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A comparative study for the boundary control of a reaction-diffusion process: MPC vs backstepping.

Abdelhamid Ouali, Matthieu Fruchard, Estelle Courtial,
Youssef Touré.

Laboratoire PRISME, Université d'Orléans, France.
(e-mail: aouali@bourges.univ-orleans.fr,
{matthieu.fruchard,estelle.courtial,youssef.toure}@univ-orleans.fr)

Abstract: In this paper, we consider a reaction-diffusion process described by a linear parabolic partial derivative equation (PDE). Two radically different control approaches are compared: the model predictive control (MPC) and the backstepping approach. The stabilization of the unstable reaction-diffusion process is first studied. Then, to deal with parameter uncertainties, an adaptive backstepping controller is developed and compared to a model predictive controller based on an internal model control (IMC) structure. Simulation results illustrate the efficiency of the two approaches in terms of precision and computational time.

Keywords: Model Predictive Control, Backstepping control, Boundary control of PDEs.

1. INTRODUCTION

The reaction-diffusion process is involved in different application fields, like physics, chemistry and biology involving transport of materials and interactions between chemical compounds. This process can be modeled by a parabolic PDE. The control of PDE systems differs strongly depending on the location of sensors and actuators. If the latter are inside the process domain, a distributed control is required, whereas the boundary control, which is more difficult to synthesize though physically more realistic, is addressed if the actuators are located along the boundary of the process domain. The present paper addresses the latter issue. Among the numerous existing approaches of PDE boundary control (optimal control, flatness, etc), we have selected the MPC approach Camacho and Bordons [1998] and the backstepping approach Krstić and Smyshlyaev [2008b]. The reason for this choice is the radical difference between the two approaches. Backstepping is a theoretical strategy, yielding to an explicit stabilizing control law. On the contrary, MPC is an efficient practical strategy which suffers from the lack of theoretical stability result in certain cases due to the implicit numerical control law.

The MPC approach has been applied either to linear parabolic PDEs Džurina et al. [2006] or to nonlinear PDE Dufour et al. [2003], Santos et al. [2005]. The inherited control, which is the solution of an optimization problem, is generically implicit. From a practical point of view, one advantage of MPC is its ability to take constraints on input and states into account. However, the resolution of the optimization problem may be time consuming.

The backstepping approach was initially developed to provide a generic procedure for synthesizing Lyapunov stabilizing control laws for triangular nonlinear ordinary differential equation (ODE) systems Kanellakopoulos et al. [1992]. However, extensions to the boundary control of lin-

ear PDEs were recently reported in Krstić and Smyshlyaev [2008b]. Despite some applications to particular nonlinear systems Vasquez and Krstić [2008], boundary control of nonlinear PDEs largely remains an outstanding problem. The advantage of backstepping is to provide explicit Lyapunov stabilizing control laws. Besides the more recent backstepping syntheses do not require any model's discretization.

From a practical point of view, modeling errors or parameter uncertainties are inevitable. To deal with this issue, one usually relies either on a dedicated Lyapunov approach or on robustness. The latter solution is achieved using an IMC structure to robustify the MPC approach. The former issue is addressed using an adaptive backstepping. Backstepping control of PDE systems with non-constant parameters were treated in Smyshlyaev and Krstić [2005]. The adaptive backstepping control of parabolic PDE systems with unknown parameters was addressed using either Lyapunov design Krstić and Smyshlyaev [2008a], passive estimator Smyshlyaev and Krstić [2007a], or swapping identifiers Smyshlyaev and Krstić [2007b].

In this paper, after a brief recap of the reaction-diffusion process, the MPC concept and the backstepping principle, we compare the advantages and disadvantages of the two approaches for the boundary control of a reaction-diffusion process. The control objective is the stabilization of the process in the ideal case (reaction rate known) and in the usual case (reaction rate unknown). The robustness of the MPC-IMC approach is addressed and an adaptive backstepping is developed in the usual case. In that case, we also provide a more reactive update law than in Krstić and Smyshlyaev [2008a] in the case of Dirichlet boundary conditions. In the usual case, the MPC strategy proves to be performant in simulations despite a computational cost lower than the backstepping's ones.

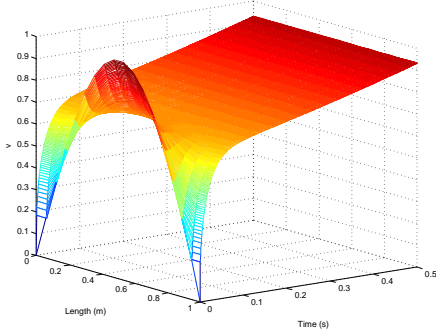


Fig. 1. Evolution of the open-loop process.

2. BACKGROUND

2.1 Reaction-diffusion process

We consider the reaction-diffusion process modeled by a linear parabolic PDE of the form:

$$(\mathcal{S}_1) \begin{cases} v_t(x, t) = v_{xx}(x, t) + \alpha v(x, t) \\ \mu_0 v_x(0, t) + (1 - \mu_0)v(0, t) = 0 \\ \mu_1 v_x(1, t) + (1 - \mu_1)v(1, t) = U(t) \\ v(x, 0) = v_0(x) \end{cases} \quad (1)$$

where the subscripts t and x denote the derivation of the state variable $v(x, t)$ with respect to time and space, respectively. The parameter α is a reaction rate, $\mu_0 v_x(0, t) + (1 - \mu_0)v(0, t) = 0$ is the free-end boundary condition and $U(t)$ is the boundary control with $(\mu_0, \mu_1) \in \{0, 1\}$. The initial condition of the system is denoted $v_0(x)$.

In open-loop, i.e. $U(t) \equiv 0$ at $x = 1$, integrating (1) using the separation of variables and the superposition principle gives, for Neumann boundary conditions ($\mu_0 = \mu_1 = 1$):

$$v(x, t) = \sum_{n=0}^{\infty} c_n e^{\lambda_n t} \cos(n\pi x) \int_0^1 \cos(n\pi x) v_0(x) dx \quad (2)$$

with the coefficients $c_0 = 1$ and $c_n = 2, \forall n \geq 1$ and the eigenvalues $\lambda_n = \alpha - n^2\pi^2$. Hence it is obvious that such a system is unstable as soon as the reaction rate $\alpha > 0^1$. As can be seen in Fig. 1, system (\mathcal{S}_1) is unstable for $U(t) \equiv 0$, $v_0(x) = \sin(\pi x)$ and $\alpha = 0.66$. The state $v(x, t)$ is dominated by the unstable eigenfunction $\phi_0(x) = 1$ and consequently diverges.

2.2 Model Predictive Control

Model Predictive Control has been extensively studied for the control of constrained linear or nonlinear processes described by ordinary differential equations. The MPC strategy is based on the receding horizon principle and is formulated as solving on-line a nonlinear optimization problem; see Camacho and Bordons [2007] for a survey. The basic concepts of MPC are the explicit use of a model to predict the process behavior over a finite prediction horizon N_p and the minimization of a cost function with respect to a sequence of N_c controls where N_c is the control horizon. The control objective is usually a trajectory or setpoint tracking. Considering $x(t)$, the state vector of the model at time t , the cost function is defined by:

¹ In the case of Dirichlet boundary conditions, eigenfunctions are $\phi_n(x) = \sin n\pi x$, so that the system is unstable for $\alpha > \pi^2$.

$$J(x, u) = F(x(t + N_p)) + \int_t^{t+N_p} L(x(\tau), u(\tau), y_{ref}(\tau)) d\tau \quad (3)$$

where L is a quadratic function and $F(x(t + N_p))$ is a terminal constraint added to ensure the stability of the closed-loop system. The classical MPC can be formulated as follows:

$$\min_u J(x, u). \quad (4)$$

Only the first element of the computed optimal sequence of controls \tilde{u} is really applied to the process. At the next sampling instant, the prediction horizon moves one step forward and the whole procedure is repeated with the updated measurements.

The main advantage of MPC is its ability to handle constraints. Constraints on states, inputs or outputs can explicitly be added to the optimization problem (4).

2.3 Backstepping control design

Backstepping control was originally developed for nonlinear EDO systems Kanellakopoulos et al. [1992]. Backstepping controller design is based on a triangular transformation of the source system into a target system in the lower triangular form. It provides an iterative choice of control Lyapunov functions and finally leads to a control law that stabilizes the state variables step by step.

In the case of boundary controlled PDEs, the principle of triangular transformation is preserved, but the objective is now to use this transformation to map the source unstable system (\mathcal{S}_1) into a stable target PDE system in closed-loop Liu [2003], Krstić and Smyshlyaev [2008b]. This method has two main advantages: the control is synthesized directly using the PDE system, i.e., with no discretization, and the resulting control law is explicit.

The purpose of the backstepping approach is to control the trajectories of system (\mathcal{S}_1) along a stable target system, by eliminating the source of instability given by the reaction term $\alpha v(x, t)$. For $\mu_0 = \mu_1 = 1$, we can consider, for instance, the following target system:

$$\begin{cases} w_t(x, t) = w_{xx}(x, t) - gw(x, t) \\ w_x(0, t) = 0 \\ w_x(1, t) = -\frac{\kappa}{2}w(1, t) \end{cases} \quad (5)$$

where parameters $\kappa > 1$ and g are used as gains to tune the rate of convergence of the Lyapunov function to zero.

A possible Lyapunov candidate function is:

$$V(t) = \frac{1}{2} \|w(x, t)\|^2 = \frac{1}{2} \int_0^1 w^2(x, t) dx. \quad (6)$$

Differentiating equation (6) with respect to time, using the chain rule and integrating by parts, gives:

$$\dot{V}(t) = w(1, t)w_x(1, t) - w(0, t)w_x(0, t) - \int_0^1 w_x^2(x, t) dx - g \int_0^1 w^2(x, t) dx. \quad (7)$$

Applying the Poincare inequality

$$\int_0^1 w^2(x, t) dx \leq 2w^2(1, t) + 4 \int_0^1 w_x^2(x, t) dx \quad (8)$$

and the boundary conditions of (5), we obtain:

$$\dot{V}(t) \leq -\left(\frac{1}{4\kappa} + g\right) \int_0^1 w^2(x, t) dx = -2\left(g + \frac{1}{4\kappa}\right)V(t). \quad (9)$$

Hence the system (5) is stable for $g > -\frac{1}{4\kappa}$.

The difficulty of the backstepping approach is to find the transformation that maps system (\mathcal{S}_1) onto system (5). We set the following integral Volterra transformation:

$$w(x, t) = v(x, t) - \int_0^x K(x, y)v(y, t)dy \quad (10)$$

where K is the function that characterizes the transformation, called kernel of the Volterra transformation.

It is important to note that this transformation is invertible, which guarantees that the stability of the target system (5) will induce the stability of system (\mathcal{S}_1) along the closed-loop trajectory of the controlled system. The integral in (10) is within the interval $[0, x]$, and induces a spatial causality that can be assimilated to the well-known triangular transformation of standard backstepping approaches developed for ODE systems.

3. CONTROL DESIGN WITH MPC APPROACH

As for all predictive strategies, a reference trajectory, a model of the dynamic process, a cost function and an optimization solver are necessary. In the sequel, the choice of these four points is discussed with regard to the two distinct control objectives: stabilization of the unstable PDE system (\mathcal{S}_1) with the parameter α known and robust control in the case of uncertainty on the parameter α .

3.1 Stabilization of the system (\mathcal{S}_1)

The reference. The control task is to regulate to zero the unstable part of the process described by equation (1). The reference y_{ref} is constant and equal to zero.

The prediction model. In order to simplify the prediction of the model, the original model is decomposed into a finite-dimensional system describing the slow dynamics and an infinite-dimensional system modeling the fast dynamics Dujic and Christofides [2006]. For this purpose, the modal decomposition technique is used. For $\mu_0 = \mu_1 = 1$ (Neumann conditions), we define the state function $\bar{v}(t)$ on the state space \mathcal{H} as

$$\bar{v}(t) = v(x, t), \quad t > 0, \quad 0 < x < 1, \quad (11)$$

and the differential operator \mathcal{F} as

$$\mathcal{F}\phi = \frac{d^2\phi}{dx^2} + \alpha\phi, \quad 0 < x < 1, \quad (12)$$

where $\phi(x)$ is a smooth function on $[0, 1]$. The boundary operator $\mathcal{B} : \mathcal{H} \mapsto \mathbb{R}$ is defined by:

$$\mathcal{B}\phi(x) = \frac{d\phi(1)}{dx} + \phi(1). \quad (13)$$

Considering (11), (12) and (13), the original system (1) can be rewritten as follows:

$$\begin{cases} \dot{\bar{v}}(t) = \mathcal{F}\bar{v}(t), & \bar{v}(0) = \bar{v}_0 \\ \mathcal{B}\bar{v}(t) = u(t). \end{cases} \quad (14)$$

The above equation has inhomogeneous boundary conditions owing to the presence of $u(t)$ in the boundary conditions. To transform this boundary control problem into an equivalent distributed control problem Fattorini [1968], Curtain [1985], we assume that a function $B(x)$ exists such that for all the $u(t), Bu(t) \in \mathcal{D}(\mathcal{F})$ satisfies:

$$\mathcal{B}Bu(t) = u(t). \quad (15)$$

The change of variable $p(t) = \bar{v}(t) - Bu(t)$ leads to the following system:

$$\begin{cases} \dot{p}(t) = \mathcal{A}p(t) + \mathcal{F}Bu(t) - B\dot{u}(t) \\ p(0) = p_0 \end{cases} \quad (16)$$

where the operator \mathcal{A} is such that $\mathcal{A}\phi(x) = \mathcal{F}\phi(x)$. The state $p(t)$ can be split into slow and fast states respectively noted $p_s(t)$ and $p_f(t)$: $p(t) = p_s(t) + p_f(t)$. The system (16) can be written as:

$$\begin{cases} \dot{p}_s(t) = \mathcal{A}_s p_s(t) + (\mathcal{F}B)_s u(t) - B_s \dot{u}(t) \\ \dot{p}_f(t) = \mathcal{A}_f p_f(t) + (\mathcal{F}B)_f u(t) - B_f \dot{u}(t) \end{cases} \quad (17)$$

where \mathcal{A}_s is a diagonal matrix of finite dimension and \mathcal{A}_f is an infinite dimensional operator. The latter represents the fast and stable dynamics, whereas \mathcal{A}_s represents the slow dynamics, which may be unstable ($\mathcal{A}_s = \text{diag}\{\lambda_k\}$, $k = 1, \dots, m$ with λ_k , the eigenvalues).

The system is still not under a suitable form to be used in an MPC strategy because of the derivative of the control. Therefore, a new variable $\bar{u}(t)$ is introduced and the system becomes:

$$\begin{pmatrix} \dot{u} \\ \dot{p}_s \\ \dot{p}_f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ (\mathcal{F}B)_s & \mathcal{A}_s & 0 \\ (\mathcal{F}B)_f & 0 & \mathcal{A}_f \end{pmatrix} \begin{pmatrix} u \\ p_s \\ p_f \end{pmatrix} + \begin{pmatrix} 1 \\ -\hat{B}_s \\ -\hat{B}_f \end{pmatrix} \bar{u} \quad (18)$$

Due to the stabilization objective, we can neglect the fast (stable) dynamics and finally the state representation of the reaction-diffusion process has the form:

$$\begin{pmatrix} \dot{u}(t) \\ \dot{p}_s(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (\mathcal{F}B)_s & \mathcal{A}_s \end{pmatrix} \begin{pmatrix} u(t) \\ p_s(t) \end{pmatrix} + \begin{pmatrix} 1 \\ -\hat{B}_s \end{pmatrix} \bar{u}(t). \quad (19)$$

The model state is noted $X_m(t) = (u(t), p_s(t))^T$.

The cost function. The control objective is to steer the state to the origin. The quadratic function L in (3) is then defined as:

$$L = (X_m)^T Q (X_m) \quad (20)$$

where Q is a symmetric positive definite matrix.

The terminal constraint is given by ($\bar{Q} > 0$):

$$F(X_m(t + N_p)) = (X_m(t + N_p))^T \bar{Q} (X_m(t + N_p)). \quad (21)$$

The solving method. In order to implement the MPC strategy, a discrete-time formulation is generally used. The optimization problem (4) becomes:

$$\min_{\bar{u}} \sum_{j=k+1}^{k+N_p} [X_m(j)]^T Q [X_m(j)] + [X_m(k + N_p)]^T \bar{Q} [X_m(k + N_p)] \quad (22)$$

subject to the model equation given by (19) in its discrete-time formulation, where k is the current time. Numerous nonlinear optimization routines are available in software libraries to solve this kind of problem.

3.2 Robust control of (\mathcal{S}_1)

We now consider the system (\mathcal{S}_1) with $\mu_0 = \mu_1 = 0$ (Dirichlet conditions); the reaction rate is an unknown parameter denoted $\hat{\alpha}$, an estimate of α . Hence, we define the model used for prediction by:

$$(\mathcal{S}_2) \begin{cases} v_t(x, t) = v_{xx}(x, t) + \hat{\alpha}v(x, t) \\ v(0, t) = 0 \\ v(1, t) = U(t) \\ v(x, 0) = v_0(x). \end{cases} \quad (23)$$

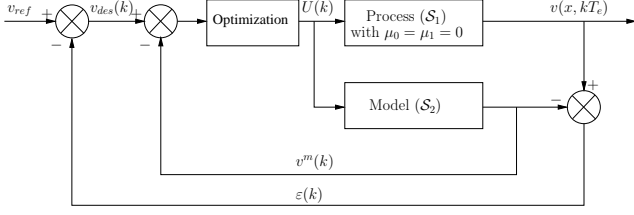


Fig. 2. MPC with IMC control structure.

To deal with this uncertainty, the well-known IMC structure is chosen Morari and Zafriou [1989].

Control Structure. (see Fig. 2)

The process block contains the reaction-diffusion system described by equation (1) with $\mu_0 = \mu_1 = 0$. The control input is the boundary control $U(t)$. The process output is the measured state $v(x, t)$. Due to the control structure, to track the reference state v_{ref} by the process output is equivalent to tracking the desired state v_{des} by the model output v^m . The spatial signal error ε represents all modeling errors and disturbances between the process and the model outputs. It is considered constant over the prediction horizon N_p but updated at each sampling instant k . The reference state is still zero.

The prediction model. The system (\mathcal{S}_2) is spatially discretized by a finite difference method yielding the state space representation in discrete-time:

$$v^m(k+1) = A_m v^m(k) + B_m U(k) \quad (24)$$

with A_m and B_m matrices of adequate dimensions.

The cost function. In order to compare the two approaches (MPC, backstepping), constraints on the input control should be considered. The robust stabilization of (\mathcal{S}_2) can be formulated into the optimization problem:

$$\min_{\tilde{U}} \sum_{j=k+1}^{k+N_p} [-\varepsilon(j) - v^m(j)]^T Q [-\varepsilon(j) - v^m(j)] \quad (25)$$

s.t.

$$\begin{cases} v^m(k+1) = A_m v^m(k) + B_m U(k) \\ \tilde{U} = [U(k), U(k+1), \dots, U(k+N_p-1)] \\ U_{min} < U(k) < U_{max}. \end{cases}$$

The simulations presented in section 5 illustrate the efficiency of the MPC strategy in the usual case.

4. CONTROL DESIGN WITH BACKSTEPPING METHOD

As stated in section 2.3, boundary control using the backstepping approach aims at mapping the original unstable system into a Lyapunov stable target PDE. When the reaction parameter is known, the sole issue is to find the kernel analytical expression. In the case of parameter uncertainties, an adaptive backstepping control law needs to be synthesized. In the case of Lyapunov design of the adaptive control law, the target PDE has to be stabilized using an appropriate update law. Despite a more complex analysis, this approach is less time consuming than passive or swapping identifiers approaches, for there is here no need for an observer.

4.1 Stabilization of the system (\mathcal{S}_1)

Since boundary backstepping control depends on the kernel expression, e.g., for Neumann boundary conditions ($\mu_0 = \mu_1 = 1$), $v_x(1, t) = U(t)$ with

$$U(t) = \int_0^1 K_x(1, y) v(y, t) dy + K(1, 1) v(1, t), \quad (26)$$

the main difficulty in the backstepping controller synthesis consists in finding the analytical expression of the kernel K of transformation (10). To do so, we differentiate (10) along the PDE of the source system (\mathcal{S}_1) and use the PDE and boundary conditions of the target system (5). We thus obtain the following kernel PDEs:

$$\begin{cases} K_{xx}(x, y) - K_{yy}(x, y) = (\alpha + g)K(x, y) & (27a) \\ K_y(x, 0) = 0 & (27b) \\ K(x, x) = -\frac{\alpha+g}{2}x. & (27c) \end{cases}$$

Setting $\xi = x + y$, $\eta = x - y$ and $g(\xi, \eta) = K(x, y)$, the PDE system (27) can be rewritten as an integral equation. Integrating (27a) *i*) with respect to η on $[0, \xi]$ then with respect to ξ on $[0, \xi]$, and *ii*) with respect to η on $[0, \eta]$ then with respect to ξ on $[0, \eta]$, and using conditions (27b)–(27c) to simplify expressions, we have the following integral equation:

$$g(\xi, \eta) = \frac{(\alpha+g)}{4} \left(-(\xi + \eta) + \int_0^\xi \int_0^\eta g(\varepsilon, \kappa) d\varepsilon d\kappa + \int_0^\eta \int_0^\varepsilon g(\varepsilon, \kappa) d\varepsilon d\kappa \right). \quad (28)$$

We use successive approximations to solve (28). We *a priori* set $g_0(\xi, \eta) = \frac{(\alpha+g)}{4}(\xi + \eta)$ and define $g_{n+1}(\xi, \eta)$ as the solution of (28) evaluated at $g = g_n$. If the sequence converges, then the same goes for the associated series $G_n = g_{n+1} - g_n$. Inductively we get:

$$G_n(\xi, \eta) = -\left(\frac{\alpha+g}{4}\right)^{n+1} \frac{(\xi + \eta)\xi^n \eta^n}{n!(n+1)!}, \quad (29)$$

so passing to the limit gives:

$$\begin{aligned} G(\xi, \eta) &= -(\xi + \eta) \sum_{n=0}^{\infty} \left(\frac{\alpha+g}{4}\right)^{n+1} \frac{\xi^n \eta^n}{n!(n+1)!} \\ &= -\frac{(\alpha+g)(\xi+\eta)}{2} \frac{I_1(\sqrt{(\alpha+g)\xi\eta})}{\sqrt{(\alpha+g)\xi\eta}} \end{aligned} \quad (30)$$

with I_n denoting the n th-order modified Bessel function $I_n(x) = \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^{n+2m}}{m!(m+n)!}$ that satisfies $\frac{d}{dx}(x^{-n} I_n(x)) = x^{-n} I_{n+1}(x)$. We hence deduce from (30) the kernel of (10):

$$K(x, y) = -(\alpha + g)x \frac{I_1(\sqrt{(\alpha + g)(x^2 - y^2)})}{\sqrt{(\alpha + g)(x^2 - y^2)}}. \quad (31)$$

Set $z(y) = (\alpha + g)(1 - y^2)$. The Lyapunov stabilizing boundary control is thus given by (26) using (31) and its derivative with respect to x :

$$\begin{aligned} U(t) &= -(\alpha + g) \int_0^1 \left(\frac{I_1(\sqrt{z(y)})}{\sqrt{z(y)}} + \frac{I_2(\sqrt{z(y)})}{(1-y^2)} \right) v(y, t) dy \\ &\quad - (\alpha + g) \frac{v(1, t)}{2}. \end{aligned} \quad (32)$$

For Dirichlet boundary conditions ($\mu_0 = \mu_1 = 0$), a similar approach leads to the following boundary controller:

$$U(t) = v(1, t) = \int_0^1 K(x, y) v(y, t) dy, \quad (33)$$

with the kernel

$$K(x, y) = -(\alpha + g)y \frac{I_1(\sqrt{(\alpha + g)(x^2 - y^2)})}{\sqrt{(\alpha + g)(x^2 - y^2)}}. \quad (34)$$

4.2 Adaptive backstepping control of the system (\mathcal{S}_1)

We now consider the process (\mathcal{S}_1) with $\mu_0 = \mu_1 = 0$ (Dirichlet conditions) and the model used for synthesizing the controller is given by system (\mathcal{S}_2).

Replacing the unknown parameter α by the estimated one $\hat{\alpha}(t)$ in the kernel (34), we have:

$$K(x, y, \hat{\alpha}) = -(\hat{\alpha} + g)y \frac{I_1(\sqrt{(\hat{\alpha} + g)(x^2 - y^2)})}{\sqrt{(\hat{\alpha} + g)(x^2 - y^2)}}. \quad (35)$$

Then, the integral Volterra transformation (10) maps system (\mathcal{S}_1) into²:

$$(\mathcal{S}_3) \begin{cases} w_t = w_{xx} + (\tilde{\alpha} - g)w + \dot{\hat{\alpha}} \int_0^x \frac{y}{2} w(y, t) dy \\ w(0) = 0 \\ w(1) = 0 \end{cases} \quad (36)$$

where $\tilde{\alpha} = \alpha - \hat{\alpha}$ is the parameter estimation error, updated by the parameter update law $\dot{\hat{\alpha}} = u_\alpha(t)$.

Proposition 1. The estimated parameter update law and boundary controller:

$$\begin{cases} u_\alpha(t) = \gamma \frac{\|w\|^2}{1 + \|w\|^2}, \quad \gamma \in (0; 4\sqrt{3}(g + \pi^2)) \end{cases} \quad (37a)$$

$$\begin{cases} v(1, t) = \int_0^1 K(x, y, \hat{\alpha}(t))v(y, t) dy \end{cases} \quad (37b)$$

with kernel (35) achieve regulation of $v(x, t)$ to zero for all $x \in [0, 1]$, for arbitrarily initial condition $v(x, 0)$ and estimate $\hat{\alpha}(0)$.

The proof, detailed in Appendix, is based on the one developed in Krstić and Smyshlyaev [2008a], but we here exploit the Dirichlet boundary conditions and gain g to obtain a wider range for the choice of the update gain γ .

5. SIMULATION RESULTS

The simulations were performed using the centered finite difference method. The optimization problem of MPC was solved by using the Matlab subroutine Quadprog.

Simulation 1: Stabilization of (\mathcal{S}_1) in the ideal case. Conditions: $v_0(x) = \sin(\pi x)$; $\alpha = 0.66$; $T_e = 7 \cdot 10^{-4}$ (s); $N_p = 10$; $N_c = 1$; $Q = \text{diag}(0.01; 0.07; 0.07)$; $g = 0.5$.

The MPC and backstepping strategies stabilize the unstable part of the reaction-diffusion process (see Fig. 3(a) and 3(b)). For both, the control input applied at the boundary $x = 1$ reaches its maximum absolute value at the beginning so as to compensate for non-null initial conditions and the instability caused by the reaction term (see Fig. 3(c)).

Concerning the computational load, the backstepping method requires less computing time (2ms) than the MPC method (4.6ms) which is due to both modal decomposition and the optimization processing. Moreover, the maximum absolute value of control is less aggressive in the case

² For the sake of readability, we have dropped the (x, t) dependency if there is no possible confusion.

of backstepping control ($U_{max} = -0.87$), which means that the latter is more efficient than the MPC method ($U_{max} = -1$) in this case.

Simulation 2: Stabilization of (\mathcal{S}_1) in the usual case. Conditions: $v_0(x) = 10 \sin(\pi x)$; $\alpha = 15$; $T_e = 12 \cdot 10^{-4}$ (s); $N_p = 40$; $N_c = 1$; $Q = \text{diag}(1; 0.2; 0.2)$; $\gamma = 60$.

We add to the optimization problem (25) the control constraint, $U \in [-8, 8]$ which is equivalent to a time scale change for adaptive backstepping control.

Contrary to Simulation 1, the control law profiles significantly differ due to the basic difference between the two approaches. Indeed, MPC stability is assumed by the robustness of the strategy to modeling errors whilst, in the adaptive backstepping case, the stability is ensured thanks to the model update through the update law. In both cases, the closed-loop process is stabilized (see Fig. 4(a) and 4(d)). The prediction horizon is chosen in order to satisfy a compromise between the stability of the closed loop and the computational time requirement. It should be pointed out that the higher the reaction rate—and in turn the instability of the process—the higher the prediction horizon that should be chosen so as to preserve the stability of the MPC-IMC design ($N_p = 40$, $N_c = 1$).

The error signal ε (Fig. 4(c)) shows a transient behavior relative to both the initial condition error (between the model and the process) and to the model error caused by the unknown reaction parameter $\hat{\alpha}$.

Fig. 4(f) shows that the parameter estimation is improved with respect to the initial value $\hat{\alpha}(0) = 1$ when the norm is non null. The nominal value is not exactly reached because the Lyapunov derivative \dot{V} in (A.6) is only negative semi-definite. The closed-loop state is yet stable in accordance with the result presented in section 4.2.

The computational time by step required for robust MPC (4.7ms) is less than for the adaptive backstepping method (5.5ms). This difference of calculation burden is explained firstly by the double spatial integral to calculate the norm needed to compute the update model in the adaptive backstepping approach, and secondly by the subsampling of the MPC prediction model.

6. CONCLUSION

Two boundary control approaches to stabilize the reaction-diffusion process have been compared for two cases. The first one (ideal case) with a known reaction rate, the second one (usual case) with an unknown reaction rate.

Stabilization, theoretically proven with the backstepping approach, was achieved in both cases. In the ideal case, the backstepping technique requires less computing time than the MPC. However, the double space integrals required to determine the norm $\|w\|$ at each step are time consuming in the usual case. The adaptive backstepping method entails a heavier computational burden than MPC strategy combined with the IMC structure. As expected, the MPC approach is robust to modeling errors. Different boundary conditions have been considered to highlight the easiest way to adapt the control strategy. The MPC approach remains a very robust and flexible control approach despite its non-explicit control.

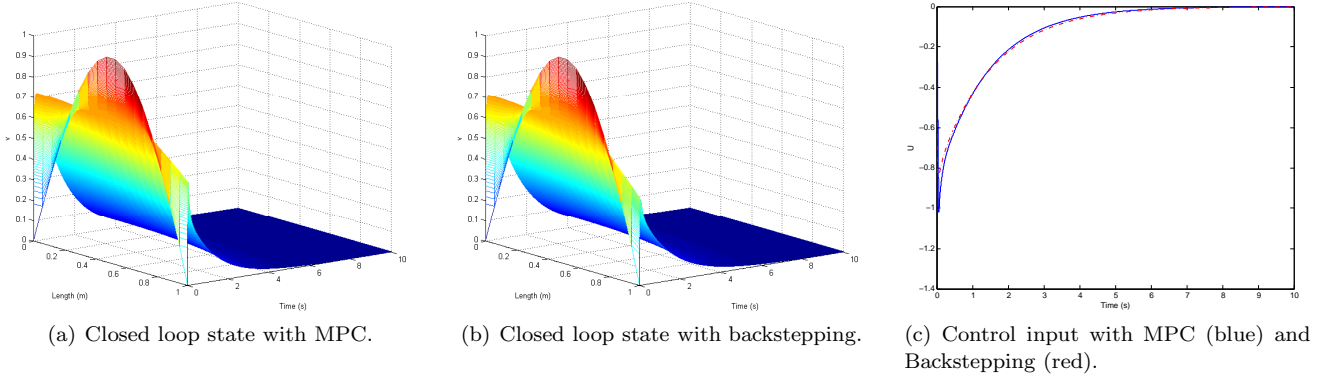


Fig. 3. Simulation 1: control of process (\mathcal{S}_1) with Neumann-Neumann boundary conditions and a known reaction parameter. MPC approach (a), backstepping control (b) and the control law $U(t) = v_x(1, t)$ (c).

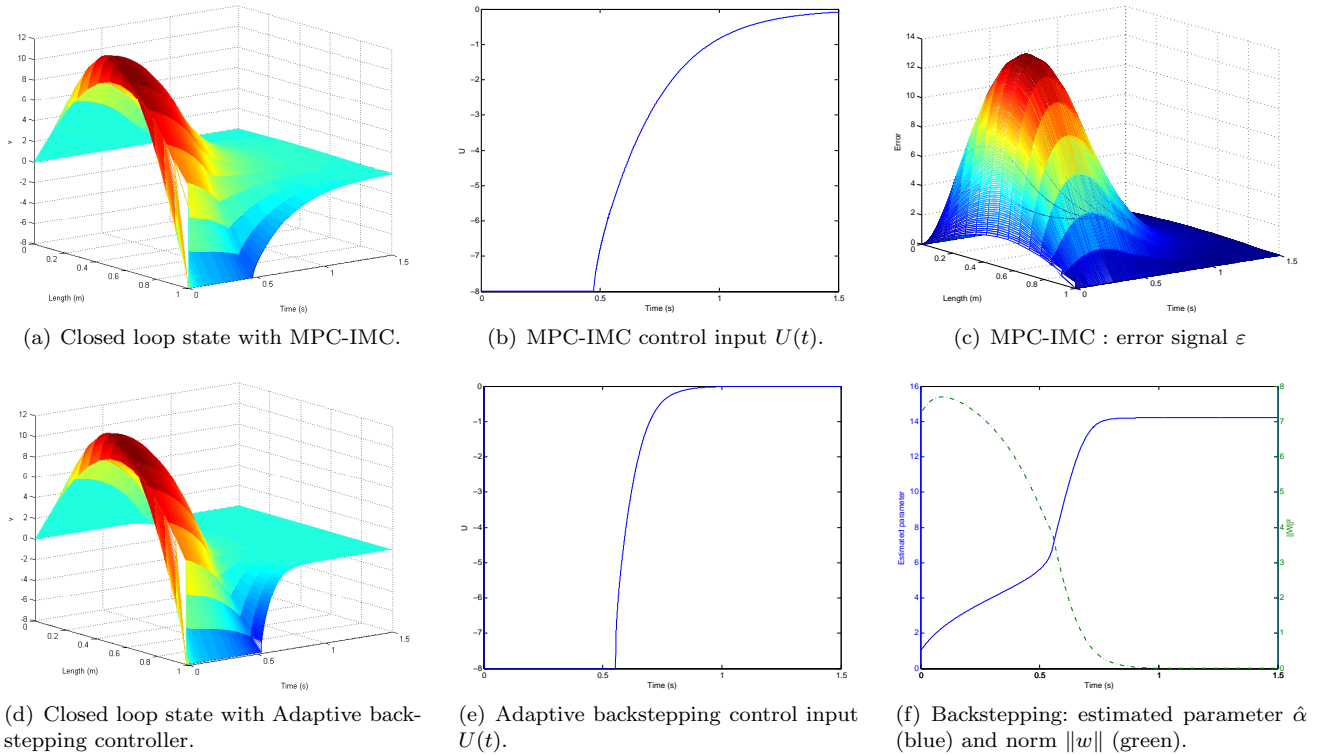


Fig. 4. Simulation 2: control of system (\mathcal{S}_1) with Dirichlet-Dirichlet boundary conditions and an unknown reaction parameter. MPC approach (a)-(c), backstepping control (d)-(f).

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Appendix A. PROOF OF PROPOSITION 1

Proof. Consider the candidate Lyapunov function:

$$V(t) = \frac{1}{2} \ln(1 + \|w\|^2) + \frac{1}{2\gamma} \tilde{\alpha}^2. \quad (\text{A.1})$$

Using the chain rule and integration by parts, we obtain along (36):

$$\frac{1}{2} \overbrace{\|w\|^2}^{\dot{}} = -\|w_x\|^2 + (\tilde{\alpha} - g)\|w\|^2 + \dot{\alpha}F(t) \quad (\text{A.2})$$

with $F(t) = \int_0^1 w(x, t) \left(\int_0^x \frac{y}{2} w(y, t) dy \right) dx$. Hence we have

$$\dot{V}(t) = \frac{u_\alpha F(t) - \|w_x\|^2 - g\|w\|^2}{1 + \|w\|^2} + \tilde{\alpha} \left(\frac{\|w\|^2}{1 + \|w\|^2} - \frac{u_\alpha}{\gamma} \right). \quad (\text{A.3})$$

Since $\tilde{\alpha}$ is unknown, we set the update law $u_\alpha(t)$ as defined by (37a), so as to cancel the last factor in (A.3). Using twice Cauchy-Schwarz inequality, we have

$$|F(t)| \leq \|w\|^2 / (4\sqrt{3}). \quad (\text{A.4})$$

Since system (\mathcal{S}_3) has homogeneous Dirichlet boundary conditions, we apply the Poincaré-Wirtinger inequality Hardy et al. [1952]:

$$\|w\|^2 \leq \|w_x\|^2 / \pi^2. \quad (\text{A.5})$$

Besides, since (37a) also implies that $|u_\alpha(t)| < \gamma$, we finally have:

$$\dot{V}(t) \leq -\left(1 + \frac{g}{\pi^2} - \frac{\gamma}{4\sqrt{3}\pi^2}\right) \frac{\|w_x\|^2}{1 + \|w\|^2}. \quad (\text{A.6})$$

Consequently, $\dot{V}(t)$ is negative semi-definite and $V(t) \leq V(0)$ is bounded for $\gamma \in (0; 4\sqrt{3}(g + \pi^2))$. In turn, $\|w\|$ and $\hat{\alpha}$ are bounded in time.

To show the boundedness of w in space and time, we first bound $\|w_x\|$. Integrating by parts and using boundary conditions, we have

$$\frac{1}{2} \overbrace{\|w_x\|^2}^{\cdot} = -\frac{u_\alpha w_x(1)}{\pi^2} \int_0^1 y w(y) dy - \|w_x\|^2 - \|w_{xx}\|^2 - \frac{u_\alpha}{4} \|w\|^2. \quad (\text{A.7})$$

Using the variation of Wirtinger's inequality, we find $-\|w_{xx}\|^2 \leq -\frac{\pi^2}{4} (\|w_x\|^2 + w_x^2(1))$. Using Cauchy-Schwarz and Young inequalities, we have $|\int_0^x y w(y) dy| \leq \frac{\|w\|}{\sqrt{3}}$ and $\|w\| w_x(1) \leq \frac{\varepsilon}{2} \|w\|^2 + \frac{1}{2\varepsilon} w_x^2(1)$. Using $|u_\alpha| < \gamma$, choosing $\varepsilon = \frac{\gamma}{\sqrt{3}\pi^2}$, and using the Poincaré-Wirtinger inequality, we finally have:

$$\frac{1}{2} \overbrace{\|w_x\|^2}^{\cdot} \leq L(t) \|w_x\|^2, \quad L(t) = \tilde{\alpha}(t) - g - \frac{\pi^2}{4} + \frac{\gamma^2}{12\pi^4}. \quad (\text{A.8})$$

Integrating (A.8) with respect to time, we have

$$\|w_x(t)\|^2 \leq \|w_x(0)\|^2 + 2 \sup_{[0,t]} |L| \int_0^t \|w_x(\tau)\|^2 d\tau. \quad (\text{A.9})$$

$V(t) \leq V(0)$ both implies that

$$\tilde{\alpha}^2(t) \leq 2\gamma V(0) \text{ and} \quad (\text{A.10a})$$

$$1 + \|w(t)\|^2 \leq (1 + \|w_0\|^2) e^{\frac{\tilde{\alpha}^2(0)}{\gamma}}. \quad (\text{A.10b})$$

Set $\sigma = 1 + \frac{g}{\pi^2} - \frac{\gamma}{4\sqrt{3}\pi^2}$. Integrating (A.6) on $[0, t]$, we have

$$\int_0^t \frac{\|w_x\|^2}{1 + \|w\|^2}(\tau) d\tau \leq \frac{[V(\tau)]_0^t}{\sigma} \leq \frac{V(0)}{\sigma}. \quad (\text{A.11})$$

Using (A.10a) to bound $\sup_{[0,t]} |L|$, and (A.10b), (A.11) in the inequality

$$\int_0^t \|w_x(\tau)\|^2 d\tau \leq \sup_{[0,t]} (1 + \|w(\tau)\|^2) \int_0^t \frac{\|w_x\|^2}{1 + \|w\|^2}(\tau) d\tau, \quad (\text{A.12})$$

we get the boundedness of $\|w_x(\tau)\|^2$:

$$\|w_x(\tau)\|^2 \leq \|w_x(0)\|^2 + \frac{\sqrt{8\gamma}}{\sigma} (1 + \|w_0\|^2) e^{\frac{\tilde{\alpha}^2(0)}{\gamma}} V^{\frac{3}{2}}(0). \quad (\text{A.13})$$

Using (A.5) and homogeneous Dirichlet condition in Agmon's inequality $\max |w(x, t)|^2 \leq w^2(0) + 2\|w\|\|w_x\|$, we finally have the boundedness of w in both time and space:

$$\max |w(x, t)|^2 \leq \frac{2}{\pi} \|w_x\|^2. \quad (\text{A.14})$$

Using both (A.2) and (A.4), it is straightforward that:

$$\frac{1}{2} \overbrace{\|w\|^2}^{\cdot} \leq \left(|\tilde{\alpha} - g| + \pi^2 + \frac{\gamma}{4\sqrt{3}} \right) \|w\|^2. \quad (\text{A.15})$$

Since $\|w\|$ is bounded, it follows that it is also uniformly continuous. Barbalat's lemma thus implies that $w(x, t)$ asymptotically converges to zero. To infer the boundedness of $\|v\|$, we use the inverse transformation of (10) to bound $\|u_x\|$ and $\|u\|$ using bounds found on $\|w_x\|$ and $\|w\|$. The

stabilization of $v(x, t)$ to zero is finally inherited from the stabilization of $w(x, t)$ to zero.

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