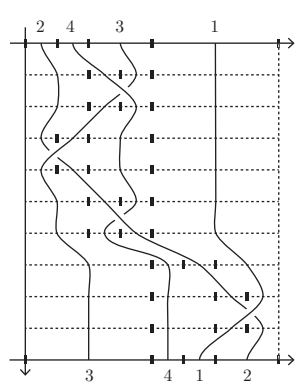
LEMMA 3.2.10. *The mapping of Theorem 3.2.7 is one-to-one: for any object $m = m(x_1, x_2)$ and morphisms $a = a(x_1, x_2, x_3)$ and $c = c(x_1, x_2)$ satisfying our constraints in Q we have one and only one morphism of operads in groupoids $\phi : PaB \to \mathsf{Q}$ such that $\phi(\mu) = m$, $\phi(\Phi) = a$ and $\phi(\Theta) = c$.*

PROOF. Any element $m \in \mathtt{Ob}\,\mathsf{Q}(2)$ is associated with a morphism of set-operads $\phi : \mathtt{Ob}\,PaB \to \mathtt{Ob}\,\mathsf{Q}$, such that $m = \phi(\mu)$, since $\Omega = \mathtt{Ob}\,PaB$ is, by definition, a free operad. We aim to determine a map on morphisms sets from a given associator $a(x_1, x_2, x_3)$ and braiding $c(x_1, x_2)$ in Q. We first observe that any morphism $\beta \in \mathtt{Mor}_{PaB(r)}(p, q)$ can be decomposed into a product of morphisms formed by operadic composites of associators and braiding morphisms.

In our argument lines, we use (an operadic interpretation of) Mac Lane's coherence theorem asserting that, if the pentagon constraint of Figure 3.4 is fulfilled, then all composites of associators going from one parenthesization $p = p(x_{s(1)}, \ldots, x_{s(r)})$ to another $q = q(x_{s(1)}, \ldots, x_{s(r)})$ (for a fixed underlying permutation s) define the same isomorphism (see [65, §VII.2]).

To begin with, since we have $\mathtt{Mor}_{PaB(r)}(p, q) = \mathtt{Mor}_{CoB(r)}(\rho(p), \rho(q)))$ by definition of the groupoids of parenthesized braids, we immediately see that any $\beta \in \mathtt{Mor}_{PaB(r)}(p, q)$ admits a decomposition $\beta = \beta_1 \cdot \ldots \cdot \beta_n$, where each factor $\beta_i \in \mathtt{Mor}_{PaB(r)}(p_i, q_i)$ consists, after forgetting about parenthesizations, of a single generating element τ_k in the colored braid coset $\mathtt{Mor}_{CoB(r)}(\rho(p_i), \rho(q_i))) \subset B_r$. If $p_i = p_i(x_{s(1)}, \ldots, x_{s(r)})$ has $s = (s(1), \ldots, s(k), s(k+1), \ldots, s(r))$ as associated permutation, then q_i has an associated permutation of the form $(s(1), \ldots, s(k+1), s(k), \ldots, s(r))$, with the factors $(s(k), s(k+1))$ switched. We pick a parenthesization gathering the factors $x_{s(k)}$ and $x_{s(k+1)}$ in the word $x_{s(1)} \cdot \ldots \cdot x_{s(r)}$. We thus consider a parenthesized word of the form $\sigma_i = \pi_i(x_{s(1)}, \ldots, \mu(x_{s(k)}, x_{s(k+1)}), \ldots, x_{s(r)})$, where $\pi_i \in \Omega(r-1)$. We can take a composite of associators α in order to go from p_i to $\sigma_i = \pi_i(x_{s(1)}, \ldots, \mu(x_{s(k)}, x_{s(k+1)}), \ldots, x_{s(r)})$, and we similarly pick a composite of associators γ going from q_i to $\tau_i = \pi_i(x_{s(1)}, \ldots, \mu(x_{s(k+1)}, x_{s(k)}), \ldots, x_{s(r)})$. We therefore have a decomposition $\beta_i = \gamma^{-1} \cdot \pi_i(x_{s(1)}, \ldots, \Theta(x_{s(k)}, x_{s(k+1)}), \ldots, x_{s(r)}) \cdot \alpha$ of each β_i, with a composite of associators involved in γ, as well as α, and with a medium factor $s_*(\pi_i \circ_k \Theta) = \pi_i(x_{s(1)}, \ldots, \Theta(x_{s(k)}, x_{s(k+1)}), \ldots, x_{s(r)})$ reduced to the application of a braiding Θ within a fixed parenthesized word. The image of β under a morphism $\phi : PaB \to \mathsf{Q}$ is determined, from this decomposition, by the preservation of products, operadic composites, and the assignments $\Phi \mapsto a(x_1, x_2, x_3)$, $\Theta \mapsto c(x_1, x_2)$.

As an example, the braid of Figure 3.2 admits a decomposition of the form



from which we deduce that the image of this braid under any morphism ϕ is given by the product

$$\begin{aligned}\alpha = m(1,a^{-1})\cdot m(1,m(1,c))\cdot m(1,a)\cdot a(1,m,1)\cdot m(m(1,c),1)\\ \cdot m(a,1)\cdot m(m(c,1),1)\cdot m(a^{-1},1)\cdot m(m(1,c),1)\cdot m(a,1).\end{aligned}$$

(To simplify, we have not specified the variable permutations occurring in this composite.)

This analysis proves the uniqueness of the morphism ϕ associated to given elements $m = m(x_1,x_2)$, $a = a(x_1,x_2,x_3)$ and $c = c(x_1,x_2)$. To prove the existence part of our assertion, we first check that the definition of a mapping $\phi : \mathtt{Mor}_{PaB(r)}(p,q) \to \mathtt{Mor}_{\mathsf{Q}}(\rho(p),\rho(q))$ from decompositions of the form considered in our argument line does not depend on choices. The Mac Lane coherence theorem implies that $\phi(\beta)$ does not depend on the choice of the associator decompositions between the parenthesized words occurring in our factorization of morphisms $\beta \in \mathtt{Mor}_{PaB(r)}(p,q)$. The result of our construction does not depend on the parenthesizations $\pi \in \Omega(r-1)$, which we chose to gather the factors of the braiding operations, too. Indeed, we can go from one parenthesization $\pi_i = \pi_i(x_1,\ldots,x_{r-1})$ to another $\rho_i = \rho_i(x_1,\ldots,x_{r-1})$ by a morphism $\alpha = \alpha(x_1,\ldots,x_{r-1})$ (formed by a composite of associators) in the parenthesized braid operad. The middle square in the commutative diagram

$$\begin{tikzcd}
& p_i(x_{s(1)},\ldots,x_{s(r)}) \arrow[dl] \arrow[dr] & \\
\pi_i(x_{s(1)},\ldots,\mu(x_{s(k)},x_{s(k+1)}),\ldots,x_{s(r)}) \arrow[rr, dotted, "\alpha\circ_k\mu"] \arrow[d, "\pi_i\circ_k\Theta"'] & & \rho_i(x_{s(1)},\ldots,\mu(x_{s(k)},x_{s(k+1)}),\ldots,x_{s(r)}) \arrow[d, "\rho_i\circ_k\Theta"] \\
\pi_i(x_{s(1)},\ldots,\mu(x_{s(k+1)},x_{s(k)}),\ldots,x_{s(r)}) \arrow[rr, dotted, "\alpha\circ_k\mu"] \arrow[dr] & & \rho_i(x_{s(1)},\ldots,\mu(x_{s(k+1)},x_{s(k)}),\ldots,x_{s(r)}) \arrow[dl] \\
& q_i(x_{s(1)},\ldots,x_{s(r)}) &
\end{tikzcd}$$

is carried to a commutative square by our morphism ϕ, for any choice of assignment $c = \phi(\Theta)$, by functoriality of the composition products of operads in categories. The external triangles are carried to commutative triangles in Q too (by the already mentioned MacLane's coherence theorem), and we conclude that both paths from $p_i = p_i(x_{s(1)},\ldots,x_{s(r)})$ to $q_i = q_i(x_{s(1)},\ldots,x_{s(r)})$ yields the same resulting morphism in Q, should we take π_i and ρ_i as both possible braiding parenthesizations.

We still have to establish that $\phi(\beta)$ does not depend on the decomposition $\beta = \beta_1\cdot\ldots\cdot\beta_n$ formed from the image of β in the coloured braid operad CoB. We are reduced to check, for this purpose, that the application of the generating relations of braids does not change the result of our construction.

In the case of the commutation relations $\tau_k\tau_l = \tau_l\tau_k$, we assume that a parenthesization of the form $\sigma_i = \pi_i(x_{s(1)},\ldots,\mu(x_{s(k)},x_{s(k+1)}),\ldots,\mu(x_{s(l)},x_{s(l+1)}),\ldots,x_{s(r)})$ is chosen when we proceed to determine the image of the factors β_i and β_{i+1} associated with the elementary braids of this relation. The identity of the result associated to the decompositions $\beta = \beta_1\cdot\ldots\cdot\beta_i\cdot\beta_{i+1}\cdot\ldots\cdot\beta_n = \beta_1\cdot\ldots\cdot\beta_{i+1}\cdot\beta_i\cdot\ldots\cdot\beta_n$ follows, in that case, from the associativity of the composition product of operads.

In the case of the braiding relations $\tau_k\tau_{k+1}\tau_k = \tau_{k+1}\tau_k\tau_{k+1}$, we assume that a parenthesization of the form $\sigma_i = \pi_i(x_{s(1)}, \ldots, \mu(\mu(x_{s(k)}, x_{s(k+1)}), x_{s(k+2)}), \ldots, x_{s(r)})$ is chosen when we proceed to determine the image of the factors associated with the elementary braids of the relation. The identity of our morphisms in Q reduces in that case to the commutation of the duodecagon of Lemma 3.2.9.

Our previous verifications imply that each mapping $\phi : PaB(r) \to \mathsf{Q}(r)$ is coherently defined as a morphism of groupoids, but we still have to check that the collection of these morphisms defines an operad morphism. Since the equivariance of our morphisms and the preservation of operadic unit are immediate, we only have to check the preservation of operadic composition products. The decomposition of morphisms, which we have used to determine ϕ, can be applied to reduce the verification of the relations $\phi(\alpha\circ_i\beta) = \phi(\alpha)\circ_i\phi(\beta)$ to the case where α (respectively, β) is an identity morphism in PaB and β (respectively, α) is produced by the application of a braiding within a parenthesized word, after sorting out the case of associator composites (which follows again from Mac Lane's coherence theorem). The relation is immediate in the case where α is the identity $\alpha = id_\pi$, for some $\pi \in \Omega(m)$, and the braiding occurs in $\beta = \rho \circ_k \Theta$, $\rho \in \Omega(n-1)$. In the symmetric case, where we have $\alpha = \pi \circ_k \Theta$ and $\beta = id_\rho$, for some $\pi \in \Omega(m-1)$ and $\rho \in \Omega(n)$, we can still use the decomposition of ρ within the Magma operad to reduce our verification to the case where $\rho = \mu$ and $n = 2$. The cases where $\beta = id_\mu$ is plugged in an input $i \neq k, k+1$ of $\alpha = \pi \circ_k \Theta$, follows from the associativity of the composition products in Q. In the remaining cases, where we plug $\beta = id_\mu$ in an input of the braiding Θ within the composite $\alpha = \pi \circ_k \Theta$, we see that the decomposition of the morphism $\Theta \circ_1 id_\mu$, involved in the construction of our map ϕ, amounts to the application of the hexagon relations of Figure 3.5 within the parenthesized braid operad. Thus, the commutation of these diagrams in Q implies the preservation of the operadic composition operation in this case. □

The verification of this lemma completes the proof of assertion (a) in Theorem 3.2.7. The second assertion of the theorem, assertion (b), is a consequence of the following lemma:

LEMMA 3.2.11. *The morphism ϕ in Theorem 3.2.7 preserves the deletion morphisms on the whole operad PaB as soon as the object $m = m(x_1, x_2)$ and the morphisms $a = a(x_1, x_2, x_3)$ and $c = c(x_1, x_2, x_3)$ satisfy the relations of assertion (b) in the theorem.*

PROOF. Assuming the relations $\partial_1 m = \partial_2 m = 1$, $\partial_1 a = \partial_2 a = \partial_3 a = id_m$, and $\partial_1 c = \partial_2 c = id_1$ in Q amounts to requiring that our morphism $\phi : PaB \to \mathsf{Q}$ preserves the deletion operations on the objects $\mu \in \mathtt{Ob}\, PaB(2)$, and on the morphisms $\Phi \in \mathtt{Mor}_{PaB(3)}(((x_1x_2)x_3), (x_1(x_2x_3)))$ and $\Theta \in \mathtt{Mor}_{PaB(2)}((x_1x_2), (x_2x_1))$. Since μ is by definition a generating element of the operad $\mathtt{Ob}\, PaB = \Omega$, the requirement on $m = \phi(\mu)$ implies that ϕ preserves deletion operations on all objects of PaB. In the proof of Lemma 3.2.10 we also observe that all morphisms of PaB can be decomposed into a product of operadic composites of the generating elements Φ and Θ. Thus our requirement on $a = \phi(\Phi)$ and $c = \phi(\Theta)$ implies that ϕ preserves deletion operations on all morphisms of PaB, and not only on these generating morphisms. □

The verification of this lemma completes the proof of Theorem 3.2.7. □

We can now give the formal definition of the pro-unipotent Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$.

3.2.12. *The Grothendieck-Teichmüller group as a group of operad automorphisms.* We consider the Malcev completion of the operad of parenthesized braids $\widehat{PaB}$ and the associated unitary operad $\widehat{PaB}_+$. Recall that the operad in pro-nilpotent groupoids $\widehat{PaB}$ has still the Magma operad $\mathtt{Ob}\,\widehat{PaB} = \mathtt{Ob}\,PaB = \Omega$ as objects by construction.

The Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$ is the group of automorphisms

$$\phi_+ : \widehat{PaB}_+ \xrightarrow{\simeq} \widehat{PaB}_+$$

of the unitary operad in pro-nilpotent groupoids $\widehat{PaB}_+$ which:

(a) are the identity on object sets;
(b) and fix the braiding morphism $\Theta \in \mathtt{Mor}_{\widehat{PaB}(2)}((x_1x_2),(x_2x_1))$.

Recall that such a morphism of unitary operads ϕ_+ is equivalent to a morphism of connected operads $\phi : \widehat{PaB} \to \widehat{PaB}$ preserving deletion operations, and as in Theorem 3.2.7, we usually give this associated morphism to determine ϕ_+. By idempotence of the completion process, a morphism $\phi : \widehat{PaB} \to \widehat{PaB}$ is equivalent to a morphism of operads in groupoids $\phi : PaB \to \widehat{PaB}$, where we now consider the plain version of the parenthesized braid operad PaB. Since we fix the image of the generating object $\underline{\mu} \in \mathtt{Ob}\,PaB(2)$ and of the braiding Θ in the definition of $GT^1(\mathbb{Q})$, our morphism $\phi : PaB \to \widehat{PaB}$ is uniquely determined by an associated element $a(x_1,x_2,x_3)$ in the morphism set $\mathtt{Mor}_{\widehat{PaB}}(((x_1x_2)x_3),(x_1(x_2x_3)))$ of the completed parenthesized braid operad $\widehat{PaB}$ and satisfying the requirements of Theorem 3.2.7 in that operad.

We go back to the completion process in order to figure out the explicit definition of such an element $a(x_1,x_2,x_3)$. We have

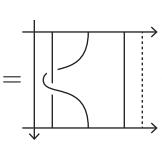
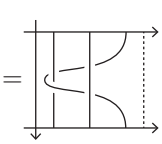
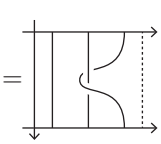
$$\begin{aligned}\mathtt{Hom}_{\widehat{PaB}}(((x_1x_2)x_3),(x_1(x_2x_3))) &= \widehat{\mathbb{G}}(\widehat{\mathbb{Q}}\{\mathtt{Mor}_{PaB}(((x_1x_2)x_3),(x_1(x_2x_3)))\}),\\ \mathtt{Mor}_{PaB}(((x_1x_2)x_3),(x_1(x_2x_3))) &= \mathtt{Mor}_{CoB}(id_3,id_3),\end{aligned}$$

and this latter morphism set, the endomorphism set of the identity permutation of $(1,2,3)$ in the groupoid of colored braids CoB, is identified with the pure braid group on 3 strands P_3. We therefore have

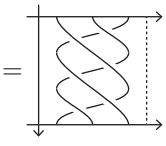
$$\mathtt{Hom}_{\widehat{PaB}}(((x_1x_2)x_3),(x_1(x_2x_3))) = \widehat{\mathbb{G}}(\widehat{\mathbb{Q}}\{P_3\}) = \widehat{P}_3.$$

Thus, our element $a(x_1,x_2,x_3)$ is determined by an element in the Malcev completion of P_3, or equivalently, by a group-like element in the completion of the Hopf algebra associated to this group $\mathbb{Q}\{P_3\}$. Intuitively, we regard such an element $a \in \widehat{P}_3$ as a formal (rational pro-nilpotent) composite of pure braids on 3 strands which we insert on the associator of Figure 3.3 (see the picture in Theorem 3.B).

The group of pure braid groups on 3 strands P_3 is generated by the elements

$A_{12} =$  , $A_{13} =$  , $A_{23} =$  ,

and is also isomorphic to the cartesian product of a central cyclic subgroup $< E >$, generated by the following element

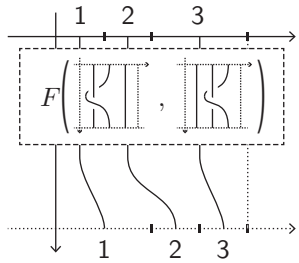
$E =$ 

with the free group generated by A_{12} and A_{23} (see for instance [55, §1.3]). We actually have $E = A_{12}A_{23}A_{13}$.

Our element $a(x_1, x_2, x_3)$ consequently splits as a product $a = E^\lambda \cdot F(A_{12}, A_{23})$, for some formal exponent $\lambda \in \mathbb{Q}$ of the central element E, and where $F(A_{12}, A_{23})$ is an element in the Malcev completion of the free group $\mathbb{F}(A_{12}, A_{23})$ generated by the pure braids A_{12} and A_{23}. Equivalently, the element $F(A_{12}, A_{23})$ is a group-like element in the completion of the Hopf algebra associated to $\mathbb{F}(A_{12}, A_{23})$.

Since the factor E is central in P_3, we can collect the occurrences of the factors E^λ in the hexagon equation satisfied by a to conclude that we necessarily have $E^\lambda = 1$. Hence, we finally obtain that $a = a(x_1, x_2, x_3)$ reduces to the element $a = F(A_{12}, A_{23})$ in the Malcev completion of the free group $\mathbb{F}(A_{12}, A_{23})$. To complete this analysis, we write down the pentagon and hexagon relations satisfied by $a = a(x_1, x_2, x_3)$ in terms of this $F = F(A_{12}, A_{23})$. This gives the following theorem:

THEOREM 3.B (V. Drinfeld [28], see also D. Bar-Natan [6]). *An element ϕ of the Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$, characterized by giving $\phi(\Theta) = c(x_1, x_2) = \Theta$ and some $\phi(\Phi) = a(x_1, x_2, x_3) \in \mathtt{Mor}_{\widehat{PaB}}(((x_1x_2)x_3), (x_1(x_2x_3)))$, is also uniquely determined by an element $F(A_{12}, A_{23})$ in the Malcev completion of the free group $\mathbb{F}(A_{12}, A_{23})$ so that*

$a(x_1, x_2, x_3) =$ 

in the rational pro-nilpotent groupoid $\widehat{PaB}(3)$. The hexagon constraints of Theorem 3.2.7 are equivalent, for this Malcev group element $F(X, Y)$, to the conjunction of the involution

$$F(x, y) \cdot F(y, x) = 1,$$

and hexagon equation

$$F(x, y) \cdot F(z, x) \cdot F(y, z) = 1,$$

where (x, y, z) is any triple of variables (x, y, z) such that $z \cdot y \cdot x = 1$. The pentagon constraint is equivalent to the equation of Figure 3.6 and the deletion constraints are equivalent to the identities $F(x, 1) = 1 = F(1, x)$. □

PROOF. The expression of $a(x_1, x_2, x_3)$ has been determined before the statement of the theorem. We now determine the expression of the hexagon and pentagon constraints associated with the braiding $c(x_1, x_2) = \Theta$ and an element of

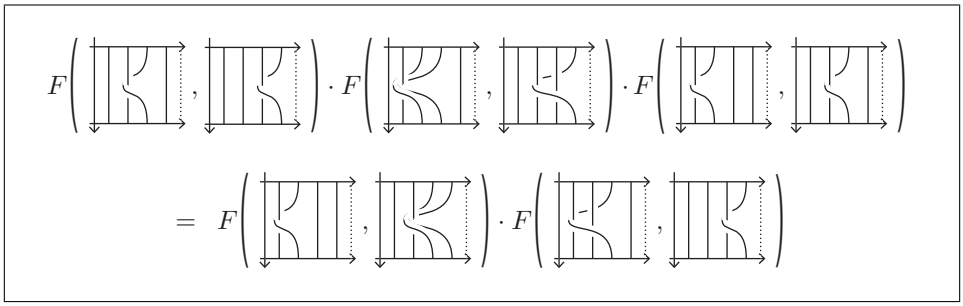


FIGURE 3.6. The pentagon constraints for the group element $F = F(A_{12}, A_{23})$ determining an element of the Grothendieck-Teichmüller group $GT^1(\mathbb{Q})$. The relation holds in the Malcev completion of the braid group $\widehat{P}_4$. The factors of this relation are obtained by applying F, which we regard as an element of the Malcev completion of free group on two generators (x, y), to the various braids $\alpha \in P_4$ represented in the picture.

this form $a(x_1, x_2, x_3)$. The equivalence between the pentagon constraint of Figure 3.4 and the equation of Figure 3.6 is immediate (we just expand the expression of $a(x_1, x_2, x_3)$ in the general relation).

To express the hexagon constraints in terms of $F(x, y)$, we apply formal identities $g^{-1} \cdot F(x, y) \cdot g = F(g^{-1}xg, g^{-1}yg)$ in the Malcev completion of the pure braid group P_3, and we rewrite the equations given by these constraints

$$m(1, c) \cdot a \cdot m(c, 1) = a \cdot c(m, 1) \cdot a$$
$$m(c, 1) \cdot a^{-1} \cdot m(1, c) = a^{-1} \cdot c(m, 1) \cdot a^{-1}$$

as

$$a = (m(1, c)^{-1} a \cdot m(1, c)) \cdot (m(c, 1) \cdot a \cdot m(c, 1)^{-1})$$
$$c(m, 1) \cdot (m(1, c)^{-1} \cdot a^{-1} \cdot m(1, c)) = (c(m, 1)^{-1} \cdot a^{-1} \cdot c(m, 1)) \cdot a^{-1}$$

in order to gather and simplify the braiding factors. Since we have

$$m(c, 1) \cdot A_{12} \cdot m(c, 1)^{-1} = A_{12}, \qquad m(c, 1) \cdot A_{23} \cdot m(c, 1)^{-1} = A_{13},$$
$$c(m, 1)^{-1} \cdot A_{12} \cdot c(m, 1) = A_{23}, \qquad c(m, 1)^{-1} \cdot A_{23} \cdot c(m, 1) = A_{12},$$
$$m(1, c)^{-1} \cdot A_{12} \cdot m(1, c) = A_{13}, \qquad m(1, c)^{-1} \cdot A_{23} \cdot m(1, c) = A_{23},$$

(draw the pictures corresponding to these conjugation relations), we obtain that the above equations are equivalent to the identities

$$F(A_{12}, A_{23}) = F(A_{13}, A_{23}) \cdot F(A_{12}, A_{13}),$$
$$F(A_{13}, A_{12})^{-1} = F(A_{13}, A_{23})^{-1} \cdot F(A_{12}, A_{23})^{-1},$$

which we can rewrite as

$$F(A_{12}, A_{13}) = F(A_{13}, A_{23})^{-1} \cdot F(A_{12}, A_{23}),$$
$$F(A_{13}, A_{12})^{-1} = F(A_{13}, A_{23})^{-1} \cdot F(A_{12}, A_{23})^{-1},$$

in the group $\widehat{P}_3$.

The elements $x = A_{12}$ and $y = A_{13}$ generate a free group in P_3 (like A_{12} and A_{23}), and the already mentioned relation $E = A_{12}A_{23}A_{13}$ implies that A_{23} agrees with the product $z = x^{-1}y^{-1}$ up to a central factor E which we can extract from any formal expression in the plain group P_3 and in the Malcev completion $\widehat{P}_3$ similarly. From these observations all together, we see that our last equations are equivalent to the system of relations

$$F(x,y) = F(y,x)^{-1} \quad \text{and} \quad F(x,y)\cdot F(z,x)\cdot F(y,z) = 1$$

given in the statement of the theorem.

The reduction of the deletion constraints of Theorem 3.2.7 to $F(x,1) = 1 = F(1,x)$ is immediate, and the proof of Theorem 3.B is therefore complete. $\square$

3.3. The Grothendieck-Teichmüller group is the group of homotopy automorphisms of E_2-operads

We consider the classifying spaces $B(\widehat{PaB})$ associated to the Malcev completion of the operad of Parenthesized braids PaB. We have already observed that the definition $\mathtt{Mor}_{PaB}(p,q) = \mathtt{Mor}_{CoB}(\rho(p),\rho(q))$ implies $\widehat{\mathtt{Mor}}_{PaB}(p,q) = \widehat{\mathtt{Mor}}_{CoB}(\rho(p),\rho(q))$ for the morphism sets of the completed operads associated to PaB and CoB. We therefore have a categorical equivalence of operads $\widehat{PaB} \xrightarrow{\sim} \widehat{CoB}$, which induces a weak-equivalence at the classifying space level, so that the operad $B(\widehat{PaB})$ forms, like $B(\widehat{CoB})$ (see Theorem 3.A), a model of the rationalization of the little 2-disc operad. We pick a cofibrant replacement of this operad $\widehat{\mathsf{Q}}_2 \overset{\sim}{\twoheadrightarrow} B(\widehat{PaB})$.

We can perform this process in the category of operads with deletion morphisms to obtain a cofibrant replacement of $B(\widehat{PaB})$ in that category, and of which associated unitary operad $\widehat{\mathsf{Q}}_{2+}$ defines a cofibrant model of the rationalization of the unitary little 2-disc operad. We obviously have $B(\widehat{PaB})_+ = B(\widehat{PaB}_+)$. Any element of the Grothendieck-Teichmüller group $\phi \in GT^1(\mathbb{Q})$ induces an isomorphism $B\phi_+ : B(\widehat{PaB})_+ \xrightarrow{\simeq} B(\widehat{PaB})_+$ which we can lift to a weak-equivalence on our cofibrant replacement:

$$\begin{tikzcd}
\widehat{\mathsf{Q}}_{2+} \arrow[r, dotted, "\exists \widetilde{B\phi}_+"] \arrow[d, two heads, "\sim"] & \widehat{\mathsf{Q}}_{2+} \arrow[d, two heads, "\sim"] \\
B(\widehat{PaB})_+ \arrow[r, "B\phi_+", "\simeq"'] & B(\widehat{PaB})_+
\end{tikzcd} .$$

We readily see that the mapping $B : \phi_+ \mapsto \widetilde{B\phi}_+$ induces a group morphism

$$B : GT^1(\mathbb{Q}) \to \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\widehat{\mathsf{Q}}_{2+})$$

from $GT^1(\mathbb{Q})$ towards the group of homotopy automorphism classes of $\widehat{\mathsf{Q}}_{2+}$, since our lifting construction has, by a general statement of the theory of model categories, a result which is homotopically unique. We moreover have:

Proposition 3.B. *The morphism $B\phi_+ : B(\widehat{PaB})_+ \to B(\widehat{PaB})_+$ associated to an element of the Grothendieck-Teichmüller group $\phi \in GT^1(\mathbb{Q})$ acts identically in homology.*

Explanations. We know from the classical homotopy theory that the spaces $B(\widehat{PaB}(r))$ satisfy $\mathtt{H}_*(B(\widehat{PaB}(r))) = \mathtt{H}_*(\mathsf{D}_2(r))$, where we consider the homology with $\mathbb{Q}$ coefficients (see [24, §V]). Consequently, we have an identity between $\mathtt{H}_*(B(\widehat{PaB}))$ and the Gerstenhaber operad Gerst_2. The proposition follows from the requirement that the elements of the Grothendieck-Teichmüller group $\phi \in GT^1(\mathbb{Q})$ act identically in arity 2 and the preliminary observation that the Gerstenhaber operad is generated by operations $\mu = \mu(x_1, x_2)$ and $\lambda = \lambda(x_1, x_2)$ in arity 2 precisely (see §1.2.13). □

Recall that we use the notation $\mathcal{O}p_1$ for the category of unitary operads in general, and the notation $\mathcal{O}p_*$, in the special case where the tensor unit of the ambient symmetric monoidal category is the final object (see §I.4). In this context, which includes the case of operads in topological spaces, the category $\mathcal{O}p_*$ actually forms a full subcategory of $\mathcal{O}p$ (with the same class of weak-equivalence). Thus, we can equally keep or omit the mark $*$ in the expression of morphism sets (and in the expression of homotopy automorphisms similarly). In the definition of the mapping B (before Proposition 3.B), this mark $*$ has omitted. From now on, we prefer to carefully recall the consideration of category structures in our formulas.

We have already mentioned that we have a functor $\mathtt{H}_* : \mathcal{T}op\mathcal{O}p \to gr\,\mathcal{O}p$, from the category of topological operads $\mathcal{T}op\mathcal{O}p$ towards the category of graded operads $gr\,\mathcal{O}p$, yielded by the classical homology of topology spaces, and an induced functor $\mathtt{H}_* : \mathcal{T}op\mathcal{O}p_* \to gr\,\mathcal{O}p_1$ on unitary operads. We easily see that homotopic operad morphisms $\phi, \psi : \mathsf{P} \to \mathsf{Q}$ induce the same morphism in homology: the underlying spaces $\mathsf{Q}^{\Delta^1}(r)$ of a path object of Q in the category of operads are path objects in topological spaces; morphisms $\phi, \psi : \mathsf{P} \to \mathsf{Q}$, which are homotopic in the category operads, are therefore homotopic as maps of topological spaces. Accordingly, the homology defines a functor $\mathtt{H}_* : \mathtt{Ho}(\mathcal{T}op\mathcal{O}p) \to gr\,\mathcal{O}p$ on the homotopy category of topological operads $\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)$, and similarly in the unitary setting, which we now consider.

Proposition 3.B implies that the mapping $B : GT^1(\mathbb{Q}) \to \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p_*)}(\widehat{\mathsf{Q}}_{2+})$ lands in the kernel of the group morphism

$$\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p_*)}(\widehat{\mathsf{Q}}_{2+}) \xrightarrow{\mathtt{H}_*} \mathtt{Aut}_{gr\,\mathcal{O}p_1}(\mathtt{H}_*(\widehat{\mathsf{Q}}_{2+}))$$

deduced from the homology functor $\mathtt{H}_* : \mathtt{Ho}(\mathcal{T}op\mathcal{O}p_*) \to gr\,\mathcal{O}p_1$.

Recall that $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p_*)}(\widehat{\mathsf{Q}}_{2+})$ represents the set of connected components of a space $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p_*}(\widehat{\mathsf{Q}}_{2+})$ (actually a monoid) associated to the operad $\widehat{\mathsf{Q}}_{2+}$. To be more explicit, we have a connected component $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p_*}(\widehat{\mathsf{Q}}_{2+})_\phi$ canonically associated to each homotopy class of homotopy automorphism $\phi : \widehat{\mathsf{Q}}_{2+} \xrightarrow{\sim} \widehat{\mathsf{Q}}_{2+}$. Motivated by the result of Proposition 3.B, we consider the space $\mathtt{hAut}^1_{\mathcal{T}op\mathcal{O}p_*}(\widehat{\mathsf{Q}}_{2+})$ formed by the sum of connected components of $\mathtt{hAut}_{\mathcal{T}op\mathcal{O}p_*}(\widehat{\mathsf{Q}}_{2+})$ associated to maps ϕ such that $\mathtt{H}_*(\phi) = Id$.

The main result of this monograph reads:

THEOREM 3.C. *Our mapping* $B : GT^1(\mathbb{Q}) \to \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p_*)}(\widehat{\mathsf{Q}}_{2+})$ *induces a group isomorphism*

$$GT^1(\mathbb{Q}) \xrightarrow{\simeq} \ker\{\underbrace{\mathtt{H}_* : \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p_*)}(\widehat{\mathsf{Q}}_{2+}) \to \mathtt{Aut}_{gr\,\mathcal{O}p_1}(\mathtt{H}_*(\widehat{\mathsf{Q}}_{2+}))}_{\pi_0\,\mathtt{hAut}^1_{\mathcal{T}op\mathcal{O}p_*}(\widehat{\mathsf{Q}}_{2+})}\},$$

and we have

$$\pi_*(\mathtt{hAut}^1_{\mathcal{T}op\mathcal{O}p_*}(\widehat{\mathsf{Q}}_{2+}) = 0$$

when $* > 0$.

We will actually start the proof of this theorem in §9, after a tour through deformation complexes.

Remarks. We will not used the following approach, but some partial results might however be obtained from constructions sketched in the present chapter and in the previous one.

In §2.2, we mention that the fundamental groupoid functor defines a left adjoint of the classifying space functor from operads in groupoids to operads in simplicial sets. The augmentation of this adjunction $\pi B(\mathsf{P}) \xrightarrow{\sim} \mathsf{P}$ is an isomorphism of operads in groupoids, for any $\mathsf{P} \in \mathcal{G}rd\mathcal{O}p$, and the weak-equivalence $\widehat{\mathsf{Q}}_{2+} \xrightarrow{\sim} \widehat{PaB}_+$, associated to our cofibrant replacement $\widehat{\mathsf{Q}}_2$, induces a categorical equivalence of operads in groupoids at the fundamental groupoid level. Thus, we can apply fundamental groupoids to retrieve a commutative diagram of operads in groupoids of the form

(a)
$$\begin{tikzcd} \pi(\widehat{\mathsf{Q}}_{2+}) \arrow[r, "\pi(\widetilde{B\phi})"] \arrow[d, "\sim"'] & \pi(\widehat{\mathsf{Q}}_{2+}) \arrow[d, "\sim"'] \\ \widehat{PaB}_+ \arrow[r, "\phi"'] & \widehat{PaB}_+ \end{tikzcd} .$$

By Proposition 3.A, we have an operad morphism $s : \widehat{PaB}_+ \to \pi(\widehat{\mathsf{Q}}_{2+})$ giving, in each arity, an inverse equivalence of the morphism $\pi\widehat{\mathsf{Q}}_{2+}(r) \xrightarrow[\sim]{p} \pi B\widehat{PaB}_+(r) = \widehat{PaB}_+(r)$. Moreover, we can establish that the natural equivalences connecting the composites sp and ps to the identity are operadic (in the sense of Proposition 3.A) too.

This construction could be used to prove the injectivity of our map from $GT^1(\mathbb{Q})$ to $\mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p_*)}(\widehat{\mathsf{Q}}_{2+})$.

One could also define the cofibrant replacement $\widehat{\mathsf{Q}}_2$ from the cotriple resolution of $\widehat{PaB}$ in the category of operads in groupoids (we refer to a next chapter §5 for detailed recollections on cotriple resolutions). Indeed, one can check that the classifying space construction preserves free operads (because free operads are made of coproducts and cartesian products). Therefore, the classifying space operad $B(\widehat{\mathsf{R}}_\bullet)$ of a cotriple resolution $\widehat{\mathsf{R}}_\bullet$ is a cotriple resolution is simplicial operads, and hence, defines a natural cofibrant replacement of $B(\widehat{PaB})$ in the category of simplicial operads. Then the problem is to relate homotopy classes of morphisms of operads in simplicial sets $B(\widehat{\mathsf{R}}_\bullet) \to B(\widehat{PaB})$ to morphisms of operads in groupoids $\pi_0(\widehat{\mathsf{R}}_\bullet) = \widehat{PaB} \to \widehat{PaB}$ in order to prove the surjectivity of our mapping.

But we will not follow such a direct approach. Instead, we will use a decomposition of an operadic mapping space $\mathtt{Map}_{\mathcal{T}op\mathcal{O}p_*}(\widehat{\mathsf{Q}}_{2+}, \widehat{\mathsf{E}}_{2+})$ (where we take $\widehat{\mathsf{E}}_2 = B(\widehat{CoB})$ instead of $B(\widehat{PaB})$ by the way) into a tower of fibrations involving classifying spaces of chord diagrams as fibers, and we will adapt arguments of [24, §IX] to prove the bijectivity of our mapping.

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