Dual logarithmic residues and free complete intersections

Michel Granger, Mathias Schulze

DUAL LOGARITHMIC RESIDUES AND FREE COMPLETE INTERSECTIONS

MICHEL GRANGER AND MATHIAS SCHULZE

Abstract. We introduce a dual logarithmic residue map for hypersurface singularities and use it to answer a question of Kyoji Saito. Our result extends a theorem of Lê and Saito by an algebraic characterization of hypersurfaces that are normal crossing in codimension one. For free divisors, we relate the latter condition to other natural conditions involving the Jacobian ideal and the normalization. We suggest a generalization of the notions of logarithmic vector fields and freeness for complete intersections. In the case of quasihomogeneous complete intersection space curves, we give an explicit description.

Contents

1. Introduction

In the landmark paper [Sai80], Kyoji Saito introduced the modules of logarithmic differential forms and of logarithmic vector fields along a reduced divisor D in a complex manifold S. These algebraic objects contain deep geometric, topological, and representation theoretic information on the singularities that is only partly understood.

The notion of freeness of a divisor, defined in terms of these logarithmic modules, generalizes that of a normal crossing divisor (see Remark 1.4.(1) below). Free divisors can be seen as the opposite extreme of isolated singularities: They have maximal, in fact Cohen-Macaulay, singular loci. Classical examples of free divisors include discriminants in the deformation theory of singularities (see for instance [Sai80, (3.19)], [Loo84, §6], [vS95]) and reflection arrangements and discriminants of Coxeter groups (see [Sai80, (3.19)], [Ter80b]). More recent examples are discriminants in certain prehomogeneous vector spaces (see [GMS11]). The freeness property is closely related to the complement of the divisor being a $K(\pi, 1)$-space (see [Sai80, (1.12)], [Del72]), although these two properties are not equivalent (see

1991 *Mathematics Subject Classification.* 32S25, 32A27, 32C37.
Key words and phrases. differential form, residue, duality, free divisor, complete intersection.

[ER95]). Even in special cases, such as that of hyperplane arrangements, freeness is not completely understood. For instance, Terao's conjecture on the combinatorial nature of freeness for arrangements is one of the central open problems in arrangement theory.

Another interesting construction based on logarithmic modules has been given much less attention: Generalizing classical residue constructions of Poincaré and Leray, Saito introduced the residue of a logarithmic differential form. Logarithmic residues of 1-forms are meromorphic functions on the normalization $\bar{D}$ of D. In contrast, a holomorphic function on $\bar{D}$ can be considered as a so-called weakly holomorphic function on D, that is, a function on the complement of the singular locus $\operatorname{Sing} D$ of D, locally bounded near points of $\operatorname{Sing} D$ (see [dJP00, Thm. 4.4.15]). While any such weakly holomorphic function is the residue of some logarithmic 1-form, the image of the residue map might be strictly larger than the ring of weakly holomorphic functions. The case of equality was related by Lê and Saito (see [Sai80, Lem. 2.13] and [LS84]) to a geometric, and to a purely topological property.

Theorem 1.1 (Lê–Saito)**.** *Let D be a reduced divisor in a complex manifold S. Then the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) hold true for the following statements:*

(1) The local fundamental groups of $S \backslash D$ are Abelian.
(2) D is normal crossing in codimension 1.
(3) The residue of any logarithmic 1-form along D is weakly holomorphic.

While most constructions in Saito's logarithmic theory and its generalizations have a dual counterpart (for instance, restriction maps in arrangement theory), a notion of a dual logarithmic residue associated to a vector field was not know to the authors. The main motivation for this article was to construct such a dual logarithmic residue (see Section 3). This duality turns out to translate condition (3) in Theorem 1.1 into the more familiar equality of the Jacobian ideal and the conductor ideal of a normalization. This will lead to a proof of the missing implication in Theorem 1.1.

Theorem 1.2. *The implication (2) $\Leftarrow$ (3) in Theorem 1.1 holds true.*

Under the additional hypothesis that D is a free divisor, there are other algebraic conditions equivalent to those in Theorem 1.1.

Theorem 1.3. *Extend the list of statements in Theorem 1.1 as follows:*

(4) The Jacobian ideal of D is reduced.
(5) D is Euler-homogeneous.
(6) D has a Cohen–Macaulay normalization.
(7) The Jacobian ideal of D equals the conductor ideal of a normalization.

Then (2) $\Leftarrow$ (4) $\Rightarrow$ (5). If D is a free divisor then (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ ((6) and (7)).

Remark 1.4.

(1) Faber [Fab11] studied condition (4) in Theorem 1.3 and raised the following question: Is any free divisor with reduced Jacobian ideal a normal crossing divisor? Faber gave a positive answer for special cases including plane curves, hyperplane arrangements, and divisors with Gorenstein Jacobian ideal. She reduced the general to the irreducible case.

(2) Saito [Sai80, (2.11)] proved the missing implication in Theorem 1.1 for plane curves. If D is holonomic in codimension 1, this yields the general case by analytic

triviality along logarithmic strata (see [Sai80, §3]). However, for example, the equation $xy(x+y)(x+yz)=0$ defines a well-known free divisor which is not holonomic in codimension 1.

(3) Saito [Sai80, (2.9) iii) ⇔ iv)] proved the equivalence of two conditions which are stronger than (2) and (3) of Theorem 1.1, respectively: Let $D_1,\dots,D_k$ denote the local irreducible components of D at a point $p\in D$. Then, in the strong version of (2), self-intersections of $D_1,\dots,D_k$ in codimension 1 near p are excluded, while in the strong version of (3), the residues are required to be sums of functions on $D_1,\dots,D_k$ near p, instead of on the corresponding normalizations $\bar{D}_1,\dots,\bar{D}_k$. For example, the Whitney umbrella is irreducible and, in codimension 1, normal crossing but not smooth due to self-intersection. However it is not free. Therefore, it does not constitute a counter-example to the question in (1).

In the last Section 5, we study a natural generalization of freeness for complete intersections. We define an analogue of the module of logarithmic vector fields and describe it explicitly in the case of homogeneous complete intersection space curves (see Proposition 5.5).

2. Freeness and Jacobian

In this section, we review Saito's logarithmic modules, the relation of freeness and Cohen–Macaulayness of the Jacobian ideal, and the duality of maximal Cohen-Macaulay fractional ideals. We switch to a local setup for the remainder of the article.

Let D be a reduced divisor defined by $\mathscr{I}_D=\mathscr{O}_S\cdot h$ in the smooth complex analytic space germ $S=(\mathbb{C}^n,0)$. Recall Saito's definition [Sai80, §1] of the $\mathscr{O}_S$-modules of logarithmic differential forms and of vector fields. We abbreviate $\Theta_S:=\operatorname{Der}_{\mathbb{C}}(\mathscr{O}_S)=\operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S,\mathscr{O}_S)$.

Definition 2.1 (Saito)**.**

$$\Omega^p(\log D):=\{\omega\in\Omega^p_S(D)\mid d\omega\in\Omega^{p+1}_S(D)\}$$
$$\operatorname{Der}(-\log D):=\{\delta\in\Theta_S\mid dh(\delta)\in\mathscr{I}_D\}$$

These modules are stalks of analogously defined coherent $\mathscr{O}_S$-sheaves which are normal: If $i\colon S\backslash\operatorname{Sing}D\hookrightarrow S$ denotes the inclusion of the complement of the singular locus of D then $i_*i^*\mathscr{F}=\mathscr{F}$ for any of the sheaves $\mathscr{F}$ in Definition 2.1. It follows that $\delta\in\operatorname{Der}(-\log D)$ if and only if δ is tangent to D at smooth points, and that $\Omega^1(\log D)$ and $\operatorname{Der}(-\log D)$ are mutually dual and hence reflexive.

Definition 2.2. A reduced divisor D is called free if $\operatorname{Der}(-\log D)$ is a free $\mathscr{O}_S$-module.

In particular, normal crossing divisors are free. By definition, there is an exact sequence

$$0\longleftarrow\mathscr{J}_D\xleftarrow{dh}\Theta_S\longleftarrow\operatorname{Der}(-\log D)\longleftarrow 0 \tag{2.1}$$

where the Fitting ideal $\mathscr{J}_D=\mathscr{F}^{n-1}_{\mathscr{O}_D}(\Omega^1_D)$ is the Jacobian ideal of D. We shall consider the singular locus $\operatorname{Sing}D$ of D equipped with the structure defined by $\mathscr{J}_D$, that is,

$$\mathscr{O}_{\operatorname{Sing}D}:=\mathscr{O}_D/\mathscr{J}_D.$$

The following fundamental result is a consequence of the sequence (2.1), the Hilbert–Burch theorem (see [Ale88, §1 Thm.] or [Ter80a, Prop. 2.4]), and the analytic triviality lemma [Sai80, (3.5)].

Theorem 2.3. *A divisor D is free if and only if it is smooth or* $\operatorname{Sing} D$ *is Cohen–Macaulay of codimension* 1. ☐

Using a theorem of Scheja [Sch64, Satz 5] one deduces the following result.

Theorem 2.4. *Any D is free in codimension* 1. ☐

We denote the ring of meromorphic functions on D by $\mathscr{M}_D$.

Definition 2.5. A fractional ideal (on D) is a finite $\mathscr{O}_D$-submodule of $\mathscr{M}_D$ which contains a non-zero divisor.

Lemma 2.6. *$\mathscr{J}_D$ is a fractional ideal.*

Proof. It follows from the Jacobian criterion, Serre's reducedness criterion, and prime avoidance, that $\mathscr{J}_D$ contains a non-zero divisor in $\mathscr{O}_D$. ☐

Corollary 2.7. *A singular divisor D is free if and only if it is reduced and $\mathscr{J}_D$ is a maximal Cohen–Macaulay module.*

Proof. This follows from Theorem 2.3, Lemma 2.6, and the depth inequalities

$$\begin{aligned}\operatorname{depth}(\mathscr{J}_D) &\geq \min\{\operatorname{depth}(\mathscr{O}_D), \operatorname{depth}(\mathscr{O}_D/\mathscr{J}_D)+1\},\\ \operatorname{depth}(\mathscr{O}_D/\mathscr{J}_D) &\geq \min\{\operatorname{depth}(\mathscr{J}_D)-1, \operatorname{depth}(\mathscr{O}_D)\},\end{aligned}$$

resulting from the exact sequence

$$0 \longrightarrow \mathscr{J}_D \longrightarrow \mathscr{O}_D \longrightarrow \mathscr{O}_D/\mathscr{J}_D \longrightarrow 0.$$

☐

Proposition 2.8. *The $\mathscr{O}_D$-dual of any fractional ideal $\mathscr{I}$ is again a fractional ideal $\mathscr{I}^\vee = \{f \in \mathscr{M}_D \mid f \cdot \mathscr{I} \subseteq \mathscr{O}_D\}$. The duality functor*

$$-^\vee = \operatorname{Hom}_{\mathscr{O}_D}(-, \mathscr{O}_D)$$

reverses inclusions. It is an involution on the class of maximal Cohen–Macaulay fractional ideals.

Proof. See [dJvS90, Prop. (1.7)]. ☐

3. Dual residues

In this section, we develop the dual picture of Saito's residue map and apply it to find inclusion relations of certain natural fractional ideals and their duals.

Let $\pi\colon \bar{D} \to D$ denote the normalization of D. Then $\mathscr{M}_D = \mathscr{M}_{\bar{D}}$ and $\mathscr{O}_{\bar{D}}$ is the ring of weakly holomorphic functions on D. Let

$$\Omega^p(\log D) \xrightarrow{\rho_D^p} \Omega_D^{p-1} \otimes_{\mathscr{O}_D} \mathscr{M}_D$$

be Saito's residue map [Sai80, §2] which is defined as follows: By [Sai80, (1.1)], any $\omega \in \Omega^p(\log D)$ can be written as

$$\omega = \frac{dh}{h} \wedge \frac{\xi}{g} + \frac{\eta}{g}. \tag{3.1}$$

where $\xi \in \Omega_S^{p-1}$, $\eta \in \Omega_S^p$, and $g \in \mathscr{O}_S$ restricts to a non-zero divisor in $\mathscr{O}_D$. Then

$$\rho_D^p(\omega) := \frac{\xi}{g}|_D \tag{3.2}$$

is well defined by [Sai80, (2.4)]. We shall abbreviate $\rho_D := \rho_D^1$ and denote by its image by

$$\mathscr{R}_D := \rho_D(\Omega^1(\log D)).$$

Using this notation, condition 3 in Theorem 1.1 becomes $\mathscr{O}_{\bar{D}} = \mathscr{R}_D$.

Example 3.1.

(1) Let $D = \{xy = 0\}$ be a normal crossing curve. Then $\frac{dx}{x} \in \Omega(\log D)$ and

$$(x+y)\frac{dx}{x} = dx + \frac{yd(xy)}{yx} - dy$$

shows that

$$\rho_D\left(\frac{dx}{x}\right) = \frac{y}{x+y}\Big|_D.$$

On the components $D_1 = \{x = 0\}$ and $D_2 = \{y = 0\}$ of the normalization $\bar{D} = D_1 \coprod D_2$, this function equals 1 and 0 respectively and is therefore not in $\mathscr{O}_D$. In particular,

$$\mathscr{R}_D = \mathscr{O}_{\bar{D}} = \mathscr{O}_{D_1} \times \mathscr{O}_{D_2} = \mathscr{J}_D^\vee$$

since $\mathscr{J}_D = \langle x, y\rangle_{\mathscr{O}_D}$ is the maximal ideal in $\mathscr{O}_D = \mathbb{C}\{x,y\}/\langle xy\rangle$. This observation will be generalized in Proposition 3.2.

(2) Conversely assume that $D_1 = \{h_1 = x = 0\}$ and $D_2 = \{h_2 = x + y^m = 0\}$ are two smooth irreducible components of D. Consider the logarithmic 1-form

$$\omega = \frac{ydx - mxdy}{x(x+y^m)} = y^{1-m}\left(\frac{dh_1}{h_1} - \frac{dh_2}{h_2}\right) \in \Omega^1(\log(D_1 + D_2)) \subset \Omega^1(\log D).$$

Its residue $\rho_D(\omega)|_{D_1} = y^{1-m}|_{D_1}$ has a pole along $D_1 \cap D_2$ unless $m = 1$. Thus, if $\mathscr{O}_{\bar{D}} = \mathscr{R}_D$ then D_1 and D_2 must intersect transversally.

(3) Assume that D contains $D' = D_1 \cup D_2 \cup D_3$ with D_1 and D_2 as in (1) and $D_3 = \{x - y = 0\}$. Consider the logarithmic 1-form

$$\omega = \frac{1}{x-y}\cdot\left(\frac{dx}{x} - \frac{dy}{y}\right) \in \Omega^1(\log D') \subset \Omega^1(\log D).$$

Its residue $\rho(\omega)|_{D_1} = -\frac{1}{y}|_{D_1}$ has a pole along $D_1 \cap D_2 \cap D_3$ and hence $\mathscr{O}_{\bar{D}} \subsetneq \mathscr{R}_D$.

Examples (2) and (3) are due to Saito (see [Sai80, (2.9) iii) $\Rightarrow$ iv)]) and will be used in the proof of Theorem 1.2.

By definition, there is a short exact residue sequence

$$0 \longrightarrow \Omega_S^1 \longrightarrow \Omega^1(\log D) \xrightarrow{\rho_D} \mathscr{R}_D \longrightarrow 0. \tag{3.3}$$

Applying $\mathrm{Hom}_{\mathscr{O}_S}(-, \mathscr{O}_S)$ to (3.3) gives an exact sequence

(3.4)

$$0 \longleftarrow \mathrm{Ext}^1_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S) \longleftarrow \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{R}_D, \mathscr{O}_S) \longleftarrow \Theta_S \longleftarrow \mathrm{Der}(-\log D) \longleftarrow 0$$

The right end of this sequence extends to the short exact sequence (2.1) and

$$-^\vee \cong \mathrm{Ext}^1_{\mathscr{O}_S}(-, \mathscr{O}_S) \tag{3.5}$$

by the change of rings spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathscr{O}_D}(-, \operatorname{Ext}^q_{\mathscr{O}_S}(\mathscr{O}_D, \mathscr{O}_S)) \underset{p}{\Rightarrow} \operatorname{Ext}^{p+q}_{\mathscr{O}_S}(-, \mathscr{O}_S). \tag{3.6}$$

This motivates the following

Proposition 3.2. *There is an exact sequence*

$$0 \longleftarrow \operatorname{Ext}^1_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S) \longleftarrow \mathscr{R}_D^\vee \xleftarrow{\sigma_D} \Theta_S \longleftarrow \operatorname{Der}(-\log D) \longleftarrow 0 \tag{3.7}$$

such that $\sigma_D(\delta)(\rho_D(\omega)) = dh(\delta) \cdot \rho_D(\omega)$*. In particular,* $\sigma_D(\Theta_S) = \mathscr{J}_D$ *as fractional ideals. Moreover,* $\mathscr{J}_D^\vee = \mathscr{R}_D$ *as fractional ideals.*

Proof. The spectral sequence (3.6) applied to $\mathscr{R}_D$ is associated with

$$\operatorname{RHom}_{\mathscr{O}_S}(\Omega^1_S \hookrightarrow \Omega^1(\log D), h \colon \mathscr{O}_S \to \mathscr{O}_S).$$

Expanding the double complex $\operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S \hookrightarrow \Omega^1(\log D), h \colon \mathscr{O}_S \to \mathscr{O}_S)$, we obtain the following diagram of long exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\operatorname{Ext}^1_{\mathscr{O}_S}(\mathscr{R}_D, \mathscr{O}_S) & \leftarrow & \operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S, \mathscr{O}_S) & \leftarrow & \operatorname{Hom}_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S) & \longleftarrow & 0 & & \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & & & \\
\operatorname{Ext}^1_{\mathscr{O}_S}(\mathscr{R}_D, \mathscr{O}_S) & \leftarrow & \operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S, \mathscr{O}_S) & \leftarrow & \operatorname{Hom}_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S) & \longleftarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S, \mathscr{O}_D) & \leftarrow & \operatorname{Hom}_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_D) & \xleftarrow{\rho_D^\vee} & \mathscr{R}_D^\vee & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
 & & 0 & \longleftarrow & \operatorname{Ext}^1_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S) & \leftarrow & \operatorname{Ext}^1_{\mathscr{O}_S}(\mathscr{R}_D, \mathscr{O}_S) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array} \tag{3.8}$$

We can define a homomorphism σ_D from the upper left $\operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S, \mathscr{O}_S)$ to the lower right $\mathscr{R}_D^\vee$ by a diagram chasing process and we find that $\delta \in \Theta_S = \operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S, \mathscr{O}_S)$ maps to

$$\sigma_D(\delta) = \big\langle h\delta, \rho_D^{-1}(-)\big\rangle|_D \in \mathscr{R}_D^\vee$$

and that (3.7) is exact. By comparison with the spectral sequence, we can check that α is the change of rings isomorphism (3.5) applied to $\mathscr{R}_D$, and that $\alpha \circ \sigma_D$ coincides with the connecting homomorphism of the top row of the diagram, which is the same as the one in (3.4).

Let $\rho_D(\omega) \in \mathscr{R}_D$ where $\omega \in \Omega^1(\log D)$. Following the definition of ρ_D in (3.2), we write ω in the form (3.1). Then we compute

$$\begin{aligned} &\sigma_D(\delta)(\rho_D(\omega)) = \\ &\quad \langle h\delta, \omega\rangle|_D = dh(\delta) \cdot \frac{\xi}{g}|_D + h \cdot \frac{\langle \delta, \eta\rangle}{g}|_D = dh(\delta) \cdot \rho_D(\omega) \end{aligned} \tag{3.9}$$

which proves the first two claims.

For the last claim, we consider the diagram dual to (3.8):

$$
\begin{array}{ccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathrm{Hom}_{\mathscr{O}_S}(\Theta_S, \mathscr{O}_S) & \to & \mathrm{Hom}_{\mathscr{O}_S}(\mathrm{Der}(-\log D), \mathscr{O}_S) & \to & \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{J}_D, \mathscr{O}_S) \\
 & & & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 0} \\
 & & 0 & \longrightarrow & \mathrm{Hom}_{\mathscr{O}_S}(\Theta_S, \mathscr{O}_S) & \to & \mathrm{Hom}_{\mathscr{O}_S}(\mathrm{Der}(-\log D), \mathscr{O}_S) & \to & \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{J}_D, \mathscr{O}_S) \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{J}_D^\vee & \xrightarrow{dh^\vee} & \mathrm{Hom}_{\mathscr{O}_S}(\Theta_S, \mathscr{O}_D) & \to & \mathrm{Hom}_{\mathscr{O}_S}(\mathrm{Der}(-\log D), \mathscr{O}_D) & & \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & & & \\
 & & \mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{J}_D, \mathscr{O}_S) & \longrightarrow & 0 & & & &
\end{array}
$$

As before, we construct a homomorphism ρ'_D from the upper right $\mathrm{Hom}_{\mathscr{O}_S}(\mathrm{Der}(-\log D), \mathscr{O}_S)$ to the lower left $\mathscr{J}_D^\vee$ such that $\beta \circ \rho'_D$ coincides with the connecting homomorphism of the top row of the diagram, where β is the change of rings isomorphism (3.5) applied to $\mathscr{J}_D$. By the diagram, $\omega \in \Omega^1(\log D) = \mathrm{Hom}_{\mathscr{O}_S}(\mathrm{Der}(-\log D), \mathscr{O}_S)$ maps to

$$\rho'_D(\omega) = \langle h\omega, dh^{-1}(-)\rangle|_D \in \mathscr{J}_D^\vee$$

which gives an exact sequence

$$0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1(\log D) \xrightarrow{\rho'_D} \mathscr{J}_D^\vee \longrightarrow 0 \tag{3.10}$$

similar to the sequence (3.3). Using (3.1) and (3.9), we compute

$$\rho'_D(\omega)(\delta(h)) = \rho'_D(\omega)(dh(\delta)) = \langle h\omega, \delta\rangle|_D = \rho_D(\omega) \cdot dh(\delta) = \rho_D(\omega) \cdot \delta(h)$$

for any $\delta(h) \in \mathscr{J}_D$ where $\delta \in \Theta_S$. Hence, $\rho'_D = \rho_D$ and the last claim follows using (3.3) and (3.10). □

Corollary 3.3. $(\Omega_D^{n-1})^\vee \cong \mathscr{J}_D^\vee$.

Proof. Let $\omega_D^\bullet$ be the complex of regular differential forms on D. By [Ale90, §4 Thm.] and [Bar78, Prop. 3],

$$\mathscr{R}_D = \omega_D^0 = \mathrm{Hom}_{\mathscr{O}_D}(\Omega_D^{n-1}, \omega_D^{n-1}) \cong \mathrm{Hom}_{\mathscr{O}_D}(\Omega_D^{n-1}, \omega_D) \cong (\Omega_D^{n-1})^\vee$$

and Proposition 3.2 yields the claim. □

Corollary 3.4. *There is a chain of fractional ideals*

$$\mathscr{J}_D \subseteq \mathscr{R}_D^\vee \subseteq \mathscr{C}_D \subseteq \mathscr{O}_D \subseteq \mathscr{O}_{\tilde{D}} \subseteq \mathscr{R}_D$$

in $\mathscr{M}_D$ *where* $\mathscr{C}_D = \mathscr{O}_{\tilde{D}}^\vee$ *is the conductor ideal of* π*. In particular,* $\mathscr{J}_D \subseteq \mathscr{C}_D$*.*

Proof. By Lemma 2.6, $\mathscr{J}_D$ is a fractional ideal contained in $\mathscr{R}_D^\vee$ by Proposition 3.2. By [Sai80, (2.7),(2.8)], $\mathscr{R}_D$ is a finite $\mathscr{O}_D$-module containing $\mathscr{O}_{\tilde{D}}$ and hence a fractional ideal. The remaining inclusions and fractional ideals are then obtained using Proposition 2.8. □

Corollary 3.5. *If* D *is free then* $\mathscr{J}_D = \mathscr{R}_D^\vee$ *as fractional ideals.*

Proof. This follows from Corollary 2.7 and Propositions 2.8 and 3.2. ☐

Corollary 3.6. *If D is free then $\mathscr{R}_D = \mathscr{O}_{\bar{D}}$ if and only if $\mathscr{J}_D = \mathscr{C}_D$ and $\bar{D}$ is Cohen–Macaulay.*

Proof. This follows from Proposition 2.8 and 3.2, Corollary 3.5, and $\mathscr{C}_D = \mathscr{O}_{\bar{D}}^\vee$. ☐

4. Lê–Saito theorem

In this section, we prove the missing implication in the Lê–Saito Theorem 1.1. We begin with some general preparations.

Lemma 4.1. *Any map $\phi\colon Y \to X$ of analytic germs with $\Omega^1_{Y/X} = 0$ is an immersion.*

Proof. The map ϕ can be embedded in a map Φ of smooth analytic germs:

$$\begin{array}{ccc} Y & \hookrightarrow & T \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\Phi} \\ X & \hookrightarrow & S. \end{array}$$

Setting $\Phi_i = x_i \circ \Phi$ and $\phi_i = \Phi_i + \mathscr{I}_Y$ for coordinates $x_1, \dots, x_n$ on S and $\mathscr{I}_Y$ the defining ideal of Y in T, we can write $\Phi = (\Phi_1, \dots, \Phi_n)$ and $\phi = (\phi_1, \dots, \phi_n)$ and hence

$$\Omega^1_{Y/X} = \frac{\Omega^1_Y}{\sum_{i=1}^n \mathscr{O}_Y d\phi_i} = \frac{\Omega^1_T}{\mathscr{O}_T d\mathscr{I}_Y + \sum_{i=1}^n \mathscr{O}_T d\Phi_i}. \tag{4.1}$$

We may choose T of minimal dimension so that $\mathscr{I}_Y \subseteq \mathfrak{m}_T^2$ and hence $d\mathscr{I}_Y \subseteq \mathfrak{m}_T\Omega^1_T$. Now (4.1) and the hypothesis $\Omega^1_{Y/X} = 0$ show that $\Omega^1_T = \sum_{i=1}^n \mathscr{O}_T d\Phi_i + \mathfrak{m}_T\Omega^1_T$ which implies that $\Omega^1_T = \sum_{i=1}^n \mathscr{O}_T d\Phi_i$ by Nakayama's lemma. But then Φ and hence ϕ is a closed embedding as claimed. ☐

Lemma 4.2. *If $\mathscr{J}_D = \mathscr{C}_D$ and $\bar{D}$ is smooth then D has smooth irreducible components.*

Proof. By definition, the ramification ideal of π is the Fitting ideal $\mathscr{R}_\pi = \mathscr{F}^0_{\mathscr{O}_{\bar{D}}}(\Omega^1_{\bar{D}/D})$. By [Pie79] and our hypotheses, we have

$$\mathscr{C}_D\mathscr{R}_\pi = \mathscr{J}_D\mathscr{O}_{\bar{D}} = \mathscr{C}_D\mathscr{O}_{\bar{D}} = \mathscr{C}_D. \tag{4.2}$$

This implies $\mathscr{R}_\pi = \mathscr{O}_{\bar{D}}$ by Nakayama's lemma, and hence $\Omega^1_{\bar{D}/D} = 0$.

Since $\bar{D}$ is normal, irreducible and connected components coincide. By localization to a connected component $\bar{D}_i$ of $\bar{D}$ and base change to $D_i = \pi(\bar{D}_i)$ (see [Har77, Ch. II, Prop. 8.2A]), we obtain $\Omega^1_{\bar{D}_i/D_i} = 0$. Then $\bar{D}_i \to D_i$ is an immersion by Lemma 4.1 and hence $D_i = \bar{D}_i$ is smooth. ☐

We are now ready to prove our main results.

Proof of Theorem 1.2. In codimension 1, D is free by Theorem 2.4 and hence $\mathscr{J}_D = \mathscr{C}_D$ by Corollary 3.6 and our hypothesis. Moreover, $\bar{D}$ is smooth in codimension 1 by normality. Therefore, the local irreducible components of D in codimension 1 are smooth by Lemma 4.2. Finally, the claim follows by [Sai80, (2.9) iii) ⇒ iv)] (see Examples 3.1.(2) and 3.1.(3)), or [Sai80, (2.11)] applied in a transversal slice. ☐

Proof of Theorem 1.3. By the Briançon–Skoda theorem, (4) ⇒ (5) and the analytic triviality lemma [Sai80, (3.5)] applied to $\mathscr{J}_D$ at smooth points of $\operatorname{Sing} D$ yields (2) ⇐ (4) (see [Fab11]).

Now assume that D is free and normal crossing in codimension 1. By the first assumption and Theorem 2.3, $\operatorname{Sing} D$ is Cohen–Macaulay of codimension 1 and, in particular, satisfies Serre's condition S_1. By the second assumption, $\operatorname{Sing} D$ also satisfies Serre's condition R_0. Then $\operatorname{Sing} D$, and hence $\mathscr{J}_D$, is reduced by Serre's reducedness criterion. This proves (2) ⇒ (4) for free D.

The last equivalence then follows from Theorems 1.1 and 1.2 and Corollary 3.6. □

5. Complete intersections

In this section, we suggest a notion of freeness for complete intersections and study it in the case of homogeneous complete intersection space curves. We also indicate a direction for generalizing the results in Section 3 to the complete intersection case.

Let $C = D_1 \cap \cdots \cap D_k$, $D_i = \{h_i = 0\}$, be a reduced complete intersection in S with normalization $\pi\colon \bar{C} \to C$. We set $D = D_1 \cup \cdots \cup D_k$, $h = h_1 \cdots h_k$, and $dh = dh_1 \wedge \cdots \wedge dh_k$. Then the analogue of (2.1) reads

$$0 \longleftarrow \mathscr{J}_C \xleftarrow{dh} \Theta_S^k \longleftarrow \operatorname{Der}^k(-\log C) \longleftarrow 0 \tag{5.1}$$

where $\Theta_S^k := \bigwedge^k \Theta_S$ and $\operatorname{Der}^k(-\log C)$ is defined by the sequence. Theorem 2.3 leads to the following natural generalization of freeness.

Definition 5.1. We call a complete intersection C free if it is smooth or if $\operatorname{Sing} C$ is Cohen–Macaulay of codimension 1.

In particular, any reduced complete intersection curve is trivially free. As in Corollary 2.7, a singular complete intersection C is free if and only if it is reduced and $\mathscr{J}_C$ is maximal Cohen–Macaulay. Thus, by (5.1), C is free if and only if it is reduced and $\operatorname{pd}\operatorname{Der}^k(-\log C) < k$, generalizing the divisor case $k = 1$.

Proposition 5.2. *Let $\omega_C^\bullet$ be the complex of regular differential forms on C. Then $\operatorname{Der}^k(-\log C)$ can be identified with the kernel of the natural map $\Omega_S^{n-k} \to \omega_C^{n-k}$.*

Proof. By [Bar78, Lem. 4], there is an isomorphism

$$\omega_C^{n-k} \xrightarrow{\frac{dh}{h}} \operatorname{Ext}^k_{\mathscr{O}_S}(\mathscr{O}_C, \Omega_S^n). \tag{5.2}$$

The latter is the dualizing module (see [Har66, Ch. III, Prop. 7.2])

$$\operatorname{Ext}^k_{\mathscr{O}_S}(\mathscr{O}_C, \Omega_S^n) = \operatorname{Hom}_{\mathscr{O}_S}(\bigwedge^k \mathscr{I}_C/\mathscr{I}_C^2, \Omega_S^n \otimes \mathscr{O}_C) = \omega_S \otimes_{\mathscr{O}_S} \omega_{C/S} = \omega_C.$$

Using a logarithmic Čech complex resolving $\mathscr{O}_C(D)$, it can also be represented as

$$\omega_C = \mathscr{O}_C \otimes_{\mathscr{O}_S} \Omega_S^n(D) \cong \mathscr{O}_C.$$

Thus, (5.2) reduces the claim to identifying $\operatorname{Der}^k(-\log C)$ with the kernel of

$$\Omega_S^{n-k} \xrightarrow{dh} \mathscr{O}_C \otimes_{\mathscr{O}_S} \Omega_S^n.$$

But identifying $\Omega_S^{n-k} = \Theta_S^k$, this is just the definition of $\operatorname{Der}^k(-\log C)$ in (5.1). □

There are two natural ways of producing elements of $\mathrm{Der}^k(-\log C)$:

$$\mathrm{Der}(-\log D)\otimes_{\mathscr{O}_S}\Theta_S^{k-1}\to\mathrm{Der}^k(-\log C),\tag{5.3}$$

$$dh_i\colon\Theta_S^{k+1}\to\mathrm{Der}^k(-\log C).\tag{5.4}$$

Lemma 5.3. *Let C be a reduced quasihomogeneous complete intersection space curve defined by $f,g\in\mathscr{O}_S$, quasihomogeneous of degrees r,s with respect to the weights a,b,c on the variables x,y,z. Then we have a resolution*

$$0\longrightarrow\mathscr{O}_S\xrightarrow{K}\mathscr{O}_S^5\xrightarrow{J}\mathscr{O}_S^5\xrightarrow{K^t}\mathscr{O}_S\longrightarrow\mathscr{O}_{\mathrm{Sing}\,C}\longrightarrow 0$$

where

$$K=\begin{pmatrix}f & g & f_yg_z-f_zg_y & f_zg_x-f_xg_z & f_xg_y-f_yg_x\end{pmatrix}^t,$$

$$J=\begin{pmatrix}0 & 0 & -rg_x & -rg_y & -rg_z\\ 0 & 0 & sf_x & sf_y & sf_z\\ rg_x & -sf_x & 0 & -cz & by\\ rg_y & -sf_y & cz & 0 & -ax\\ rg_z & -sf_z & -by & ax & 0\end{pmatrix}.$$

Proof. This follows from the description of codimension 3 Gorenstein algebras in terms of Pfaffians of skew-symmetric matrices due to Buchsbaum and Eisenbud [BE82]. □

Remark 5.4. In particular, Lemma 5.3 shows that $\mathrm{Sing}\,C$ is Gorenstein, which is well-known (see [KW84]).

Proposition 5.5. *Let C be a reduced quasihomogeneous complete intersection space curve defined by $f,g\in\mathscr{O}_S$, quasihomogeneous of degrees r,s with respect to the weights a,b,c on the variables x,y,z. Then we have a resolution*

$$0\longrightarrow\mathscr{O}_S^2\xrightarrow{K'}\mathscr{O}_S^5\xrightarrow{J'}\mathrm{Der}^2(-\log C)\longrightarrow 0\tag{5.5}$$

where

$$K'=\begin{pmatrix}f & g & f_yg_z-f_zg_y & f_zg_x-f_xg_z & f_xg_y-f_yg_x\\ 0 & 0 & ax & by & cz\end{pmatrix}^t,$$

$$J'=\begin{pmatrix}rg_x & -sf_x & 0 & -cz & by\\ rg_y & -sf_y & cz & 0 & -ax\\ rg_z & -sf_z & -by & ax & 0\end{pmatrix}.$$

In particular, $\mathrm{Der}^2(-\log C)$ *is generated by the images of the maps* (5.3) *and* (5.4).

Proof. Surjectivity of J' and $J'\circ K'=0$ follow immediately from Lemma 5.3. The columns of J' correspond to

$$rdg(\partial_x\wedge\partial_y\wedge\partial_z),\ -sdf(\partial_x\wedge\partial_y\wedge\partial_z),\ \chi\wedge\partial_x,\ \chi\wedge\partial_y,\ \chi\wedge\partial_z,\tag{5.6}$$

where $\chi=ax\partial_x+by\partial_y+cz\partial_z\in\mathrm{Der}(-\log D)$ is the Euler vector field. This proves the last claim. Wedging the elements in (5.6) with χ gives

$$rsg\cdot\partial_x\wedge\partial_y\wedge\partial_z,\ -rsf\cdot\partial_x\wedge\partial_y\wedge\partial_z,\ 0,\ 0,\ 0.$$

Then using the first column of K' reduces any relation of the columns of J' to a relation of the last 3 columns of J', which is clearly in the span of the second row of K'. This proves exactness of (5.5) in the middle, and injectivity of K' is obvious. □

Remark 5.6. Recalling the proof of Lemma 5.3, the Buchsbaum–Eisenbud theorem plays the role of a kind of Saito criterion in the situation of Proposition 5.5.

Example 5.7. For the non-reduced homogeneous complete intersection

$$C = \{xz - y^2 = y^2 - z^2 = z^2 - w^2 = 0\},$$

$\mathrm{Der}^k(-\log C)$ is not generated by the images of the maps in (5.3) and (5.4).

There is the following generalization of the first part of Definition 2.1 due to Aleksandrov and Tsikh [AT01, Def. 2.2]. We set $\hat{D}_i = D_1 \cup \cdots \cup \widehat{D_i} \cup \cdots \cup D_k$ and abbreviate $\tilde{\Omega}^p_S = \sum_i \Omega^p_S(*\hat{D}_i)$.

Definition 5.8 (Aleksandrov–Tsikh)**.**

$$\Omega^p(\log C) = \{\omega \in \Omega^p_S(*D) \mid \forall j = 1, \ldots, k\colon h_j\omega \in \tilde{\Omega}^p_S, h_j d\omega \in \tilde{\Omega}^{p+1}_S\}$$

As opposed to what the notation suggests, these modules depend on D. Aleksandrov and Tsikh [AT01, Thm. 2.4] construct a generalized residue sequence

$$0 \longrightarrow \tilde{\Omega}^k_S \longrightarrow \Omega^k(\log C) \xrightarrow{\rho_C} \mathscr{R}_C = (\Omega^{n-k}_C)^\vee \longrightarrow 0 \tag{5.7}$$

where ρ^p_C and $\rho_C := \rho^k_C$ are formally defined as in (3.1) and (3.2), but with $\eta \in \tilde{\Omega}^p_S$.

Proposition 5.9. $\mathscr{O}_{\bar{C}} \subseteq \mathscr{R}_C$.

Proof. This can be proved as in [Sai80, (2.8)] using [Tsi87, Thm. 1]. □

Dualizing (5.7) and applying the change of rings spectral sequence (3.6) with $\mathscr{O}_D$ replaced by $\mathscr{O}_C$, yields an exact complex

(5.8)

$$0 \longleftarrow \mathrm{Ext}^k_{\mathscr{O}_S}(\Omega^k(\log C), \mathscr{O}_S) \longleftarrow \mathscr{R}^\vee_C \longleftarrow \Theta^k_S \longleftarrow \mathrm{Ext}^{k-1}_{\mathscr{O}_S}(\Omega^k(\log C), \mathscr{O}_S) \longleftarrow 0$$

analogous to (3.7). How to generalize our arguments in Section 3 is unclear however.

Acknowledgments

We are grateful to David Mond and Eleonore Faber for helpful discussions and comments. The foundation for this paper was laid during a "Research in Pairs" stay at the "Mathematisches Forschungsinstitut Oberwolfach" in summer 2011.

References

[Ale88] A. G. Aleksandrov, *Nonisolated Saito singularities*, Mat. Sb. (N.S.) **137(179)** (1988), no. 4, 554–567, 576. MR 981525 (90b:32024) 2

[Ale90] ———, *Nonisolated hypersurface singularities*, Theory of singularities and its applications, Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 211–246. MR 1089679 (92b:32039) 3

[AT01] Aleksandr G. Aleksandrov and Avgust K. Tsikh, *Théorie des résidus de Leray et formes de Barlet sur une intersection complète singulière*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 11, 973–978. MR 1872457 (2002m:32004) 5, 5

[Bar78] Daniel Barlet, *Le faisceau $\omega^{\cdot}_X$ sur un espace analytique X de dimension pure*, Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977), Lecture Notes in Math., vol. 670, Springer, Berlin, 1978, pp. 187–204. MR 521919 (80i:32037) 3, 5

[BE82] David A. Buchsbaum and David Eisenbud, *Gorenstein ideals of height* 3, Seminar D. Eisenbud/B. Singh/W. Vogel, Vol. 2, Teubner-Texte zur Math., vol. 48, Teubner, Leipzig, 1982, pp. 30–48. MR 686456 (84i:13017) 5

[Del72] Pierre Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302. MR 0422673 (54 #10659) 1 [dJP00] Theo de Jong and Gerhard Pfister, *Local analytic geometry*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2000, Basic theory and applications. MR 1760953 (2001c:32001) 1

[dJvS90] T. de Jong and D. van Straten, *Deformations of the normalization of hypersurfaces*, Math. Ann. **288** (1990), no. 3, 527–547. MR MR1079877 (92d:32050) 2

[ER95] Paul H. Edelman and Victor Reiner, *Not all free arrangements are* $K(\pi, 1)$, Bull. Amer. Math. Soc. (N.S.) **32** (1995), no. 1, 61–65. MR 1273396 (95g:52017) 1

[Fab11] Eleonore Faber, *Normal crossings in local analytic geometry*, Ph.D. thesis, Universität Wien, 2011. 1, 4

[GMS11] Michel Granger, David Mond, and Mathias Schulze, *Free divisors in prehomogeneous vector spaces*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 5, 923–950. MR 2795728 1

[Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR 0222093 (36 #5145) 5

[Har77] ———, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116) 4

[KW84] Ernst Kunz and Rolf Waldi, *Über den Derivationenmodul und das Jacobi-Ideal von Kurvensingularitäten*, Math. Z. **187** (1984), no. 1, 105–123. MR 753425 (85j:14033) 5.4

[Loo84] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Society Lecture Note Series, vol. 77, Cambridge University Press, Cambridge, 1984. MR MR747303 (86a:32021) 1

[LS84] Lê Dũng Tráng and Kyoji Saito, *The local* π_1 *of the complement of a hypersurface with normal crossings in codimension* 1 *is abelian*, Ark. Mat. **22** (1984), no. 1, 1–24. MR 735874 (86a:32019) 1

[Pie79] Ragni Piene, *Ideals associated to a desingularization*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 503–517. MR 555713 (81a:14001) 4

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR MR586450 (83h:32023) 1, 2, 3, 2, 2, 3, 3, 3.1, 3, 4, 5

[Sch64] Günter Scheja, *Fortsetzungssätze der komplex-analytischen Cohomologie und ihre algebraische Charakterisierung*, Math. Ann. **157** (1964), 75–94. MR 0176466 (31 #738) 2

[Ter80a] Hiroaki Terao, *Arrangements of hyperplanes and their freeness. I*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 293–312. MR MR586451 (84i:32016a) 2

[Ter80b] ———, *Free arrangements of hyperplanes and unitary reflection groups*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 8, 389–392. MR 596011 (82e:32018a) 1

[Tsi87] A. K. Tsikh, *Weakly holomorphic functions on complete intersections, their holomorphic continuation*, Mat. Sb. (N.S.) **133(175)** (1987), no. 4, 429–445, 559. MR 911801 (89b:32018) 5

[vS95] D. van Straten, *A note on the discriminant of a space curve*, Manuscripta Math. **87** (1995), no. 2, 167–177. MR MR1334939 (96e:32032) 1

M. Granger, Université d'Angers, Département de Mathématiques, LAREMA, CNRS UMR n^{o}6093, 2 Bd Lavoisier, 49045 Angers, France

E-mail address: granger@univ-angers.fr

M. Schulze, Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States

E-mail address: mschulze@math.okstate.edu