Virtual Poincaré polynomial of the link of a real algebraic variety

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VIRTUAL POINCARÉ POLYNOMIAL OF THE LINK OF A REAL ALGEBRAIC VARIETY

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ABSTRACT. The Euler characteristic of the link of a real algebraic variety is an interesting topological invariant in order to discuss local topological properties. We prove in the paper that an invariant stronger than the Euler Characteristic is well defined for the link of an algebraic variety: its virtual Poincaré polynomial.

INTRODUCTION

Let $X \subset \mathbf{R}^n$ be a real algebraic variety and a a point in X. Let $S(a, \epsilon)$ denote the sphere with centre a and radius $\epsilon > 0$ in $\mathbf{R}^n$. The local conic structure theorem implies that, for ϵ small enough, the topological type of $X \cap S(a, \epsilon)$ is independent of ϵ; this intersection is called the link of X at a, denoted by $lk(X, a)$. The link is known to be a very interesting topological invariant, as illustrated by Sullivan Theorem [16] that asserts that the Euler characteristic of the link at any point of a real algebraic subvariety of $\mathbf{R}^n$ is even. This evenness even leads to a topological characterisation of affine real algebraic varieties in dimension less than two. In higher dimension only necessary conditions are known (see [1, 11]).

Actually, the link is not only a topological invariant but also a semialgebraic topological invariant: for ϵ, ϵ' small enough one may find a semialgebraic homeomorphism between $X \cap S(a, \epsilon)$ and $X \cap S(a, \epsilon')$ (see [4] for a more precise statement). But can we find a stronger invariance than the semialgebraic topological one for the link of a real algebraic subvariety? The link is again a real algebraic variety, so one may wonder whether the link is an algebraic or Nash invariant (Nash means here analytic and semialgebraic).

In this paper, we focus on a finer invariant for real algebraic varieties than the Euler characteristic: the virtual Poincaré polynomial. The virtual Poincaré polynomial, denoted by β, has been introduced in [12] as an algebraic invariant for Zariski constructible real algebraic sets. Then the invariance of the virtual Poincaré polynomial has been established under Nash diffeomorphisms [6], and finally in [13] under regular homeomorphisms. We prove in Corollary 2.3 that the virtual Poincaré polynomial of the link of an affine real algebraic variety is well defined. More generally, we state in Theorem 2.1) that the virtual Poincaré polynomial of the fibres of a regular map between real algebraic varieties is constant along the connected components of an algebraic stratification of the target variety.

The proof of Theorem 2.1 is different from the proof of the fact that the Euler characteristic of the link is well defined. If one knows that the link is invariant under semialgebraic homeomorphisms, we are not able to prove that this is still

the case under the stronger class of regular homeomorphisms. To reach our aim, we pass through resolution of singularities [8] and a crucial property of the virtual Poincaré polynomial. Similarly to the Euler characteristic (with compact supports), the virtual Poincaré polynomial is an additive invariant: $\beta(X) = \beta(Y) + \beta(X \setminus Y)$ for $Y \subset X$ a closed subvariety of X. This enables to express the virtual Poincaré polynomial of the link at a point of the algebraic variety X as a sum of terms coming from a given resolution of the singularities of X. Then using Nash triviality results, in the smooth case [5] together with the normal crossing case [7] and applying the invariance of the virtual Poincaré polynomial under Nash diffeomorphisms to selected varieties coming from the resolution process enable to reach our goal.

The paper is organised as follows. In the first section we recall the basic properties of the virtual Poincaré polynomial as we need it. We adapt also the Nash triviality results from [5] and [7] to our particular setting. In the second section we state the invariance of the virtual Poincaré polynomial of the fibres of a regular map between real algebraic varieties, and deduce from it Corollary 2.3.

1. Preliminaries

1.1. **Virtual Poincaré polynomial.** For real algebraic varieties, the best additive and multiplicative invariant known is the virtual Poincaré polynomial [12]. It assigns to a Zariski constructible real algebraic set a polynomial with integer coefficients in such a way that the coefficients coincide with the Betti numbers with $\mathbb{Z}_2$-coefficients for proper non singular real algebraic varieties.

Proposition 1.1. ([12]) *Take $i \in \mathbb{N}$. The Betti number $\beta_i(\cdot) = \dim H_i(\cdot, \frac{\mathbb{Z}}{2\mathbb{Z}})$, considered on compact non singular real algebraic varieties, admits an unique extension as an additive map β_i to the category of Zariski constructible real algebraic sets, with values in $\mathbb{Z}$. Namely*
$$\beta_i(X) = \beta_i(Y) + \beta_i(X \setminus Y)$$
for $Y \subset X$ a closed subvariety of X.
  Moreover the polynomial $\beta(\cdot) = \sum_{i \geq 0} \beta_i(\cdot) u^i \in \mathbb{Z}[u]$ is multiplicative:
$$\beta(X \times Y) = \beta(X)\beta(Y)$$
for Zariski constructible real algebraic sets.

The invariant β_i is called the i-*th* virtual Betti number, and the polynomial β the virtual Poincaré polynomial. By evaluation of the virtual Poincaré polynomial at $u = -1$ one recovers the Euler characteristic with compact supports [12].

Proposition 1.2. ([6, 13]) *Let X and Y be Nash diffeomorphic compact real algebraic varieties. Then their virtual Poincaré polynomial coincide.*

Remark 1.3. In the non compact case the result is no longer true. For example an hyperbola in the plane in Nash diffeomorphic to the union of two lines, but the virtual Poincaré polynomial of the hyperbola is $u - 1$ whereas the virtual Poincaré polynomial of the two lines is $2u$.

A crucial property of the virtual Poincaré polynomial is that the degree of $\beta(X)$ is equal to the dimension of the Zariski constructible real algebraic set X. In particular, and contrary to the Euler characteristics with compact supports, the virtual Poincaré polynomial cannot be zero for a non-empty set.

1.2. **Nash triviality.** The Nash triviality Theorem of M. Coste and M. Shiota [5] gives the local Nash triviality for proper Nash maps with smooth fibres. We state it below in the form we need it later and show how to derive it from Theorem A in [5].

Theorem 1.4. *Let X and Y be a real algebraic sets and $p : X \to Y$ be a regular map. Assume X is non singular. There exists a real algebraic subset $Z \subset Y$ of strictly positive codimension in Y such that any two fibres of p over a connected component of $Y \setminus Z$ are Nash diffeomorphic.*

Proof. Note that it suffices to work in the category of Nash manifolds and Nash mappings. Actually, if we remove from Y its algebraic singular points $\operatorname{Sing} Y$, then $Y \setminus \operatorname{Sing} Y$ becomes a Nash manifold. Moreover if we obtain a semialgebraic subset Z' of Y such that $p^{-1}(y_1)$ and $p^{-1}(y_2)$ are Nash diffeomorphic for any points y_1 and y_2 in one connected component of $Y - Z'$ then the Zariski closure Z of Z' fulfills the requirements.

Now we treat the case $\dim X \geq \dim Y$ and $\dim p(X) = \dim Y$ since otherwise we can remove $p(X)$ from Y. We may assume that $p(X)$ is equal to Y by removing if necessary $\overline{p(X)} - \operatorname{Int} p(X)$ from Y, where the closure and interior are those in Y. We may assume moreover that p is a submersion because the critical value set is semialgebraic and of smaller dimension by Sard's Theorem. Finally, we may assume that X and Y are included in $\mathbf{R}^m \times \mathbf{R}^n$ and $\mathbf{R}^n$ respectively, and that p is the restriction to X of the projection $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ by replacing X with the graph of p. Then $p : X \to Y$ satisfies the conditions of Theorem A in [5]. □

Let M be a Nash manifold and $X \subset M$ be a Nash subset of M. Then X has only normal crossing if for each $a \in M$ there exists a local coordinate system $(x_1, \ldots, x_n)$ at $a \in M$ such that X is a union of some coordinate spaces in a neighbourhood of a in M.

The Nash isotopy Lemma (Theorem I in [7]) of T. Fukui, S. Koike and M. Shiota extends the Nash triviality obtained in Theorem 1.4 to the normal crossing situation. A subvariety N of a non singular real algebraic variety M has only normal crossings if at any point of N there exists a local system of coordinates such that N is a union of some coordinates spaces.

We state below the Nash isotopy Lemma as we need it later, and prove how to derive it from Theorem I in [7].

Theorem 1.5. *Let M be a non singular real algebraic variety and $N_1, \ldots, N_k$ be non singular real algebraic subvarieties of M such that $\cup_{i=1}^k N_i$ has only normal crossing. Let Y be a real algebraic variety and $p : M \to Y$ be a proper regular map. There exists a real algebraic subset $Z \subset Y$ of strictly positive codimension in Y such that for any connected component P of $Y \setminus Z$, there exist $y \in P$ and a Nash diffeomorphism*

$$\phi : (M; N_1, \ldots, N_k) \to \left(M \cap p^{-1}(y); N_1 \cap p^{-1}(y), \ldots, N_k \cap p^{-1}(y)\right) \times P$$

such that $p \circ \phi^{-1} : \left(M \cap p^{-1}(y)\right) \times P \to P$ is the canonical projection.

In particular, any two fibres of p along P are Nash diffeomorphic.

Proof of Theorem 1.5. Similarly to the proof of Theorem 1.4, there exists an algebraic subvariety $Z \subset Y$ such that $Y \setminus Z$ is smooth and p together with its restrictions to $N_{i_1} \cap \cdots \cap N_{i_s}$, for $0 \leq i_1 < \cdots < i_s \leq k$, are submersions onto $Y \setminus Z$. To apply

Theorem I in [7], it remains to prove that the connected components of $Y \setminus Z$ are Nash diffeomorphic to an open simplex.

Let X be a bounded semialgebraic set included in $\mathbf{R}^n$. Then the triangulation theorem of semialgebraic sets states that there exist a finite simplicial complex K and a semialgebraic homeomorphism from the underlying polyhedron $|K|$ to $\overline{X}$ such that $\pi^{-1}(X)$ is the union of some open simplexes in K (Theorem 9.2.1 in [3]). Moreover we can choose π so that the restriction to each open simplex is a Nash embedding. Hence if we remove the union of $\pi(\sigma)$ for simplexes σ in K of dimension smaller than $\dim X$, then each connected component of X become Nash diffeomorphic to an open simplex.

In case X is not bounded in $\mathbf{R}^n$, embed $\mathbf{R}^n$ in the sphere S^n and then embed S^n in $\mathbf{R}^{n+1}$ by Nash embeddings, so that X becomes bounded in R^{n+1} and we may apply the preceding argument. □

1.3. **Resolution of singularities.** The desingularisation Theorem of H. Hironaka [8] (or [2] for the form used in this paper) transforms a singular variety into a non singular one together with normal crossings with the exceptional divisors of the resolution. In view of the proof of Theorem 2.1 in section 2, it will enable us to use both Nash Triviality Theorem and Nash Isotopy Lemma after a resolution of the singularities of the source space in the singular case.

We recall that a smooth submanifold $N \subset M$ of a manifold M and a normal crossing divisor $D \subset M$ have simultaneously only normal crossings if, locally, there exist local coordinates such that N is the intersection of some coordinate hyperplanes and D is the union of some coordinate hyperplanes.

Theorem 1.6. *([2]) Let M be a non singular real algebraic variety and $X \subset M$ be a real algebraic subvariety of M. There exists a finite sequence of blowings-up*

$$M = M^s \xrightarrow{\pi^{s-1}} M^{s-1} \longrightarrow \cdots \xrightarrow{\pi^1} M^1 = M$$

with smooth centres $C^i \subset M^i$ such that, if $X^1 = X$ and $E^1 = \emptyset$, and if we denote by X^{j+1} the strict transform of X^j by π^j and by E^{j+1} the exceptional divisor $E^{j+1} = (\pi^j)^{-1}(C^j \cup E^j)$, then

(1) each C^j is included in X^j, and is of dimension smaller than $\dim X^j$,
(2) each C^j and E^j have simultaneously only normal crossings,
(3) X^s is non singular,
(4) X^s and E^s have simultaneously only normal crossings.

Remark 1.7. The fact that the centre C^j together with the divisor E^j have simultaneously only normal crossings at any stage of the resolution process will enable to use Theorem 1.5 in the proof of Theorem 2.1.

2. Virtual Poincaré polynomial of the link

Next result states the constancy of the virtual Poincaré polynomial of the fibres of a regular map along the connected components of an algebraic stratification of the target space.

Theorem 2.1. *Let $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ be a real algebraic sets and $f : X \to Y$ be a regular map. There exist a real algebraic subset $Z \subset Y$ of strictly positive*

codimension in Y such that for any a, b in the same connected component of $Y \setminus Z$, the virtual Poincaré of $f^{-1}(a)$ is equal to the virtual Poincaré of $f^{-1}(b)$.

Remark 2.2. In the statement of Theorem 2.1 we cannot avoid taking the connected components of $Z \setminus Y$, as illustrated by the square map from the line to itself. Indeed, the virtual Poincaré polynomial of the fibres is equal to 2 along strictly positive real numbers and 0 along strictly negatives ones.

Before giving the proof of Theorem 2.1, we apply it to define the virtual Poincaré polynomial of the link at a point of a real algebraic variety. Let $X \subset \mathbf{R}^n$ be a real algebraic variety and $a \in X$ be a point in X. Let $S(a, \epsilon)$ denote the sphere with centre a and radius $\epsilon > 0$ in $\mathbf{R}^n$. The set $lk(X, a) = X \cap S(a, \epsilon)$ is the link of X at a.

Corollary 2.3. *Let $X \subset \mathbf{R}^n$ be a real algebraic variety and $a \in X$ a point in X. There exists $\delta > 0$ such that for $\epsilon, \epsilon' > 0$, the virtual Poincaré polynomial of $X \cap S(a, \epsilon)$ is equal to the virtual Poincaré polynomial of $X \cap S(a, \epsilon')$, provided that $\epsilon < \delta$ and $\epsilon' < \delta$.*

Proof. We apply Theorem 2.1 to the map $d_a : X \to \mathbf{R}$ defined by taking the square of the Euclidean distance to a in $\mathbf{R}^n$, namely $d_a(x) = ||x - a||^2$. Then there exist a finite number of points $-\infty = b_0 < b_1 < \cdots < b_r < b_{r+1} = +\infty$ in $\mathbf{R}$ such that the virtual Poincaré polynomial of the fibres of d_a are constant along the intervals $]b_i, b_{i+1}[$ for $i = 0, \ldots, r$. In particular there exists δ such that $\epsilon > 0$ and $\epsilon' > 0$ belong to the same interval if $\epsilon < \delta$ and $\epsilon' < \delta$. $\square$

Remark 2.4. If X has at most an isolated singularity at a, then Theorem 1.4 asserts that the link is a Nash invariant, and therefore its virtual Poincaré polynomial is well defined by Proposition 1.2. If the singular component containing a is a curve, then the link of X at a has an isolated singularity and the link is a blow-Nash invariant by Theorem IIb in [10]. This implies again that its virtual Poincaré polynomial is well defined.

In the general case, the question of the existence of continuous moduli for blow-Nash equivalence remains open, and Corollary 2.3 says only that the virtual Poincaré polynomial of the link is well defined.

Proof of Theorem 2.1. We reduce the proof to the proper case using the graph of p. In the proper case, we express the virtual Poincaré polynomial of the fibres of p in terms of a resolution of the singularities of p. Then, we use Theorem 1.4 and Theorem 1.5 to obtain Nash triviality results along the resolution of p in the smooth and in the normal crossing case.

Reduction to the proper case

We first reduce the proof of theorem 2.1 to the case of a proper map f. Let $G \subset \mathbf{R}^n \times \mathbf{R}^m$ denote the graph of f and consider its Zariski closure $\overline{G}$ in $\mathbf{P}^n(\mathbf{R}) \times \mathbf{R}^m$. Then the projection $p : \overline{G} \to Y$ together with its restriction to $\overline{G} \setminus G$ are proper regular maps.

By additivity of the virtual Poincaré polynomial, it suffices now to prove theorem 2.1 for $p : \overline{G} \to Y$ and $p : \overline{G} \setminus G \to Y$.

The proper case

We assume that f is proper. Choose a resolution of the singularities of $X \subset \mathbf{R}^n$ as in Theorem 1.6. Denote by D^{j+1} the exceptional divisor $D^{j+1} = (\pi^j)^{-1}(C^j)$ of the j-th blowing-up of M^j along C^j, by $\tilde{E}^j$ the intersection $\tilde{E}^j = X^j \cap E^j$ of the exceptional divisor E^j with the strict transform X^j, and by $\tilde{D}^{j+1}$ the intersection $\tilde{D}^{j+1} = X^{j+1} \cap D^{j+1}$. Denote by σ^j the composition map

$$\sigma^j = f \circ \pi^1 \circ \cdots \circ \pi^j : M^{j+1} \to Y,$$

which is proper. Let X_y^{j+1} denote the fibre over $y \in Y$ of the restriction of σ^j to X^{j+1}, namely

$$X_y^{j+1} = (\sigma^j_{|X^{j+1}})^{-1}(y).$$

Denote similarly by C_y^{j+1}, $\tilde{D}_y^{j+1}$ and $\tilde{E}_y^{j+1}$ the fibres over $y \in Y$ of the restrictions of σ^j to C^{j+1}, $\tilde{D}^{j+1}$ and $\tilde{E}^{j+1}$ respectively.

The proof proceeds by induction on the number of blowings-up. We detail the first two blowings-up in order to illustrate the difference in the use of Theorems 1.4 and 1.5.

First blowing-up.
Consider the blowing-up of $M^1 = \mathbf{R}^n$ with centre C^1. By Theorem 1.4, there exists $Z^1 \subset Y$ of codimension at least one in Y such that the restriction $f_{|C^1}$ of f to C^1 is Nash trivial along any connected component of $Y \setminus Z^1$, and C_y^1 is smooth for any $y \in Y \setminus Z^1$, since C^1 is non singular (note that the fibres might be empty in case f is not a submersion). In particular for $a, b \in Y \setminus Z^1$ in the same connected component of $Y \setminus Z^1$, the virtual Poincaré polynomial of C_a^1 and C_b^1 coincide by Proposition 1.2 since C_a^1 and C_b^1 are Nash diffeomorphic compact real algebraic sets.

Even if E^2 is smooth in M^2, its intersection $\tilde{E}^2$ with X^2 is no longer smooth in general, and we can not apply Theorem 1.4 to find a Nash trivialisation of $\tilde{E}^2$ at this stage.

At the level of virtual Poincaré polynomials, by additivity one obtains

$$\beta(X_y^1) = \beta(C_y^1) + \beta(X_y^2) - \beta(\tilde{E}_y^2)$$

for $y \in Y$, where moreover $\beta(C_a^1) = \beta(C_b^1)$ for a, b in the same connected component of $Y \setminus Z^1$.

Second blowing-up.
Consider the blowing-up of M^2 with centre C^2. As C^2 is non singular, there exists $Z_1^2 \subset Y$ of codimension at least one in Y such that the restriction $\sigma^2_{|C^2}$ of σ^2 to C^2 is Nash trivial along $Y \setminus Z_1^2$ by Theorem 1.4. As a consequence $\beta(C_a^2) = \beta(C_b^2)$ for $a, b \in Y \setminus Z_1^2$ by Proposition 1.2.

This blowing-up gives rise the following commutative diagrams

$$
\begin{array}{ccc}
\tilde{D}_y^3 & \longrightarrow & X_y^3 \\
\pi_y^2 \downarrow & & \downarrow \pi_y^2 \\
C_y^2 & \longrightarrow & X_y^2
\end{array}
\qquad\qquad
\begin{array}{ccc}
D_y^3 \cap (\pi^2)^{-1}(\tilde{E}_y^2) & \longrightarrow & (\pi_y^2)^{-1}(\tilde{E}_y^2) \\
\pi_y^2 \downarrow & & \downarrow \pi_y^2 \\
C_y^2 \cap E_y^2 & \longrightarrow & \tilde{E}_y^2
\end{array}
$$

for $y \in Y$ (note that $C_y^2 \cap E_y^2 = C_y^2 \cap \tilde{E}_y^2$ since C^2 is included in X^2). Note that $C^2 \cap E^2$ is non singular since C^2 and E^2 have simultaneously normal crossings, so

we can apply Theorem 1.4 to obtain a Nash trivialisation of the intersection $C^2 \cap E^2$ outside a codimension at least one subset $Z_2^2 \subset Y$ of Y. In particular the virtual Poincaré polynomials of $C_a^2 \cap E_a^2$ and $C_b^2 \cap E_b^2$ coincide for a, b in the same connected component of $Y \setminus Z_2^2$ by Proposition 1.2.

Note moreover that $(\pi^2)^{-1}(\tilde{E}_y^2) \setminus \left(D^3 \cap (\pi^2)^{-1}(\tilde{E}_y^2)\right)$ is isomorphic to $\tilde{E}^3 \setminus \tilde{D}^3$ since π^2 is an isomorphism from $M^3 \setminus D^3$ to $M^2 \setminus C^2$. Therefore at the level of virtual Poincaré polynomials

$$\beta((\pi^2)^{-1}(\tilde{E}_y^2)) - \beta(D^3 \cap (\pi^2)^{-1}(\tilde{E}_y^2)) = \beta(\tilde{E}^3) - \beta(\tilde{D}^3).$$

In particular we obtain the following expression for the virtual Poincaré polynomial of X_y^1:

$$\beta(X_y^1) = \beta(C_y^1) + \beta(C_y^2) - \beta(C_y^2 \cap E_y^2) + \beta(X_y^3) - \beta(\tilde{E}_y^3),$$

with $\beta(C_a^1) = \beta(C_b^1)$, $\beta(C_a^2) = \beta(C_b^2)$ and $\beta(C_a^2 \cap E_a^2) = \beta(C_b^2 \cap E_b^2)$ for a, b in the same connected component of $Y \setminus Z^2$, where $Z^2 = Z^1 \cup Z_1^2 \cup Z_2^2$.

If $s \geq 4$, assume that after j blowings-up, with $j \in \{2, \ldots, s-1\}$, we are in such a situation that

$$\beta(X_y^1) = \sum_{i=1}^{j} \left(\beta(C_y^i) - \beta(C_y^i \cap E_y^i)\right) + \beta(X_y^{j+1}) - \beta(\tilde{E}_y^{j+1}),$$

with $\beta(C_a^i) = \beta(C_b^i)$ and $\beta(C_a^i \cap E_a^i) = \beta(C_b^i \cap E_b^i)$ for a, b in the same connected component of $Y \setminus Z^j$ and $i \in \{1, \ldots, j\}$, where $Z^j \subset Y$ is a subset of codimension at least one in Y.

We proceed similarly to the second blowing-up. As $C^{j+1} \subset M^{j+1}$ is non singular, there exists $Z_1^{j+1} \subset Y$ of codimension at least one in Y such that the restriction $\sigma_{|C^{j+1}}^{j+1}$ of σ^{j+1} to C^{j+1} is Nash trivial along $Y \setminus Z_1^{j+1}$ by Theorem 1.4. As a consequence $\beta(C_a^{j+1}) = \beta(C_b^{j+1})$ for $a, b \in Y \setminus Z_1^{j+1}$ by Proposition 1.2.

This blowing-up gives rise the following commutative diagrams

$$
\begin{array}{ccc}
\tilde{D}_y^{j+2} \longrightarrow X_y^{j+2} \\
\pi_y^{j+1} \downarrow \qquad \downarrow \pi_y^{j+1} \\
C_y^{j+1} \longrightarrow X_y^{j+1}
\end{array}
\qquad
\begin{array}{ccc}
D_y^{j+2} \cap (\pi^{j+1})^{-1}(\tilde{E}_y^{j+1}) \longrightarrow (\pi_y^{j+1})^{-1}(\tilde{E}_y^{j+1}) \\
\pi_y^{j+1} \downarrow \qquad\qquad \downarrow \pi_y^{j+1} \\
C_y^{j+1} \cap E_y^{j+1} \longrightarrow \tilde{E}_y^{j+1}
\end{array}
$$

for $y \in Y$ (note that $C_y^{j+1} \cap E_y^{j+1} = C_y^{j+1} \cap \tilde{E}_y^{j+1}$ since C^{j+1} is included in X^{j+1}). The difference with the case of the second blowing-up is now that $C^{j+1} \cap E^{j+1}$ is not necessarily non singular since E^{j+1} has normal crossings, so we can not apply directly Theorem 1.4. However, as C^{j+1} is smooth and has simultaneously only normal crossings with E^{j+1} by Theorem 1.6, we may apply Theorem 1.5 with M^{j+1}, C^{j+1} and E^{j+1} in order to obtain a Nash trivialisation of the intersection $C^{j+1} \cap E^{j+1}$ outside a codimension at least one subset $Z_2^{j+1} \subset Y$ of Y. In particular the virtual Poincaré polynomials of $C_a^{j+1} \cap E_a^{j+1}$ and $C_b^{j+1} \cap E_b^{j+1}$ coincide for a, b in the same connected component of $Y \setminus Z_2^{j+1}$ by Proposition 1.2.

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Finally, we obtain by additivity of the virtual Poincaré polynomial, in a similarly way than in the case of the second blowing-up, that

$$\beta(X_y^1) = \sum_{i=1}^{j+1} \left(\beta(C_y^i) - \beta(C_y^i \cap E_y^i) \right) + \beta(X_y^{j+2}) - \beta(\tilde{E}_y^{j+2}),$$

with $\beta(C_a^i) = \beta(C_b^i)$ and $\beta(C_a^i \cap E_a^i) = \beta(C_b^i \cap E_b^i)$ for a, b in the same connected component of $Y \setminus Z^{j+1}$ and $i \in \{1, \ldots, j+1\}$, where $Z^{j+1} = Z_1^{j+1} \cup Z_2^{j+1} \subset Y$ is a subset of codimension at least one in Y.

By induction, we obtain after the last blowing-up that the virtual Poincaré polynomial of X_y^1 is equal to:

$$\beta(X_y^1) = \sum_{i=1}^{s-1} \left(\beta(C_y^i) - \beta(C_y^i \cap E_y^i) \right) + \beta(X_y^s) - \beta(\tilde{E}_y^s),$$

with $\beta(C_a^i) = \beta(C_b^i)$ and $\beta(C_a^i \cap E_a^i) = \beta(C_b^i \cap E_b^i)$ for a, b in the same connected component of $Y \setminus Z^{s-1}$ and $i \in \{1, \ldots, s-1\}$, where $Z^{s-1} \subset Y$ is a subset of codimension at least one in Y.

The singularities of the variety X being resolved in X^s, we are in position to apply once more Theorem 1.4 to $\sigma_{|X^s}^s : X^s \to Y$ and Theorem 1.5 to M^s, $\tilde{E}^s$ and σ_s since $\tilde{E}^s$ has only normal crossings. We obtain that way a new subset $Z^s \subset Y$ of codimension at least one in Y such that $\beta(X_a^s) = \beta(X_b^s)$ and $\beta(\tilde{E}_a^s) = \beta(\tilde{E}_b^s)$ for a, b in the same connected component of $Y \setminus Z^s$.

Finally, for a, b in the same connected component of $Y \setminus Z$, where $Z = Z^{s-1} \cup Z^s$, we obtain that $\beta(X_a^1) = \beta(X_b^1)$ and the proof is achieved. $\qquad\square$

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