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Schrödinger manifolds

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Abstract

This article propounds, in the wake of influential work of Fefferman and Graham about Poincaré extensions of conformal structures, a definition of a (Poincaré-)Schrödinger manifold whose boundary is endowed with a conformal Bargmann structure above a non-relativistic Newton-Cartan spacetime. Examples of such manifolds are worked out in terms of homogeneous spaces of the Schrödinger group in any spatial dimension, and their global topology is carefully analyzed. These archetypes of Schrödinger manifolds carry a Lorentz structure together with a preferred null Killing vector field; they are shown to admit the Schrödinger group as their maximal group of isometries. The relationship to similar objects arising in the non-relativisitc AdS/CFT correspondence is discussed and clarified.

Keywords: Conformal structures; Fefferman-Graham construction; Schrödinger group; Non-relativistic holography

MSC: 53A30; 53C30; 53C50; 53C80; 81R20

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Contents

1 Introduction **3**

2 Schrödinger equation and conformal Bargmann structures **7**

2.1 Covariant Schrödinger equation 8

2.2 Symmetries of the Schrödinger equation 9

2.3 Conformal Bargmann structures 9

3 General definition of Schrödinger manifolds **10**

3.1 Formal theory of Poincaré metrics & conformal infinity according to Fefferman-Graham 10

3.2 Schrödinger manifolds & conformal Bargmann structures as null infinity 13

4 Global structure of the Schrödinger group **14**

4.1 Flat conformal Bargmann structure and Schrödinger Lie algebra . . . 14

4.2 Schrödinger group as a subgroup of conformal group 15

4.3 A nilpotent coadjoint orbit of the conformal group 20

5 Homogeneous Schrödinger manifolds **21**

5.1 A special family of Schrödinger-homogeneous spaces 21

5.2 Distinguished Schrödinger-invariant structures 24

5.3 Conformal infinity and conformal Bargmann structures 26

5.4 Main result: homogeneous Schrödinger manifolds 31

6 Conclusion **34**

1 Introduction

The notion of "non-relativistic conformal symmetry" goes back to Jacobi and Lie as highlighted in, e.g., [43, 15]. In the early seventies, Jackiw [31], Niederer [40], and Hagen [25] rediscovered this symmetry within the quantum mechanical context; the maximal kinematical symmetry group of the free Schrödinger equation has since then been coined the "Schrödinger group". One of the key features of this symmetry is a specific action of dilations according to which time is dilated twice as much as space (the *dynamical exponent* is $z = 2$). A remarkable relationship between the Schrödinger Lie algebra and the relativistic conformal Lie algebra (in suitable dimensions) was then unveiled in [8]. The Schrödinger symmetry also happened to play a central rôle in the physics of strongly anisotropic critical systems [28], and in the description of ageing phenomena [29, 30]. At a geometrical level, the (center-free) Schrödinger group has been interpreted as the group of those "conformal transformations" of a Newton-Cartan spacetime that also permute its unparametrized geodesics [9]. It soon became patent that an adapted framework to deal intrinsically with non-relativistic conformal symmetries is provided by Bargmann structures [12] defined on $(\mathbb{R}, +)$- or circle-bundles over Newton-Cartan spacetimes. Let us recall that a Bargmann manifold (akin to generalized *pp*-waves [19]) is such a principal fibre-bundle, endowed with a Lorentz metric, whose fundamental vector field is null and covariantly constant. This definition entailed that the conformal automorphisms of a Bargmann structure constitute a Lie group which turns out to be actually isomorphic with the Schrödinger group, yielding henceforth a clear-cut geometrical status to the latter [13, 14]; see also [42]. We refer to [43] for a modern and recent review of the Schrödinger and Schrödinger-Virasoro symmetries.

A few years ago, Son [44] and, independently, Balasubramanian and McGreevy [4] have put forward a geometrical realization of the Schrödinger group as a group of

isometries of some Lorentz metric on a two-parameter spacetime extension, in the framework of the AdS/CFT correspondence initiated by Maldacena [37, 1]. From then on, this *non-relativistic holography* has triggered much interest within a wide range of subjects, for instance, in ageing-gravity duality [39], non-relativistic field theory [23], string theory and black hole physics [38, 36]. In condensed matter physics, the Schrödinger group turns out to be also the dynamical symmetry group of the two-body interactions in ultracold fermionic atoms [4, 44]. See also recent work [5] on the conformal symmetries of the unitary Fermi gas.

It has been pointed out in [38, 7, 26] that non-relativistic conformal symmetries for backgrounds arising in string theory and black hole geometry should be best viewed as asymptotic symmetries of AdS spacetime associated with a definite notion of conformal boundary. These authors also called attention to work of Fefferman-Graham [20] which should, conversely, provide efficient geometrical means to deal with expansions of these asymptotic symmetries to the bulk spacetime, endowed with a *Poincaré metric*. This is precisely the viewpoint we will espouse in this article in an effort to adapt the Fefferman-Graham (FG) construct to the particular instance of the Schrödinger symmetry. We mention, in this vein, another approach using the alternative notion of *ambient space* [20] used in [35] to describe conformal *pp*-waves.

Let us now recall that the above-mentioned extension of the AdS/CFT correspondence to non-relativistic field theory is based on using the (locally defined) metric [4, 44, 7, 36]

$$\widehat{\mathrm{g}} = \frac{1}{r^2}\left[\sum_{i=1}^{d}(dx^i)^2 + 2dtds + dr^2 - \frac{dt^2}{r^2}\right] \tag{1.1}$$

on a $(d+3)$-dimensional relativistic spacetime, whose key property is that its group of *isometries* is the *Schrödinger group* of non-relativistic *conformal* transformations of $(d+1)$-dimensional Galilei spacetime, coordinatized by $(x^1, \ldots, x^d, t)$.

The purpose of this article is to provide an appropriate geometrical interpretation of such a manifold which, as we will show, turns out to be an instance of what we will call a "Schrödinger manifold".

We now summarize the main outcome and results of this article.

Our approach strongly relies, on the one hand, on the general notion of a conformal Bargmann structure above non-relativistic spacetime (Definition 2.4), and, on the other hand, on an adaptation to this non-relativistic conformal structure of the FG formal theory of Poincaré metrics. This standpoint will help us introduce, via Definition 3.3, the novel notion of *Schrödinger manifold*, endowed with both a Poincaré metric and a null Killing vector field, and whose conformal boundary corresponds precisely to our original conformal Bargmann structure. Such Schrödinger manifolds are, indeed, exemplified by the Poincaré metric $\mathrm{g}^+ = \widehat{\mathrm{g}} + dt^2/r^4$ and the null Killing vector field $\partial/\partial s$, read off Equation (1.1). This is the content of Theorem 5.14, the main upshot of our article. We will furthermore prove (Proposition 5.2) that this emblematic example actually stems from a certain homogeneous space, $\widehat{M}$, of the Schrödinger group $\mathrm{Sch}(d+1,1)$, the latter being the maximal group of isometries of $\widehat{M}$ (Proposition 5.4).

The article is organized as follows.

Section 2 introduces the basics of conformal non-relativistic geometry, namely the definition of a *conformal Bargmann structure* above a Newton-Cartan structure on $(d+1)$-dimensional spacetime. We recall, and put in a geometrical guise, the covariant Schrödinger equation and its relationship with the conformal Laplace (Yamabe) operator acting on densities. The Schrödinger group is then naturally introduced in terms of the automorphisms of a conformal Bargmann structure.

Our definition of *Schrödinger manifolds* is presented in Section 3. It fundamentally relies on the construction of the "Poincaré" formal deformation of a con-

formal (pseudo-)Riemannian structure due to Fefferman and Graham. Emphasis will be put on the Lorentzian case, relevant to deform conformal Bargmann structures. The rôle of a special null Killing vector field will also be highlighted in the definition of Schrödinger manifolds.

Section 4 is concerned with the global structure of the Schrödinger group. The Schrödinger Lie algebra, $\mathrm{sch}(d+1,1)$, spanned by the vector fields (4.2) of the flat Bargmann structure is chosen to be integrated inside the conformal group $\mathrm{O}(d+2,2)$ viewed as the group of isometries of the ambient vector space $\mathbb{R}^{d+4}$, endowed with the metric (4.3). The *Schrödinger group* $\mathrm{Sch}(d+1,1)$ is then defined as the stabilizer of some nilpotent element, Z_0, in the Lie algebra $\mathrm{o}(d+2,2)$. Its relationship with the manifold of null geodesics of compactified Minkowski space $\mathrm{Ein}_{d+1,1}$ is revealed.

Section 5 gathers the main results of the article. Much in the spirit of the Klein program, we seek examples of Schrödinger manifolds as homogeneneous spaces of the Schrödinger group $\mathrm{Sch}(d+1,1)$ itself. The outcome is given by Propositions 5.2 and 5.8. These *homogeneneous Schrödinger manifolds* turn out to be open submanifolds of AdS_{d+3}, and their topology is completely worked out (see Figure 1 for an illustration). We furthermore show that the Schrödinger group is actually their maximal group of isometries. The AdS/CFT metric (1.1) acquires, hence, a global status as the canonical metric of our Schrödinger-homogeneous space arising as a (Poincaré-)Schrödinger metric inherited from the FG construction.

In Section 6 we summarize the content of the article and draw several conclusions. We also offer perspectives related, among others, to open problems regarding the existence and uniqueness of Schrödinger manifolds.

2 Schrödinger equation and conformal Bargmann structures

Let us recall that a *Bargmann structure* [12] is a principal H-bundle $\pi : M \to \mathcal{M}$ over a $(d+1)$-dimensional smooth manifold $\mathcal{M}$, where $H \cong (\mathbb{R}, +)$ or $\mathrm{U}(1)$; its total space, M, is assumed to carry a Lorentz metric, g, the fundamental vector field, ξ, of the H-action being null, $\mathrm{g}(\xi, \xi) = 0$, and covariantly constant with respect to the Levi-Civita connection, $\nabla\xi = 0$.

It has been proved [12] that a Bargmann structure (M, g, ξ) projects onto a "Newton-Cartan" (NC) structure on non-relativistic spacetime $\mathcal{M} = M/H$ [34]. The nowhere vanishing 1-form $\theta = \mathrm{g}(\xi)$ associated with ξ via the metric, g, is closed; it therefore descends onto the *time* axis $T = M/\ker(\theta)$ as a 1-form which we call the "clock" of the structure. Bargmann structures are interpreted as *generalized pp-waves* in general relativity; see, e.g., [19, 33, 13].

We recall that the canonical *flat* Bargmann structure on $M = \mathbb{R}^{d+2}$, with $H = (\mathbb{R}, +)$, is given by

$$\boxed{\mathrm{g} = \sum_{i=1}^{d} dx^i \otimes dx^i + 2dt \odot ds \qquad \& \qquad \xi = \frac{\partial}{\partial s}} \tag{2.1}$$

where we have put $t = x^{d+1}$, and $s = x^{d+2}$; we will use the shorthand notation $\mathbb{R}^{d+1,1} = (\mathbb{R}^{d+2}, \mathrm{g})$; also will "$\odot$" denote the symmetrized tensor product. Here $(x^1, \ldots, x^d)$ are "spatial" coordinates, and t stands for the absolute time coordinate on Galilei spacetime such that

$$\theta = dt \tag{2.2}$$

while s is a *Kaluza-Klein-type* coordinate, homogeneous to an action per mass.

2.1 Covariant Schrödinger equation

We first introduce the useful notion of λ-densities spanning the $\mathrm{Diff}(M)$-module $\mathcal{F}_\lambda(M)$ whose elements can be locally written as $\Psi = f|\mathrm{Vol}|^\lambda$, with $f \in C^\infty(M, \mathbb{C})$, if Vol is a volume element of M. The associated $\mathrm{Vect}(M)$-module structure of $\mathcal{F}_\lambda(M)$ is then defined via the Lie derivative $\mathsf{L}^\lambda_X f = X(f) + \lambda \mathrm{Div}(X) f$, for all $X \in \mathrm{Vect}(M)$.

Let (M, g) be a n-dimensional pseudo-Riemannian manifold. We recall that the *Yamabe operator*, or conformal Laplacian [6, 17, 18], is the conformally-invariant differential operator $\Delta^{\mathrm{conf}}_{\mathrm{g}} : \mathcal{F}_{\frac{n-2}{2n}}(M) \to \mathcal{F}_{\frac{n+2}{2n}}(M)$ defined by $\Delta^{\mathrm{conf}}_{\mathrm{g}} = \Delta_{\mathrm{g}} - \frac{n-2}{4(n-1)} R(\mathrm{g})$, where $R(\mathrm{g})$ denotes the scalar curvature of the Levi-Civita connection of (M, g).

Proposition 2.1. *Given a Bargmann manifold (M, g, ξ) of dimension $n = d + 2$, the system*

$$\Delta^{\mathrm{conf}}_{\mathrm{g}} \Psi = 0 \qquad \& \qquad \frac{\hbar}{i} \mathsf{L}^\lambda_\xi \Psi = m \Psi \tag{2.3}$$

with $\lambda = \frac{d}{2d+4}$ descends as the covariant Schrödinger equation of mass m on the associated Newton-Cartan spacetime.

The proof of Proposition 2.1 relies essentially on the derivation given in [12], and on the fact that the fundamental vector field, ξ, is divergencefree, $\mathrm{Div}(\xi) = 0$. In the latter reference, the NC field equations, $\mathrm{Ric}(\mathrm{g}) = 4\pi G \varrho\, \theta \otimes \theta$, where ϱ stands for the mass density of the sources, have been assumed to hold, implying thus $R(\mathrm{g}) = 0$.

Remark 2.2. The structural group H may be compact in some special instances, e.g., $H = \mathrm{U}(1)$ for a Taub-NUT like solution of NC field equations. This leads, in view of the second equation in (2.3), to the quantization of mass [13]. From now on, we shall be mainly concerned with the case $H = (\mathbb{R}, +)$.

2.2 Symmetries of the Schrödinger equation

Denote by $[\Phi \mapsto \Phi_\lambda]$ the action of $\mathrm{Diff}(M)$ on $\mathcal{F}_\lambda(M)$. A *symmetry* of the Schrödinger equation is a local diffeomorphism $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ such that

$$\Delta_{\mathrm{g}}^{\mathrm{conf}} \circ \Phi_\lambda = \Phi_\mu \circ \Delta_{\mathrm{g}}^{\mathrm{conf}} \qquad \& \qquad \mathsf{L}_\xi^\lambda \circ \Phi_\lambda = \Phi_\lambda \circ \mathsf{L}_\xi^\lambda \tag{2.4}$$

with the weights $\lambda = \frac{d}{2d+4}$, and $\mu = \frac{d+4}{2d+4}$.

Proposition 2.3. *[13] The symmetries of the Schrödinger equation form the "Schrödinger (pseudo-)group"* $\mathrm{Sch}(M, \mathrm{g}, \xi) = \mathrm{Conf}_{\mathrm{loc}}(M, \mathrm{g}) \cap \mathrm{Aut}(M, \xi)$ *consisting of those* $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ *such that*

$$\boxed{\Phi^* \mathrm{g} = \Omega_\Phi^2 \cdot \mathrm{g} \qquad \& \qquad \Phi_* \xi = \xi} \tag{2.5}$$

for some $\Omega_\Phi \in C^\infty(M, \mathbb{R}_+^*)$ *depending on* Φ.[1] *These* $\Phi \in \mathrm{Sch}(M, \mathrm{g}, \xi)$ *permute, hence, the solutions of the Schrödinger equation (2.3) according to*

$$\Psi \mapsto (\Phi_\lambda)_* \Psi \tag{2.6}$$

with $\lambda = \frac{d}{2d+4}$. *This group descends onto NC spacetime* $\mathcal{M}$ *as the "center-free Schrödinger (pseudo-)group"* $\mathrm{Sch}(M, \mathrm{g}, \xi)/H$.

2.3 Conformal Bargmann structures

In view of Proposition 2.3, one of the fundamental geometrical objects associated with the Schrödinger equation is clearly the conformal class $[\mathrm{g}]$ of the Bargmann metric g on (extended spacetime) M. Indeed, given any $\overline{\mathrm{g}} \in [\mathrm{g}]$, one duly has $\mathrm{g}(\xi, \xi) = \overline{\mathrm{g}}(\xi, \xi) = 0$; now, to further insure $\nabla \xi = \overline{\nabla} \xi = 0$, i.e., that g and $\overline{\mathrm{g}}$ are Bargmann-equivalent, one finds

$$\mathrm{g} \sim \overline{\mathrm{g}} \qquad \Longleftrightarrow \qquad \overline{\mathrm{g}} = \Omega^2 \mathrm{g} \quad \& \quad d\Omega \wedge \theta = 0 \tag{2.7}$$

[1] We will confine considerations to conformal diffeomorphisms of (M, g) that commute with the H-action on M, hence satisfying (2.5). We will *not* consider, here, the larger (pseudo-)group of all conformal transformations Φ of (M, g) that permute the H-orbits, i.e., such that $\xi \wedge \Phi_* \xi = 0$.

which infers that the conformal factor, Ω, be a function of the time axis, T.

Definition 2.4. *A "conformal Bargmann structure" is an equivalence class* $(M, [\mathrm{g}], \xi)$ *of Bargmann manifolds for the equivalence relation (2.7).*

Such a structure basically involves a *conformal class* of Lorentz metrics on a principal fibre bundle $\pi : M \to \mathcal{M}$ with structure group $(\mathbb{R}, +)$ — or $\mathrm{U}(1)$ — whose fundamental vector field is lightlike, and parallel.

The Schrödinger group, as defined in Proposition 2.3, is therefore isomorphic to the group of automorphisms of the conformal Bargmann structure defined in Proposition 2.4, namely $\mathrm{Sch}(M, \mathrm{g}, \xi) \cong \mathrm{Aut}(M, [\mathrm{g}], \xi)$.

3 General definition of Schrödinger manifolds

We have, so far, unveiled new geometrical structures involving conformal structures in the presence of a null, parallel, and nowhere vanishing vector field admitting a clear-cut physical interpretation via the definition of the mass in the Schrödinger equation (2.3). Accordingly, our main goal will now be to specialize the FG definition of "Poincaré metrics" associated with conformal structures to our particular, non-relativistic, framework featuring conformal Bargmann structures.

3.1 Formal theory of Poincaré metrics & conformal infinity according to Fefferman-Graham

In their quest of *conformal invariants* of a conformal structure of signature (p, q), Fefferman and Graham [20] have devised two equivalent constructs:

1. the ambient metric on a pseudo-Riemannian manifold of signature $(p+1, q+1)$,

2. the Poincaré metric on a pseudo-Riemannian manifold of signature $(p+1, q)$.

In order to make contact with the aforementioned physics literature, involving local expressions for the metric in Poincaré patches, we restrict further considerations to item 2. Let us hence recall the general definition [20] of Poincaré metrics that will appear as the cornerstone of the subsequent study.

Start with a manifold M and a conformal class $[\mathrm{g}]$ of metrics of signature (p, q), such that $n = p + q > 2$. Consider now a manifold M^+ such that $M = \partial M^+$. Let $r \in C^\infty(M^+)$ verify $r > 0$ in $\mathrm{Int}(M^+)$, and $r = 0$ & $dr \neq 0$ on ∂M^+ (this function is called a *defining function* for M). A metric g^+ of signature $(p+1, q)$ on M^+ is "conformally compact" iff $r^2\,\mathrm{g}^+$ extends smoothly to M^+ and $r^2\,\mathrm{g}^+|_{TM}$ is non-degenerate [41].

Definition 3.1. *We say that (M^+, g^+) has $(M, [\mathrm{g}])$ as conformal infinity whenever $r^2\,\mathrm{g}^+|_{TM} \in [\mathrm{g}]$.*

Definition 3.2. *[20] A Poincaré metric for $(M, [\mathrm{g}])$ is a pair (M^+, g^+) where M^+ is an open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}^+$ such that*

- *(M^+, g^+) has $(M, [\mathrm{g}])$ as conformal infinity*
- *(M, g^+) is an asymptotic solution of Einstein's equation $\mathrm{Ric}(\mathrm{g}^+) + k\mathrm{g}^+ = 0$ (normalization condition: $k = n$).*

Poincaré metrics admit the local expression

$$\mathrm{g}^+ = \frac{1}{r^2}\left[\sum_{i,j=1}^{n} \mathrm{g}^+_{ij}(x, r)dx^i \otimes dx^j + dr \otimes dr\right] \tag{3.1}$$

where the $\mathrm{g}^+_{ij}(x, r)$ are formal power series in the parameter r.

It has been proven [20] that a Poincaré metric for a given pair $(M, [\mathrm{g}])$ exists and is unique, up to diffeomorphisms fixing M, for n odd provided the $\mathrm{g}^+_{ij}(x, r)$ are even functions of r. If n is even, $\mathrm{Ric}(\mathrm{g}^+) + k\mathrm{g}^+ = \mathcal{O}(r^{n-2})$ uniquely determines g^+ (modulo $\mathcal{O}(r^n)$), again up to diffeomorphisms fixing M, for which the $\mathrm{g}^+_{ij}(x, r)$ are even functions of r (modulo $\mathcal{O}(r^n)$).

Let us illustrate this construction by the well-known example of the Einstein space $\mathrm{Ein}_{n-1,1} = \partial(\mathrm{AdS}_{n+1})$, the archetype of Poincaré metric being provided by the Anti de-Sitter (AdS) metric whose conformal infinity is the Einstein conformal structure (compactified Minkowski space). Here, $M = \mathrm{Ein}_{n-1,1}$, and $M^+ = \mathrm{AdS}_{n+1}$.

Start with $\mathbb{R}^{n+2}$, where $n = d+2$, with the following metric

$$G = \sum_{i=1}^{d} dx^i \otimes dx^i + 2dx^{d+1} \odot dx^{d+2} + 2dx^{d+3} \odot dx^{d+4} \tag{3.2}$$

of signature $(n, 2)$, and consider the unit hyperboloid

$$\mathrm{AdS}_{n+1} = \{X \in \mathbb{R}^{n,2} \,|\, \overline{X}X = -1\} \tag{3.3}$$

where $\overline{X} \equiv G(X)$ as a shorthand notation. The induced metric $\mathrm{g}^+ = G|_{T\mathrm{AdS}_{n+1}}$ is Lorentzian (of signature $(n, 1)$), and of constant sectional curvature. View now $\mathrm{AdS}_{n+1} \subset \mathbb{P}^{n+1}(\mathbb{R})$ as the projectivized open ball $\mathcal{B} = \{X \in \mathbb{R}^{n,2} \,|\, \overline{X}X < 0\}$. Its conformal boundary is the Einstein space

$$\mathrm{Ein}_{n-1,1} = \mathbb{P}\mathcal{Q} \tag{3.4}$$

which is the projectivization of the null cone $\mathcal{Q} = \{Q \in \mathbb{R}^{n,2} \setminus \{0\} \,|\, \overline{Q}Q = 0\} \cong \mathbb{R}^*_+ \times (S^{n-1} \times S^1)$, and is endowed with the conformal class $[\mathrm{g}]$ of Lorentzian metrics inherited from $G|_{T\mathcal{Q}}$; see, e.g., [21]. Conformal infinity of $(\mathrm{AdS}_{n+1}, \mathrm{g}^+)$ is therefore $(\mathrm{Ein}_{n-1,1}, [\mathrm{g}])$, and both

$$\mathrm{AdS}_{n+1} \cong \mathbb{R}^n \times S^1, \tag{3.5}$$

$$\mathrm{Ein}_{n-1,1} \cong (S^{n-1} \times S^1)/\mathbb{Z}_2 \tag{3.6}$$

are homogeneous spaces of $\mathrm{O}(n, 2)$.

We refer to [22] for a comprehensive review of the geometry of AdS spacetimes.

3.2 Schrödinger manifolds & conformal Bargmann structures as null infinity

In the wake of the previously reviewed work [20], we will introduce the new notion of a (Poincaré-)Schrödinger manifold whose "conformal infinity" is a given conformal Bargmann structure $(M, [\mathrm{g}], \xi)$ in the sense of Definition 2.4.

Definition 3.3. *A (Poincaré-)Schrödinger manifold for a conformal Bargmann structure $(M, [\mathrm{g}], \xi)$ is a triple $(\widehat{M}, \widehat{\mathrm{g}}, \widehat{\xi})$ with $M = \partial\widehat{M}$, where $\widehat{\mathrm{g}}$ is a Lorentz metric, and $\widehat{\xi}$ a nowhere vanishing lightlike Killing vector field such that*

1. $\widehat{\xi}\,|_{T^*M} = \xi$
2. $\widehat{\mathrm{g}}^{-1}|_{T^*M} = \mu\,\xi \otimes \xi$ *(normalization condition: $\mu = 1$)*
3. $\mathrm{g}^+ = \widehat{\mathrm{g}} + \mu\,\widehat{\theta} \otimes \widehat{\theta}$ *(where $\widehat{\theta} = \widehat{\mathrm{g}}(\widehat{\xi})$) is a Poincaré metric for $(M, [\mathrm{g}])$.*

Let us explain and justify the different items of Definition 3.3.

The generator ξ of the structure group of the principal H-bundle $\pi : M \to M/H$ extends smoothly to $\widehat{M}$. In fact, the null vector field ξ enters the definition of the character of H associated with the mass in the Schrödinger equation (2.3); as such, it ought to give rise to a unique null, Killing vector field $\widehat{\xi}$ of $(\widehat{M}, \widehat{\mathrm{g}})$, leading to a null Kaluza-Klein metric, $\widehat{\mathrm{g}}$, on $\widehat{M}$. This justifies our first axiom.

In Axiom 2, the real constant μ is, in fact, quite arbitrary; it allows, via the null vector field ξ, for extra terms in the metric $\widehat{\mathrm{g}}$ with higher order singularities at the conformal boundary, M.[2]

The third axiom, in Definition 3.3, resorts explicitly to the conformal class of Bargmann metrics; it is thus devised to make use of the FG approach to Poincaré metrics which is at our disposal (see Definition 3.2). However, we will not address

[2]See, e.g., Equations (5.16) and (5.17) below showing that the metric $\widehat{\mathrm{g}}$ exhibits a singular behavior $\sim r^{-4}$ at conformal infinity, namely $r^4\,\widehat{\mathrm{g}}\big|_{TM} \in [\theta \otimes \theta]$.

here the problem of the existence and uniqueness of (Poincaré-)Schrödinger structures of Definition 3.3. Instead, we will provide explicit examples. We duly recover the FG axioms if $\widehat{\xi}$ is ignored, or if $\mu = 0$.

Proposition 3.4. *Let us set $\widehat{\theta} = \widehat{\mathrm{g}}(\widehat{\xi})$, then the family of symmetric tensor fields $\widetilde{\mathrm{g}} = \widehat{\mathrm{g}} + \mu\,\widehat{\theta} \otimes \widehat{\theta}$ parametrized by $\mu \in \mathbb{R}$ defines on $\widehat{M}$ a family of Lorentzian metrics for which $\widetilde{\xi} = \widehat{\xi}$ is a null, nowhere vanishing, Killing vector field.*

Proof. If $\{\lambda_1, \ldots, \lambda_{d+1}, +\lambda, -\lambda\}$ denotes the spectrum of the Gram matrix of $\widehat{\mathrm{g}}$ with respect to some basis, then the spectrum of the corresponding Gram matrix of $\widetilde{\mathrm{g}}$ is given by $\{\lambda_1, \ldots, \lambda_{d+1}, \frac{1}{2}\mu + \sqrt{\lambda^2 + \left(\frac{1}{2}\mu\right)^2}, \frac{1}{2}\mu - \sqrt{\lambda^2 + \left(\frac{1}{2}\mu\right)^2}\}$, with the same Lorentz signature. Since $\widehat{\xi}$ is null for $\widehat{\mathrm{g}}$, i.e., $\widehat{\theta}(\widehat{\xi}) = 0$, then $\widetilde{\xi} = \widehat{\xi}$ is clearly $\widetilde{\mathrm{g}}$-null. Moreover, the fact that $\widehat{\xi}$ is a Killing vector field for $\widehat{\mathrm{g}}$ entails that the same is true for $\widetilde{\mathrm{g}}$. □

Had we put $\widetilde{\xi} = \alpha\,\widehat{\xi}$ for some $\alpha \in C^\infty(\widehat{M}, \mathbb{R}^*)$ in Proposition 3.4, we would have found that necessarily $d\alpha = 0$, i.e., $\alpha \in \mathbb{R}^*$.

We will resort to Proposition 3.4 in Section 5 where we supply paragons of Schrödinger manifolds.

4 Global structure of the Schrödinger group

4.1 Flat conformal Bargmann structure and Schrödinger Lie algebra

The conformal automorphisms of a Bargmann structure (M, g, ξ) — which we will later on identify, for the flat structure (2.1), to the so-called "Schrödinger group" — have been introduced in Proposition 2.3. As read off Equations (2.5), they consist in (local) diffeomorphisms, Φ, of M such that $\Phi^*\mathrm{g} = \Omega_\Phi^2 \cdot \mathrm{g}$ and $\Phi_*\xi = \xi$ for some smooth, positive, function Ω_Φ. The latter turns out to be necessarily (the pull-back of) a function of the time axis, T. See Equation (2.7).

Accordingly, at the Lie algebraic level, the infinitesimal conformal automorphisms of such a structure span the so-called *Schrödinger Lie algebra*, which is therefore the Lie algebra of those vector fields Z of M such that

$$\mathsf{L}_Z\, \mathrm{g} = \varphi_Z \cdot \mathrm{g} \qquad \& \qquad \mathsf{L}_Z\, \xi = 0 \tag{4.1}$$

for some smooth function φ_Z, again necessarily defined on T.

The Schrödinger Lie algebra, $\mathrm{sch}(d+1,1)$, of the flat Bargmann structure (2.1) is therefore isomorphic to the Lie algebra of all smooth vector fields $Z = [x \mapsto \delta x]$ of $\mathbb{R}^{d+2}$ satisfying (4.1), i.e., of vector fields of the form

$$\delta x = \Lambda x + \Gamma - \frac{1}{2}\alpha\, \mathrm{g}(x,x)\xi + \alpha\, \mathrm{g}(\xi,x)x + \chi x \tag{4.2}$$

where $\Lambda \in \mathrm{so}(d+1,1)$, $\Gamma \in \mathbb{R}^{d+2}$, and $\alpha, \chi \in \mathbb{R}$ are such that $\Lambda\xi + \chi\xi = 0$. We find that $\dim(\mathrm{sch}(d+1,1)) = \frac{1}{2}(d^2+3d+8)$, so that $\dim(\mathrm{sch}(4,1)) = 13$ in the standard case $d = 3$. Homogeneous Galilei transformations are generated by Λ, Bargmann translations by Γ, while α and χ generate inversions and dilations, respectively. The centre, $\mathrm{h} \cong \mathbb{R}$, of $\mathrm{sch}(d+1,1)$ is generated by "vertical" translations Γ, i.e., such that $\xi \wedge \Gamma = 0$. The quotient $\mathrm{sch}(d+1,1))/\mathrm{h}$ acts therefore on Galilei spacetime $E \cong \mathbb{R}^{d+1}$ as the Lie algebra of flat NC infinitesimal automorphisms; it is sometimes called the *center-free* Schrödinger Lie algebra, and is isomorphic to $(\mathrm{so}(d)\times\mathrm{sl}(2,\mathbb{R}))\ltimes(\mathbb{R}^d\times\mathbb{R}^d)$.

4.2 Schrödinger group as a subgroup of conformal group

Taking advantage of the content of the preceding section, let us focus attention on the global structure of the Schrödinger group, $\mathrm{Sch}(d+1,1)$, of the flat (conformal) Bargmann structure (2.1). The latter will be naturally chosen so as to integrate $\mathrm{sch}(d+1,1)$ *inside* the "conformal group" of $\mathbb{R}^{d+1,1}$.

Therefore, in view of (2.5), we will characterize the *Schrödinger group* as a subgroup of the group, $\mathrm{O}(d+2,2)$, of all linear isometries of $\mathbb{R}^{d+2,2} = \mathbb{R}^{d+1,1} \oplus \mathbb{R}^{1,1}$

endowed with the metric (3.2) that we split according to

$$G = \left[\sum_{i=1}^{d} dx^i \otimes dx^i + 2dx^{d+1} \odot dx^{d+2}\right] + 2dx^{d+3} \odot dx^{d+4} \tag{4.3}$$

in order to render explicit the Bargmann metric g as given by (2.1); this metric reads in matrix guise,[3]

$$G = \begin{pmatrix} \mathrm{g} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.4}$$

We need, at this stage, a new geometric object, namely a preferred element, Z_0, of the Lie algebra, $\mathrm{o}(d+2,2)$, of $\mathrm{O}(d+2,2)$.

Definition 4.1. *We will call "special null vector" any $Z_0 \in \mathrm{o}(d+2,2)$ such that: (i) $(Z_0)^2 = 0$, and (ii) $Z_0 \neq 0$.*

The following Lemma is classical; see, e.g., [45, 24].

Lemma 4.2. *A special null vector is of the general form $Z_0 = P_0 \wedge Q_0$ for some $P_0, Q_0 \in \mathbb{R}^{d+2,2} \setminus \{0\}$ such that $G(P_0, P_0) = G(Q_0, Q_0) = G(P_0, Q_0) = 0$.[4] The set of these vectors form a single adjoint orbit of $\mathrm{O}(d+2,2)$.[5]*

Our choice of origin of the orbit of special null vectors is performed by selecting $P_0 = e_{d+2}$, and $Q_0 = e_{d+3}$ where $e_i = \partial/\partial x^i$ for all $i = 1, \ldots, d+4$. It thus reads

$$\boxed{Z_0 = \begin{pmatrix} 0 & 0 & \xi \\ -\xi^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{o}(d+2,2)} \tag{4.5}$$

where $\xi \in \mathbb{R}^{d+2} \setminus \{0\}$ is as in (2.1), the superscript "$*$" standing for the g-adjoint; thus, $\xi^* = \mathrm{g}(\xi)$ is the covector $\xi^* = \theta (= dt)$, interpreted as the *Galilei clock* (see Section 2). This Z_0 will henceforth be identified with the null generator, ξ, of "vertical translations" on Bargmann space $\mathbb{R}^{d+1,1}$ [10, 27].

[3]The matrix (4.4) is the Gram matrix of some chosen basis that will not be further specified, unless otherwise stated.

[4]We will often use the identification $\mathrm{o}(d+2,2) \cong \bigwedge^2 \mathbb{R}^{d+2,2}$.

[5]This nilpotent orbit has two connected components; in the case $d = 2$ each one is symplectomorphic to the manifold of regularized Keplerian motions [45, 24, 11].

Proposition 4.3. *The Lie algebra* $\mathrm{sch}(d+1,1)$ *is isomorphic to the Lie algebra of the group*

$$\boxed{\mathrm{Sch}(d+1,1) = \{A \in \mathrm{O}(d+2,2) \,|\, AZ_0 = Z_0 A\}} \tag{4.6}$$

which we call the "Schrödinger group".

Proof. Straightforward computation shows that the stabilizer of Z_0 in $\mathrm{O}(d+2,2)$ consists of matrices of the form

$$A = \begin{pmatrix} L & a\xi & C \\ B^* & b & d \\ -a\xi^* & 0 & e \end{pmatrix} \tag{4.7}$$

where $L \in \mathrm{End}(\mathbb{R}^{d+2})$, $B, C \in \mathbb{R}^{d+2}$, and $a, b, d, e \in \mathbb{R}$ satisfy

$$0 = L\xi - e\xi \tag{4.8}$$

$$0 = L^*\xi - b\xi \tag{4.9}$$

$$\mathbf{1} = L^*L - a(\xi B^* + B\xi^*) \tag{4.10}$$

$$0 = L^*C - ad\xi + eB \tag{4.11}$$

$$1 = a\xi^*C + be \tag{4.12}$$

$$0 = \xi^*(B + C) \tag{4.13}$$

$$0 = C^*C + 2de \tag{4.14}$$

where, again, L^* stands for the g-adjoint of the linear operator L.

In view of (2.1) and (4.4), let us put $\xi = e_{d+2}$, where $(e_1, \ldots, e_{d+2})$ is the "canonical" basis of $\mathbb{R}^{d+2}$; let us complete it in $\mathbb{R}^{d+2} \oplus \mathbb{R}^2$ with the canonical basis (e_{d+3}, e_{d+4}) of $\mathbb{R}^2$. Define then (with a slight abuse of notation) $A_i = Ae_i$ for all $i = 1, \ldots, d+4$, where A is as in (4.7). Upon specifying

$$X = A_{d+4} = \begin{pmatrix} C \\ d \\ e \end{pmatrix}, \qquad Y = A_{d+3} = \begin{pmatrix} a\xi \\ b \\ 0 \end{pmatrix}, \tag{4.15}$$

we trivially check that

$$\overline{X}X = \overline{Y}Y = \overline{X}Y - 1 = 0 \qquad \& \qquad Z_0 Y = 0, \tag{4.16}$$

where $\overline{X} = G(X)$ is, as before, the G-adjoint of $X \in \mathbb{R}^{d+2,2}$.

The group law of $\mathrm{Sch}(d+1,1)$, plainly given by matrix multiplication using (4.7), translates as the group action $\mathrm{Sch}(d+1,1) \ni A : (X,Y) \mapsto (X',Y')$ given by

$$(X',Y') = (AX, AY) \tag{4.17}$$

on the $(d+4)$-dimensional manifold defined by the constraints (4.16).

We then find, using (4.7), that vectors in the Lie algebra of $\mathrm{Sch}(d+1,1)$ are of the form

$$Z = \begin{pmatrix} \Lambda & \alpha\xi & \Gamma \\ -\Gamma^* & \chi & 0 \\ -\alpha\xi^* & 0 & -\chi \end{pmatrix} \tag{4.18}$$

where $\Lambda \in \mathrm{so}(d+1,1)$, $\Gamma \in \mathbb{R}^{d+2}$, and $\alpha, \chi \in \mathbb{R}$ are such that $\Lambda\xi + \chi\xi = 0$ (see (4.8)).

Let us now prove that the Lie algebra of $\mathrm{Sch}(d+1,1)$ is indeed isomorphic to $\mathrm{sch}(d+1,1)$, whose action on flat Bargmann space is given by (4.2).

Assuming $e(= \overline{X}Q_0) \neq 0$, in view of (4.12), (4.14), and (4.15) we can write

$$X = \frac{1}{r}\begin{pmatrix} x \\ -\frac{1}{2}x^*x \\ 1 \end{pmatrix}, \qquad Y = r\begin{pmatrix} q\xi \\ 1 - q\xi^*x \\ 0 \end{pmatrix} \tag{4.19}$$

where $x = C/e \in \mathbb{R}^{d+2}$, $q = ae \in \mathbb{R}$, and $r = 1/e \in \mathbb{R}^*$. We deduce from (4.17) that the Schrödinger group acts projectively on *Bargmann space* $\mathbb{R}^{d+1,1}$ according to $A : x \mapsto x'$, viz.,

$$x' = \frac{Lx - \frac{1}{2}a(x^*x)\xi + C}{e - a\xi^*x} \tag{4.20}$$

where $A \in \mathrm{Sch}(d+1,1)$ is as in (4.7). We, likewise, get the transformation law

$$r' = \frac{r}{e - a\xi^*x} \tag{4.21}$$

with the same notation as before.

As for the infinitesimal action of the Schrödinger group on $\mathbb{R}^{d+1,1}$, it can be computed, using (4.20), by $\delta x = \delta x'|_{A=\mathbf{1},\delta A=Z}$, where Z is as in (4.18); we then find $\delta x = \Lambda x + \Gamma - \frac{1}{2}\alpha(x^*x)\xi + \alpha(\xi^*x)x + \chi x$, which exactly matches Equation (4.2). Note that we get from (4.21) $\delta r = (\alpha\xi^*x + \chi)r$.

The proof that the Lie algebra of $\mathrm{Sch}(d+1,1)$ is isomorphic to $\mathrm{sch}(d+1,1)$ is complete. □

Proposition 4.4. *The Schrödinger group (4.6) has two connected components,*

$$\pi_0(\mathrm{Sch}(d+1,1)) = \mathbb{Z}_2. \tag{4.22}$$

Proof. Let us express the matrix Z_0' of the central element Z_0 given by (4.5) in a new basis of $\mathbb{R}^{d+2,2}$ whose Gram matrix is

$$G' = \begin{pmatrix} \mathbf{1}_{\mathbb{R}^d} & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \tag{4.23}$$

where $D = \mathrm{diag}(1,-1)$. The sought expression is therefore

$$Z_0' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & U \\ 0 & V & 0 \end{pmatrix} \tag{4.24}$$

where

$$U = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \& \qquad V = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.25}$$

The group $\mathrm{O}(d+2,2))$ has four connected components, and the generators $\{I, P, T, PT\}$ of $\pi_0(\mathrm{O}(d+2,2)) = \pi_0(\mathrm{O}(d+2)) \times \pi_0(\mathrm{O}(2)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ can be defined — up to conjugation — by

$$I = \begin{pmatrix} \mathbf{1}_{\mathbb{R}^d} & 0 & 0 \\ 0 & \mathbf{1}_{\mathbb{R}^2} & 0 \\ 0 & 0 & \mathbf{1}_{\mathbb{R}^2} \end{pmatrix}, \quad P = \begin{pmatrix} S & 0 & 0 \\ 0 & \mathbf{1}_{\mathbb{R}^2} & 0 \\ 0 & 0 & \mathbf{1}_{\mathbb{R}^2} \end{pmatrix}, \quad T = \begin{pmatrix} \mathbf{1}_{\mathbb{R}^d} & 0 & 0 \\ 0 & \mathbf{1}_{\mathbb{R}^2} & 0 \\ 0 & 0 & D \end{pmatrix} \tag{4.26}$$

where $S \in \mathrm{O}(d)$ is such that $S^2 = \mathbf{1}_{\mathbb{R}^d}$ and $\det(S) = -1$.

It is a trivial matter to check that the only non-zero commutators are $[T, Z_0']$ and $[PT, Z_0']$, proving, via the definition (4.6) of the Schrödinger group, that, indeed, $\pi_0(\mathrm{Sch}(d+1,1))$ is generated by I, and P. □

4.3 A nilpotent coadjoint orbit of the conformal group

We highlight that the (non-relativistic) Schrödinger group is, interestingly, associated with a special homogeneous *symplectic* manifold of the (relativistic) conformal group.

As we have seen in Proposition 4.3, the Schrödinger group, $\mathrm{Sch}(d+1,1)$, is the stabilizer of $Z_0 \in \mathrm{o}(d+2,2)$, given by (4.5), for the adjoint action of $\mathrm{O}(d+2,2)$. The (co)adjoint orbit

$$\boxed{\mathcal{O}_{Z_0} = \mathrm{O}(d+2,2)/\mathrm{Sch}(d+1,1)} \tag{4.27}$$

is therefore a $2(d+1)$-dimensional *symplectic manifold* we now describe as follows.

Consider the left-invariant Maurer-Cartan 1-form $\Theta = A^{-1}dA$, and the 1-form $\varpi = -\frac{1}{2}\mathrm{Tr}(Z_0\Theta)$ of $\mathrm{O}(d+2,2)$. A classical result tells us that $d\varpi$ descends to $\mathcal{O}_{Z_0}$ as the canonical Kirillov-Kostant-Souriau symplectic 2-form, ω, of $\mathcal{O}_{Z_0}$, viz., $d\varpi = (\mathrm{O}(d+2,2) \to \mathcal{O}_{Z_0})^*\omega$. Indeed, let us put again $\xi = e_{d+2}$, and $A_i = Ae_i$ for $i = 1, \ldots d+4$ whenever $A \in \mathrm{O}(d+2,2)$; with the help of (4.5) we get $\varpi = \overline{P}dQ$, where $P = A_{d+2}$, and $Q = A_{d+3}$ are nonzero, and such that $\overline{P}P = \overline{P}Q = \overline{Q}Q = 0$. The 2-form

$$d\varpi = d\overline{P} \wedge dQ \tag{4.28}$$

clearly descends to the slit null tangent bundle, $NT\mathcal{Q}\backslash\mathcal{Q}$, of the null quadric

$$\mathcal{Q} = \{Q \in \mathbb{R}^{d+2,2} \backslash \{0\} \,|\, \overline{Q}Q = 0\}. \tag{4.29}$$

It defines the sought symplectic structure, ω, on

$$\mathcal{O}_{Z_0} = (NT\mathcal{Q}\backslash\mathcal{Q})/\mathrm{SL}(2,\mathbb{R}) \tag{4.30}$$

interpreted as the *manifold of null geodesics of conformally compactified Minkowski spacetime* $\mathbb{P}\mathcal{Q} = \mathrm{Ein}_{d+1,1}$; see (3.4) and (3.6). (The leaves of the distribution $\ker(d\varpi)$ of $NT\mathcal{Q}\backslash\mathcal{Q}$ project to $\mathbb{P}\mathcal{Q}$ as the null geodesics of its conformally flat structure.) Note that $\mathcal{O}_{Z_0} = \mathcal{O}^+_{Z_0} \cup \mathcal{O}^-_{Z_0}$ with $\mathcal{O}^\pm_{Z_0} \cong TS^{d+1}\backslash S^{d+1}$, topologically [45, 24, 11].

5 Homogeneous Schrödinger manifolds

We are now led to the following query: what $\mathrm{Sch}(d+1,1)$-homogeneous space would host a *genuine*, well-behaved, Lorentz metric whose isometries constitute the whole Schrödinger group (4.6)?

Let us first consider the distinguished element $Z_0 \in \mathrm{o}(d+2,2)$ represented as in (4.5) and the associated vector field $\delta_{Z_0} : Q \mapsto Z_0 Q$ on the quadric

$$\mathrm{AdS}_{d+3}(\sqrt{-2\lambda}) = \{Q \in \mathbb{R}^{d+2,2} \,|\, \overline{Q}Q = 2\lambda\} \tag{5.1}$$

with a given $\lambda < 0$ (see (3.3)).

Lemma 5.1. *The vector field δ_{Z_0} of $\mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$ nowhere vanishes.*

Proof. In view of (4.5), we find

$$\delta_{Z_0} : \begin{pmatrix} x \\ \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \beta\xi \\ -\xi^* x \\ 0 \end{pmatrix} \tag{5.2}$$

where $x \in \mathbb{R}^{d+1,1}$, and $\alpha, \beta \in \mathbb{R}$ are such that $\overline{Q}Q = x^*x + 2\alpha\beta = 2\lambda$, and where the metric (3.2) has been used. Suppose, for the moment, that $\delta_{Z_0}Q = 0$ for some $Q \in \mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$, i.e., that $\beta = 0$, and $\xi^* x = 0$. We readily get $x^*x = 2\lambda < 0$. We hence find that $x \in \mathbb{R}^{d+1,1}$ is at the same time g-orthogonal to the null vector $\xi \neq 0$, and timelike: contradiction! Thus, $\delta_{Z_0}Q \neq 0$ for all $Q \in \mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$. □

5.1 A special family of Schrödinger-homogeneous spaces

Let us resort to the definition (4.15) of the vectors $X, Y \in \mathcal{Q}$ (the last two column vectors of the Schrödinger matrix (4.7)), and posit

$$\boxed{Q = X + \lambda Y} \tag{5.3}$$

where $\lambda \in \mathbb{R}^*$ is fixed.

We contend, and will prove right below, that the set

$$\boxed{\widehat{M}_\lambda = \{X + \lambda Y \in \mathbb{R}^{d+2,2} \,|\, \overline{X}X = \overline{Y}Y = \overline{X}Y - 1 = 0, Z_0 Y = 0\}} \tag{5.4}$$

of these Q, with $\lambda < 0$, is actually a homogeneous manifold of the Schrödinger group, and an open submanifold $\widehat{M}_\lambda \subset \mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$.

Proposition 5.2. *For every $\lambda < 0$, the manifold (5.4) is a connected, $(d+3)$-dimensional, homogeneous space of the Schrödinger group, viz.,*

$$\boxed{\widehat{M}_\lambda \cong \mathrm{Sch}(d+1,1)/(\mathrm{E}(d) \times \mathbb{R})} \tag{5.5}$$

where $\mathrm{E}(d) = \mathrm{O}(d) \ltimes \mathbb{R}^d$ is the Euclidean group of $\mathbb{R}^d$. These manifolds have topology

$$\boxed{\widehat{M}_\lambda \cong (\mathbb{R}^{d+2} \setminus \{0\}) \times S^1.} \tag{5.6}$$

Proof. From the very definition (5.4), each manifold $\widehat{M}_\lambda$ is the image of the surjection $\pi_\lambda : \mathrm{Sch}(d+1,1) \to \widehat{M}_\lambda$ given by $\pi_\lambda(A) = \lambda A_{d+3} + A_{d+4}$. The left-action of the Schrödinger group clearly passes to the quotient according to (4.17), and $\widehat{M}_\lambda$ is therefore diffeomorphic to a homogeneous space $\mathrm{Sch}(d+1,1)/K$. Let us prove that $K \cong \mathrm{E}(d) \times \mathbb{R}$.

The coordinate system chosen in (4.19) provides us with the local expression

$$Q = \frac{1}{r} \begin{pmatrix} \widehat{x} \\ -\frac{1}{2}\widehat{x}^* \widehat{x} + \lambda r^2 \\ 1 \end{pmatrix} \tag{5.7}$$

where

$$\widehat{x} = x + \lambda r^2 q \xi \in \mathbb{R}^{d+2} \tag{5.8}$$

and $r \neq 0$. Consider now the "origin", Q_0, defined by $\widehat{x} = 0$ and $r = 1$ in (5.7). Then look for the subgroup, K, of all $A \in \mathrm{Sch}(d+1,1)$ such that $AQ_0 = Q_0$. In view of (4.7) we readily find $C = -\lambda a \xi$, $d = \lambda(1-b)$, and $e = 1$. Moreover, the

constraints (4.8)–(4.14) yield $L\xi = L^*\xi = \xi$, $B = -C = \lambda a\xi$, $b = 1$, $d = 0$, and $L^*L = \mathbf{1} + 2\lambda a^2\xi\xi^*$. The equations $L\xi = L^*\xi = \xi$ help us write

$$L = \begin{pmatrix} R & u & 0 \\ 0 & 1 & 0 \\ v & w & 1 \end{pmatrix} \tag{5.9}$$

where $R \in \mathrm{End}(\mathbb{R}^d)$, $u \in \mathbb{R}^d$, $v \in (\mathbb{R}^d)^*$, and $w \in \mathbb{R}$. At last, the extra constraint $L^*L = \mathbf{1}+2\lambda a^2\xi\xi^*$ entails that $R^tR = \mathbf{1}$, $v = -u^tR$ (where the superscript "t" stands for transposition), and $w = -\frac{1}{2}u^tu+\lambda a^2$. The isotropy group $K \subset \mathrm{Sch}(d+1,1)$ of Q_0 is therefore parametrized by the triples $(R, u, a) \in \mathrm{O}(d) \times \mathbb{R}^d \times \mathbb{R}$, and easily found to be isomorphic to the direct product $K \cong \mathrm{E}(d)\times\mathbb{R}$. Since $\dim(K) = \frac{1}{2}d(d+1)+1$, we indeed get $\dim(\widehat{M}_\lambda) = \frac{1}{2}(d^2 + 3d + 8) - \frac{1}{2}(d^2 + d + 2) = d + 3$.

We now work out the topology of our $\mathrm{Sch}(d+1,1)$-homogeneous space $\widehat{M}_\lambda$ given by (5.4). We will suitably use a frame of $\mathbb{R}^{d+2,2}$ with Gram matrix (4.23) where the distinguished element $Z_0 \in \mathrm{o}(d+2,2)$ is represented by the matrix Z_0' in (4.24).

Let us write the components of the $Q = X+\lambda Y$, defined by (5.3), in this frame. Solving the equations in (5.4) for Y, namely $\overline{Y}Y = 0$, and $Z_0Y = 0$, we get

$$Y = \begin{pmatrix} 0 \\ a' \\ -a' \\ b' \\ b' \end{pmatrix} \in \mathbb{R}^{d+2,2} \setminus \{0\}$$

with $a', b' \in \mathbb{R}$ (and $a'^2 + b'^2 > 0$).

As for the remaining equations satisfied by

$$X = \begin{pmatrix} x \\ a \\ u \\ b \\ v \end{pmatrix} \in \mathbb{R}^{d+2,2} \setminus \{0\}$$

with $x \in \mathbb{R}^d$, and $a, b, u, v \in \mathbb{R}$, we obtain

$$\overline{X}X = 0 \iff x^tx + a^2 + b^2 = u^2 + v^2 > 0 \tag{5.10}$$

$$\overline{X}Y = 1 \iff a'(u + a) + b'(b - v) = 1. \tag{5.11}$$

Noticing that the dilations $(X, Y) \mapsto (\alpha X, \alpha^{-1} Y)$ with $\alpha \in \mathbb{R}^*$ do preserve $\widehat{M}_\lambda$, we claim that the latter dilation invariance and the conditions (5.10) and (5.11) leave us with $d+6-3 = d+3$ free parameters, e.g., x, a, b, u, v, a'. Then, Equation (5.11) yields a' as a function of a, b, u, v. The only remaining constraint on X is therefore given by Equation (5.10). This entails that $X \in \mathcal{Q}$, hence $\widehat{M}_\lambda$ has the same topology as $\mathcal{Q} \cong (\mathbb{R}^{d+2} \setminus \{0\}) \times S^1$, and is thus connected. □

5.2 Distinguished Schrödinger-invariant structures

With these preparations, we are ready to introduce Schrödinger-invariant tensors on $\widehat{M}_\lambda$.

Denote by $\widehat{\mathrm{g}}_\lambda = (\widehat{M}_\lambda \hookrightarrow \mathbb{R}^{d+4})^* G$ the induced symmetric tensor on $\widehat{M}_\lambda$, viz.,

$$\boxed{\widehat{\mathrm{g}}_\lambda(\delta Q, \delta' Q) = \delta \overline{Q}\, \delta' Q} \tag{5.12}$$

for all $\delta Q, \delta' Q \in T_Q \widehat{M}_\lambda$. This tensor, $\widehat{\mathrm{g}}_\lambda$, is clearly $\mathrm{Sch}(d+1,1)$-invariant. In view of (4.6), the same remains true for the one-form $\widehat{\theta}$ of $\widehat{M}_\lambda$ defined by

$$\boxed{\widehat{\theta}(\delta Q) = -\overline{Q}\, Z_0\, \delta Q} \tag{5.13}$$

for all $\delta Q \in T_Q \widehat{M}_\lambda$.

We easily find that $d\widehat{\theta}(\delta Q, \delta' Q) = -2\delta \overline{Q} Z_0 \delta' Q$. By means of the fact that Z_0 has rank 2 (as clear from Lemma 4.2 stating that $Z_0 = P_0 \wedge Q_0$ where P_0 and Q_0 span a totally null plane in $\mathbb{R}^{d+2,2}$), and by some straightforward computation, we get

$$\widehat{\theta} \wedge d\widehat{\theta} = 0. \tag{5.14}$$

Remark 5.3. *Local* expressions for (5.12) and (5.13) are easily deduced from (5.7); we get $\widehat{\mathrm{g}}_\lambda(\delta Q, \delta' Q) = r^{-2}\left[\mathrm{g}(\delta \widehat{x}, \delta' \widehat{x}) - 2\lambda\, \delta r \delta' r\right]$ or, alternatively,

$$\widehat{\mathrm{g}}_\lambda = \frac{1}{r^2}\left[\sum_{i,j=1}^{d+2} \mathrm{g}_{ij}\, d\widehat{x}^i \otimes d\widehat{x}^j - 2\lambda\, dr \otimes dr\right] \tag{5.15}$$

together with

$$\widehat{\theta} = \frac{\theta}{r^2} \tag{5.16}$$

where $\theta = \sum_{i=1}^{d+2} \mathrm{g}_{ij}\, \xi^i d\widehat{x}^j (= dt)$ is the Galilei clock of the flat Bargmann structure. The metric (5.15) is the well-known expression of the $\mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$ metric in Poincaré coordinates; see, e.g., [1].

Theorem 5.4. *For every $\lambda < 0$, the manifold $\widehat{M}_\lambda$ admits a family of Lorentz metrics*

$$\boxed{\widehat{\mathrm{g}}_{\lambda,\mu} = \widehat{\mathrm{g}}_\lambda - \mu\, \widehat{\theta} \otimes \widehat{\theta}} \tag{5.17}$$

given by (5.12) and (5.13), parametrized by $\mu \in \mathbb{R}$. The Schrödinger group is the group of isometries of $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu})$.

Proof. The signature of the metrics $\widehat{\mathrm{g}}_\lambda$ and $\widehat{\mathrm{g}}_{\lambda,\mu}$ is clearly Lorentzian since $\lambda < 0$. Then the group of isometries of $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_\lambda)$ is, by construction, a subgroup of the group $\mathrm{O}(d+2,2)$ of isometries of $\mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$, which furthermore preserves the constraint $Z_0 Y = 0$ in (5.4). It is thus the stabilizer of Z_0 in $\mathrm{O}(d+2,2)$, i.e., the Schrödinger group $\mathrm{Sch}(d+1,1)$ in view of (4.6). The extra term, $-\mu\, \widehat{\theta} \otimes \widehat{\theta}$, in (5.17) being $\mathrm{Sch}(d+1,1)$-invariant, Proposition 3.4 helps us complete the proof. □

Remark 5.5. The expression (5.17) is — up to an overall multiplicative constant factor — the most general twice-symmetric tensor constructed by means of the only data at our disposal, namely the "ambient" metric G given by (3.2), and the central element $Z_0 \in \mathrm{sch}(d+1,1)$ defined in (4.5).

Remark 5.6. In view of Proposition 5.2, the manifold (5.4) is $(d+3)$-dimensional, it is thus an open submanifold $\widehat{M}_\lambda \subset \mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$.

There exists a privileged vector field on $\widehat{M}_\lambda$, namely

$$\boxed{\widehat{\xi} : Q \mapsto \delta_{Z_0} Q = Z_0 Q} \tag{5.18}$$

where $Z_0 \in \mathrm{o}(d+2,2)$ is defined by (4.5).

Proposition 5.7. *The vector field $\widehat{\xi}$ defined by (5.18) is a nowhere vanishing, light-like, Killing vector field of $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu})$.*

Proof. The restriction $\widehat{\xi}$ to $\widehat{M}_\lambda$ of the vector field $\delta_{Z_0} : Q \mapsto Z_0 Q$ of $\mathbb{R}^{d+2,2}$ is tangent to $\widehat{M}_\lambda$ at the point Q since $\delta_{Z_0}(\overline{Q}Q) = 2\overline{Q}\delta_{Z_0}Q = 2\overline{Q}Z_0Q = 0$ as a consequence of the G-skewsymmetry of Z_0. Let us furthermore show that $Z_0Q \neq 0$ for all $Q \in \widehat{M}_\lambda$. Resorting to (5.4), we find $Z_0Q = Z_0X$; using (4.5) and (4.15), we get

$$Z_0X = \begin{pmatrix} e\xi \\ -\xi^*C \\ 0 \end{pmatrix} \tag{5.19}$$

and claim that the latter vector nowhere vanishes since $\xi \neq 0$. Indeed, suppose that $e = 0$; then Equation (4.12) would necessarily yield $\xi^*C \neq 0$, implying $Z_0X \neq 0$, whence $\delta_{Z_0}Q \neq 0$ for all $Q \in \widehat{M}_\lambda$.

The vector field (5.18) is actually a Killing vector field of the metric (5.17) since it generates the 1-parameter additive group $s \mapsto \exp(sZ_0) = \mathrm{Id} + sZ_0 \in \mathrm{Sch}(d+1,1)$, i.e., a group of isometries of $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu})$ as a consequence of Theorem 5.4.

We finally check that $\mathrm{g}_{\lambda,\mu}(\widehat{\xi}, \widehat{\xi}) = 0$. By Equations (5.12) and (5.13), we get $\mathrm{g}_{\lambda,\mu}(\delta_{Z_0}Q, \delta_{Z_0}Q) = \overline{Z_0Q}\, Z_0Q - \mu(\overline{Q}Z_0^2Q)^2 = 0$ since $Z_0 + \overline{Z_0} = Z_0^2 = 0$. □

5.3 Conformal infinity and conformal Bargmann structures

Resorting to Definition (5.4), we will consider the limit $\lambda \to 0$ as a route to conformal infinity of $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu}, \widehat{\xi})$, our candidate to the status of Schrödinger manifold.

Observe that, in view of Lemma 5.1, there holds $Z_0X \neq 0$ in (5.4). So, the limiting manifold $\widehat{M}_0 = \lim_{\lambda\to 0} \widehat{M}_\lambda$ is an open submanifold of the null cone $\mathcal{Q}$. The construction (3.4) of the Einstein space therefore prompts the following definition for conformal infinity of the previous structure, namely $M = \widehat{M}_0 \,/\, \mathbb{R}^*$, i.e.,

$$\boxed{M = \{X \in \mathbb{R}^{d+2,2} \,|\, \overline{X}X = 0, Z_0X \neq 0\} \,/\, \mathbb{R}^*} \tag{5.20}$$

where $X \sim X'$ iff $X' = \alpha X$ for some $\alpha \in \mathbb{R}^*$.

Proposition 5.8. *The manifold (5.20) is diffeomorphic to the following* $(d+2)$*-dimensional homogeneous space of the Schrödinger group*

$$\boxed{M \cong \mathrm{Sch}(d+1,1)/(\mathrm{E}(d) \times T\mathbb{R}^*)} \tag{5.21}$$

and has topology

$$\boxed{M \cong (\mathbb{R}^{d+1} \times S^1)/\mathbb{Z}_2.} \tag{5.22}$$

Proof. If $X \in \widehat{M}_0$, the same is true for AX for any $A \in \mathrm{Sch}(d+1,1)$ since $Z_0 AX = AZ_0 X \neq 0$, see Definition (4.6). This enables us to choose, e.g.,

$$X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \widehat{M}_0 \tag{5.23}$$

in the frame whose Gram matrix is as in (4.4).

Now, M being the projectivization of $\widehat{M}_0$, let us determine the stabilizer, S, of the direction of X in (5.23). Seek thus the form of those $A \in \mathrm{Sch}(d+1,1)$ such that $AX = \alpha X$, for some $\alpha \in \mathbb{R}^*$. Using (4.7), we get $C = 0$, $d = 0$, and $e = \alpha$. Equations (4.11) and (4.12) entail $B = 0$, and $b = 1/e$. From Equation (4.10) we get $L^* L = \mathbf{1}$, hence

$$A = \begin{pmatrix} L & a\xi & 0 \\ 0 & e^{-1} & 0 \\ -a\xi^* & 0 & e \end{pmatrix} \tag{5.24}$$

with $L \in \mathrm{O}(d+1,1)$ satisfying the constraint (4.8), $a \in \mathbb{R}$, and $e \in \mathbb{R}^*$.

In order to implement the latter constraint $L\xi = e\xi$, and fully characterize $A \in S$, let us choose the constant g-null vector ξ to be of the form

$$\xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1,1} \tag{5.25}$$

as in the coordinate system used in (2.1). This entails that

$$L = \begin{pmatrix} R & -e^{-1}Rv & 0 \\ 0 & e^{-1} & 0 \\ v^t & -\frac{1}{2}e^{-1}v^t v & e \end{pmatrix} \tag{5.26}$$

with $R \in \mathrm{O}(d)$, and $v \in \mathbb{R}^d$. The matrix group law for this stabilizer readily yields $S = (\mathrm{O}(d) \ltimes \mathbb{R}^d) \times (\mathbb{R}^* \ltimes \mathbb{R})$ proving (5.21). We can therefore confirm that $\dim(M) = \frac{1}{2}(d^2 + 3d + 8) - \frac{1}{2}(d^2 + d + 4) = d + 2$.

We now work out the topology of M, our $\mathrm{Sch}(d{+}1, 1)$-homogeneous space (5.20). To that end, use a frame with Gram matrix (4.23) where the distinguished element $Z_0 \in \mathrm{o}(d + 2, 2)$ is represented by the matrix Z_0' given by (4.24). Then look for all $X \in \mathcal{Q} \setminus \widehat{M}_0$, i.e., for those X lying in the null cone $\mathcal{Q}$, and outside $\widehat{M}_0$. This amounts to finding all solutions

$$X = \begin{pmatrix} x \\ a \\ u \\ b \\ v \end{pmatrix} \in \mathbb{R}^{d+2,2} \setminus \{0\}$$

with $x \in \mathbb{R}^d$, and $a, b, u, v \in \mathbb{R}$ of both equation $\overline{X}X = 0$, viz., $x^t x + a^2 + b^2 = u^2 + v^2$, and $Z_0 X = 0$, i.e., $u = -a$, and $v = b$. Since $X \neq 0$, we get $x^t x = 0$. This leaves us with $x = 0$, and $a^2 + b^2 > 0$; hence $\mathcal{Q} \setminus \widehat{M}_0 \cong (\{\mathsf{pt}\} \times S^1) \times \mathbb{R}^*_+$, which reveals that, in this forbidden domain, the fibre above $(a, b) \neq 0$ is a point, $\{\mathsf{pt}\}$. Thanks to (3.6), and (5.20), we obtain $M \cong \big((S^{d+1} \setminus \{\mathsf{pt}\}) \times S^1\big)/\mathbb{Z}_2$, i.e., $M \cong (\mathbb{R}^{d+1} \times S^1)/\mathbb{Z}_2$. $\square$

Remark 5.9. As a consequence of (5.22), the manifold (5.20) has the topology of a Möbius band as shown in [9, 16]. It will be interpreted as an extended spacetime, fibered above the time axis $T \cong \mathbb{P}^1(\mathbb{R})$; see Section 2.

Let us show that M is actually endowed with a conformal Bargmann structure (see Section 2.3) inherited from its very definition (5.20).

Consider then $\widehat{\mathrm{g}}_\lambda = \widehat{\mathrm{g}}_{\lambda,0}$ where $\widehat{\mathrm{g}}_{\lambda,\mu}$ is as in (5.17). The induced twice symmetric covariant tensor field $\mathrm{g}_0 = \widehat{\mathrm{g}}_0|_{T\widehat{M}_0}$ on $\widehat{M}_0 \subset \mathcal{Q}$ is degenerate, and $\ker(\mathrm{g}_0)$ is spanned by $\mathcal{E}$, the restriction to $\widehat{M}_0$ of the Euler vector field of the quadric $\mathcal{Q}$. We find that $\mathsf{L}_{\mathcal{E}}\, \mathrm{g}_0 = 2\mathrm{g}_0$, which entails that g_0 defines but a conformal class $[\mathrm{g}]$ of Lorentz metrics on $M = \mathbb{P}\widehat{M}_0$ (just as in the $\mathrm{Ein}_{d+1,1}$ case dealt with in Section 3.1).

We have thus proved the following result.

Proposition 5.10. *The quadratic form* g_0 *on* $\widehat{M}_0$ *defines a conformal class* $[\mathrm{g}]$ *of Lorentz metrics on* M*.*

Let us derive, at this stage, a remarkable global representative of $[\mathrm{g}]$ constructed via a nowhere vanishing function F_0 of $\widehat{M}_0$, which is homogeneous of degree 2, e.g., via the function

$$F_0(X) = (\overline{X}P_0)^2 + (\overline{X}Q_0)^2 \tag{5.27}$$

associated with the distinguished element $Z_0 = P_0 \wedge Q_0 \in \mathrm{o}(d+2,2)$ of Lemma 4.2. Indeed $F_0(X) \neq 0$ is equivalent to the defining condition $Z_0 X \neq 0$ of $\widehat{M}_0$.

We will also denote by $\pi : \widehat{M}_0 \to M$ the projection where $\pi(X) = [X]$ is the ray through $X \in \widehat{M}_0$.

Lemma 5.11. *A representative* $\mathrm{g}_{F_0} \in [\mathrm{g}]$ *associated with the choice (5.27) reads as*

$$\mathrm{g}_{F_0}(\delta[X], \delta'[X]) = \frac{\delta\overline{X}\,\delta' X}{F_0(X)}. \tag{5.28}$$

Proof. Clearly, the quadratic form g_0/F_0 is dilation-invariant, and hence passes to the quotient M as a representative $\mathrm{g}_{F_0} \in [\mathrm{g}]$. Putting $[X] = X/\sqrt{F_0(X)}$ for $X \in \widehat{M}_0$, and using the fact that $\overline{X}X = 0$, we end up with Equation (5.28). □

Moreover the action of the Schrödinger group on M, given by $A : [X] \mapsto [AX]$ for all $A \in \mathrm{Sch}(d+1,1)$ is well-defined; we further check, via Equation (5.28), that it is indeed a conformal action since it preserves $[\mathrm{g}]$.

Considering then the 1-form $\widehat{\theta}_0$ induced by $\widehat{\theta}$ on $\widehat{M}_0$, we find that $\widehat{\theta}_0(\mathcal{E}) = 0$, and $\mathsf{L}_{\mathcal{E}}\,\widehat{\theta}_0 = 2\widehat{\theta}_0$. This implies, with the above choice, that the dilation-invariant 1-form $\widehat{\theta}_0/F_0$ descends to M as the 1-form θ_{F_0} given by

$$\theta_{F_0}(\delta[X]) = -\frac{\overline{X}Z_0\delta X}{F_0(X)}. \tag{5.29}$$

Let us then prove that θ_{F_0} is closed. As a first step, we obtain $d\theta_{F_0}(\delta[X], \delta'[X]) = \left(-2F_0(X)\,\delta\overline{X}Z_0\delta'X + \delta F_0(X)\overline{X}Z_0\delta'X - \delta' F_0(X)\overline{X}Z_0\delta X\right)/(F_0(X))^2$. Then, using the fact that $Z_0 = P_0 \wedge Q_0$ (see Lemma 4.2), and the expression (5.27), one finds

$$d\theta_{F_0} = 0. \tag{5.30}$$

We claim that $\theta_{F_0} \in [\theta]$, where θ is the Bargmann clock introduced in Section 2.

Proposition 5.12. *The vector field $\delta_{Z_0} : X \mapsto Z_0 X$ of $\widehat{M}_0$ descends to the quotient M defined in (5.20) as a nowhere vanishing, null, vector field ξ, viz.,*

$$\xi_{[X]} = D\pi(X)Z_0X. \tag{5.31}$$

Proof. The derivation δ_{Z_0} preserves the constraint $\overline{X}.X = 0$; it thus defines a vector field of $\widehat{M}_0$ which is nowhere zero because of the definition (5.20). we readily check that δ_{Z_0} is invariant against dilations $X \mapsto \alpha X$ with $\alpha \in \mathbb{R}^*$. The push-forward, ξ, of δ_{Z_0} to $M = \mathbb{P}\widehat{M}_0$ is therefore a nowhere vanishing vector field. Finally, (5.28) yields $\mathrm{g}_{F_0}(\xi_{[X]}, \xi_{[X]}) = \overline{\delta_{Z_0}X}\,\delta_{Z_0}X/F_0(X) = -\overline{X}Z_0^2X/F_0(X) = 0$. □

Let us end by proving that the vector field ξ is indeed covariantly constant with respect to the Levi-Civita connection, ∇, of g_{F_0}.

Applying the general formula $\nabla\theta = \frac{1}{2}d\theta + \frac{1}{2}\mathsf{L}_\xi\,\mathrm{g}$, where $\theta = \mathrm{g}(\xi)$, we readily find, using Equation (5.30), that $\nabla\theta_{F_0} = \frac{1}{2}\mathsf{L}_\xi\,\mathrm{g}_{F_0}$. Now, Equation (5.28) helps us compute $\mathsf{L}_\xi\,\mathrm{g}_{F_0}(\delta'[X], \delta''[X]) = \delta_{Z_0}\left(\delta'\overline{X}\,\delta''X/F_0(X)\right) - \overline{[\delta_{Z_0}, \delta']X}\,\delta''X/F_0(X) - \delta'\overline{X}\,[\delta_{Z_0}, \delta'']X/F_0(X) = -(\delta'\overline{X}\,\delta''X)\delta_{Z_0}F_0(X)/(F_0(X))^2 = 0$ since $\delta_{Z_0}F_0(X) = 0$ in view of $Z_0P_0 = Z_0Q_0 = 0$. We thus get $\nabla\theta_{F_0} = 0$, hence $\nabla\xi = 0$.

We have thereby proved the following proposition.

Proposition 5.13. *The triple $(M, [\mathrm{g}], \xi)$ is a conformal Bargmann structure in the sense of Definition 2.4.*

5.4 Main result: homogeneous Schrödinger manifolds

Consider the triple $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu}, \widehat{\xi})$ where $\widehat{M}_\lambda \subset \mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$ defined by (5.4) is the Schrödinger-homogeneous space (5.6) endowed with the metric (5.17) and the vector field (5.18). Consider next the conformal Bargmann structure $(M, [\mathrm{g}], \xi)$ where M defined by (5.20) is the Schrödinger homogeneous space (5.21) endowed via (5.28) with the conformal class $[\mathrm{g}]$ of Bargmann metrics, and where ξ is the fundamental vector field (5.31) of the group generated by Z_0.

Theorem 5.14. *The triple $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu}, \widehat{\xi})$ is the (Poincaré-)Schrödinger manifold, in the sense of Definition 3.3, with conformal Bargmann boundary $(M, [\mathrm{g}], \xi)$ provided $\lambda = -\frac{1}{2}$, and $\mu = 1$.*

Proof. Our objective is thus to demonstrate that the preceding data fulfill all items of our Definition 3.3 of Schrödinger manifolds.

Let us first review some previous results expressed in local coordinate systems adapted to the Schrödinger symmetry pervading our construction. To this purpose, and in order to make contact with the FG construction, we find it convenient to work now on the open domain where $Q \in \widehat{M}_\lambda$ admits the local form (5.7). Owing to Equation (5.8), write

$$\left\{ \begin{array}{lcl} \widehat{x}^i & = & x^i \quad (i = 1, \ldots, d) \\ \widehat{t} & = & t \\ \widehat{s} & = & s + \lambda r^2 q. \end{array} \right. \tag{5.32}$$

Now, positing

$$\widehat{r} = r\sqrt{-2\lambda} \tag{5.33}$$

we find

$$Q = \frac{\sqrt{-2\lambda}}{\widehat{r}} \begin{pmatrix} \widehat{x} \\ -\frac{1}{2}\widehat{x}^* \widehat{x} - \frac{1}{2}\widehat{r}^2 \\ 1 \end{pmatrix} \in \widehat{M}_\lambda \tag{5.34}$$

in view of (5.7). This implies that

$$X = \frac{1}{r}\begin{pmatrix} x \\ -\frac{1}{2}x^*\,x \\ 1 \end{pmatrix} \in \widehat{M}_0 \tag{5.35}$$

in the limit $\widehat{r} \to 0$ corresponding to $\lambda \to 0$, hence that a representative $[X]$ of the ray $\mathbb{R}^*X \in \mathbb{P}\widehat{M}_0$ is given, with $x \in \mathbb{R}^{d+1,1}$, by

$$[X] = \begin{pmatrix} x \\ -\frac{1}{2}x^*\,x \\ 1 \end{pmatrix} \in M. \tag{5.36}$$

Collecting the expressions of Section 5.2, we assert that the metrics $\widehat{\mathrm{g}}_{\lambda,\mu}$ given by (5.15), (5.17), and the vector field $\widehat{\xi}$ as defined by (5.18), viz.,

$$\boxed{\widehat{\mathrm{g}}_{\lambda,\mu} = \frac{-2\lambda}{\widehat{r}^2}\left[\sum_{i,j=1}^{d+2} \mathrm{g}_{ij}\, d\widehat{x}^i \otimes d\widehat{x}^j + d\widehat{r}\otimes d\widehat{r} + 2\lambda\mu\,\frac{d\widehat{t}\otimes d\widehat{t}}{\widehat{r}^2}\right] \quad \& \quad \widehat{\xi} = \frac{\partial}{\partial \widehat{s}}} \tag{5.37}$$

constitute a family of Lorentz metrics (for $\lambda < 0$), while $\widehat{\xi}$ is a nowhere zero null Killing vector field.

The crux of the matter is that $\widehat{r} = r\sqrt{-2\lambda}$ defined by (5.33) is definitely (see (3.1) and (5.37)) our *defining function* for M, the conformal boundary of $\widehat{M}_\lambda$ (coordinatized as in (5.36)). Moreover Equations (5.32) and (5.33) entail that, locally, $\partial/\partial\widehat{s} = \partial/\partial s$, a relationship which is consistent with the limit $\widehat{r} \to 0$. The vector field $\widehat{\xi} = \partial/\partial\widehat{s}$ of $\widehat{M}_\lambda$ therefore goes smoothly over to $M = \partial\widehat{M}_\lambda$ as the vector field $\xi = \partial/\partial s$. This justifies, in local terms, item 1 of Definition 3.3, the latter being globally accounted for by Proposition 5.12.

Furthermore, straightforward computation using (5.37) shows that

$$\widehat{\mathrm{g}}^{\lambda,\mu} = \frac{-\widehat{r}^2}{2\lambda}\left[\sum_{i,j=1}^{d+2} \mathrm{g}^{ij}\,\frac{\partial}{\partial\widehat{x}^i}\otimes\frac{\partial}{\partial\widehat{x}^j} + \frac{\partial}{\partial\widehat{r}}\otimes\frac{\partial}{\partial\widehat{r}}\right] + \mu\,\frac{\partial}{\partial\widehat{s}}\otimes\frac{\partial}{\partial\widehat{s}}. \tag{5.38}$$

This readily yields $\widehat{\mathrm{g}}^{\lambda,\mu}|_{T^*M} = \mu\,\xi\otimes\xi$ in the limit $\widehat{r} \to 0$, insuring that item 2 of Definition 3.3 holds true prior to imposing the normalization condition $\mu = 1$.

Using then the form of the Poincaré metric g^+ of Definition 3.3, we easily deduce from (5.37) that $\mathrm{g}^+_\lambda = \widehat{\mathrm{g}}_\lambda$ as given by Equation (5.15). This proves that it is only the conformal class, $[\mathrm{g}]$, of the metric $\mathrm{g} = \lim_{\lambda\to 0}(r^2\,\mathrm{g}^+_\lambda)$ that goes over to the boundary $M = \mathbb{P}\widehat{M}_0$ of $\widehat{M}_\lambda$. To sum up, we find that the triple $(M, [\mathrm{g}], \xi)$ where

$$\boxed{\mathrm{g} = \sum_{i=1}^{d} dx^i \otimes dx^i + 2dt \odot ds \left(= \sum_{i,j=1}^{d+2} \mathrm{g}_{ij}\, dx^i \otimes dx^j \right) \quad \& \quad \xi = \frac{\partial}{\partial s}} \tag{5.39}$$

is a representative of our Schrödinger-homogeneous conformal Bargmann structure (see (2.1)) expressed in the adapted local coordinate system provided by (5.36). Direct computation moreover shows that

$$\mathrm{Ric}(\mathrm{g}^+_\lambda) + (d+2)\mathrm{g}^+_\lambda = \frac{(d+2)(1+2\lambda)}{2\lambda}\mathrm{g}^+_\lambda \tag{5.40}$$

which enables us to conclude that g^+_λ is indeed a Poincaré metric on $\widehat{M}_\lambda$, consistently with Definition 3.2, if the right-hand side of Equation (5.40) vanishes, i.e., if $\lambda = -\frac{1}{2}$. Item 3 of the definition 3.3 of Schrödinger manifolds is therefore fulfilled.

The proof of Theorem 5.14 is complete. □

We refer to Figure 1 for a graphical representation of our construction.

Remark 5.15. We duly recover the metric (1.1) of the "AdS/CFT" correspondence from the metric given by (5.37) by imposing the special values $\lambda = -\frac{1}{2}$, and $\mu = 1$ yielded by Theorem 5.14.

Remark 5.16. It can be checked that there holds, in full generality,

$$\mathrm{Ric}(\widehat{\mathrm{g}}_{\lambda,\mu}) - \frac{d+2}{2\lambda}\,\widehat{\mathrm{g}}_{\lambda,\mu} = -\mu\frac{d+4}{2\lambda}\,\widehat{\theta}\otimes\widehat{\theta}. \tag{5.41}$$

These equations are interpreted as Einstein's equations $\widehat{\mathrm{Ric}} - \frac{1}{2}\widehat{R}\,\widehat{\mathrm{g}} + \Lambda\,\widehat{\mathrm{g}} = \mathrm{T}$ with a cosmological constant $\Lambda = (d+1)(d+2)/(4\lambda)$, and sources given in terms of the "null fluid" stress-energy-momentum tensor $\mathrm{T} = -\mu(d+4)/(2\lambda)\,\widehat{\theta}\otimes\widehat{\theta}$. See also [14].

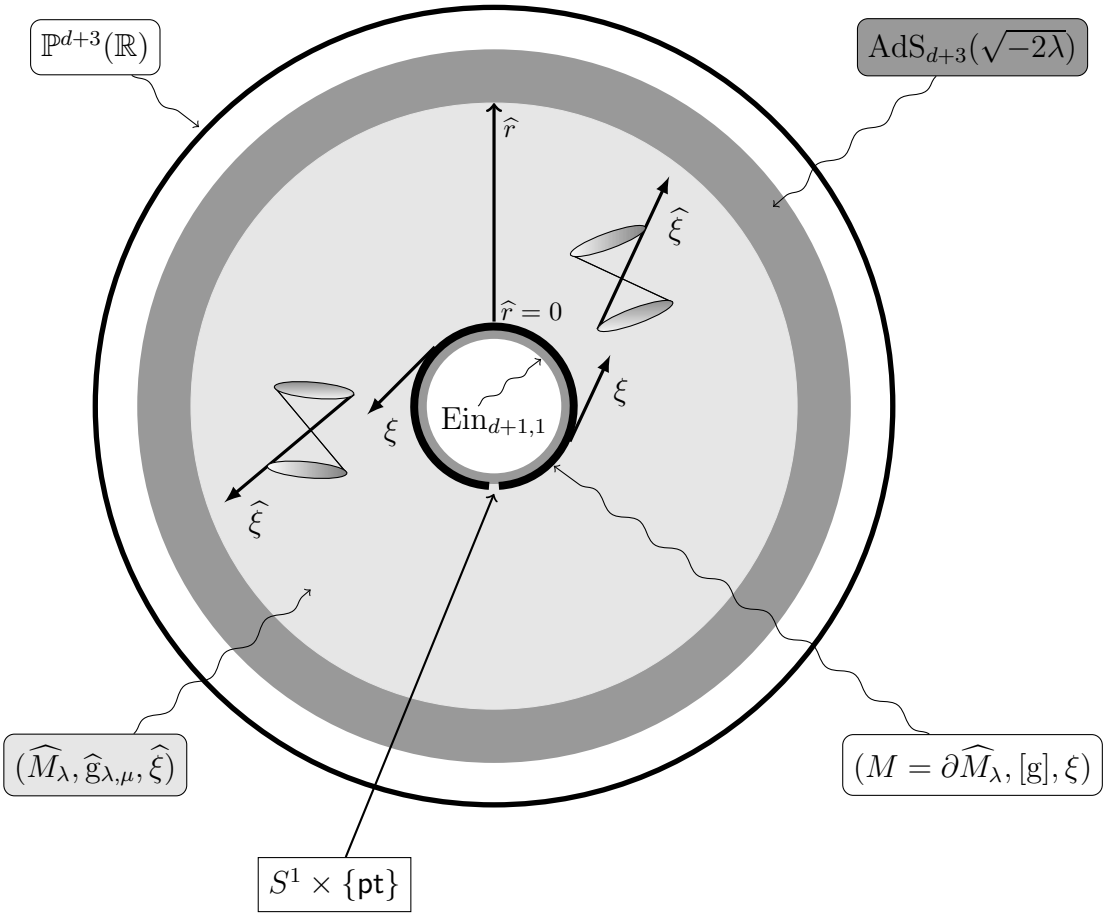

Figure 1: Schrödinger manifold & conformal Bargmann boundary

Remark 5.17. We learn from Equation (5.14) that the distribution $\ker(\widehat{\theta})$ is actually integrable. This is the very condition found in [32] to achieve a null-Killing dimensional reduction. Our Schrödinger-homogeneous manifolds $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu}, \widehat{\xi})$ thus provide examples of those manifolds considered by Julia and Nicolai [32].

6 Conclusion

This article has been triggered by the seemingly contradictory emergence of non-relativisitc Schrödinger "isometries" within the framework of an *a priori* relativistic AdS/CFT correspondence.

A closer look at the literature referred to in the introduction made it clear that the metric (1.1) appearing in the physics of non-relativistic holography should be related to the structure of what has been called a Bargmann extension of non-relativistic spacetime; see Section 2 which also offers a general definition of the Schrödinger group. This hint has been first investigated in [14]. Our task, here, was thus to put this observation on more global geometrical grounds.

From this vantage point, we have chosen to specialize the construction of a Poincaré metric, due to Fefferman and Graham, to the case where conformal infinity is moreover endowed with a conformal Bargmann structure governed by a null, parallel, vector field. This has led us to our definition 3 of Schrödinger manifolds.

Let us insist that the general proof of the existence and uniqueness (in suitable dimensions) of Schrödinger prolongations of Bargmann manifold structures has not been envisaged here, being clearly beyond the scope of this article. This will be deferred to subsequent work.

Nevertheless, the purpose of this article is to supply explicit examples of such Schrödinger manifolds that would help us understand the origin of the above-mentioned metric, with Schrödinger isometries, in a non-relativistic avatar of the AdS/CFT correspondence. Accordingly, we have found it useful to characterize, in the "flat" case, the Schrödinger group, $\mathrm{Sch}(d+1,1)$, as the stabilizer within $\mathrm{O}(d+2,2)$ of a distinguished nilpotent element, Z_0, of the Lie algebra, $\mathrm{o}(d+2,2)$. Our construction interestingly confers, as awaited and in a clear-cut fashion, a non-relativistic status to the Schrödinger group within a purely relativistic framework.

Our main result, namely Theorem 5.14, provides us with examples of Schrödinger manifolds, $(\widehat{M}_\lambda, \widehat{\mathrm{g}}_{\lambda,\mu}, \widehat{\xi})$; the canonical one is fixed by the normalization conditions $\lambda = -\frac{1}{2}$, and $\mu = 1$. Note that $\widehat{M}_\lambda$ is actually a homogeneous space of the Schrödinger group $\mathrm{Sch}(d{+}1,1)$, and, besides, an open submanifold of $\mathrm{AdS}_{d+3}(\sqrt{-2\lambda})$.

In a appropriate coordinate system on $\widehat{M}_{-\frac{1}{2}}$, the metric $\widehat{\mathrm{g}}_{-\frac{1}{2},1}$ turns out to match exactly the Balasubramanian-McGreevy and Son metric (1.1).

Let us stress that it finally appears that the Schrödinger group $\mathrm{Sch}(d+1,1)$ is, as expected, the maximal group of isometries of our Schrödinger manifolds. This definitely firms up the claims of [4, 44].

There remains, however, to understand, in completely general terms, the relationship between the Schrödinger group defined as the group of automorphisms of a conformal Bargmann structure and the group of automorphisms of an associated (Poincaré-)Schrödinger structure. This program for future work should indeed take advantage of a key result of Anderson [2] about the isometric extensions of the automorphisms of conformal infinity of a conformally compact Einstein manifold.

We finally expect that the definition of Schrödinger manifolds put forward in this article, and the explicit examples that have been worked out, will foster new research in the very attractive domain of non-relativistic AdS/CFT correspondence.

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