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# A note on online and recursively arbitrarily vertex-partitionable balloons

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## Abstract

In this note, we study arbitrarily vertex-partitionable balloons. In particular, we show that an online arbitrarily vertex-partitionable balloon has maximum degree at most 5. Moreover, we exhibit some infinite families of online and recursively vertex-partitionable balloons with four and five branches.

## 1 Preliminaries

In this note, the specific class of *balloons* is studied:

**Definition 1** We denote by  $B(b_1, \dots, b_k)$  the graph of order  $2 + \sum_{1 \leq i \leq k} b_i$  consisting of two vertices  $r_1, r_2$ , with  $k$  vertex-disjoint paths  $P_1, \dots, P_k$  of order  $b_1, \dots, b_k$  joining  $r_1$  to  $r_2$ . The graph  $B(b_1, \dots, b_k)$  is called a  $k$ -balloon with roots  $r_1$  and  $r_2$  and branches  $P_1, \dots, P_k$ .

We will also need the following definition.

**Definition 2** We denote by  $PB(b_1, \dots, \overline{b_i}, \dots, \underline{b_j}, \dots, b_k)$  the partial  $k$ -balloon with branches of sizes  $b_1, \dots, b_k$  and roots  $r_1, r_2$  where the branches denoted  $\overline{b_i}$  are not adjacent to  $r_1$  and the ones denoted  $\underline{b_j}$  are not adjacent to  $r_2$ .

Let  $n, \tau_1, \dots, \tau_k$  be positive integers such that  $\sum_{1 \leq i \leq k} \tau_i = n$ . Then  $\tau = (\tau_1, \dots, \tau_k)$  is called a *decomposition* of  $n$ .

We study the problem of partitioning the vertex set of graphs according to certain conditions. The following definitions have been introduced in [1], [4] and [3], respectively.

**Definition 3** Let  $G$  be a connected graph on  $n$  vertices.

- $G$  is said to be arbitrarily vertex-partitionable (*AP for short*) if for every decomposition  $\tau = (\tau_1, \dots, \tau_k)$  of  $n$ , there exists a partition  $V_1, \dots, V_k$  of  $V(G)$  such that each part  $V_i$  has order  $\tau_i$  and induces a connected subgraph of  $G$ .

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- $G$  is said to be online arbitrarily vertex-partitionable (*OLAP for short*) if for every decomposition  $\tau = (\tau_1, \dots, \tau_k)$  of  $n$ , there exists a partition  $V_1 \dots, V_k$  of  $V(G)$  such that each part  $V_i$  has order  $\tau_i$ , induces a connected subgraph of  $G$ , and  $V(G) \setminus V_i$  induces an OLAP subgraph of  $G$ .
- $G$  is said to be recursively arbitrarily vertex-partitionable (*RAP for short*) if for every decomposition  $\tau = (\tau_1, \dots, \tau_k)$  of  $n$ , there exists a partition  $V_1 \dots, V_k$  of  $V(G)$  such that each part  $V_i$  has order  $\tau_i$ , and both  $V_i$  and  $V(G) \setminus V_i$  induce a RAP subgraph of  $G$ .

It is worth noting that every graph with a hamiltonian path is RAP, every RAP graph is OLAP, every OLAP graph is AP, and every AP graph contains a (quasi)-perfect matching.

**Observation 4 ([3])** Let  $B = B(b_1, \dots, b_k)$  be an AP balloon with  $b_1 \leq \dots \leq b_k$ . If  $n$  is odd,  $B$  contains at most three branches of odd size. If  $n$  is even,  $B$  contains at most two branches of odd size.

**Lemma 5 ([2])** Let  $B(b_1, \dots, b_k)$  be an AP balloon with  $b_1 \leq \dots \leq b_k$ . Then for each  $i \leq k$  we have  $2b_i \geq \sum_{j < i} b_j$ .

## 2 OLAP balloons have maximum degree at most 5

It is shown in [3] that every RAP balloon has maximum degree at most 5. We show that this holds as well for OLAP balloons.

**Theorem 6** Let  $G$  be an OLAP  $k$ -balloon. Then  $k \leq 5$ .

**Proof** The proof is by contradiction. Let  $G$  be a minimum counterexample to the theorem:  $G$  is an OLAP  $k$ -balloon ( $k \geq 6$ ) with least possible order.

For each  $\lambda < n$ , it is possible to find a subset of vertices  $V_\lambda$  with  $|V_\lambda| = \lambda$ ,  $G[V_\lambda]$  is connected, and  $G[V \setminus V_\lambda]$  is OLAP. Since by the minimality of  $G$ ,  $G' = G[V \setminus V_\lambda]$  cannot be a partial  $k$ -balloon (otherwise adding an edge to  $G'$  would provide an OLAP  $k$ -balloon with order strictly less than  $G$ ), for each  $\lambda$ , one of the following conditions must be fulfilled:

1.  $G[V \setminus V_\lambda]$  is a  $k'$ -balloon with  $k' = k - 1$  (i.e.  $V_\lambda$  is a branch of  $G$ ), or
2.  $G[V \setminus V_\lambda]$  is a path (i.e.  $V_\lambda$  consists of a root of  $G$  together with  $k - 2$  branches), or
3.  $G[V \setminus V_\lambda]$  is an OLAP caterpillar (i.e.  $V_\lambda$  consists of a root of  $G$  together with  $k - 2$  branches but one vertex), or
4.  $G[V \setminus V_\lambda]$  is the tripode  $T_3(2, 4, 6)$  (i.e.  $V_\lambda$  consists of a root of  $G$  together with  $k - 2$  branches but two vertices of the same branch)

We claim that  $G$  has branches of sizes respectively 1, 2, 3, 4, 5 and 6. We show this using contradiction. Let  $\lambda \in \{1, 2, 3, 4, 5, 6\}$  and suppose there is no branch of size  $\lambda$  in  $G$ .

- $\lambda \in \{1, 2, 3\}$ : for any choice of  $V_\lambda$ ,  $G[V \setminus V_\lambda]$  is either a partial  $k$ -balloon or a tree with a vertex of degree 4 (since at most two branches can be completely included in  $V_\lambda$ ), a contradiction.
- $\lambda = 4$ : similarly as in the previous case, for any choice of  $V_\lambda$ ,  $G[V \setminus V_\lambda]$  is either a partial  $k$ -balloon or a tree with a vertex of degree 3. By Observation 4,  $G[V \setminus V_\lambda]$  cannot be a caterpillar, thus it is  $T_3(2, 4, 6)$  and  $G = B_6(1, 1, 1, 2, 4, 6)$ , a contradiction since  $G$  has a branch of size 4.

- $\lambda = 5$ : by the previous cases we know that  $G$  has branches of sizes 1, 2, 3 and 4.  $G[V \setminus V_\lambda]$  cannot be a path since by Observation 4, five vertices cannot cover four branches and a root. If  $G[V \setminus V_\lambda]$  is an OLAP caterpillar, then necessarily  $k = 6$ , the caterpillar is  $Cat(4, 5)$  and  $G$  contains three branches of size 1:  $G = B_6(1, 1, 1, 2, 3, 4)$ . But then following Observation 4,  $G$  is not OLAP. Finally, if  $G[V \setminus V_\lambda]$  is  $T_3(2, 4, 6)$ ,  $V_\lambda$  must either totally cover the branch of size 3, or the branch of size 2 plus a vertex of the one of size 3. But then it is only possible to totally cover two branches, and the remaining part has a vertex of degree at least 4 and is not  $T_3(2, 4, 6)$ .
- $\lambda = 6$ : by the previous cases we know that  $G$  has branches of sizes 1, 2, 3, 4 and 5. Now suppose there is one additional branch  $b_i$  ( $i \leq 6$ ) such that  $b_i \leq 5$ . By Lemma 5 there can be at most one such additional branch, and then  $b_i = 1$ . Thus  $G = B_k(1, 1, 2, 3, 4, 5, \dots)$ . But now the only potential choice for  $V_\lambda$  would be the three smallest branches, a root and one additional vertex. If  $k > 6$ ,  $G[V \setminus V_\lambda]$  is a tree with a vertex of degree at least 4, which is not OLAP. Otherwise it is  $T_3(2, 4, 5)$ , which is not OLAP either. Thus,  $G = B_k(1, 2, 3, 4, 5, b_6, \dots)$  with  $b_6 \geq 7$ . But again for  $\lambda = 6$  it is not possible to decompose it. Thus  $G$  is not OLAP, a contradiction.

So,  $G = B_k(1, 2, 3, 4, 5, 6, \dots)$ . But using Lemma 5,  $G$  is not AP, a contradiction.  $\square$

### 3 Families of OLAP balloons

In this section, we show that there are infinitely many OLAP  $k$ -balloons, for each  $k \leq 5$ . The case where  $k \leq 3$  is trivial since then any  $k$ -balloon has a hamiltonian path and is therefore RAP.

#### 3.1 OLAP 4-balloons

**Proposition 7**  $B(1, 1, n, m)$  is OLAP iff at most one of  $n$  and  $m$  is odd.

**Proof** Easily partitionable for  $\lambda = 1, 3$ . For  $\lambda = 2$ , one can prove by induction that  $B(1, 1, \overline{2k}, m)$  and  $B(1, 1, n, \overline{2k})$  is OLAP.  $\square$

**Proposition 8** Let  $a \leq b \leq c$  be such that  $Cat(b + 1, a + c + 2)$  or  $Cat(c + 1, a + b + 2)$  or  $Cat(a + 1, b + c + 2)$  is OLAP. Then  $B(1, a, b, c)$  is OLAP.

**Proof** These caterpillars are spanning subgraphs of  $B(1, a, b, c)$ .  $\square$

**Proposition 9**  $B(1, 2, 2, k)$ ,  $k \geq 3$ , is OLAP iff  $k \equiv 0, 1 \pmod{3}$ .

**Proof** Easily partitionable for  $\lambda \neq 3$ . If  $k \equiv 0, 1 \pmod{3}$ , also OK for  $\lambda = 3$  because of  $Cat(3, k + 1)$  which is OLAP.

Now suppose  $k \equiv 2 \pmod{3}$ . For  $\lambda = 3$ , for the balloon to be partitionable we should have either:

- $PB(1, 2, 2, \overline{k - 3})$  OLAP.
- $PB(1, 2, 2, \underline{a}, \underline{b})$  ( $a + b = k - 3$ ) OLAP

But in both cases these partial balloons are not  $(3, \dots, 3)$ -partitionable.  $\square$

**Proposition 10**  $B(1, 2, 3, k)$ ,  $k \geq 3$ , is OLAP.

**Proof** Straightforward. □

**Proposition 11**  $PB(2, 2, 2, \bar{k})$ , is OLAP iff  $k \equiv 0 \pmod{3}$ .

**Proof** By induction on  $k$ . □

**Observation 12**  $B(2, 2, 2, 3)$ ,  $B(2, 2, 3, 4)$ ,  $B(2, 2, 3, 6)$ ,  $B(2, 2, 3, 18)$ ,  $B(2, 2, 4, 6)$ ,  $B(2, 3, 4, 6)$ , are OLAP.

### 3.2 OLAP 5-balloons

The proof of the following result is omitted.

**Proposition 13** Let  $G$  be a graph of Table 1. Then  $G$  is OLAP.

$B(1, 1, 1, 2, k)$ with $k \equiv 0, 4 \pmod{6}$ $B(1, 1, 1, 4, k)$ with $k \in \{6, 8, 10, 18\}$ $B(1, 1, 1, 6, k)$ with $k \in \{8, 10, 12, 14\}$ $B(1, 1, 1, 8, 10)$
$B(1, 1, 2, 2, 3)$ $B(1, 1, 2, 2, 6)$ $B(1, 1, 2, 3, k)$ with $k \equiv 0 \pmod{2}$ $B(1, 1, 2, 4, 6)$ $B(1, 1, 2, 5, k)$ with $k \equiv 0, 4 \pmod{6}$ $B(1, 1, 3, 4, 6)$ $B(1, 1, 4, 4, 6)$
$B(1, 2, 2, 3, 4)$ $B(1, 2, 2, 4, 6)$ $B(1, 2, 3, 4, 5)$ $B(1, 2, 3, 4, 6)$ $B(1, 2, 3, 4, 7)$
$B(2, 2, 3, 4, 6)$

Table 1: Some OLAP 5-balloons

## 4 Families of RAP balloons

In this section, we show that there are infinitely many RAP  $k$ -balloons, for each  $k \leq 5$ .

### 4.1 RAP 4-balloons

**Proposition 14**  $PB(1, 1, 1, \bar{k})$ ,  $B(1, 1, 1, k)$ ,  $PB(1, 1, 3, \bar{k})$  and  $B(1, 1, 3, k)$  are RAP iff  $k$  is even.

**Proof** If  $k$  is odd, they are not AP by Observation 4. On the other hand,  $Cat(2, k + 3)$  is a spanning subgraph of  $PB(1, 1, 1, \bar{k})/B(1, 1, 1, k)$  and  $Cat(2, k + 5)$  is a spanning subgraph of  $PB(1, 1, 3, \bar{k})/B(1, 1, 3, k)$ . These caterpillars are RAP when  $k$  is even. □

**Proposition 15**  $PB(1, 1, 2, \bar{k})$  is RAP for any  $k \geq 1$ .

**Proof** Observe that it is true for  $k \leq 3$  since  $Cat(3, 5)$  and  $Cat(4, 5)$  are spanning subgraphs of  $PB(1, 1, 2, \bar{1})$ ,  $PB(1, 1, 2, \bar{2})$  and  $PB(1, 1, 2, \bar{3})$ , respectively.

Now, suppose the proposition is true for all  $i \leq k - 1$  and consider  $PB(1, 1, 2, \bar{k})$ . For  $\lambda \leq 4$ ,  $PB(1, 1, 2, \bar{k})$  can be partitioned into  $P_\lambda$  and  $P_{n-\lambda}$ . For  $\lambda = 5$ , it can be partitioned into  $Cat(2, 3)$  and  $P_{k+1}$ . For  $\lambda = 6$ , it can be partitioned into  $B(1, 1, 2)$  and  $P_k$ . Finally, for  $\lambda \geq 7$ , it can be partitioned into  $PB(1, 1, 2, \overline{\lambda-6})$  and  $P_{k-\lambda+6}$ , the first being RAP by induction hypothesis.  $\square$

**Proposition 16**  $PB(1, 2, 2, \bar{k})$  and  $B(1, 2, 2, k)$  are RAP iff  $k \equiv 1, 2 \pmod{3}$ .

**Proof** On the one hand, observe that  $Cat(3, k+4)$  is a spanning subgraph of  $PB(1, 2, 2, \bar{k})$  and  $B(1, 2, 2, k)$ , which is RAP when  $k \equiv 1, 2 \pmod{3}$ . On the other hand, if  $k \equiv 0 \pmod{3}$ , they are not  $(3, \dots, 3)$ -partitionable.  $\square$

**Proposition 17**  $PB(1, 2, 3, \bar{k})$  is RAP for any  $k \geq 3$ .

**Proof** For any  $\lambda \leq \frac{n}{2}$ ,  $PB(1, 2, 3, \bar{k})$  must be partitionable into two vertex-disjoint RAP subgraphs of orders  $\lambda$  and  $n - \lambda$ .

- $\lambda = 1$ :  $P_1 + P_{n-1}$
- $\lambda = 2$ :  $P_2 + P_{n-2}$
- $\lambda = 3$ :  $P_3 + P_{n-3}$
- $\lambda = 4$ :  $P_4 + P_{k+4}$
- $\lambda = 5$ :  $P_5 + P_{k+3}$
- $\lambda = 6$ :  $P_6 + P_{k+2}$
- $\lambda = 7$ :  $Cat(3, 4) + P_{k+1}$
- $\lambda = 8$ :  $B(1, 2, 3) + P_k$

Note that by the previous cases, the proposition is true for  $n \leq 16$  (i.e.  $k \leq 8$ ).

Now, let  $\lambda \geq 9$ , suppose it is true for any  $i \leq k - 1$ , and consider  $PB(1, 1, 2, 3, \bar{k})$ . Now one can partition the graph into  $P_{n-\lambda}$  and  $PB(1, 2, 3, \overline{\lambda-8})$ , which is RAP by induction hypothesis.  $\square$

## 4.2 RAP 5-balloons

**Proposition 18**  $PB(1, 1, 2, 3, \overline{2k})$  is RAP for any  $k \geq 0$ .

**Proof** For any  $\lambda \leq \frac{n}{2}$ ,  $PB(1, 1, 2, 3, \overline{2k})$  must be partitionable into two vertex-disjoint RAP subgraphs of orders  $\lambda$  and  $n - \lambda$ .

- $\lambda = 1$ :  $P_1 + B(1, 2, 3, \overline{2k})$  (RAP by Proposition 17)
- $\lambda = 2$ :  $P_2 + B(1, 1, 3, \overline{2k})$  (RAP by Proposition 14)
- $\lambda = 3$ :  $P_3 + B(1, 1, 2, \overline{2k})$  (RAP by Proposition 15)
- $\lambda = 4$ :  $P_4 + Cat(4, 2k+1)$
- $\lambda = 5$ :  $Cat(2, 3) + P_{2k+4}$

- $\lambda = 6$ :  $P_6 + Cat(2, 2k + 1)$
- $\lambda = 7$ :  $Cat(3, 4) + P_{2k+2}$

Note that by the previous cases, the proposition is true for  $n \leq 14$  (i.e.  $k \leq 2$ ).

Now, let  $\lambda \geq 8$ , suppose it is true for any  $i \leq k - 1$ , and consider  $PB(1, 1, 2, 3, \overline{2k})$ .

- $\lambda$  even:
  - if  $\lambda \leq 2k$ , one can partition the graph into  $P_\lambda$  and  $PB(1, 1, 2, 3, \overline{2k - \lambda})$  by placing  $P_\lambda$  at the end of the last branch, and use the induction hypothesis.
  - if  $\lambda > 2k$ , then  $n - \lambda \leq 7$  (since both  $2k$  and  $\lambda$  are even), thus one can use the cases where  $\lambda \leq 7$ .
- $\lambda$  odd. Then partition the graph into  $PB(1, 1, 2, 3, \overline{\lambda - 9})$  (note that it is RAP by induction hypothesis since  $\lambda - 9$  is even and positive) and  $P_{2k - \lambda + 9}$ .

□

This proves the existence of an infinite family of RAP 5-balloons:

**Corollary 19**  $B(1, 1, 2, 3, 2k)$  is RAP for any  $k \geq 1$ .

In [3], it is shown that  $B(1, 1, 1, 2, 4)$  and  $B(1, 1, 2, 2, 3)$  are RAP.

**Proposition 20** Other RAP 5-balloons are  $B(1, 2, 2, 3, 4)$  and  $B(2, 2, 3, 4, 6)$ .

## References

- [1] O. Baudon, D. Barth and J. Puech. Decomposable trees: a polynomial algorithm for tripodes. *Discrete applied mathematics*, 119(3):205–216, 2002.
- [2] O. Baudon, F. Foucaud, J. Przybyło and M. Woźniak. Structure of  $k$ -connected Arbitrarily Partitionable Graphs. Manuscript, 2011.
- [3] O. Baudon, F. Gilbert and M. Woźniak. Recursively arbitrarily vertex-decomposable graphs. Submitted, 2010.
- [4] M. Hornák, Z. Tuza and M. Woźniak. On-line arbitrarily vertex decomposable trees. *Discrete Applied Mathematics* 155(11):1420–1429, 2007.