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POLYMER DYNAMICS IN THE DEPINNED PHASE: METASTABILITY WITH LOGARITHMIC BARRIERS

PIETRO CAPUTO, HUBERT LACOIN, FABIO MARTINELLI, FRANÇOIS SIMENHAUS, AND FABIO LUCIO TONINELLI

Abstract. We consider the stochastic evolution of a $(1+1)$-dimensional polymer in the depinned regime. At equilibrium the system exhibits a double well structure: the polymer lies (essentially) either above or below the repulsive line. As a consequence, one expects a metastable behavior with rare jumps between the two phases combined with a fast thermalization inside each phase. However, the energy barrier between these two phases is only logarithmic in the system size L and therefore the two relevant time scales are only polynomial in L with no clear-cut separation between them. The whole evolution is governed by a subtle competition between the diffusive behavior inside one phase and the jumps across the energy barriers. Our main results are: (i) a proof that the mixing time of the system lies between $L^{\frac{5}{2}}$ and $L^{\frac{5}{2}+2}$; (ii) the identification of two regions associated with the positive and negative phase of the polymer together with the proof of the asymptotic exponentiality of the tunneling time between them with rate equal to a half of the spectral gap.

2000 *Mathematics Subject Classification: 60K35, 82C20*

Keywords: reversible Markov chains, polymer pinning model, metastability, spectral gap, mixing time, coupling, quasi-stationary distribution.

1. Introduction, model and results

Random polymers are commonly used in statistical mechanics to model a variety of interesting physical phenomena. A rich class of models with a non-trivial behavior is obtained by considering a simple random walk path interacting with a defect line in the thermodynamic limit when the length of the path tends to infinity. The equilibrium of these so-called polymer pinning models has been studied in depth in the mathematical literature, and the associated localization/delocalization phase transition is, nowadays, a well understood phenomenon, even in the presence of non-homogeneous interactions; see [9] for a recent survey.

Markovian stochastic dynamics of random pinned polymers, on the other hand, have received much less attention from a mathematical point of view. Besides its importance in bio-physical applications (see e.g. [5, 6] and references therein), the stochastic evolution of polymer models poses new challenging probabilistic problems from many points of view and the connection between the equilibrium and dynamical properties of the model is still largely unexplored. In particular, we feel that the problem of how the polymer relaxes to the stationary distribution (time scales, overcoming of energy barriers, metastability, patterns leading to equilibrium) still lacks a satisfactory solution even in the simplest homogeneous models; see [7] for some initial results in this direction.

In this paper we consider the dynamics of a homogeneous polymer model interacting with a repulsive defect line with two main motivations in mind:

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(i) the repulsive regime is characterized by a relaxation to equilibrium occurring on a time scale certainly much larger [7] than the usual diffusive one which is typical e.g. of the neutral case[1] [21]. The new scale is clearly the result of a subtle competition in the polymer evolution between diffusive behavior and jumps against energy barriers.

(ii) The whole relaxation mechanism should show certain typical features of metastable evolution but in a very *atypical* context[2] in which the relevant relaxation time scales are only polynomial in the size L of the system (*i.e.* the energy barriers are only logarithmic in L), with little separation between the mixing time inside one phase and the global mixing time. A signature of this fact can be found in the anomalous growth with L of the global mixing time, a growth which is much more rapid than the naive guess based on the usual rule $T_{\rm mix} \approx \exp(\Delta E)$, with ΔE the so-called activation energy. In order to appreciate the novelty of such a situation it is useful to compare it to another well known case, namely the Glauber dynamics for the low temperature Ising model in a square box with free boundary [14], for which a very precise analysis of the metastable behavior was possible exactly because of a sharp separation, at an exponential level, between the two time scales.

1.1. **Dynamics of the polymer pinning model.** Let $\Omega = \Omega_{2L}$ denote the set of all lattice paths (polymers) starting at 0 and ending at 0 after $2L$ steps, $L \in \mathbb{N}$:

$$\Omega = \{\eta \in \mathbb{Z}^{2L+1} : \ \eta_{-L} = \eta_L = 0 \, , \ \eta_{x+1} = \eta_x \pm 1 \, , \ \ x = -L, \dots, L-1\} \, .$$

The stochastic dynamics is defined by the natural spin-flip continuous time Markov chain with state space Ω. Namely, sites $x = -L+1, \dots, L-1$ are equipped with independent rate 1 Poisson clocks. When site x rings, the height η_x of the polymer at x is updated according to the rules: if $\eta_{x-1} = \eta_{x+1} \pm 2$ then do nothing; if $\eta_{x-1} = \eta_{x+1} = h$, and $|h| \neq 1$, then set $\eta_x = h \pm 1$ with equal probabilities; if $\eta_{x-1} = \eta_{x+1} = +1$, then set $\eta_x = 0$ with probability $\frac{\lambda}{\lambda+1}$ and $\eta_x = 2$ with probability $\frac{1}{\lambda+1}$; similarly, if $\eta_{x-1} = \eta_{x+1} = -1$, then set $\eta_x = 0$ with probability $\frac{\lambda}{\lambda+1}$ and $\eta_x = -2$ with probability $\frac{1}{\lambda+1}$. Here $\lambda > 0$ is a parameter describing the strength of the attraction ($\lambda > 1$) or repulsion ($\lambda < 1$) between the polymer and the line $\eta \equiv 0$. The infinitesimal generator of the Markov chain is given by

$$\mathcal{L} f(\eta) = \sum_{x=-L+1}^{L-1} r_{x,+}(\eta) \left[f(\eta^{x,+}) - f(\eta) \right] + \sum_{x=-L+1}^{L-1} r_{x,-}(\eta) \left[f(\eta^{x,-}) - f(\eta) \right] \, , \tag{1.1}$$

where: f is a function $\Omega \mapsto \mathbb{R}$; $\eta^{x,\pm}$ denotes the configuration which coincides with η at every site $y \neq x$ and equals $\eta_x \pm 2$ at site x; the rates $r_{x,\pm}$ are zero unless $\eta^{x,\pm} \in \Omega$, while if $\eta^{x,\pm} \in \Omega$ and $\eta_{x-1} = \eta_{x+1} = h$ they satisfy $r_{x,\pm} = \frac{1}{2}$ for $h \neq \pm 1$, and $r_{x,\mp} = \frac{\lambda}{\lambda+1} = 1 - r_{x,\pm}$, for $h = \pm 1$.

The process defined above is the *heat bath* dynamics for the homogeneous *polymer pinning model*, with equilibrium measure $\pi = \pi^\lambda_{2L}$ on Ω defined by

$$\pi^\lambda_{2L}(\eta) = \frac{\lambda^{N(\eta)}}{Z^\lambda_{2L}} \, , \tag{1.2}$$

where $N(\eta) = \#\{x \in \{-L+1, \dots, L-1\} \ : \ \eta_x = 0\}$ denotes the number of zeros in the path $\eta \in \Omega$ and $Z^\lambda_{2L} = \sum_{\eta' \in \Omega} \lambda^{N(\eta')}$. For every $\lambda > 0$ and $L \in \mathbb{N}$, $\pi = \pi^\lambda_{2L}$ is the unique reversible invariant measure for the Markov chain.

[1] In the neutral case (absence of an interaction between the polymer and the line) our process is nothing but the usual (finite) symmetric simple exclusion model.

[2] Over the years there have been many different formulations of "metastability"; see [18, 20, 4]. We also refer to the recent contributions [2, 3] where, as in our case, energy barriers are only logarithmic in the characteristic size of the system. We feel however that our situation does not fit completely in any of the mentioned contexts.

1.2. **Relaxation to equilibrium.** The equilibrium properties of the polymer pinning model have been studied in detail, cf. e.g. [10] or [9, Section 2] for an extensive review. In particular, it is well known that, under the measure π_{2L}^{λ}, for $\lambda > 1$ the path is strongly localized with a non-vanishing density of zeros, while for $\lambda < 1$ the path is delocalized with $\sqrt{L}$ height fluctuations and with the number of zeros stochastically dominated by a geometric random variable with λ-dependent parameter. The dynamical counterpart of this localization/delocalization transition has not been fully understood yet. Some progress in this direction has been reported in [7], where various bounds on the spectral gap and mixing time of the Markov chain were obtained, together with estimates on the decay of time correlations.

We recall that the spectral gap is the smallest nonzero eigenvalue of $-\mathcal{L}$, and one is often interested in the *relaxation time* $T_{\rm rel} = 1/\text{gap}$ which governs decay to equilibrium in $\mathbb{L}_2(\pi)$, while the *mixing time* $T_{\rm mix}(\delta)$, for $\delta \in (0,1)$, is the smallest time t such that

$$\max_{\eta} \|P_t(\eta,\cdot) - \pi\| \leqslant \delta\,, \tag{1.3}$$

where $P_t(\eta,\cdot)$ denotes the distribution of the Markov chain at time t with initial state $\eta \in \Omega$, and $\|\mu - \nu\|$ is the usual total variation distance between two probability measures. When $\delta = \frac{1}{2e}$ we often write simply $T_{\rm mix}$ instead of $T_{\rm mix}(\delta)$. With these conventions one has $T_{\rm rel} \leqslant T_{\rm mix}$ always, and the inequality is strict in general.

A dynamical phase transition occurs when we move from the localized regime $\lambda > 1$ to the delocalized regime $\lambda < 1$. It was shown in [7], see Theorem 3.4 and Theorem 3.5 there, that for $\lambda \geqslant 1$ one has $T_{\rm rel} = O(L^2)$ and $T_{\rm mix} = O(L^2 \log L)$, while for $\lambda < 1$ one has

$$T_{\rm rel} \geqslant L^{\frac{5}{2}-\varepsilon}\,, \tag{1.4}$$

for all $\varepsilon > 0$, provided L is large enough.

We refer to [7] for results and conjectures concerning the localized regime $\lambda > 1$. Here, we consider the delocalized regime, i.e. in the repulsive case $\lambda < 1$. The first question we address concerns an upper bound on the relaxation time $T_{\rm rel}$ and the mixing time $T_{\rm mix}$. It is worth noting that even a crude polynomial bound is non-trivial. We refer to [15, 7] for polynomial bounds for the model with a horizontal wall at level zero, i.e. when lattice paths are constrained to be non-negative. On the other hand, without the wall constraint, the equilibrium measure π is known to be concentrated, as $L \to \infty$, on configurations in which the density of monomers in the upper (lower) half plane is approximately one (zero). However, a mathematical working definition of the plus or minus phase for the polymer is not so obvious and we have been forced to introduce a mesoscopic parameter ℓ (*i.e.* $L \gg \ell \gg 1$) and define $\Omega^{\pm}$ by

$$\Omega^+ = \{\eta \in \Omega \,:\, \eta_x > 0\,,\ -L+\ell < x < L-\ell\}\,, \qquad \Omega^- = -\Omega^+\,, \tag{1.5}$$

where for any set A of polymer configurations $-A := \{\eta \in \Omega \,:\, -\eta \in A\}$.

The presence of the two phases associated to $\Omega^{\pm}$ dramatically changes the relaxation scenario, with a bottleneck at the set $\Omega \setminus (\Omega^+ \cup \Omega^-)$. As explained in [7, Section 6], one may suspect that $T_{\rm rel} \sim L^{\frac{5}{2}}$ is the correct asymptotic behavior in the delocalized regime. Let us briefly recall the heuristic reasoning behind this prediction.

The time to reach equilibrium can be roughly thought of as the time needed to switch from, say, Ω^- to Ω^+. A point x such that $\eta_x = 0$ and $\eta_{x-1} \neq \eta_{x+1}$ is called a *crossing* of the polymer. Note that any zero (and therefore any crossing) x must belong to the set E_L of points in the segment $\{-L,\dots,L\}$ which have the same parity as L. Since there are typically very few zeros at equilibrium, one may consider the extreme case where at most one crossing ξ is allowed at all

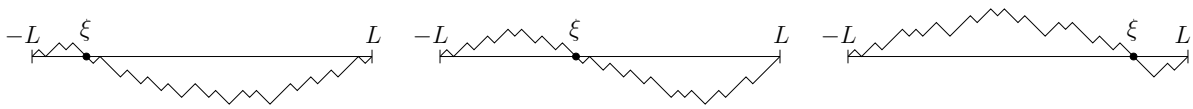

FIGURE 1. From left to right a snapshot sequence of the motion of a single crossing ξ which allows the system to switch from a mostly negative to a mostly positive configuration.

times. In this case, the time evolution of ξ should be essentially described by a suitable birth and death process or random walk on E_L; see Figure 1.

From equilibrium considerations, one knows that this random walk should have reversible invariant measure ρ roughly proportional to

$$\rho(x) \propto L^{3/2}(L+x)^{-3/2}(L-x)^{-3/2}\,, \qquad x \in E_L\,, \tag{1.6}$$

and that its relaxation time can be bounded from above and below by constant multiples of $L^{5/2}$; see Lemma 2.2 and Lemma 4.6 below for more details. Notice that, although the measure ρ gives uniformly (in L) positive mass to the two *attractors* $x^{\pm} = \pm L$, the drift which pushes the random walk away from the saddle $x = 0$ is proportional to the inverse of the distance from the attractors.

This heuristics is turned into the rigorous bound (1.4) by using a suitable test function in the variational principle that characterizes the spectral gap; see [7, Section 6]. However, it seems very hard to give a rigorous upper bound on $T_{\rm rel}$ of the same order of magnitude. We obtain a bound that can be off by at most two powers of L.

Theorem 1.1. *For any $\lambda < 1$,*

$$\limsup_{L\to\infty} \frac{\log T_{\rm mix}}{\log L} \leqslant \frac{5}{2} + 2\,.$$

The main tool for the proof of Theorem 1.1 is the analysis of an effective dynamics for the crossings of the polymer. To describe this, we introduce the variable $\sigma \in \{-1,+1\}^{O_L}$, where $O_L = \{-L,\dots,L\} \setminus E_L$ denotes the sites with the same parity of $L+1$. If η is a configuration of the polymer, then $\eta_x \neq 0$ at any $x \in O_L$, and we define $\sigma(\eta)$ by $\sigma_x = \mathrm{sign}(\eta_x)$. The projection of π on $\mathcal{S} = \{-1,+1\}^{O_L}$ is then

$$\nu(\sigma) = \sum_{\eta\,:\,\eta\sim\sigma} \pi(\eta)\,, \tag{1.7}$$

where the sum is over all configurations η compatible with the signs σ. The field ν has non-trivial long range correlations. Consider the heat bath dynamics for the variables σ: sites $x \in O_L$ are equipped with independent rate 1 Poisson clocks; when site x rings we replace σ_x by σ'_x where the new sign σ'_x is distributed according to the conditional probability $\nu(\cdot\,|\,\sigma_y\,,\ y \neq x)$, i.e. the probability (1.7) conditioned on the value of $\sigma_y\,,\ y \neq x$. Denote by $T^{\mathcal{S}}_{\rm rel}$ the corresponding relaxation time. For this process, the exponent 5/2 can be shown to be optimal.

Theorem 1.2. *For any $\lambda < 1$,*

$$\lim_{L\to\infty} \frac{\log T^{\mathcal{S}}_{\rm rel}}{\log L} = \frac{5}{2}\,.$$

The proof of Theorem 1.1 and Theorem 1.2 combines several different tools which play a prominent role in the analysis of convergence to equilibrium of Markov chains: decomposition methods, spectral gap analysis, comparison inequalities, and coupling estimates. An outline of the main steps of the proof is given at the beginning of Section 4.

1.3. **Metastability.** Recall the definition (1.5) of the two sets $\Omega^\pm$, and define the associated phases as the restricted equilibrium measures $\pi^\pm := \pi(\cdot\,|\,\Omega^\pm)$, so that (cf. Section 2)

$$\Big\|\pi - \frac{1}{2}(\pi^+ + \pi^-)\Big\| = o(1)\,. \tag{1.8}$$

The notation $o(1)$ refers to asymptotics as $L \to \infty$. In the thermodynamic limit, we expect relaxation to equilibrium within each phase to occur on time scales $T^\pm_{\rm rel}$ such that $T_{\rm rel} \gg T^\pm_{\rm rel}$, while on a time scale proportional to $T_{\rm rel}$ one should see the system jump from one phase to the other according to i.i.d. exponentially distributed times. A strong indication of this metastable behavior comes from the following theorem. Below, we use $\eta(t)$ to denote the state at time t of the Markov chain with generator (1.1).

Theorem 1.3. *There exists a set $S^+ \subset \Omega^+$ such that $\pi(S^+) = 1/2 + o(1)$, and that uniformly in $\eta \in S^+$ and uniformly in $t > 0$:*

$$\mathbb{P}^\eta\left(\tau^- > t\right) = e^{-t/(2T_{\rm rel})} \,+\, o(1)\,,$$

where

$$\tau^- = \inf\{t \geqslant 0 :\ \eta(t) \in S^-\}\,, \qquad S^- = -S^+\,.$$

Here $\mathbb{P}^\eta$ stands for the law of the process with initial state $\eta \in \Omega$. By symmetry, Theorem 1.3 also implies that uniformly in $\eta \in S^- = -S^+$ and uniformly in $t > 0$,

$$\mathbb{P}^\eta\left(\tau^+ > t\right) = e^{-t/(2T_{\rm rel})} \,+\, o(1)\,,$$

where $\tau^+ = \inf\{t \geqslant 0 :\ \eta(t) \in S^+\}$.

Remark 1.4. From the proof of Theorem 1.3 it will be clear that the set S^+ is *increasing* w.r.t. the natural partial order among polymer configurations defined in Section 2.5, so that in particular the maximal configuration (in the sequel denoted by $\wedge$) is in S^+.

If we define the renormalized process

$$\omega_s = \mathbf{1}\left(\eta(sT_{\rm rel}) \in \Omega^+\right) - \mathbf{1}\left(\eta(sT_{\rm rel}) \in \Omega^-\right)\,,$$

we expect that, starting from *any* configuration in $\Omega^\pm$, $\{\omega_s,\ s \geqslant 0\}$ converges to the simple two-state Markov chain with switching rate (from ± 1 to ∓ 1) equal to $1/2$, whose spectral gap equals 1. Such a strong uniform result seems very hard to obtain for our model. The difficulty is that, in contrast with familiar metastability results [18], here there is no clear-cut separation of time scales: while (1.4) and Proposition 2.6 below imply $T_{\rm rel} \gg T^+_{\rm rel}$, the ratio $T_{\rm rel}/T^+_{\rm rel}$ is only polynomially large in L. However, we do have a detailed description of the renormalized process when the initial condition is the maximal configuration. Namely, define the maximal element of Ω as $\eta_{\max} = \wedge$, i.e. $\wedge_x = x + L$ for $x \leqslant 0$ and $\wedge_x = L - x$ for $x \geqslant 0$, and let $T^\wedge_{\rm mix}(\varepsilon)$ denote the first time t such that $\|P_t(\wedge, \cdot) - \pi\| \leqslant \varepsilon$.

Theorem 1.5. *For any $\delta > 0$, uniformly in $t \geqslant L^{2+\delta}$*

$$\Big\|P_t(\wedge,\cdot) - \Big[\,\frac{1 + e^{-t/T_{\rm rel}}}{2}\,\pi^+ + \frac{1 - e^{-t/T_{\rm rel}}}{2}\,\pi^-\Big]\Big\| = o(1), \tag{1.9}$$

and uniformly in $t \geqslant 0$

$$\Big\|\nu_t^{\pi^+} - \Big[\,\frac{1 + e^{-t/T_{\rm rel}}}{2}\,\pi^+ + \frac{1 - e^{-t/T_{\rm rel}}}{2}\,\pi^-\Big]\Big\| = o(1) \tag{1.10}$$

where $\nu_t^{\pi^+}$ denotes the law at time t of the process started from the initial distribution π^+. Moreover, for any $\varepsilon \in (0, 1/2)$ one has

$$T_{\text{mix}}^{\wedge}(\varepsilon) = T_{\text{rel}} \log\left(\frac{1}{2\varepsilon}\right)(1 + o(1)). \tag{1.11}$$

Remark 1.6. Theorem 1.5 shows in particular that, when the dynamics is started from either π^+ or $\wedge$, there is no *cut-off* phenomenon [13], i.e. the variation distance from equilibrium does not fall abruptly to zero, but rather does so smoothly (on the timescale T_{rel}). That is another signature of the metastable behavior of our system and it is in contrast with what one expects for the neutral or attractive case $\lambda \geqslant 1$.

One of the key features of metastability is that, once the system decides to jump from e.g. S^+ to S^-, then it does so very quickly on the time scale of the mixing time. We verify that this is indeed the case for most starting configurations inside $S^+ \cup S^-$.

Let $\mathcal{T}$ denote the random time spent outside $S^+ \cup S^-$ up to the hitting time of S^-:

$$\mathcal{T} := \int_0^{\tau^-} \mathbf{1}_{\{\eta(s) \in (S^+ \cup S^-)^c\}} \, \mathrm{d}s. \tag{1.12}$$

From Theorems 1.3 and 1.5 one easily deduces that, for most initial conditions in S^+, $\tau^- \gg \mathcal{T}$:

Corollary 1.7. *There exists a subset $\tilde{S}^+$ of the set S^+ of Theorem 1.3 satisfying $\pi(\tilde{S}^+) = 1/2 + o(1)$ such that, uniformly on $\eta \in \tilde{S}^+$, $\mathcal{T} = o(\tau^-)$ in probability, i.e. there exists a sequence δ_L tending to zero as $L \to \infty$ such that for every $\eta \in \tilde{S}^+$,*

$$\mathbb{P}^\eta\left[\mathcal{T} \geqslant \delta_L \tau^-\right] \leqslant \delta_L. \tag{1.13}$$

Along the same lines of the proof of the Corollary, one can establish the weak convergence of the renormalized process ω_s to the two-state Markov chain, provided that the initial configuration is inside a suitable subset of $\tilde{S}^+ \cup \tilde{S}^-$ with almost full measure. We decided to omit details for shortness.

1.4. **Organization of the paper.** The rest of the paper consists of three sections. Section 2 starts with standard material and then proceeds with the introduction of some essential tools to be used in the proof of the main results, including general results for monotone systems that can be of independent interest. This section contains also some new results concerning the relaxation within one phase and the properties of the principal eigenfunction of the generator. The metastability results are discussed in Section 3. Here, we start with the proof of Theorem 1.5. In later subsections we develop the construction needed for the proof of Theorem 1.3. Finally, Section 4 proves Theorem 1.1 and Theorem 1.2. This section is broken into several subsections corresponding to the various steps of the proof. A high level description of the arguments involved is given at the beginning of the section.

Notational conventions. Whenever we write $o(L^p)$ or $O(L^p)$ for some $p \in \mathbb{R}$ it is understood that this refers to the thermodynamic limit $L \to \infty$. Also, we use the notation $f(L) = \Omega(L^p)$ when there exists a constant $c > 0$ such that $f(L) \geqslant c\, L^p$ for all sufficiently large L. For positive functions f, g, we use the notation $f(L) \gg g(L)$ whenever $\liminf_{L\to\infty} f(L)/g(L) = +\infty$, and $f(L) \sim g(L)$ when $\lim_{L\to\infty} f(L)/g(L) = 1$. Also, we write $f \asymp g$ if there exists some constant $c > 0$ such that $c^{-1} g \leqslant f \leqslant c\, g$.

2. Some tools

We begin with some generalities about reversible Markov chains. Then, we recall the definition of the polymer dynamics and derive some consequences of monotonicity. Next, we give some estimates on convergence to equilibrium in the "plus" phase. Finally, we characterize in detail an eigenfunction of $\mathcal{L}$ with eigenvalue $-$gap.

2.1. **Preliminaries.** We will consider reversible continuous time Markov chains with finite state space X, defined by the infinitesimal generator $\mathcal{L}$ acting on functions $f : X \mapsto \mathbb{R}$,

$$[\mathcal{L}f](x) = \sum_{y\in X} c(x,y)[f(y) - f(x)]\,, \tag{2.1}$$

where $c(\cdot,\cdot)$ is a bounded non-negative function on $X \times X$ satisfying $\pi(x)c(x,y) = \pi(y)c(y,x)$, for a probability measure π on X. In the applications below, the rates $c(x,y)$ will always be such that the Markov chain is irreducible and the reversible invariant measure π is positive on X. We refer e.g. to [1, 13] for more details on reversible Markov chains.

Let $\nu_t^x = \mathbb{P}(v_t^x \in \cdot)$ denote the law of the state v_t^x of the Markov chain at time t with initial condition $x \in X$. We shall investigate the rate of convergence of ν_t^x to π. If the initial condition x is distributed according to a probability μ on X, we write $\nu_t^\mu = \sum_{x\in X} \mu(x)\nu_t^x$ for the distribution at time t. As usual, one can associate a semi-group $\{P_t,\, t \geqslant 0\}$ to the generator $\mathcal{L}$ in such a way that $[P_t f](x) = [e^{t\mathcal{L}} f](x) = \sum_{y\in X} \nu_t^x(y) f(y)$. We also use the notation $P_t(x,y) = \nu_t^x(y)$, and $\nu_t^\mu = \mu P_t$.

The mixing time of the Markov chain is defined by

$$T_{\mathrm{mix}}(\varepsilon) = \inf\Big\{t > 0\,,\ \max_{x\in X} \|\nu_t^x - \pi\| \leqslant \varepsilon\Big\}\,, \tag{2.2}$$

where

$$\|\mu - \nu\| = \frac{1}{2}\sum_x |\mu(x) - \nu(x)|$$

is the total variation distance. We shall use the convention that T_{mix} stands for $T_{\mathrm{mix}}(\frac{1}{2e})$. It is well known that with this notation one has

$$\|\nu_t^x - \pi\| \leqslant e^{-\lfloor t/T_{\mathrm{mix}}\rfloor}\,, \tag{2.3}$$

for all $t \geqslant 0$, where $\lfloor a\rfloor$ denotes the integer part of $a \geqslant 0$. The spectral gap and the relaxation time of the process are defined by

$$\mathrm{gap} = \min_{f:X\mapsto\mathbb{R}} \frac{\mathcal{E}(f,f)}{\mathrm{Var}_\pi(f)}\,, \qquad T_{\mathrm{rel}} = \frac{1}{\mathrm{gap}}\,, \tag{2.4}$$

where for $f : X \mapsto \mathbb{R}$,

$$\mathcal{E}(f,f) = \sum_{x\in X} \pi(x) f(x)[-\mathcal{L}f](x) = \frac{1}{2}\sum_{x,y\in X} \pi(x)c(x,y)[f(y) - f(x)]^2 \tag{2.5}$$

is the quadratic form of the generator, a.k.a. the Dirichlet form, while $\mathrm{Var}_\pi(f)$ stands for the variance $\pi(f^2) - \pi(f)^2$. Thus, gap is the lowest non-zero eigenvalue of $-\mathcal{L}$. The following bound relating total variation distance and relaxation time is an immediate consequence of reversibility and Schwarz' inequality:

$$\|\nu_t^\mu - \pi\| \leqslant \frac{1}{2}\, e^{-t/T_{\mathrm{rel}}} \sqrt{\mathrm{Var}_\pi(f)}\,, \tag{2.6}$$

where $f(\sigma) = \mu(\sigma)/\pi(\sigma)$ and μ is a probability on X. Another standard relation between total variation and relaxation time is the identity

$$\text{gap} = -\lim_{t\to\infty} \frac{1}{t} \log \max_{x,y} \|\nu_t^x - \nu_t^y\|. \tag{2.7}$$

Combining (2.3), (2.7) and (2.6), one can obtain the following well known relations:

$$T_{\text{rel}} \leqslant T_{\text{mix}} \leqslant \big(1 - \log \pi_*\big) T_{\text{rel}}\,, \qquad \text{where } \pi_* = \min_{x\in X} \pi(x)\,. \tag{2.8}$$

2.2. **A general decomposition bound on the spectral gap.** We shall need a continuous time version of a general decomposition bound obtained by Jerrum et al. [11]. Consider the continuous time reversible Markov chain defined by (2.1). Suppose the space X is partitioned in the disjoint union of subspaces $X_1, \dots, X_m$, for some $m \in \mathbb{N}$ and define the generators

$$[\mathcal{L}_i f](x) = \sum_{y\in X} c_i(x,y)[f(y) - f(x)]\,, \qquad c_i(x,y) = c(x,y)\mathbf{1}(y \in X_i)\,, \quad x \in X_i\,.$$

Then $\mathcal{L}_i$ is the generator of the Markov chain restricted to X_i, its reversible invariant measure being given by $\pi_i = \pi(\cdot \,|\, X_i)$. Let $\lambda_{\min}$ denote the minimum of the spectral gaps of the Markov chains generated by $\mathcal{L}_i$, $i = 1, \dots, m$. Next, let $\overline{\mathcal{L}}$ denote the infinitesimal generator defined by

$$[\overline{\mathcal{L}}\varphi](i) = \sum_{j=1}^{m} \bar{c}(i,j)[\varphi(j) - \varphi(i)]\,,$$

for $\varphi \in \mathbb{R}^m$, where

$$\bar{c}(i,j) = \sum_{x\in X_i\,,\, y\in X_j} \pi(x \,|\, X_i)\, c(x,y)\,.$$

This defines a continuous time Markov chain on $\{1, \dots, m\}$ with reversible invariant measure $\bar{\pi}(i) = \pi(X_i)$. Let $\bar{\lambda}$ denote the gap of this chain. A straightforward adaptation of [11, Theorem 1] yields the following estimate.

Proposition 2.1. *Define* $\gamma = \max_i \max_{x\in X_i} \sum_{y\in X\setminus X_i} c(x,y)$ *. Then, with the notation of* (2.4),

$$\text{gap} \geqslant \min\Big\{\frac{\bar{\lambda}}{3}, \frac{\bar{\lambda}\,\lambda_{\min}}{\bar{\lambda} + 3\gamma}\Big\}\,. \tag{2.9}$$

2.3. **Killed process and quasi-stationary distribution.** Here we recall some standard facts about killed processes, their generators and quasi-stationary distributions for reversible Markov chains; we refer to [1] for an introduction. Given a reversible Markov chain with generator $\mathcal{L}$ as above and a subset $\Gamma \subset X$, we consider the process killed upon entering Γ, with sub-probability law defined by

$$\nu_t^{x,\Gamma}(B) = \mathbb{P}^x(v_t \in B\,;\ \tau_\Gamma > t)\,, \qquad x \in \Gamma^c\,, \tag{2.10}$$

where $B \subset X$, v_t denotes the state of the Markov chain with generator $\mathcal{L}$ at time t, $\mathbb{P}^x$ denotes the law of the process started at x, and τ_Γ denotes the hitting time of the set Γ. The associated semi-group P_t^Γ is given by

$$[P_t^\Gamma f](x) = [e^{t\mathcal{L}^\Gamma} f](x) = \sum_{y\in\Gamma^c} \nu_t^{x,\Gamma}(y) f(y)\,, \qquad x \in \Gamma^c\,, \tag{2.11}$$

where the killed generator $\mathcal{L}^\Gamma$ satisfies, for every $x \in \Gamma^c$:

$$[\mathcal{L}^\Gamma f](x) = [\mathcal{L}(f\mathbf{1}_{\Gamma^c})](x) = [\mathcal{L}f](x) - \sum_{y\in\Gamma} c(x,y) f(y)\,. \tag{2.12}$$

We assume that P_t^Γ is irreducible. Then $\mathcal{L}^\Gamma$ is a negative definite, self-adjoint operator in $\mathbb{L}_2(\pi)$, and its top eigenvalue $-\gamma_\Gamma$ is characterized by

$$\gamma_\Gamma = \min_{\substack{f:X\mapsto\mathbb{R},\\ f\mathbf{1}_\Gamma=0}} \frac{\left\langle -\mathcal{L}^\Gamma f, f\right\rangle_\pi}{\pi(f^2)} = \min_{\substack{f:X\mapsto\mathbb{R},\\ f\mathbf{1}_\Gamma=0}} \frac{\mathcal{E}(f,f)}{\pi(f^2)}\,, \tag{2.13}$$

where we use $\langle\cdot,\cdot\rangle_\pi$ for the scalar product in $\mathbb{L}_2(\pi)$, and $\mathcal{E}(f,f)$ is defined by (2.5).

Let g_Γ denote the (unique, positive on Γ^c) eigenfunction of $\mathcal{L}^\Gamma$ associated to $-\gamma_\Gamma$. Extending g_Γ to all $x\in X$ by setting $g_\Gamma(x)=0$ for $x\in\Gamma$, one defines the *quasi-stationary* distribution ν^Γ, i.e. the probability on X given by

$$\nu_\Gamma(y) = \frac{\pi(y)g_\Gamma(y)}{\pi(g_\Gamma)}\,, \quad y\in X\,. \tag{2.14}$$

An equivalent characterization of ν_Γ is as the limit

$$\nu_\Gamma(B) = \lim_{t\to\infty} \mathbb{P}^x(v_t\in B\,|\,\tau_\Gamma>t)\,, \tag{2.15}$$

where $B\subset X$, and the chosen initial point $x\in\Gamma^c$ is arbitrary. The fundamental property of the quasi-stationary distribution is that, starting from ν_Γ, the hitting time τ_Γ is exponentially distributed with parameter γ_Γ:

$$\mathbb{P}^{\nu_\Gamma}(\tau_\Gamma>t) = e^{-\gamma_\Gamma t}\,, \tag{2.16}$$

where $\mathbb{P}^{\nu_\Gamma}$ stands for the law of the process when the initial state is distributed according to ν_Γ. Another way of expressing quasi-stationarity is $\nu^\Gamma P_t^\Gamma = e^{-\gamma_\Gamma t}\nu^\Gamma$, for all $t\geqslant 0$. A general property of γ_Γ (cf. Lemma 3.1 below) is that $\gamma_\Gamma\geqslant\pi(\Gamma)$ gap.

2.4. **Polymer model.** Let $\Omega=\Omega_{2L}$ stand for the space of all lattice paths defined in the introduction. A partial order in Ω is given by

$$\eta\leqslant\eta' \iff \eta_x\leqslant\eta'_x\,, \quad x=-L,\dots,L\,. \tag{2.17}$$

Given $\zeta,\xi\in\Omega$ such that $\zeta\leqslant\xi$ we define the restricted space $\Omega^{\zeta,\xi}$ of all paths $\eta\in\Omega$ such that $\zeta\leqslant\eta\leqslant\xi$. The dynamics is defined by the continuous time Markov chain with state space $\Omega^{\zeta,\xi}$, with infinitesimal generator $\mathcal{L}^{\zeta,\xi}$ given by (1.1) where the rates $r_{x,\pm}(\eta)$ are replaced by

$$r_{x,\pm}^{\zeta,\xi}(\eta) = r_{x,\pm}(\eta)\mathbf{1}(\eta^{x,\pm}\in\Omega^{\zeta,\xi}) \tag{2.18}$$

This process is the heat bath dynamics associated to the probability measure $\pi_{2L}^{\lambda,\zeta,\xi}$ on $\Omega^{\zeta,\xi}$ defined as in (1.2) with the normalization now given by

$$Z_{2L}^{\lambda,\zeta,\xi} = \sum_{\eta'\in\Omega^{\zeta,\xi}} \lambda^{N(\eta')}\,. \tag{2.19}$$

Equivalently, $\pi_{2L}^{\lambda,\zeta,\xi}=\pi_{2L}^\lambda(\cdot\,|\,\Omega^{\zeta,\xi})$. This is referred to as the polymer model with top/bottom constraints (ζ is the bottom, ξ is the top). For simplicity, when no confusion arises, we often omit the superscripts λ,ζ,ξ and the subscript L from our notation in what follows. We write v_t^η for the state of the Markov chain at time t when the initial configuration is some η, and let ν_t^η denote its distribution. When the initial condition η is distributed according to a probability measure μ on Ω we write ν_t^μ as in Section 2.1.

Note that the generator $\mathcal{L}^{\zeta,\xi}$ can be written in the form (2.1) by setting $c(\eta,\eta')=r_{x,\pm}^{\zeta,\xi}(\eta)\mathbf{1}(\eta'=\eta^{x,\pm})$, and $\pi=\pi_{2L}^{\lambda,\zeta,\xi}$ is reversible. While this holds for every value $\lambda>0$ of the parameter describing the strength of the interaction, we will only consider the case $\lambda<1$ below, which corresponds to a strictly delocalized regime for the polymer.

The minimal path $\vee$ and maximal path $\wedge$ for the order (2.17) are defined by $\vee_x = -x - L$ for $x \leqslant 0$, $\vee_x = -L + x$ for $x \geqslant 0$, and $\wedge = -\vee$. Clearly, if $\zeta = \vee$ and $\xi = \wedge$, then $\Omega^{\zeta,\xi} = \Omega$. This case is referred to as the polymer model with no top/bottom constraint.

The following well known estimates will be often used in our proofs. We refer e.g. to [9, Section 2] for the proof of Lemma 2.2 below, as well as for other known properties of the delocalized equilibrium measure. Let $Z_{2L} = Z_{2L}^{\lambda}$ denote the partition function (2.19) with no top/bottom boundaries and write $Z_{2L}^{+} = Z_{2L}^{+,\lambda}$ for the partition function (2.19) with $\xi = \wedge$ and ζ given by the minimal non-negative element of Ω, i.e. $\zeta_x = 0$ if $x \in E_L$ (x has the same parity as L) and $\zeta_x = 1$ if $x \in O_L$ (x has opposite parity w.r.t. L); Z_{2L}^{+} is the partition function of the polymer with a horizontal wall at height zero. Recall that $N = N(\eta)$ stands for the number of zeros in the path η lying strictly between $-L$ and L. Considering reflections of the path between consecutive zeros one obtains

$$2Z_{2L}^{+,\lambda} = Z_{2L}^{\lambda/2}\,. \tag{2.20}$$

Lemma 2.2. *Consider the polymer with no top/bottom constraint with $\lambda \in (0,1)$. There exist constants $c_i = c_i(\lambda) > 0$, $i = 1,2$ such that*

$$2^{-2L} Z_{2L}^{\lambda} \sim c_1\, L^{-3/2}\,, \tag{2.21}$$

and

$$\pi(N(\eta) > k) \leqslant c_2\, e^{-k/c_2}\,, \quad \forall\, k \in \mathbb{N}\,. \tag{2.22}$$

An immediate implication of (2.20) and (2.21) is that

$$2^{-2L} Z_{2L}^{+,\lambda} \sim c_+\, L^{-3/2}\,, \tag{2.23}$$

for some constant $c_+ > 0$ as soon as $\lambda < 2$. Moreover, (2.21) and (2.23) imply the bounds

$$\begin{aligned} \pi\left(\eta_y \geqslant 0 \ \ \forall y \in \{-L,\dots,x\}\,,\ \text{and}\ \ \eta_x = 0\right) &\asymp \pi\left(\eta_x = 0\right) \\ &\asymp L^{3/2}(L+x)^{-3/2}(L-x)^{-3/2}\,, \end{aligned} \tag{2.24}$$

for every $x \in E_L$.

2.5. **Monotonicity.** An important property satisfied by the Markov chains introduced above is the monotonicity with respect to the partial order (2.17). A convenient way of stating the monotonicity property is that there exists a coupling $\mathbb{P}$ of the trajectories of the Markov chains corresponding to distinct initial conditions such that if $\eta \leqslant \eta'$ then $\mathbb{P}$ almost surely $v_t^{\eta} \leqslant v_t^{\eta'}$ for all $t \geqslant 0$. More generally, one can define a coupling $\mathbb{P}$ of trajectories corresponding to distinct top/bottom constraints and distinct initial conditions such that if $\zeta \leqslant \zeta'$, $\xi \leqslant \xi'$, and $\eta \leqslant \eta'$, then $\mathbb{P}$ almost surely $v_t^{\eta;\zeta,\xi} \leqslant v_t^{\eta';\zeta',\xi'}$ for all $t \geqslant 0$. Recall that a function $f : \Omega \mapsto \mathbb{R}$ is said to be increasing if $f(\eta) \leqslant f(\eta')$ whenever $\eta \leqslant \eta'$. An event A is increasing if the indicator function $\mathbf{1}_A$ is increasing. The monotonicity property of the dynamics implies the so-called FKG property of the equilibrium measures $\pi = \pi_{2L}^{\lambda,\zeta,\xi}$: for every pair of increasing functions $f, g : \Omega \mapsto \mathbb{R}$, one has $\pi(fg) \geqslant \pi(f)\pi(g)$. We refer to [7, Section 2] for a more detailed discussion of the monotone coupling and the consequences of monotonicity.

Lemma 2.3. *Let μ be a probability on Ω and write $f(\eta) = \mu(\eta)/\pi(\eta)$, and $f_t(\eta) = \nu_t^{\mu}(\eta)/\pi(\eta)$, $t > 0$. If f is increasing then, for every $t > 0$, f_t is increasing. As a consequence, there exists an increasing event A such that*

$$\|\nu_t^{\mu} - \pi\| = \nu_t^{\mu}(A) - \pi(A)\,. \tag{2.25}$$

Proof. Write $\nu_t^\mu(\eta) = \sum_{\eta_0\in\Omega} \mu(\eta_0)P_t(\eta_0,\eta)$, where $P_t(\cdot,\cdot)$ stands for the kernel of the Markov chain. Reversibility then gives

$$f_t(\eta) = \sum_{\eta_0\in\Omega} f(\eta_0)\pi(\eta_0)P_t(\eta_0,\eta)/\pi(\eta) = \sum_{\eta_0\in\Omega} f(\eta_0)P_t(\eta,\eta_0)\,. \tag{2.26}$$

Next, let $\mathbb{P}$ denote the monotone coupling introduced above and let $\mathbb{E}$ denote expectation w.r.t. $\mathbb{P}$. Then, (2.26) coincides with $\mathbb{E}[f(v_t^\eta)]$, and if $\eta \leqslant \eta'$,

$$f_t(\eta') - f_t(\eta) = \mathbb{E}[f(v_t^{\eta'}) - f(v_t^\eta)] = \mathbb{E}[f(v_t^{\eta'}) - f(v_t^\eta)\,;\; v_t^{\eta'} \geqslant v_t^\eta]\,.$$

Thus, f_t is increasing whenever f is. Finally, it is well known that the total variation distance can be written in the form (2.25) where $A = \{\eta : \nu_t^\mu(A) \geqslant \pi(A)\}$. Since $A = \{f_t \geqslant 1\}$, A is increasing whenever f is. □

Lemma 2.4 compares arbitrary initial conditions to the extremal initial conditions. Lemma 2.5 states a useful sub-multiplicativity property satisfied by extremal evolutions. For lightness of notation, we state these results only in the case of no top/bottom boundaries, i.e. $\zeta = \vee$, $\xi = \wedge$, but the same applies for general ζ, ξ with exactly the same proof.

Lemma 2.4. *For any $t > 0$ and any $\eta, \eta' \in \Omega$:*

$$\|\nu_t^\eta - \nu_t^{\eta'}\| \leqslant 4L^2\,\|\nu_t^\wedge - \nu_t^\vee\|\,.$$

As a consequence,

$$\text{gap} = -\lim_{t\to\infty} \frac{1}{t} \log \|\nu_t^\wedge - \nu_t^\vee\| \;\;.$$

Proof. Let $\mathbb{P}$ denote the monotone coupling as above. Then,

$$\begin{aligned}
\|\nu_t^\eta - \nu_t^{\eta'}\| &\leqslant \mathbb{P}(v_t^\eta \neq v_t^{\eta'}) \leqslant \mathbb{P}(v_t^\wedge \neq v_t^\vee) \\
&\leqslant \sum_{x=-L+1}^{L-1} \sum_{h=-L}^{L-1} [\mathbb{P}((v_t^\wedge)_x > h) - \mathbb{P}((v_t^\vee)_x > h)] \\
&\leqslant 4L^2\,\|\nu_t^\wedge - \nu_t^\vee\|\,.
\end{aligned}$$

The second point follows from the first one and the classical characterization (2.7) of the spectral gap. □

Lemma 2.5. *For any $s, t \geqslant 0$,*

$$\|\nu_{t+s}^\wedge - \nu_{t+s}^\vee\| \leqslant \|\nu_t^\wedge - \nu_t^\vee\|\,\|\nu_s^\wedge - \nu_s^\vee\|\,.$$

Proof. With the same argument of Lemma 2.3, for some increasing event A

$$\|\nu_{t+s}^\wedge - \nu_{t+s}^\vee\| = \nu_{t+s}^\wedge(A) - \nu_{t+s}^\vee(A)\,.$$

Let ρ be a coupling beween $\nu_t^\wedge$ and $\nu_t^\vee$ at fixed time $t \geqslant 0$. Then

$$\begin{aligned}
\nu_{t+s}^\wedge(A) - \nu_{t+s}^\vee(A) &= \int (\nu_s^\eta(A) - \nu_s^\sigma(A))d\rho(\eta,\sigma) \\
&= \int (\nu_s^\eta(A) - \nu_s^\sigma(A))\mathbf{1}(\sigma \neq \eta)d\rho(\eta,\sigma) \\
&\leqslant (\nu_s^\wedge(A) - \nu_s^\vee(A))\rho(\sigma \neq \eta) \\
&\leqslant \|\nu_s^\wedge - \nu_s^\vee\|\,\rho(\sigma \neq \eta)\,.
\end{aligned}$$

To conclude, we take ρ as the maximal coupling, i.e. such that $\rho(\sigma \neq \eta) = \|\nu_t^\wedge - \nu_t^\vee\|$. □

2.6. **Relaxation in one phase.** Here we obtain results concerning the polymer dynamics in the phase $\pi^+ = \pi(\cdot|\Omega^+)$ with Ω^+ defined after (1.5); cf. Proposition 2.6 below. Then, we show that the polymer started at the maximal configuration $\wedge$ relaxes first to the restricted equilibrium π^+ in a time $O(L^{2+\delta})$ for arbitrarily small $\delta > 0$, while it takes much longer to reach the full equilibrium π; cf. Lemma 2.7 and Lemma 2.8 below.

Recall the definition (1.5) of the subspace $\Omega^+ \subset \Omega$, where $L \gg \ell$, and ℓ diverges as $L \to \infty$; see (2.29) below. The corresponding restricted equilibrium is given by $\pi^+ = \pi(\cdot|\Omega^+)$. Note that this is a particular instance of the polymer equilibrium $\pi_{2L}^{\lambda,\zeta,\xi}$ with top/bottom boundaries: the top is $\xi = \wedge$ while the bottom $\zeta = \zeta(\Omega^+)$ is the lowest element of Ω^+. Similarly, one defines $\Omega^- = -\Omega^+$, i.e. use (1.5) with $\eta_x > 0$ replaced by $\eta_x < 0$, and the equilibrium π^- is defined accordingly.

Since $\lambda < 1$, the equilibrium bounds (2.24) imply that

$$\pi(\Omega^+) = \pi(\Omega^-) = \frac{1}{2} + O(\ell^{-1/2})\,; \tag{2.27}$$

see e.g. [7, Section 2]. In particular, if ℓ diverges as $L \to \infty$, then

$$\big\|\pi - \frac{1}{2}(\pi^+ + \pi^-)\big\| = o(1)\,. \tag{2.28}$$

What follows depends only marginally on the precise dependence of ℓ on L, provided that $L \gg \ell \gg 1$. For the sake of simplicity we shall fix its value as

$$\ell(L) = \lfloor (\log L)^{\frac{1}{4}} \rfloor\,. \tag{2.29}$$

This choice turns out to be convenient in the proof of Proposition 2.6 below, but we point out that any choice of the form $\ell(L) = O(L^\varepsilon)$ for small $\varepsilon > 0$ would be sufficient to obtain the same conclusion with a little more work.

We start by establishing a mixing time upper bound for the dynamics constrained to stay in Ω^+, i.e. the process evolving with bottom boundary given by $\zeta = \zeta(\Omega^+)$. To avoid confusion we shall write μ_t^η (instead of ν_t^η) for the law at time t of this Markov chain with state space Ω^+ and initial condition η. We write $\mathcal{L}^+$ for its generator and gap^+ for the associated spectral gap.

Proposition 2.6. *For every $\varepsilon > 0$, there exists $L_0 = L_0(\varepsilon)$ such that for all $L \geqslant L_0$, for all $t \geqslant 0$ and all initial conditions $\eta \in \Omega^+$:*

$$\|\mu_t^\eta - \pi^+\| \leqslant 4L^2 \exp\big(-t/L^{2+\varepsilon}\big)\,. \tag{2.30}$$

In particular, $\mathrm{gap}^+ \geqslant L^{-2-\varepsilon}$.

Proof. The last statement follows from (2.30) and (2.7). To prove (2.30) we establish that for every $\varepsilon > 0$, there is a constant $L_0 = L_0(\varepsilon) > 0$ such that, taking $T = L^{2+\varepsilon}$, we have

$$\|\mu_T^\wedge - \mu_T^\zeta\| \leqslant 1 - L^{-\varepsilon}\,, \tag{2.31}$$

for all $L \geqslant L_0(\varepsilon)$, where ζ stands for the minimal element $\zeta = \zeta(\Omega^+)$ of Ω^+. Once (2.31) is available, we obtain (2.30) (with a new value of ε) from Lemma 2.4, since Lemma 2.5 (which is also valid for the restricted dynamic) and (2.31) imply

$$\begin{aligned}\|\mu_t^\wedge - \mu_t^\zeta\| &\leqslant \Big(\|\mu_T^\wedge - \mu_T^\zeta\|\Big)^{\lfloor t/T \rfloor} \\ &\leqslant \exp\big(-\lfloor t/T \rfloor L^{-\varepsilon}\big)\big) \leqslant 2\exp\big(-t/L^{2+2\varepsilon}\big)\,,\end{aligned}$$

for any L large enough.

To prove (2.31), we divide the sites $x \in \{-L, \dots, L\}$ in three overlapping regions:

$$I_1 = \{-L, \dots, -L + \ell^2\}\,,\ I_2 = \{-L + \ell, \dots, L - \ell\}\,,\ \text{ and } I_3 = \{L - \ell^2, \dots, L\}\,,$$

where $\ell = \ell(L)$ is given by (2.29). Let $T_2 = L^{2+\varepsilon_1}$, $T_1 = L^{\varepsilon_1}$ with some $\varepsilon_1 > 0$ such that $T = L^{2+\varepsilon} \geqslant T' := 2T_2 + T_1$. We shall prove (2.31) with T replaced by T' (this implies the claim since by Lemma 2.4 the left hand side of (2.31) is monotone as a function of T). Call $\mu_{T'}^{\eta_0,c}$ the law of the "censored" process obtained as follows. Start from η_0 at time 0 and, for time $t \in [0, T_2]$ reject all the updates involving $x \notin I_2$. For time $t \in (T_2, T_2 + T_1]$ reject all updates involving $x \notin I_1 \cup I_3$, and for time $t \in (T_2 + T_1, 2T_2 + T_1 = T']$ reject all the updates involving $x \notin I_2$. From the Peres-Winkler censoring inequality [19, Theorem 16.5] one has that $\mu_{T'}^{\wedge,c}$ stochastically dominates $\mu_{T'}^{\wedge}$. Similarly, $\mu_{T'}^{\zeta,c}$ is stochastically dominated by $\mu_{T'}^{\zeta}$. On the other hand, as in Lemma 2.3, one has

$$\|\mu_{T'}^{\wedge} - \mu_{T'}^{\zeta}\| = \mu_{T'}^{\wedge}(A) - \mu_{T'}^{\zeta}(A)\,,$$

where $A \subset \Omega^+$ is an increasing event. Therefore,

$$\|\mu_{T'}^{\wedge} - \mu_{T'}^{\zeta}\| \leqslant \mu_{T'}^{\wedge,c}(A) - \mu_{T'}^{\zeta,c}(A) \leqslant \|\mu_{T'}^{\wedge,c} - \mu_{T'}^{\zeta,c}\|\,,$$

and the lemma follows once we show that

$$\|\mu_{T'}^{\wedge,c} - \mu_{T'}^{\zeta,c}\| \leqslant 1 - L^{-\varepsilon}\,. \tag{2.32}$$

To prove (2.32) we shall couple the two configurations $\eta_{T'}^{\wedge,c}$, $\eta_{T'}^{\zeta,c}$ with law $\mu_{T'}^{\wedge,c}$, $\mu_{T'}^{\zeta,c}$ respectively. From the analysis of the polymer with the wall [7, Section 4], it is not hard to infer that uniformly in the boundary values at $-L + \ell - 1$ and $L - \ell + 1$ the system evolving in the region I_2 has a mixing time $O(L^2 \log L)$. Therefore, after a time $T_2 = L^{2+\varepsilon_1}$, up to $O(L^{-p})$ corrections for a large constant $p > 0$, for any event E, $\mu_{T_2}^{\wedge,c}(E)$ coincides with the equilibrium probability of E in I_2 with boundary conditions $\wedge$ at $-L + \ell - 1$ and $L - \ell + 1$. The same applies to $\mu_{T_2}^{\zeta,c}(E)$ provided the equilibrium is taken with boundary conditions ζ at $-L+\ell-1$ and $L-\ell+1$. Choose the event E that the configuration is minimal (in Ω^+) at both $-L + \ell^2 - 1$ and $L - \ell^2 + 1$ (i.e. $\eta_{-L+\ell^2-1} = \eta_{L-\ell^2+1} = 0$ if $L - \ell^2$ is odd, and $\eta_{-L+\ell^2-1} = \eta_{L-\ell^2+1} = 1$ if $L - \ell^2$ is even). From known equilibrium estimates [9, Section 2], it is not difficult to show that at equilibrium, with either of the two boundary conditions considered above, the probability of E is bounded below by $c_1\,\ell^{-6}$ for some constant $c_1 > 0$ depending only on λ. Therefore, using an independent coupling in the time-lag $[0, T_2]$, we have that the event E occurs for both $\eta_{T_2}^{\wedge,c}$, $\eta_{T_2}^{\zeta,c}$ with probability at least $c\,\ell^{-12}$.

Next, conditioned on the event E we see that from time T_2 up to time $T_2 + T_1$ the two processes evolve (in the regions I_1 and I_3 only) with the same boundary conditions (equal to the minimal configuration at both $-L + \ell^2$ and $L - \ell^2$). Since $T_1 = L^{\varepsilon}$ and the mixing time of the system in the regions I_1 and I_3 (which evolve independently) is certainly at most $e^{O(\ell^2)}$, with probability $1 + O(L^{-p})$ (conditionally on event E) we have that the two configurations coincide in both regions I_1, I_2 at time $T_2 + T_1$. Therefore if we let the system run only in I_2 now for an additional time T_2 we have a probability close to 1 to have the two configurations coinciding everywhere. It follows that, conditionally on the event E, there is a probability of, say, at least 1/2 of no discrepancy between $\eta_{T'}^{\wedge,c}$ and $\eta_{T'}^{\zeta,c}$. Therefore, letting $\mathbb{P}$ denote the coupling described above,

$$\begin{aligned}
\|\mu_{T'}^{\wedge,c} - \mu_{T'}^{\zeta,c}\| &\leqslant \mathbb{P}(\eta_{T'}^{\wedge,c} \neq \eta_{T'}^{\zeta,c}) \\
&\leqslant \mathbb{P}(\eta_{T'}^{\wedge,c} \neq \eta_{T'}^{\zeta,c} \,|\, E)\mathbb{P}(E) + 1 - \mathbb{P}(E) \\
&\leqslant 1 - \frac{1}{2}\,\mathbb{P}(E)\,.
\end{aligned}$$

Since $\mathbb{P}(E) \geqslant c/\ell^{12} \geqslant 2/L^{\varepsilon}$, for $L \geqslant L_0(\varepsilon)$, this implies (2.32). □

We now go back to the model with no top/bottom boundaries, that is the law ν_t^η corresponds to the evolution with $\xi = \wedge$, $\zeta = \vee$. The next result is crucially based on estimates obtained in [7, Section 6] for the delocalized regime $\lambda < 1$.

Lemma 2.7. *Uniformly in* $t \leqslant L^{5/2}(\log L)^{-9}$,

$$\nu_t^\wedge(\Omega^+) = 1 + o(1)\,.$$

Proof. Define the event $A = \{ \sum_{x=-L}^{L} \eta_x < L^{3/2}(\log L)^{-3} \}$. Proposition 6.2 in [7] proves that $\nu_t^\wedge(A) = o(1)$ uniformly in $t \leqslant L^{5/2}(\log L)^{-9}$. Since $\Omega^- \subset A$ we have

$$\nu_t^\wedge(\Omega^-) = o(1)\,, \qquad t \leqslant L^{5/2}(\log L)^{-9}\,. \tag{2.33}$$

Next, let us check that

$$\nu_t^\wedge(\Omega^+ \,|\, (\Omega^-)^c) \geqslant \pi(\Omega^+ \,|\, (\Omega^-)^c)\,, \qquad t \geqslant 0\,. \tag{2.34}$$

To this end, observe that since $(\Omega^-)^c$ is increasing, using Lemma 2.3 the function

$$f(\sigma) = \mathbf{1}_{(\Omega^-)^c}(\sigma)\,\frac{\nu_t^\wedge(\sigma \,|\, (\Omega^-)^c)}{\pi(\sigma \,|\, (\Omega^-)^c)} = \mathbf{1}_{(\Omega^-)^c}(\sigma)\,\frac{\nu_t^\wedge(\sigma)}{\pi(\sigma)}\,\frac{\pi((\Omega^-)^c)}{\nu_t^\wedge((\Omega^-)^c)}\,,$$

is increasing. Since $\Omega^+ \subset (\Omega^-)^c$ is increasing, with the FKG property for π, this implies (2.34). From (2.33) and (2.34) we obtain

$$\nu_t^\wedge(\Omega^+) \geqslant (1+o(1))\nu_t^\wedge(\Omega^+ \,|\, (\Omega^-)^c) \geqslant (1+o(1))\pi(\Omega^+ \,|\, (\Omega^-)^c) = 1 + o(1)\,,$$

where the last bound follows from (2.27). □

Lemma 2.8. *For any* $\varepsilon > 0$, *uniformly in* $t \in [L^{2+\varepsilon}, L^{5/2}(\log L)^{-9}]$:

$$\|\nu_t^\wedge - \pi^+\| = o(1)\,,$$

where π^+ *is defined by* $\pi^+ = \pi(\cdot \,|\, \Omega^+)$.

Proof. Using Lemma 2.7 it is enough to prove

$$\|\nu_t^\wedge(\cdot \,|\, \Omega^+) - \pi^+\| = o(1)\,,$$

uniformly in $t \in [L^{2+\varepsilon}, L^{5/2}(\log L)^{-9}]$. Consider the function $f : \Omega \mapsto \mathbb{R}$ given by

$$f(\sigma) = 1_{\Omega^+}(\sigma)\,\frac{\nu_t^\wedge(\sigma \,|\, \Omega^+)}{\pi^+(\sigma)} = \mathbf{1}_{\Omega^+}(\sigma)\,\frac{\nu_t^\wedge(\sigma)}{\pi(\sigma)}\,\frac{\pi(\Omega^+)}{\nu_t^\wedge(\Omega^+)}\,.$$

Since Ω^+ is increasing, Lemma 2.3 shows that f is increasing. Therefore, the event

$$A = \{\sigma \in \Omega^+ \,:\, \nu_t^\wedge(\sigma \,|\, \Omega^+) > \pi^+(\sigma)\}$$

is increasing. Using monotonicity we have

$$\nu_t^\wedge(A \,|\, \Omega^+) = \nu_t^\wedge(A)/\nu_t^\wedge(\Omega^+) \leqslant \mu_t^\wedge(A)/\nu_t^\wedge(\Omega^+)\,,$$

where $\mu_t^\wedge$ denotes the evolution constrained to stay in Ω^+; see Proposition 2.6. Therefore,

$$\|\nu_t^\wedge(\cdot \,|\, \Omega^+) - \pi^+\| = \nu_t^\wedge(A \,|\, \Omega^+) - \pi^+(A) \leqslant \frac{\mu_t^\wedge(A)}{\nu_t^\wedge(\Omega^+)} - \pi^+(A)\,.$$

The conclusion now follows from Lemma 2.7 and Proposition 2.6. □

The full power of Lemma 2.8 will be seen in the next sections. One of its consequences is the fact that the mixing time T_{mix} can be bounded in terms of the relaxation time via

$$T_{\mathrm{mix}} \leqslant L^{2+\varepsilon} + c\,T_{\mathrm{rel}} \log L\,, \tag{2.35}$$

for some constant $c > 0$. Indeed, (2.35) follows quite easily from Lemma 2.4, Lemma 2.8 and (2.6); see Lemma 4.1 below for a more subtle application of the same reasoning. Note that, since $T_{\rm rel} \gg L^{5/2-\varepsilon}$, the bound (2.35) improves considerably the standard estimate (2.8) by replacing the factor $-\log \pi_* = O(L)$ with a factor $O(\log L)$.

2.7. **Characterization of the principal eigenfunction.** What follows refers to the model with no top/bottom boundaries. Recall that $\wedge$, (resp. $\vee$) denotes the maximal (resp. minimal) configuration in Ω. A function $g : \Omega \mapsto \mathbb{R}$ is called antisymmetric if $g(-\eta) = -g(\eta)$ for all $\eta \in \Omega$. The following result gives a precise characterization of one eigenfunction corresponding to $-$gap.

Proposition 2.9. *There exists an increasing antisymmetric eigenfunction g of $\mathcal{L}$, such that $\|g\|_{\mathbb{L}_2(\pi)} = 1$. It satisfies*

$$\mathcal{L}g = -\text{gap } g. \tag{2.36}$$

Moreover, when L tends to infinity

$$g(\wedge) = \|g\|_{\mathbb{L}_\infty} = 1 + o(1)$$

and

$$\|g - (\mathbf{1}_{\Omega^+} - \mathbf{1}_{\Omega^-})\|_{\mathbb{L}_1(\pi)} = o(1). \tag{2.37}$$

Proof. By decomposing $\mathbf{1}_\wedge - \mathbf{1}_\vee$ on a basis of eigenfunctions of $\mathcal{L}$ one sees that

$$g := \lim_{t\to\infty} \frac{P_t(\mathbf{1}_\wedge - \mathbf{1}_\vee)}{\|P_t(\mathbf{1}_\wedge - \mathbf{1}_\vee)\|_{\mathbb{L}_2(\pi)}}$$

is an eigenfunction with unit $\mathbb{L}_2$ norm. It is increasing and antisymmetric as $P_t(\mathbf{1}_\wedge - \mathbf{1}_\vee)$ is antisymmetric and increasing for all t (P_t preserves monotonicity and symmetries). To prove (2.36), it suffices to show that the projection of $\mathbf{1}_\wedge - \mathbf{1}_\vee$ on the eigenspace of $\mathcal{L}$ associated to $-$gap is non-zero. To do so, first observe that by reversibility,

$$\frac{1}{\pi(\wedge)}\|P_t(\mathbf{1}_\wedge - \mathbf{1}_\vee)\|_{\mathbb{L}_1(\pi)} = 2\,\|\nu_t^\wedge - \nu_t^\vee\|.$$

Then, by the second point of Lemma 2.4

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\log\|P_t(\mathbf{1}_\wedge - \mathbf{1}_\vee)\|_{\mathbb{L}_2(\pi)} &= \lim_{t\to\infty} \frac{1}{t}\log\|P_t(\mathbf{1}_\wedge - \mathbf{1}_\vee)\|_{\mathbb{L}_1(\pi)} \\ &= \lim_{t\to\infty} \frac{1}{t}\log\|\nu_t^\wedge - \nu_t^\vee\| = -\text{gap}\,,\end{aligned}$$

(where we used equivalence of the norms in finite dimensional spaces).

We now estimate the $\mathbb{L}_\infty$ norm of g. Let $\varepsilon > 0$ be small and fixed, and let t_0 be such that $L^{2+\varepsilon} < t_0 < L^{5/2-\varepsilon}$. The function g is an eigenfunction for $P_{t_0} = \exp(t_0\mathcal{L})$, with eigenvalue $e^{-t_0\text{gap}}$. Therefore

$$e^{-t_0\text{gap}} g(\wedge) = \mathbb{E}\left[g(v_{t_0}^\wedge)\right] \leqslant \pi^+(g) + 2g(\wedge)\|\nu_{t_0}^\wedge - \pi^+\|,$$

where the last inequality follows from the fact that for any two measures μ, ν and any function f,

$$|\mu(f) - \nu(f)| \leqslant 2\|f\|_{\mathbb{L}_\infty}\|\mu - \nu\|\,.$$

Hence, by Lemma 2.8 and the fact that $\text{gap}^{-1} \gg t_0$ (cf. (1.4)):

$$g(\wedge) \leqslant \frac{\pi^+(g)}{e^{-t_0\text{gap}} - 2\|\nu_{t_0}^\wedge - \pi^+\|} = \pi^+(g)(1 + o(1)). \tag{2.38}$$

Moreover, by symmetry and Jensen's inequality

$$1 = \pi(g^2) \geqslant 2 \sum_{\eta \in \Omega^+} \pi(\eta) g(\eta)^2 = 2\pi(\Omega^+)\pi^+(g^2) \geqslant 2\pi(\Omega^+)\pi^+(g)^2 \,,$$

so that

$$\pi^+(g) \leqslant (2\pi(\Omega^+))^{-1/2} = 1 + o(1).$$

Therefore $g(\wedge) \leqslant 1 + o(1)$ (and it is trivial to notice that $\|g\|_{\mathbb{L}_\infty} \geqslant \|g\|_{\mathbb{L}_2(\pi)} = 1$).

We turn to the proof of (2.37). First notice that by (2.38) one has $\pi^+(g) \geqslant (1+o(1))g(\wedge) \geqslant 1+o(1)$ so that

$$\pi^+(g) = 1 + o(1). \tag{2.39}$$

Next, we prove that the variation of g within Ω^+ is small:

$$\mathrm{Var}_{\pi^+}(g) = o(1). \tag{2.40}$$

Indeed, $\mathrm{Var}_{\pi^+}(g) = \pi^+(g^2) - \pi^+(g)^2 \leqslant g(\wedge)^2 - \pi^+(g)^2$ and the claim follows from (2.38). Next,

$$\|g - \mathbf{1}_{\Omega^+} + \mathbf{1}_{\Omega^-}\|_{\mathbb{L}_1(\pi)} \leqslant \|\mathbf{1}_{\Omega^+}(g-1)\|_{\mathbb{L}_1(\pi)} + \|\mathbf{1}_{\Omega^-}(g+1)\|_{\mathbb{L}_1(\pi)} + \|g\mathbf{1}_{\Omega\setminus(\Omega^+\cup\Omega^-)}\|_{\mathbb{L}_1(\pi)} \,.$$

The first two terms of the right-hand side are equal by symmetry. Adding and subtracting $\pi^+(g)$, and using Schwarz' inequality,

$$\|\mathbf{1}_{\Omega^+}(g-1)\|_{\mathbb{L}_1(\pi)} \leqslant \pi(\Omega^+)\left[|\pi^+(g) - 1| + \sqrt{\mathrm{Var}_{\pi^+}(g)}\right] = o(1)\,,$$

where the conclusion follows from (2.39) and (2.40). The third term $\|g\mathbf{1}_{\Omega\setminus(\Omega^+\cup\Omega^-)}\|_{\mathbb{L}_1(\pi)}$ is smaller than $\|g\|_{\mathbb{L}_\infty}\pi(\Omega\setminus(\Omega^+\cup\Omega^-)) = o(1)$. □

3. Metastability

In this section we first prove Theorem 1.5, which is mainly a consequence of the technical lemmas of the previous section and then move to the proof of Theorem 1.3 and its corollary.

3.1. **Proof of Theorem 1.5.** We use the notation $T = L^{2+\delta}$. Equation (1.11) is an easy consequence of (1.9) and (2.28). Indeed, assuming (1.9), for $t \geqslant T$, one has

$$\|P_t(\wedge,\cdot) - \pi\| = \left\|\left[\frac{1+e^{-t/T_{\mathrm{rel}}}}{2}\pi^+ + \frac{1-e^{-t/T_{\mathrm{rel}}}}{2}\pi^-\right] - \pi\right\| + o(1) = \frac{1}{2}e^{-t/T_{\mathrm{rel}}} + o(1). \tag{3.1}$$

To prove (1.9), one writes

$$\begin{aligned}\Big\|\nu_t^\wedge - \Big[\frac{1+e^{-t/T_{\mathrm{rel}}}}{2}\pi^+ + \frac{1-e^{-t/T_{\mathrm{rel}}}}{2}\pi^-\Big]\Big\| \\ \leqslant \big\|\nu_T^\wedge P_{t-T} - \pi^+ P_{t-T}\big\| + \Big\|\pi^+ P_{t-T} - \Big[\frac{1+e^{-t/T_{\mathrm{rel}}}}{2}\pi^+ + \frac{1-e^{-t/T_{\mathrm{rel}}}}{2}\pi^-\Big]\Big\|.\end{aligned} \tag{3.2}$$

This is just triangular inequality, combined with the observation that $\nu_T^\wedge P_{t-T} = \nu_t^\wedge$. The first term on the right hand side is smaller than $\|\nu_T^\wedge - \pi^+\|$ (as P_{t-T} contracts the norm) which is itself small, by Lemma 2.8 and the definition of T. It remains to estimate the second term i.e. to prove (1.10).

To do this, we use the fact that the density of π^+ w.r.t. π is very close to $g+1$, where g is the eigenfunction described in Proposition 2.9, so that the density of $\pi^+ P_t$ must be close to

$P_t(g+1)$. Using reversibility, one can express the densities as follows: $\frac{\mathrm{d}\pi^+ P_t}{\mathrm{d}\pi} = P_t \frac{\mathrm{d}\pi^+}{\mathrm{d}\pi}$. Then we rewrite the second term in (3.2) as an $\mathbb{L}_1$ norm (omitting a harmless factor 1/2)

$$\begin{aligned}\Big\| P_{t-T}\frac{\mathrm{d}\pi^+}{\mathrm{d}\pi} - \frac{1}{2\pi(\Omega^+)}\left(\mathbf{1}_{\Omega^+}+\mathbf{1}_{\Omega^-}\right) - \frac{1}{2\pi(\Omega^+)}\, e^{-t/T_{\mathrm{rel}}}\left(\mathbf{1}_{\Omega^+}-\mathbf{1}_{\Omega^-}\right)\Big\|_{\mathbb{L}_1(\pi)} \\ \leqslant \Big\| P_{t-T}\frac{\mathrm{d}\pi^+}{\mathrm{d}\pi} - \frac{1}{2\pi(\Omega^+)}\left(\mathbf{1}_{\Omega^+}+\mathbf{1}_{\Omega^-}\right) - \frac{1}{2\pi(\Omega^+)}\, e^{-t/T_{\mathrm{rel}}} g\Big\|_{\mathbb{L}_1(\pi)} \\ + \frac{1}{2\pi(\Omega^+)}\, \| e^{-t/T_{\mathrm{rel}}}\left(\mathbf{1}_{\Omega^+}-\mathbf{1}_{\Omega^-}-g\right)\|_{\mathbb{L}_1(\pi)}.\end{aligned}$$

The last term above is small by Proposition 2.9. From (2.27) we know that $2\pi(\Omega^+) = 1 + o(1)$. One can then estimate the first term

$$\begin{aligned}\Big\| P_{t-T}\frac{\mathrm{d}\pi^+}{\mathrm{d}\pi} - \frac{1}{2\pi(\Omega^+)}\left(\mathbf{1}_{\Omega^+}+\mathbf{1}_{\Omega^-}\right) - \frac{1}{2\pi(\Omega^+)}\, e^{-t/T_{\mathrm{rel}}} g\Big\|_{\mathbb{L}_1(\pi)} \\ \leqslant \|\left(\mathbf{1}_{\Omega^+}+\mathbf{1}_{\Omega^-}\right) - \mathbf{1}\|_{\mathbb{L}_1(\pi)} + \Big\| P_t\Big(\frac{\mathrm{d}\pi^+}{\mathrm{d}\pi} - \mathbf{1} - g\Big)\Big\|_{\mathbb{L}_1(\pi)} + o(1)\,, \quad (3.3)\end{aligned}$$

where we used the triangular inequality, the fact that $P_t\mathbf{1} = \mathbf{1}$, and

$$P_{t-T} g = e^{-(t-T)/T_{\mathrm{rel}}} g = (e^{-t/T_{\mathrm{rel}}} + o(1)) g\,,$$

which follows from $T = o(T_{\mathrm{rel}})$. On the right hand-side of (3.3), the first term is small by (2.28) and the second is bounded by $\|\frac{\mathrm{d}\pi^+}{\mathrm{d}\pi} - \mathbf{1} - g\|_{\mathbb{L}_1(\pi)}$, which is small by Proposition 2.9. □

3.2. **Proof of Theorem 1.3.** Theorem 1.5 gives some intuition on why the result should be true, and it will be used to determine the time of the jump from one state to the other. However, one needs another key ingredient to get the result, namely the description of the quasi-stationary distribution. The reason for this is that starting from the quasi-stationary distribution, a killed process dies exactly at exponential rate; see Section 2.3. Therefore, most of our effort will focus on stochastic comparison with the quasi-stationary distribution. Let us first give a brief roadmap to help the reader through the proof of Theorem 1.3.

Step 1. The sets $S^{\pm}$ of Theorem 1.3 for which we have the desired exponential hitting time description are constructed by successively refining a first attempt. One first defines $S^{0,\pm}$ as the sets of polymer configurations where the eigenfunction g in Proposition 2.9 is positive (negative) and one verifies that their equilibrium probability is $\frac{1}{2} + o(1)$. Then one examines the Dirichlet problem associated to the process killed in $S^{0,-}$ ($S^{0,+}$) and one proves that the corresponding eigenvalue γ_0 is of the same order as the spectral gap apart from a crucial unspecified multiplicative factor in $[1/2, 1]$. Similarly one verifies that the corresponding quasi-stationary measure is very close to the equilibrium measure π conditioned to be in $S^{0,+}$ ($S^{0,-}$). In this way we get the exponentiality of the hitting time of e.g. $S^{0,-}$ starting from π^+ with a rate which is, modulo a multiplicative factor in $[1/2, 1]$, the spectral gap (see Lemma 3.3).

Step 2. Next one appropriately defines new sets $S^{1,\pm} \subset S^{0,\pm}$ in order to guarantee that this time the corresponding Dirichlet eigenvalue γ_1 is equal to $(\frac{1}{2} + o(1))$gap, and that the hitting time of $S^{1,\mp}$ starting from equilibrium conditioned to $S^{1,\pm}$ is exponential (with the correct rate). Again one of the key points is to show that π^+ is close to the quasi-stationary distribution associated the process killed on entering $S^{1,-}$, and that the equilibrium probability of $S^{1,\pm}$ is still $\frac{1}{2} + o(1)$.

Step 3. Finally, one defines the sets $S^{2,\pm} \subset S^{1,\pm}$ in such a way that: a) the hitting time of $S^{2,\mp}$ starting from *any* configuration in $S^{2,\pm}$ (and not just from the conditional equilibrium) is also exponential with the correct rate $\frac{1}{2}$gap; b) the equilibrium probability of $S^{2,\pm}$ is still $\frac{1}{2}+o(1)$.

It is now time to begin the implementation of the above strategy. Let

$$S^{0,+} := \{\eta \in \Omega,\ g(\eta) > 0\}\,, \tag{3.4}$$

where g is the eigenfunction defined by Proposition 2.9 and set $S^{0,-} = -S^{0,+}$. From Proposition 2.9,

$$\|\mathbf{1}_{S^{0,+}} - \mathbf{1}_{\Omega^+}\|_{\mathbb{L}_1(\pi)} = o(1). \tag{3.5}$$

In particular, $\pi(S^{0,\pm}) = 1/2 + o(1)$. Let $S^- \subset S^{0,-}$ be a decreasing event. We consider the quasi-stationary distribution $\nu^+ := \nu_{S^-}$ of the process killed when it hits S^-. Let $P_t^* = P_t^{S^-}$, resp. $\mathcal{L}^* = \mathcal{L}^{S^-}$, denote the semi-group, resp. the generator, associated to this process (see Section 2.3), $-\gamma_{S^-} = -\gamma$ be the largest eigenvalue of $\mathcal{L}^*$ and $\tau^- = \tau_{S^-}$ be the hitting time of S^-. From (2.16):

$$\mathbb{P}^{\nu^+}(\tau^- > t) = e^{-\gamma t}. \tag{3.6}$$

Our first step is to prove that if S^- has non-negligible measure, then γ is of the same order of the gap. More precisely:

Lemma 3.1. *For any $S^- \subset S^{0,-}$, one has $\pi(S^-) \leqslant \gamma\, T_{\mathrm{rel}} \leqslant 1$.*

Proof. The bound $\pi(S^-) \leqslant \gamma\, T_{\mathrm{rel}}$ is rather standard, but we include its proof for the sake of completeness. Let $f_0 = g_{S^-}$ denote the minimizer in the variational principle defining $\gamma = \gamma_{S^-}$; see (2.13). Then

$$\mathrm{Var}_\pi(f_0) = \langle f_0, f_0\rangle_\pi - \left\langle f_0, \mathbf{1}_{(S^-)^c}\right\rangle_\pi^2 \geqslant \langle f_0, f_0\rangle_\pi\, \pi(S^-)\,,$$

where we used the Cauchy-Schwarz inequality for $\left\langle f_0, \mathbf{1}_{(S^-)^c}\right\rangle_\pi^2$. Therefore

$$\gamma = \frac{\mathcal{E}(f_0, f_0)}{\pi(f_0^2)} \geqslant \pi(S^-)\, \frac{\mathcal{E}(f_0, f_0)}{\mathrm{Var}_\pi(f_0)} \geqslant \pi(S^-)\,\mathrm{gap}\,.$$

As for the bound $\gamma\, T_{\mathrm{rel}} \leqslant 1$, γ_{S^-} being a non-decreasing function of S^- (for the inclusion), it is sufficient to prove the result for the maximal case $S^- = S^{0,-}$. Let g be the eigenfunction defined in Proposition 2.9. From (2.12), for all $\eta \in (S^{0,-})^c$

$$-(\mathcal{L}^* g_{|_{(S^{0,-})^c}})(\eta) = -(\mathcal{L}g)(\eta) + \sum_{\eta' \in S^{0,-}} c(\eta, \eta') g(\eta') \leqslant -(\mathcal{L}g)(\eta) = \mathrm{gap}\ g(\eta)\,,$$

where we use the fact that $g(\eta') < 0$ for $\eta' \in S^{0,-}$. Plugging this into (2.13), and using $g_{|_{(S^{0,-})^c}} \geqslant 0$, one gets

$$\gamma \leqslant \frac{\left\langle -\mathcal{L}^* g_{|_{(S^{0,-})^c}}, g_{|_{(S^{0,-})^c}}\right\rangle_\pi}{\pi\big(g^2_{|_{(S^{0,-})^c}}\big)} \leqslant \mathrm{gap}\,.$$

□

Next, we prove that the quasi-stationary distribution ν^+ for the process killed on S^- is very close to π^+ if S^- has probability close to 1/2.

Lemma 3.2. *Uniformly for all decreasing events $S^- \subset S^{0,-}$,*

$$\|\nu^+ - \pi^+\| \leqslant (2 - 4\pi(S^-)) + o(1)\,.$$

Proof. We use triangular inequality to get

$$\|\nu^+ - \pi^+\| \leqslant \|\nu^+ - \pi(\cdot\ |(S^-)^c)\| + \|\pi^+ - \pi(\cdot\ |(S^-)^c)\|. \tag{3.7}$$

We start with the first term. First, from (2.15) one has the characterization

$$\nu^+ = \lim_{t\to\infty} \frac{\delta_\wedge P_t^*}{\delta_\wedge P_t^*(\Omega)}\,.$$

Since the operator P_t^* preserves monotonicity (S^- is decreasing), arguing as in [19, Lemma 16.6], the density $\frac{\mathrm{d}[\delta_\wedge P_t^*]}{\mathrm{d}\pi}$ is seen to be an increasing function for every fixed $t \geqslant 0$. Hence, passing to the limit $t \to \infty$, $\mathrm{d}\nu^+/\mathrm{d}\pi$ is an increasing function. Therefore,

$$A := \left\{\eta \in (S^-)^c,\ \text{such that}\ \frac{\nu^+(\eta)\pi((S^-)^c)}{\pi(\eta)} > 1\right\}$$

is an increasing event. From standard properties of the total variation distance

$$\|\nu^+ - \pi(\cdot|(S^-)^c\| = \nu^+(A) - \pi(A\ |(S^-)^c).$$

We shall prove that $\nu^+(A)$ is smaller than $\pi^+(A) + o(1)$ by the use of monotonicity and a chain of comparisons. Recall the notation $T = L^{2+\delta}$ ($\delta \in (0, 1/4)$). We first compare ν^+ to $\nu^+ P_T$: remark that

$$\nu^+ P_T = \nu^+ P_T^* + \nu^+(P_T - P_T^*)$$

where the two terms of the decomposition are positive measures. From quasi-stationarity one has $\nu^+ P_T^* = e^{-\gamma T}\nu^+$ and therefore the total mass of the second term above is $1 - e^{-\gamma T}$. Hence

$$\|\nu^+ P_T - \nu^+\| = \frac{1}{2}\left\|\frac{\mathrm{d}[\nu^+(P_T - P_T^*)]}{\mathrm{d}\pi} - \frac{\mathrm{d}\nu^+}{\mathrm{d}\pi}(1 - e^{-\gamma T})\right\|_{\mathbb{L}_1(\pi)} \leqslant 1 - e^{-\gamma T} = o(1). \tag{3.8}$$

The last equality comes from Lemma 3.1 and the fact that $T_{\mathrm{rel}} \gg T$. Next, $\delta_\wedge P_T$ stochastically dominates $\nu^+ P_T$ so that $[\nu^+ P_T](A) \leqslant [\delta_\wedge P_T](A)$. Hence, from Lemma 2.8 and (3.8):

$$\begin{aligned}\nu^+(A) &\leqslant \nu^+ P_T(A) + o(1) \leqslant \delta_\wedge P_T(A) + o(1)\\ &\leqslant \pi^+(A) + \|\delta_\wedge P_T - \pi^+\| + o(1) = \pi^+(A) + o(1).\end{aligned}$$

Therefore, going back to (3.7)

$$\begin{aligned}\|\nu^+ - \pi^+\| &\leqslant \nu^+(A) - \pi(A\ |(S^-)^c) + \|\pi^+ - \pi(\cdot\ |(S^-)^c)\| \qquad (3.9)\\ &\leqslant \pi^+(A) - \pi(A\ |(S^-)^c) + \|\pi^+ - \pi(\cdot\ |(S^-)^c)\| + o(1) \leqslant 2\|\pi^+ - \pi(\cdot\ |(S^-)^c)\| + o(1)\,.\end{aligned}$$

To estimate the right-hand side of (3.9), notice that

$$\|\pi^+ - \pi(\cdot\ |(S^-)^c)\| \leqslant \|\pi^+ - \pi(\cdot\ |(S^{0,-})^c)\| + \|\pi(\cdot\ |(S^{0,-})^c) - \pi(\cdot\ |(S^-)^c)\|\,, \tag{3.10}$$

and the first term is $o(1)$ by Proposition 2.9. Moreover, since $S^- \subset S^{0,-}$

$$\begin{aligned}\|\pi(\cdot\ |(S^{0,-})^c) - \pi(\cdot\ |(S^-)^c)\| &= \frac{\pi((S^-)^c) - \pi((S^{0,-})^c)}{\pi((S^-)^c)}\\ &= \frac{1/2 - \pi(S^-) + o(1)}{1 - \pi(S^-)} \leqslant 1 - 2\pi(S^-) + o(1)\,. \qquad (3.11)\end{aligned}$$

Combining (3.9), (3.10) and (3.11), the desired result follows. □

Now one uses the fact that ν^+ and π^+ are close in total variation distance to estimate the jumping time to S^- starting from either $\wedge$ or from π^+. For the rest of this section, one defines, in analogy with τ^-, the hitting times $\tau^{i,-}$ (resp. $\tau^{i,+}$), ($i = 0, 1, 2$) of the sets $S^{i,-}$ (resp. $S^{i,+}$) to be defined.

Lemma 3.3. *Uniformly for all $t \geqslant 0$ and all decreasing $S^- \subset S^{0,-}$, setting $\gamma = \gamma_{S^-}$:*

(i) $|\mathbb{P}^\wedge[\tau^- > t] - e^{-\gamma t}| \leqslant (2 - 4\pi(S^-)) + o(1)$

(ii) $|\mathbb{P}^{\pi^+}[\tau^- > t] - e^{-\gamma t}| \leqslant (2 - 4\pi(S^-)) + o(1)$.

In particular, for $S^- = S^{0,-}$, setting $\gamma_0 = \gamma_{S^{0,-}}$:

(iii) $\mathbb{P}^\wedge\left[\tau^{0,-} > t\right] = e^{-\gamma_0 t} + o(1)$

(iv) $\mathbb{P}^{\pi^+}\left[\tau^{0,-} > t\right] = e^{-\gamma_0 t} + o(1)$.

Proof. Item (ii) follows from (3.6) and Lemma 3.2. Indeed,

$$|\mathbb{P}^{\pi^+}\left[\tau^- > t\right] - \mathbb{P}^{\nu^+}\left[\tau^- > t\right]| \leqslant \|\pi^+ - \nu^+\|\,.$$

For item (i) (lower bound), we use the fact that S^- is a decreasing event to get that

$$\mathbb{P}^\wedge\left[\tau^- > t\right] \geqslant \mathbb{P}^{\nu^+}\left[\tau^- > t\right] = e^{-\gamma t}.$$

For the upper bound it is sufficient to prove the result for $t \geqslant T = L^{2+\delta}$, since $\gamma^{-1} \gg T$ (a consequence of Lemma 3.1 and (1.4)). One defines

$$\bar{\tau}^- = \inf\{t \geqslant T,\ \eta(t) \in S^-\}.$$

Then, by the Markov property and quasi-stationarity

$$\begin{aligned}\mathbb{P}^\wedge\left[\tau^- > t\right] \leqslant \mathbb{P}^\wedge\left[\bar{\tau}^- > t\right] = \mathbb{P}^{\delta_\wedge P_T}\left[\tau^- > t - T\right] &\leqslant \|\delta_\wedge P_T - \nu^+\| + e^{-\gamma(t-T)} \\ &\leqslant \|\delta_\wedge P_T - \pi^+\| + \|\pi^+ - \nu^+\| + e^{-\gamma t} + o(1)\,, \quad (3.12)\end{aligned}$$

where we use $e^{-\gamma(t-T)} = e^{-\gamma t} + o(1)$, which follows from $\gamma^{-1} \gg T$. The result then follows from Lemma 2.8 and Lemma 3.2. Items (iii) and (iv) are consequences of (i) and (ii) and the fact that $\pi(S^{0,-}) = 1/2 + o(1)$. □

From the previously stated results, one may conclude that there exists δ_L, a decreasing sequence tending to zero when L tends to infinity, such that for every $t > 0$:

$$\begin{aligned}\mathbb{P}^\wedge\left[\tau^{0,-} > t\right] &\leqslant e^{-\gamma_0 t} + \delta_L, \\ \mathbb{P}^{\pi^+}\left[\tau^{0,-} > t\right] &\geqslant e^{-\gamma_0 t} - \delta_L, \\ \pi(\Omega^+) &\geqslant 1/2 - \delta_L \\ \pi^+(S^{0,+}) &\geqslant 1 - \delta_L.\end{aligned} \tag{3.13}$$

Given such a δ_L, one defines $S^{1,+}$ to be

$$S^{1,+} := S^{0,+} \cap \{\eta \in \Omega^+ \,:\, \mathbb{P}^\eta\left[\tau^{0,-} > t\right] \geqslant e^{-t\gamma_0} - 3(\delta_L)^{1/4}\,,\ \forall t > 0\ \},$$

and $S^{1,-} = -S^{1,+}$.

Lemma 3.4. *The set $S^{1,+}$ satisfies*

$$\begin{aligned}\pi(S^{1,+}) &= \frac{1}{2} + o(1)\,, \quad \textit{and} \\ \mathbb{P}^\eta\left[\tau^{0,-} > t\right] &= e^{-t\gamma_0} + o(1)\,, \quad \textit{uniformly in } t \geqslant 0\,,\ \textit{and } \eta \in S^{1,+}.\end{aligned} \tag{3.14}$$

Proof. The lower bound in the second point follows from the definition of $S^{1,\pm}$. For the upper bound, it is just a consequence of the fact that

$$\mathbb{P}^\xi\left[\tau^{0,-} > t\right] \leqslant \mathbb{P}^\wedge\left[\tau^{0,-} > t\right] \leqslant e^{-\gamma_0 t} + \delta_L\,, \tag{3.15}$$

for any $\xi \in \Omega$, by monotonicity, where the last bound is the first line in (3.13).

We turn to a proof of the first point. For $t \geqslant 0$, one defines

$$S^{1,+,t} := \{\eta \in \Omega^+ : \mathbb{P}^\eta \left[\tau^{0,-} > t\right] \geqslant e^{-t\gamma_0} - 2\delta_L^{1/4}\}.$$

From the second line in (3.13), and using (3.15) for all $\xi \in S^{1,+,t}$:

$$e^{-\gamma_0 t} - \delta_L \leqslant \mathbb{P}^{\pi^+}\left[\tau^{0,-} > t\right] \leqslant \pi^+(S^{1,+,t})(e^{-\gamma_0 t} + \delta_L) + (1 - \pi^+(S^{1,+,t}))(e^{-\gamma_0 t} - 2\delta_L^{1/4}). \quad (3.16)$$

This gives $-2\delta_L \leqslant 2\delta_L^{1/4}\pi^+(S^{1,+,t})) - 2\delta_L^{1/4}$, i.e.

$$\pi^+(S^{1,+,t}) \geqslant 1 - \delta_L^{3/4}. \quad (3.17)$$

Next, define $S_i := S^{1,+,i\delta_L^{1/4}\gamma_0^{-1}}$, $i \in \mathbb{N}$. We claim that for all L sufficiently large:

$$S^{1,+} \supset \bigcap_{i=1}^{\lfloor \delta_L^{-1/2} \rfloor} S_i \cap S^{0,+}. \quad (3.18)$$

Indeed, let $\eta \in \bigcap_{i=1}^{\lfloor \delta_L^{-1/2} \rfloor} S_i$. If $t \leqslant \delta_L^{-1/4}\gamma_0^{-1}/2$, then clearly $\eta \in S_{\lceil t\gamma_0\delta_L^{-1/4}\rceil}$ if L is large enough. In particular,

$$\mathbb{P}^\eta[\tau^{0,-} > t] \geqslant \mathbb{P}^\eta[\tau^{0,-} > \lceil t\gamma_0\delta_L^{-1/4}\rceil\delta_L^{1/4}\gamma_0^{-1}] \geqslant e^{-\gamma_0 t - \delta_L^{1/4}} - 2\delta_L^{1/4} \geqslant e^{-\gamma_0 t} - 3\delta_L^{1/4}. \quad (3.19)$$

If on the other hand $t \geqslant \delta_L^{-1/4}\gamma_0^{-1}/2$, then

$$e^{-t\gamma_0} - 3\delta_L^{1/4} \leqslant 0 \leqslant \mathbb{P}^\eta[\tau^{0,-} > t],$$

provided δ_L is small enough, i.e. L is large enough. This proves (3.18). Moreover one has, from (3.17) and the fourth line of (3.13),

$$\pi^+\Big(\cap_{i=1}^{\lfloor \delta_L^{-1/2} \rfloor} S_i \cap S^{0,+}\Big) \geqslant 1 - \delta_L^{1/4} - \delta_L.$$

From the third line of (3.13), one gets that

$$\pi\Big(\cap_{i=1}^{\lfloor \delta_L^{-1/2} \rfloor} S_i \cap S^{0,+}\Big) \geqslant (1 - \delta_L^{1/4} - \delta_L)(1/2 - \delta_L).$$

This last estimate together with (3.18) implies the first statement of the lemma. □

The previous results allow us to compute the value of $\gamma_1 = \gamma_{S^{1,-}}$.

Lemma 3.5. *Let $-\gamma_1$ be the largest eigenvalue of $\mathcal{L}^{S^{1,-}}$, the generator of the process killed when it hits $S^{1,-}$. We have*

$$\gamma_1 T_{\text{rel}} = 1/2 + o(1).$$

It is important to recall that, in contrast to γ_1, the eigenvalue γ_0 of the process killed in $\mathcal{L}^{S^{0,-}}$ was estimated only up to a factor 2 (cf. Lemma 3.1).

Proof. The inequality $\gamma_1 T_{\text{rel}} \geqslant 1/2 + o(1)$ comes from Lemma 3.4 and Lemma 3.1. Recall the definitions

$$\tau^{1,-} = \inf\left\{t \geqslant 0, \eta(t) \in S^{1,-}\right\},$$
$$\tau^{0,+} = \inf\left\{t \geqslant 0, \eta(t) \in S^{0,+}\right\}.$$

According to Theorem 1.5, (3.5), Lemma 3.1, Lemma 3.3(i) and Lemma 3.4, one can find a new sequence δ_L going to zero such that

$$\begin{aligned}
\nu_t^\wedge((S^{0,+})^c) &\leqslant (1-e^{-t/T_{\rm rel}})/2+\delta_L \quad \text{for every} \quad t>0\,,\\
\mathbb{P}^\xi\left[\tau^{0,+}>t\right] &\geqslant e^{-\gamma_0 t}-\delta_L \quad \text{for every} \quad t\geqslant 0 \quad \text{and} \quad \xi\in S^{1,-}\,,\\
\mathbb{P}^\wedge\left[\tau^{1,-}>t\right] &\leqslant e^{-\gamma_1 t}+\delta_L \qquad \text{for every} \quad t\geqslant 0\,.
\end{aligned}$$

Next, define

$$\bar\tau^{0,+}=\inf\left\{t\geqslant \tau^{1,-}\ :\ \eta(t)\in S^{0,+}\right\}$$

to be the first time the process enters $S^{0,+}$ after entering for the first time $S^{1,-}$. One has

$$\begin{aligned}
\frac{1-e^{-t/T_{\rm rel}}}{2}+\delta_L &\geqslant \nu_t^\wedge((S^{0,+})^c)\geqslant \mathbb{P}^\wedge\left[\tau^{1,-}<t,\ \bar\tau^{0,+}>t\right]\\
&\geqslant \mathbb{E}^\wedge\left[\mathbf{1}\{\tau^{1,-}<t\}\mathbb{P}^{\eta(\tau^{1,-})}\left[\tau^{0,+}>t\right]\right]\geqslant \mathbb{P}^\wedge\left[\tau^{1,-}<t\right]\min_{\eta\in S^{1,-}}\mathbb{P}^\eta\left[\tau^{0,+}>t\right]\\
&\geqslant (1-e^{-\gamma_1 t}-\delta_L)(e^{-t\gamma_0}-\delta_L). \qquad (3.20)
\end{aligned}$$

We use this inequality for $t=\gamma_1^{-1}\delta_L^{1/3}$ and get (using the fact that $x-x^2\leqslant 1-\exp(-x)\leqslant x$ for x small enough)

$$\delta_L^{1/3}/(2\gamma_1 T_{\rm rel})\geqslant (\delta_L^{1/3}-\delta_L^{2/3}-\delta_L)(1-\delta_L^{1/3}\gamma_0/\gamma_1-\delta_L)-\delta_L.$$

As $1\leqslant \gamma_0/\gamma_1\leqslant 3$ (for L large enough, cf. Lemma 3.1), all of this gives us

$$\gamma_1 T_{\rm rel}\leqslant \frac{1}{2[(1-\delta_L^{1/3}-\delta_L^{2/3})(1-3\delta^{1/3}-\delta_L)-\delta_L^{2/3}]},$$

which ends the proof. □

Once again assume that δ_L is a sequence going to zero, this time such that one has for every $t>0$

$$\begin{aligned}
\mathbb{P}^\wedge\left[\tau^{1,-}>t\right] &\leqslant e^{-\gamma_1 t}+\delta_L,\\
\mathbb{P}^{\pi^+}\left[\tau^{1,-}>t\right] &\geqslant e^{-\gamma_1 t}-\delta_L,\\
\pi(S^{1,+}) &\geqslant 1/2-\delta_L.
\end{aligned} \qquad (3.21)$$

Note that the sets $S^{1,\pm}$ are not yet good candidates for the sets $S^\pm$ of Theorem 1.3, the reason being that (3.21) and Lemma 3.5 say that the hitting time of $S^{1,-}$ is exponential with the correct rate, but only if one starts from either the maximal configuration or from π^+, while we want this to hold uniformly in the initial condition in S^+. We need therefore a final step in order to fix this problem. We set

$$S^{2,+}:=\left\{\eta\in S^{1,+}\ :\ \mathbb{P}^\eta\left[\tau^{1,-}>t\right]\geqslant e^{-\gamma_1 t}-3\delta_L^{1/4}\ \ \forall t\geqslant 0\right\},$$

and define $S^{2,-}=-S^{2,+}$. The same computations of Lemma 3.4 prove

Lemma 3.6. $\pi(S^{2,+})=1/2+o(1)$.

Now we are ready to finish the proof of Theorem 1.3, with $S^+:=S^{2,+}$. Let $-\gamma_2$ be the largest eigenvalue of the generator $\mathcal{L}^{S^{2,-}}$ of the process killed when it reaches $S^{2,-}$. From Lemma 3.1, one has $\pi(S^{2,+})\,\mathrm{gap}\leqslant\gamma_2\leqslant\gamma_1$. Therefore, Lemma 3.6 yields

$$\gamma_2=(1/2+o(1))\,\mathrm{gap}\,. \qquad (3.22)$$

Let $\tau^{2,-}$ be the hitting time of $S^{2,-}$. For any $\eta \in S^{2,+}$ (this is actually true for any η in Ω), we get from monotonicity, Lemma 3.3(i), Lemma 3.6 and (3.22)

$$\mathbb{P}^\eta(\tau^{2,-} > t) \leqslant \mathbb{P}^\wedge(\tau^{2,-} > t) \leqslant e^{-\gamma_2 t} + o(1) + (2 - 4\pi(S^{2,-})) \leqslant e^{-\frac{t}{2T_{\mathrm{rel}}}} + o(1),$$

where $o(1)$ is uniform in t. On the other side, the definition of $S^{2,+}$ and the obvious bound $\tau^{2,-} \geqslant \tau^{1,-}$ give that for any $\eta \in S^{2,+}$

$$\mathbb{P}^\eta(\tau^{2,-} > t) \geqslant \mathbb{P}^\eta(\tau^{1,-} > t) \geqslant e^{-\gamma_1 t} + o(1) \geqslant e^{-\frac{t}{2T_{\mathrm{rel}}}} + o(1),$$

where the last inequality comes from Lemma 3.5. This ends the proof of Theorem 1.3. □

3.3. **Proof of Corollary 1.7.** We use the same notation as in the previous proof. We set $\Gamma := (S^{2,+} \cup S^{2,-})^c$, and denote the local time spent by the Markov chain $(\eta_s)_{s \geqslant 0}$ in Γ by

$$H_t = H_t(\Gamma) := \int_0^t \mathbf{1}_{\eta(s)\in\Gamma} \mathrm{d}s. \tag{3.23}$$

Notice that, if $\tilde{\pi}^{2,+} := \pi(\cdot \mid S^{2,+})$, then $\|\tilde{\pi}^{2,+} - \pi^+\| = o(1)$. Equation (1.10) implies that there exists a sequence δ_L going to zero such that

$$\mathbb{P}^{\tilde{\pi}^{2,+}}(\eta_s \in \Gamma) \leqslant \delta_L, \quad \forall s \geqslant 0.$$

Integrating between zero and $\delta_L^{-1/2} T_{\mathrm{rel}}$ one gets

$$\mathbb{E}^{\tilde{\pi}^{2,+}}\left[H_{\delta_L^{-1/2}T_{\mathrm{rel}}}\right] \leqslant \delta_L^{1/2} T_{\mathrm{rel}}\,.$$

We set

$$S^{3,+} := \left\{\eta \in S^{2,+} : \mathbb{E}^\eta\left[H_{\delta_L^{-1/2}T_{\mathrm{rel}}}\right] \leqslant \delta_L^{1/4} T_{\mathrm{rel}}\right\}.$$

By Markov's inequality we see that

$$\tilde{\pi}^{2,+}(S^{3,+}) \geqslant 1 - \delta_L^{1/4}$$

which immediately implies that $\pi(S^{3,+}) = 1/2 + o(1)$ since $\pi(S^{2,+}) = 1/2 + o(1)$. Using Markov's inequality again we obtain

$$\mathbb{P}^\eta\left[H_{\delta_L^{-1/2}T_{\mathrm{rel}}} \geqslant \delta_L^{1/8} T_{\mathrm{rel}}\right] \leqslant \delta_L^{1/8}, \quad \forall \eta \in S^{3,+}\,.$$

Moreover, by Theorem 1.3, there exists a sequence δ'_L going to zero such that

$$\mathbb{P}^\eta\left[\tau^{2,-} \in [0, \delta_L^{1/16} T_{\mathrm{rel}}] \cup [\delta_L^{-1/2} T_{\mathrm{rel}}, \infty)\right] \leqslant \delta'_L, \quad \forall \eta \in S^{2,+}.$$

On the event $\{\tau^{2,-} \in [\delta_L^{1/16} T_{\mathrm{rel}}, \delta_L^{-1/2} T_{\mathrm{rel}}]\}$, one has $H_{\tau^{2,-}} \leqslant H_{\delta_L^{-1/2}T_{\mathrm{rel}}}$ and hence, for every $\eta \in S^{3,+}$,

$$\mathbb{P}^\eta\left[H_{\tau^{2,-}} \geqslant \delta_L^{1/16} \tau^{2,-}\right] \leqslant \mathbb{P}^\eta\left[H_{\delta_L^{-1/2}T_{\mathrm{rel}}} \geqslant \delta_L^{1/8} T_{\mathrm{rel}}\right] + \delta'_L \leqslant \delta_L^{1/8} + \delta'_L\,.$$

□

4. Mixing time upper bound

In this section we prove Theorem 1.1. Our approach will also yield a proof of Theorem 1.2. The main ideas of the proof can be sketched as follows.

Step 1. Lemma 2.8 shows that after a burn-in time $O(L^{2+\varepsilon})$ the distribution $\nu_t^\wedge$ has a smooth density w.r.t. the equilibrium π. The first step consists in using this fact together with (2.6) and Lemma 2.4 to reduce the mixing time upper bound to a lower bound on the spectral gap of the chain.

Step 2. To bound the spectral gap we decompose the polymer configurations using the variables $\sigma_x = \mathrm{sign}(\eta_x)$ introduced in (1.7). From the decomposition estimates in Proposition 2.1, we shall roughly obtain that the spectral gap of the chain is bounded below by $\mathrm{gap}_* \times \mathrm{gap}_+$, where gap_* denotes the spectral gap of the heat bath dynamics for the variables σ, while gap_+ stands for the spectral gap of the polymer with a wall (i.e. the polymer constrained to be non-negative). From [7], we know that $\mathrm{gap}_+ = \Omega(L^{-2})$.

Step 3. To prove a lower bound on gap_* we shall perform a second decomposition, this time by fixing the number of crossings (i.e. the number of sign switches) in the configuration σ. Another application of the bound from Proposition 2.1 will then show that gap_* is roughly bounded below by a product of two spectral gaps, say $\mathrm{gap}_*^{(1)}$ and $\mathrm{gap}_*^{(2)}$. Here $\mathrm{gap}_*^{(1)} = \min_n \mathrm{gap}_{*,n}$, where $\mathrm{gap}_{*,n}$ is the spectral gap of the dynamics on the variables σ constrained to have n crossings, while $\mathrm{gap}_*^{(2)}$ denotes the spectral gap of a birth and death chain associated to the number of crossings. We establish a lower bound $\mathrm{gap}_*^{(2)} = \Omega(1)$. Moreover, we show that when $n = 1$, one has $\mathrm{gap}_{*,1} = \Omega(L^{-5/2})$. To prove a similar bound for every n we introduce a new dynamics involving a fixed number n of crossings: with rate 1, independently, each crossing equilibrates its position between the two neighboring crossings positions. If $\mathrm{gap}_{\mathrm{eq}}^n$ denotes the spectral gap of this process, a comparison argument shows that $\mathrm{gap}_{*,n} \geqslant \mathrm{gap}_{*,1} \times \mathrm{gap}_{\mathrm{eq}}^n$.

Step 4. The final step consists in obtaining the lower bound $\mathrm{gap}_{\mathrm{eq}}^n = \Omega(L^{-\varepsilon})$. The first observation is that if $n \leqslant \varepsilon \log L$ this estimate can be obtained by means of a direct coupling argument. The proof of the estimate for larger values of n is based on a block dynamics argument which allows us to reduce the problem to the case of $n \leqslant \varepsilon \log L$ crossings. The analysis of the block dynamics uses a further coupling argument. It is worth observing that the coupling arguments used here make crucial use of the heavy tailed nature of the distribution of excursions at equilibrium; see Lemma 2.2.

Before starting the actual proof, let us pause for a few remarks. The lower bound on gap_* described in Step 3 and Step 4 above is sharp (up to $O(L^\varepsilon)$ corrections). As detailed in Section 4.5 below, Step 3 and Step 4 will essentially prove Theorem 1.2. On the other hand, the final bound $T_{\mathrm{mix}} = O(L^{5/2+2+\varepsilon})$ for Theorem 1.1 is likely to be off by a factor $O(L^2)$. As explained in Step 1 above, this comes from the use of a decomposition estimate that involves the product $\mathrm{gap}_* \times \mathrm{gap}_+$ rather than the minimum $\min\{\mathrm{gap}_*, \mathrm{gap}_+\}$, as it would be the case if one could efficiently decouple the mode associated to the variables σ from the rest.

The following four subsections will develop the four steps described above in the given order. However, we warn the reader that, because of various technical obstacles, the above plan will not be followed very strictly and several detours will be needed.

4.1. **Reduction to spectral gap.** We start with the implementation of Step 1. For later purposes it is necessary to consider a variant of the original dynamics which avoids (very unlikely) configurations with too many crossings or too many zeros between consecutive crossings.

Call χ the number of crossings in a configuration η:

$$\chi(\eta) = \sum_{x=-L+2}^{L-2} \mathbf{1}(\eta_x = 0\,,\ \eta_{x-1} \neq \eta_{x+1})\,. \tag{4.1}$$

Note that only sites $x \in E_L \setminus (\{-L\} \cup \{L\})$ appear in the summation. Define $\gamma_x = \mathbf{1}(\eta_x = 0\,,\ \eta_{x-1} \neq \eta_{x+1})$, so that $\chi = \sum_{x=-L+2}^{L-2} \gamma_x$, and write

$$\xi_0 = -L\,, \quad \xi_{\chi+1} = L\,, \tag{4.2}$$

and if $1 \leqslant j \leqslant \chi$, let ξ_j denote the position in $\{-L+2, \dots, L-2\}$ of the j-th "1" in the sequence $\{\gamma_{-L+2}, \dots, \gamma_{L-2}\}$. Thus, $\xi_1, \dots, \xi_\chi$ denote the positions of the internal crossings. Finally, denote by $N(\xi_i, \xi_{i+1})$ the number of zeros in the path η strictly between ξ_i and ξ_{i+1}. See Figure 2.

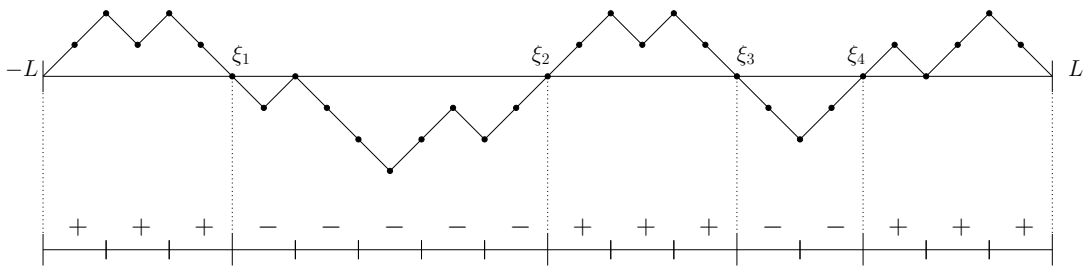

FIGURE 2. A configuration η of the polymer with $\chi(\eta) = 4$ internal crossings in positions $\xi_1, \xi_2, \xi_3, \xi_4$. Note that in this case $N(\xi_0, \xi_1) = N(\xi_2, \xi_3) = N(\xi_3, \xi_4) = 0$, and $N(\xi_1, \xi_2) = N(\xi_4, \xi_5) = 1$. Below, the corresponding configuration of signs $\sigma = \mathrm{sign}(\eta)$.

Fix a constant $c_o > 0$ and define the event

$$\Omega^o = \Big\{\eta \in \Omega \,:\, \chi(\eta) \leqslant c_o \log L\,,\ \text{and} \max_{i=0,\dots,\chi} N(\xi_i, \xi_{i+1}) \leqslant c_o \log L\Big\}\,. \tag{4.3}$$

Clearly $\eta \in (\Omega^o)^c$ implies that there are at least $c_o \log L$ zeros in the path η. Since $\lambda < 1$, the bound of Lemma 2.2 shows that the number of zeros is exponentially integrable at equilibrium. Therefore, for any $p > 0$, taking $c_o = c_o(p)$ large enough, we have

$$\pi(\Omega^o) = 1 + O(L^{-p})\,. \tag{4.4}$$

The reason for introducing the restricted set Ω^o will be apparent in the sequel. For the moment, we point out that the restriction $\chi \leqslant c_o \log L$ is essential for our estimates in Section 4.4, while both restrictions $\chi \leqslant c_o \log L$ and $N(\xi_i, \xi_{i+1}) \leqslant c_o \log L$ will be needed in the estimate of the parameter γ appearing in the decomposition of the spectral gap; see the proof of Proposition 4.2 and Proposition 4.4 below.

Next, consider the polymer process restricted to the set Ω^o, i.e. the continuous time Markov chain with state space Ω^o, and generator (1.1), where the rates $r_{x,\pm}(\eta)$ are replaced by

$$r^o_{x,\pm}(\eta) = r_{x,\pm}(\eta)\, \mathbf{1}(\eta^{x,\pm} \in \Omega^o)\,. \tag{4.5}$$

Let also π^o denote its reversible invariant measure, which is easily seen to coincide with $\pi(\cdot \,|\, \Omega^o)$. Let T^o_{rel} denote the relaxation time of the process defined above.

Lemma 4.1. *Assume that $T^o_{\rm rel} = O(L^p)$ for some $p > 0$. There exists $c > 0$, such that for any $\varepsilon > 0$ and for all $L \geqslant L_0(\varepsilon)$:*

$$T_{\rm mix} \leqslant L^{2+\varepsilon} + c\, T^o_{\rm rel} \log L\,. \tag{4.6}$$

Proof. Let $t = L^{2+\varepsilon/2} + s$, with $s = c_1 T^o_{\rm rel}$ for some c_1 to be fixed below. We prove that

$$\|\nu_t^\wedge - \nu_t^\vee\| \leqslant 1/2\,. \tag{4.7}$$

From Lemma 2.4 and Lemma 2.5, (4.7) implies that $T_{\rm mix} \leqslant c_2(L^{2+\varepsilon/2} + s)\log L$, for some other constant c_2, which implies the lemma. To prove (4.7), we first introduce some notation. We write $\nu_u^{\eta,o}$ for the distribution at time u of the state of the Markov chain restricted to Ω^o, when the initial configuration is some $\eta \in \Omega^o$. If $\eta \notin \Omega^o$ we define $\nu_u^{\eta,o} = \delta_\eta$ for all u. Next, we write $\nu_u^{\mu,o} = \sum_\eta \mu(\eta)\nu_u^{\eta,o}$ for a probability measure μ on Ω. Using symmetry we can write

$$\|\nu_t^\wedge - \nu_t^\vee\| \leqslant 2\|\nu_t^\wedge - \nu_s^{\pi^+}\| + 2\|\nu_s^{\pi^+} - \nu_s^{\pi^+,o}\| + \|\nu_s^{\pi^+,o} - \nu_s^{\pi^-,o}\|\,. \tag{4.8}$$

We start with the observation that

$$\|\nu_t^\wedge - \nu_s^{\pi^+}\| = \|\nu_s^{\nu_{t-s}^\wedge} - \nu_s^{\pi^+}\| \leqslant \|\nu_{t-s}^\wedge - \pi^+\| = o(1)\,,$$

where the first bound is obtained by writing $\nu_s^{\nu_{t-s}^\wedge} - \nu_s^{\pi^+} = \int(\nu_s^\eta - \nu_s^{\eta'})\rho(\eta,\eta')$ with ρ the maximal coupling of $\nu_{t-s}^\wedge$ and π^+, and the last bound follows from Lemma 2.8.

We turn to the last term in (4.8). Let $\pi^{+,o} = \pi^+(\cdot\,|\,\Omega^o)$ and observe that

$$\|\pi^+ - \pi^{+,o}\| = o(1)\,.$$

This last bound follows easily from (4.4) and (2.27). Moreover, the bound (2.6) applied to the process restricted to Ω^o yields

$$\|\nu_u^{\pi^{+,o},o} - \pi^o\| \leqslant c\, e^{-u/T^o_{\rm rel}}\,,$$

for some $c > 0$ and for all $u > 0$. Therefore, the third term in (4.8) can be made smaller than, say, $1/4$ by taking c_1 large enough in the definition of the time s. It remains to prove that the second term in (4.8) is $o(1)$. Since the initial condition is sampled from the same distribution π^+ we can couple the two processes $(\nu_u^{\pi^+})_{u\geqslant 0}$ and $(\nu_u^{\pi^+,o})_{u\geqslant 0}$ in such a way that they coincide until the first time when the unrestricted process exits from the set Ω^o. (Note that this time can be zero.) Thus,

$$\|\nu_s^{\pi^+} - \nu_s^{\pi^+,o}\| \leqslant \sum_{\eta_0\in\Omega} \pi^+(\eta_0)\mathbb{P}(\exists\, u \leqslant s\,:\, v_u^{\eta_0} \notin \Omega^o)\,.$$

From (2.27) we know that $\pi^+(\eta_0) \leqslant (2 + o(1))\pi(\eta_0)$, so that the time-invariance with a union bound implies

$$\|\nu_s^{\pi^+} - \nu_s^{\pi^+,o}\| \leqslant 2sL(1 - \pi(\Omega^o))(2 + o(1))\,,$$

where we use the fact that the average number of updates up to time s is bounded by $2sL$. Since, by assumption, $s = O(L^p)$ for some p, we can use (4.4) to conclude that $\|\nu_s^{\pi^+} - \nu_s^{\pi^+,o}\| = o(1)$. □

The following three subsections will focus on the upper bound $T^o_{\rm rel} = O(L^{5/2+2+\varepsilon})$. Once this bound is established, Theorem 1.1 will follow immediately from Lemma 4.1.

4.2. **Decomposing along crossings configurations.** Recall the definition of the variables $\sigma \in \{-1,+1\}^{O_L}$ given by $\sigma_y = \mathrm{sign}(\eta_y)$, where O_L is the set of sites in $\{-L,\dots,L\}$ with the same parity as $L+1$. Note that the field σ specifies uniquely the field ξ defined after (4.2), while ξ specifies σ up to a global sign switch; see Figure 2. The space Ω^o can be decomposed into disjoint subspaces

$$\Omega^o = \cup_\sigma \Omega^o_\sigma\,, \tag{4.9}$$

where Ω^o_σ denotes the set of $\eta \in \Omega^o$ such that $\mathrm{sign}(\eta_x) = \sigma_x$ for all $x \in O_L$. Let $\mathcal{S}_o$ denote the set of all $\sigma \in \{-1,+1\}^{O_L}$ such that $\Omega^o_\sigma \neq \emptyset$, i.e. the set of $\sigma \in \{-1,+1\}^{O_L}$ such that $\chi(\sigma) \leqslant c_o \log L$, where $\chi(\cdot)$, defined in (4.1), is seen as a function of $\sigma = \mathrm{sign}(\eta)$. Consider the continuous time Markov chain on $\mathcal{S}_o$ with infinitesimal generator

$$\mathcal{G}\varphi(\sigma) = \sum_{x\in O_L} \theta_x(\sigma)\,[\varphi(\sigma^x) - \varphi(\sigma)] \tag{4.10}$$

where $\varphi : \mathcal{S}_o \mapsto \mathbb{R}$, σ^x is the configuration σ flipped at x, i.e. it is defined as σ everywhere except at x where it equals $-\sigma_x$, and the rates $\theta_x(\sigma)$ are given by

$$\theta_x(\sigma) = \sum_{\eta\in\Omega^o_\sigma} \pi(\eta\,|\,\Omega^o_\sigma)\big[r^o_{x,+}(\eta)\,\mathbf{1}(\sigma(\eta^{x,+}) = \sigma^x) + r^o_{x,-}(\eta)\,\mathbf{1}(\sigma(\eta^{x,-}) = \sigma^x)\big]\,. \tag{4.11}$$

The rates $r^o_{x,\pm}$ are given in (4.5). Note that the measure

$$\nu_o(\sigma) = \sum_{\eta\in\Omega^o_\sigma} \pi(\eta\,|\,\Omega^o)\,, \tag{4.12}$$

is the reversible distribution, i.e. $\nu_o(\sigma)\theta_x(\sigma) = \nu_o(\sigma^x)\theta_x(\sigma^x)$ holds for all $x \in O_L$ and $\sigma \in \mathcal{S}_o$.

In words, the process with generator $\mathcal{G}$ is described as follows. Attach independent rate 1 Poisson clocks to all sites $x \in O_L$. Let σ be the current configuration. When site x rings, choose a configuration η sampled from the distribution $\pi(\cdot\,|\,\Omega^o_\sigma)$ and set $\eta' = \eta^{x,+}$ with probability $r^o_{x,+}(\eta)$, $\eta' = \eta^{x,-}$ with probability $r^o_{x,-}(\eta)$, and $\eta' = \eta$ with probability $1 - r^o_{x,+}(\eta) - r^o_{x,-}(\eta)$. Finally, update σ to σ' given by $\sigma'_x = \mathrm{sign}(\eta'_x)$. Let $\mathrm{gap}^{\mathcal{S}_o}$ denote the spectral gap of this Markov chain.

Proposition 4.2. *There exists $c > 0$ such that for all L:*

$$T^o_{\mathrm{rel}} \leqslant c\,L^2(\log L)^3\,(\mathrm{gap}^{\mathcal{S}_o})^{-1}\,.$$

Proof. We apply Proposition 2.1 with the decomposition (4.9). To each σ we can associate the continuous time Markov chain with state space Ω^o_σ, defined by the generator (1.1) with the rates $r_{x,+}(\eta)$ replaced by

$$r^\sigma_{x,\pm}(\eta) = r^o_{x,\pm}(\eta)\,\mathbf{1}(\eta^{x,\pm} \in \Omega^o_\sigma)\,, \tag{4.13}$$

where the rates $r^o_{x,\pm}(\eta)$ are defined in (4.5), with reversible equilibrium measure $\pi^o(\cdot\,|\,\Omega^o_\sigma)$. Call gap^σ the spectral gap of this Markov chain. For a given σ, this corresponds to independent continuous time Markov chains for each interval $\{\xi_i,\dots,\xi_{i+1}\}$, where the crossing positions ξ_i have been defined in (4.2). On a given interval $\{\xi_i,\dots,\xi_{i+1}\}$, we have a polymer dynamics with a horizontal wall constraint (polymer above or below the wall depending on the sign of the field σ inside that interval). Moreover, within each interval the polymer is constrained to have less than $c_o \log L$ zeros. Let $\mathrm{gap}^{\sigma,i}$ denote the spectral gap of this process on the interval $\{\xi_i,\dots,\xi_{i+1}\}$. From the independence recalled above, one has

$$\mathrm{gap}^\sigma = \min_{i=0,\dots,\chi} \mathrm{gap}^{\sigma,i}\,, \tag{4.14}$$

where χ is the number of interior crossings defined in (4.1). It follows from Lemma 4.3 below and (2.8) that $(\mathrm{gap}^\sigma)^{-1} = O(L^2 \log L)$, uniformly in $\sigma \in \mathcal{S}_o$.

Next, observe that the generator $\mathcal{G}$ in (4.10) coincides with the generator $\overline{\mathcal{L}}$ from Proposition 2.1 for the present choice of the decomposition. We can then use the bound of Proposition 2.1 with $\bar\lambda = \mathrm{gap}^{\mathcal{S}_o}$, and $\lambda_{\min} = \min_{\sigma\in\mathcal{S}_o} \mathrm{gap}^\sigma$:

$$T^o_{\mathrm{rel}} \leqslant c\,(\mathrm{gap}^{\mathcal{S}_o})^{-1} \max\{1, L^2 \log L(\mathrm{gap}^{\mathcal{S}_o} + \gamma)\}\,, \tag{4.15}$$

where $c > 0$ is a constant, and

$$\gamma = \max_{\sigma\in\mathcal{S}_o} \max_{\eta\in\Omega^o_\sigma} \sum_{x=-L+1}^{L-1} \big[r^o_{x,+}(\eta)\,\mathbf{1}(\sigma(\eta^{x,+}) = \sigma^x) + r^o_{x,-}(\eta)\,\mathbf{1}(\sigma(\eta^{x,-}) = \sigma^x)\big]\,. \tag{4.16}$$

It is immediate to check that $\mathrm{gap}^{\mathcal{S}_o}$ is smaller than a constant, so that $T^o_{\mathrm{rel}} \leqslant c\,L^2 \log L\,(1 + \gamma)\,(\mathrm{gap}^{\mathcal{S}_o})^{-1}$, by (4.15). It remains to give an upper bound $\gamma = O((\log L)^2)$. From the definition (4.16), we see that γ is bounded above by the maximum over $\eta \in \Omega^o$ of the number of sites $x \in O_L$ such that $\eta_{x-1} = \eta_{x+1} = 0$. By definition of the set Ω^o, the latter quantity is bounded by $(c_o \log L)^2$ in our setting. This ends the proof. □

We turn to the lower bound on the gaps defined in (4.14), that was needed in the proof of Proposition 4.2. In Lemma 4.3, we prove upper bounds for mixings times and (2.8) allows us to get the corresponding lower bounds for the spectral gap. Consider the polymer dynamics under the constraints $N(\eta) \leqslant M$ (where $N(\eta)$ is the total number of zeros and M is a positive constant) and $\eta \geqslant 0$, i.e. let $\Gamma_{+,M}$ denote the set

$$\Gamma_{+,M} = \big\{\eta \in \Omega\,:\; N(\eta) \leqslant M\,,\ \text{and } \eta_x \geqslant 0 \text{ for all } x\big\}\,,$$

and write ρ^η_t for the law at time t of the polymer dynamics restricted to $\Gamma_{+,M}$. This is the continuous time Markov chain with generator (1.1) with rates replaced by $r_{x,\pm}(\eta)\mathbf{1}(\eta^{x,\pm} \in \Gamma_{+,M})$ and with reversible measure $\pi(\cdot\,|\,\Gamma_{+,M})$.

Lemma 4.3. *There exists a constant $c > 0$ such that for any $\eta \in \Gamma_{+,M}$ and for any $M \geqslant c \log L$:*

$$\|\rho^\eta_T - \pi(\cdot\,|\,\Gamma_{+,M})\| = o(1)\,, \quad T = c\,L^2 \log L\,.$$

Proof. Let μ^η_t denote the evolution without the constraint $N(\eta) \leqslant M$, i.e. the Markov chain with generator (1.1) with rates given by $r_{x,\pm}(\eta)\mathbf{1}(\eta^{x,\pm} \geqslant 0)$ and with reversible measure $\pi^w = \pi(\cdot\,|\,\eta \geqslant 0)$. The mixing time of this "wall" constrained model has been analyzed in [7, Theorem 3.1], where it is shown that, for some constant $c > 0$, for all initial $\eta \geqslant 0$:

$$\|\mu^\eta_t - \pi^w\| \leqslant c\,L^2 \exp\Big(-\frac{t}{c\,L^2}\Big)\,. \tag{4.17}$$

As in (4.4), standard equilibrium estimates imply that for any $p > 0$ one can choose $c > 0$ such that for all $M \geqslant c \log L$:

$$\pi^w(N > M) = \|\pi^w - \pi(\cdot\,|\,\Gamma_{+,M})\| = O(L^{-p})\,. \tag{4.18}$$

Next, observe that by monotonicity μ^η_t is stochastically dominated by ρ^η_t. Let τ^η denote the hitting time of the set $N(\cdot) > M$ for the process with law $(\mu^\eta_t)_{t\geqslant 0}$, and introduce the event

$$G_t := \{\eta \geqslant 0\,:\; \mathbb{P}(\tau^\eta \leqslant t) \leqslant L^{-1}\}\,.$$

Note that, for fixed t, G_t is an increasing event, and therefore $\mu^\eta_s(G_t) \leqslant \rho^\eta_s(G_t)$, for any $s, t \geqslant 0$.

Let us fix now $t = c_1 L^2 \log L$. For any $p > 0$, using the time-invariance and a union bound with the fact that the expected number of updates up to time t is bounded by $2tL$, one has

$$\sum_\eta \pi^w(\eta)\mathbb{P}\big(\tau^\eta \leqslant t\big) \leqslant 2tL\,\pi^w(N > M) = O(L^{4-p})\,,$$

where we use (4.18). Markov's inequality then implies that

$$\pi^w(G_t^c) = O(L^{5-p})\,.$$

From (4.17) we then deduce that if c_1 is sufficiently large (in the definition of t):

$$\rho_t^\eta(G_t^c) \leqslant \mu_t^\eta(G_t^c) = O(L^{5-p})\,. \tag{4.19}$$

On the other hand, writing $\rho_{2t}^\eta(A) - \pi^w(A) = \sum_{\eta'} \rho_t^\eta(\eta')(\rho_t^{\eta'}(A) - \pi^w(A))$ for any set A one has

$$\|\rho_{2t}^\eta - \pi^w\| \leqslant \sum_{\eta' \in G_t} \rho_t^\eta(\eta')\|\rho_t^{\eta'} - \pi^w\| + \rho_t^\eta(G_t^c)\,.$$

To estimate the first term above, note that the processes with laws $(\mu_t^\eta)_{t\,\geqslant\,0}, (\rho_t^\eta)_{t\,\geqslant\,0}$ can be coupled in such a way that they coincide until time τ^η. Therefore, by definition of G_t, for any $\eta' \in G_t$:

$$\|\rho_t^{\eta'} - \pi^w\| \leqslant \mathbb{P}(\tau^{\eta'} \leqslant t) + \|\mu_t^{\eta'} - \pi^w\| = O(L^{-1})\,, \tag{4.20}$$

where we have used again (4.17) to bound the last term above. In conclusion, using

$$\|\rho_{2t}^\eta - \pi(\cdot\,|\,\Gamma_{+,M})\| \leqslant \|\pi^w - \pi(\cdot\,|\,\Gamma_{+,M})\| + \|\rho_{2t}^\eta - \pi^w\|\,,$$

together with (4.18), (4.19), and (4.20), we arrive at $\|\rho_{2t}^\eta - \pi(\cdot\,|\,\Gamma_{+,M})\| = O(L^{-1})$ which implies the desired estimate. □

Thanks to Lemma 4.1 and Proposition 4.2, Theorem 1.1 will follow from the estimate

Proposition 4.4. *For any $\varepsilon > 0$ and for all $L \geqslant L_0(\varepsilon)$:*

$$\mathrm{gap}^{\mathcal{S}_o} \geqslant L^{-\frac{5}{2}-\varepsilon}\,. \tag{4.21}$$

The following two subsections are devoted to the proof of Proposition 4.4.

4.3. **Decomposing according to the number of crossings.** We first decompose $\mathcal{S}_o$ according to whether the first excursion has positive or negative sign, i.e.

$$\mathcal{S}_o = \mathcal{S}^+ \cup \mathcal{S}^-\,, \qquad \mathcal{S}^\pm = \{\sigma \in \mathcal{S}_o \,:\, \sigma_{-L+1} = \pm 1\}\,. \tag{4.22}$$

An application of Proposition 2.1 with the decomposition (4.22) yields

$$\mathrm{gap}^{\mathcal{S}_o} \geqslant \min\{\bar\lambda/3, \bar\lambda\lambda_{\min}/(\bar\lambda + 3\gamma)\}\,, \tag{4.23}$$

where $\lambda_{\min} = \mathrm{gap}^{\mathcal{S}^+}$ denotes the gap of the process restricted to $\mathcal{S}^+$ (by symmetry, this equals the gap of the process restricted to $\mathcal{S}^-$), while $\bar\lambda$ is the gap of the symmetric two state Markov chain with transition rate

$$c(+,-) = c(-,+) = \sum_{\sigma\in\mathcal{S}^+} \nu_o(\sigma\,|\,\mathcal{S}^+)\theta_{-L+1}(\sigma)\,, \quad \gamma = \max_{\sigma\in\mathcal{S}^+} \theta_{-L+1}(\sigma)\,,$$

where ν_o is given in (4.12), and $\theta_{-L+1}(\sigma)$ is the probability of a flip at $x = -L+1$ in σ; see (4.11). The probability measure ν_o is defined in (4.12). Clearly, $\gamma \leqslant 1$. Since at equilibrium the polymer has a uniformly positive probability of taking the value $\eta_{-L+2} = 0$, cf. (2.24), it follows easily that $c(+,-) \geqslant c(\lambda) > 0$ for some L-independent constant $c(\lambda)$. Thus, $\bar\lambda$ is of order 1, and for some $c = c(\lambda) > 0$ one obtains

$$\mathrm{gap}^{\mathcal{S}_o} \geqslant c\,\mathrm{gap}^{\mathcal{S}^+}\,. \tag{4.24}$$

Next, consider the number of crossings χ defined in (4.1). Since χ is a function of the signs σ only, we can write $\chi(\sigma)$ for the number of crossings in a given $\sigma \in \mathcal{S}_o$. Thus, the space $\mathcal{S}^+$ is partitioned as

$$\mathcal{S}^+ = \cup_{n=0}^m \mathcal{S}^{+,n}\,, \tag{4.25}$$

where $m = \lfloor c_o \log L \rfloor$, and $\mathcal{S}^{+,n} = \{\sigma \in \mathcal{S}^+ \,:\, \chi(\sigma) = n\}$. We apply Proposition 2.1, this time with the decomposition (4.25). Thus, $\mathrm{gap}^{\mathcal{S}^+}$ can be bounded below as in (4.23) where $\lambda_{\min} = \min_{0 \leqslant n \leqslant m} \mathrm{gap}^{+,n}$ now stands for the minimum of gaps of the processes restricted to $\mathcal{S}^{+,n}$, while $\bar\lambda$ stands for the gap of the random walk on $\{0,\dots,m\}$ with transition rates

$$c(n, n\pm 1) = \sum_{\sigma\in\mathcal{S}^{+,n}} \sum_{x\in O_L} \nu_o(\sigma\,|\,\mathcal{S}^{+,n})\theta_x(\sigma)\mathbf{1}(\sigma^x \in \mathcal{S}^{+,n\pm1})\,, \tag{4.26}$$

$$c(n, n\pm 2) = \sum_{\sigma\in\mathcal{S}^{+,n}} \sum_{x\in O_L} \nu_o(\sigma\,|\,\mathcal{S}^{+,n})\theta_x(\sigma)\mathbf{1}(\sigma^x \in \mathcal{S}^{+,n\pm2})\,. \tag{4.27}$$

Note that the transition $n \to n+1$ is necessarily obtained by creating a new crossing at the rightmost position $\xi_n = L-2$, while the transition $n \to n+2$ can be obtained by creating two adjacent crossings between two existing consecutive crossings. These are the only transitions that are induced by the single flips $\sigma \to \sigma^x$. By construction, the reversible invariant measure of the chain defined by the rates (4.26), (4.27) is given by

$$\mu(n) := \nu_o(\chi = n\,|\,\mathcal{S}^+) = \pi(\chi = n\,|\,\Omega^o)\,, \qquad n = 0,\dots,m\,.$$

Lemma 4.5. *The gap $\bar\lambda$ of the chain defined by* (4.26), (4.27) *satisfies $c \geqslant \bar\lambda \geqslant c^{-1}$ for some $c = c(\lambda) > 0$ uniformly in $L \geqslant 2$.*

Proof. By Lemma 2.2, $\pi(\chi = n)$ is exponentially decaying in n, and the same applies to $\mu(n) = \pi(\chi = n\,|\,\Omega^o)$, up to $O(L^{-p})$ corrections; see (4.4). To prove an upper bound on $\bar\lambda$ one can take the test function χ counting the number of crossings in the variational principle defining $\bar\lambda$. The variance of χ w.r.t. $\mu(\cdot)$ is a positive constant. On the other hand the Dirichlet form can be bounded from above by

$$4\sum_{n=0}^{m-1} \mu(n)[c(n,n+1) + c(n,n+2)]\,. \tag{4.28}$$

Observe that uniformly in $\sigma \in \mathcal{S}^{+,n}$, one has

$$\sum_{x\in O_L} \theta_x(\sigma)\big[\mathbf{1}(\sigma^x \in \mathcal{S}^{+,n\pm1}) + \mathbf{1}(\sigma^x \in \mathcal{S}^{+,n\pm2})\big] = O(n)\,. \tag{4.29}$$

Indeed, each excursion can only contribute $O(1)$ to this sum since at equilibrium, in the delocalized phase, the expected number of zeros between consecutive crossings is finite (depending on $\lambda < 1$); see Lemma 2.2. Now, (4.29) implies that (4.28) is bounded above by $c\sum_n \mu(n)\,n = O(1)$. This ends the proof of $\bar\lambda \leqslant c$ for some constant c.

To prove a lower bound on $\bar\lambda$, we can neglect the additional rates (4.27). Then, we are left with a birth and death chain on the set $\{0,\dots,m\}$ (m being the maximal number of crossings allowed) with jump rates $c(n, n\pm1)$ and reversible measure $\mu(n)$ as above. It is well known (see e.g. [12]) that the spectral gap of this chain can be estimated via Cheeger's inequality $\bar\lambda \geqslant c\,\Phi^2$, where c is a universal constant and Φ is defined by

$$\Phi = \min\{\Phi(\ell,\ell')\,,\ 0 \leqslant \ell < \ell' \leqslant m :\ \mu([\ell,\ell']) \leqslant 1/2\}$$

$$\Phi(\ell,\ell') := \frac{\mu(\ell)c(\ell,\ell-1) + \mu(\ell')c(\ell',\ell'+1)}{\mu([\ell,\ell'])}\,,$$

where $\mu([\ell,\ell']) = \sum_{n=\ell}^{\ell'} \mu(n)$. Thus, one has to prove a lower bound on Φ. Below, we prove that the birth rate $c(n,n+1)$ in (4.26) is bounded away from zero uniformly: for a universal constant $a>0$ one has

$$c(n,n+1) \geqslant a\,, \qquad 0 \leqslant n \leqslant m-1. \tag{4.30}$$

By reversibility, it is easily checked that the same holds for the death rate $c(n,n-1)$, for $1 \leqslant n \leqslant m$. To prove a lower bound on Φ, consider two cases: either 1) $\Phi = \Phi(\ell,\ell')$ with $\ell > 0$ or 2) $\Phi = \Phi(\ell,\ell')$ with $\ell = 0$. In the first case, one has that $\Phi \geqslant a\mu(\ell)/\mu([\ell,\ell'])$ is uniformly positive, since μ decays exponentially. In the second case, one has $\Phi \geqslant a\mu(\ell')/\mu([0,\ell'])$, which is uniformly positive because the requirement $\mu([0,\ell']) \leqslant 1/2$ and the decay of μ force ℓ' to be uniformly bounded.

It remains to prove (4.30). We have

$$c(n,n+1) \geqslant c\,\nu_o(\xi_n = L-4 \,|\, \mathcal{S}^{+,n})\,, \tag{4.31}$$

for some constant $c = c(\lambda) > 0$, where ξ_n denotes the rightmost internal crossing. We introduce some extra notation to characterize more explicitly the measures $\nu_{n,o} := \nu_o(\cdot\,|\,\mathcal{S}^{+,n})$. Let $\Omega_j^{+,o}$ denote the set of polymers

$$\Omega_j^{+,o} = \{\eta \geqslant 0 :\ \eta_0 = \eta_j = 0\,,\ \eta_{x+1} = \eta_x \pm 1\,,\ x = 0,\dots,j-1\,,\ \text{and } N(\eta) \leqslant c_o \log L\}\,.$$

Note that these paths start at 0 and end at j, so that j must be even for this set to be non-empty. Define the probability law $\rho_{+,o}$ on the set of even positive integers by

$$\rho_{+,o}(j) = \frac{2^{-j} Z_j^{+,o}}{z_{+,o}}\,, \text{ where } \ Z_j^{+,o} = \sum_{\eta \in \Omega_j^{+,o}} \lambda^{N(\eta)}\,,$$

where $z_{+,o}$ is the normalization. From Lemma 2.2, (2.23) one has that $2^{-j} Z_j^{+,o} = O(j^{-3/2})$ for large j, so that the probability $\rho_{+,o}$ is well defined. Then, it is not hard to check that the measure $\nu_{n,o}$ introduced above is given by

$$\nu_{n,o}(x_1,\dots,x_n) = \frac{\rho_{+,o}(x_1+L)\rho_{+,o}(x_2-x_1)\cdots\rho_{+,o}(x_n-x_{n-1})\rho_{+,o}(L-x_n)}{Z_{n,L}^o} \tag{4.32}$$

where $\nu_{n,o}(x_1,\dots,x_n) = \nu_{n,o}(\xi_1 = x_1,\dots,\xi_n = x_n)$, if $-L < x_1 < \cdots < x_n < L$ is any allowed configuration of the crossing positions $\xi_1,\dots,\xi_n$, and $Z_{n,L}^o$ is the normalizing constant. In particular, $\nu_{n,o}$ is a product measure $\rho_{+,o}^{\otimes(n+1)}$ over $n+1$ positive increments $x_{i+1}-x_i$ (with $x_0 = -L, x_{n+1} = L$), conditioned to have $\sum_{i=0}^n (x_{i+1}-x_i) = 2L$.

Going back to (4.31), with the notation in (4.32) we have

$$\nu_o(\xi_n = L-4 \,|\, \mathcal{S}^{+,n}) = \frac{\rho_{+,o}(4)\rho_{+,o}^{\otimes(n)}(\sum_{i=1}^n(\xi_i-\xi_{i-1}) = 2L-4)}{\rho_{+,o}^{\otimes(n+1)}(\sum_{i=1}^{n+1}(\xi_i-\xi_{i-1}) = 2L)}\,.$$

Since $\sqrt{L} \gg m \geqslant n$, and $\rho_{+,o}(4) > 0$, with the same arguments of Lemma 4.12 below one easily sees that $\nu_o(\xi_n = L-4\,|\,\mathcal{S}^{+,n}) > 0$ uniformly. (We omit the details here to avoid repetitions). This ends the proof of (4.30) for $n \geqslant 1$. A bound on $c(0,1)$ can be obtained similarly. □

We return to the application of Proposition 2.1 with decomposition (4.25). The constant γ is now given by

$$\gamma = \max_n \max_{\sigma \in \mathcal{S}^{+,n}} \sum_{x \in O_L} \theta_x(\sigma)\big[\mathbf{1}(\sigma^x \in \mathcal{S}^{+,n\pm1}) + \mathbf{1}(\sigma^x \in \mathcal{S}^{+,n\pm2})\big]\,.$$

Recall (4.29). Since $n \leqslant m = O(\log L)$ we obtain that $\gamma = O(\log L)$. In conclusion, using (4.24) and Lemma 4.5, we arrive at

$$\mathrm{gap}^{\mathcal{S}_o} \geqslant c\,(\log L)^{-1} \min_{1 \leqslant n \leqslant m} \mathrm{gap}^{+,n}\,. \tag{4.33}$$

Thanks to (4.33), the proof of Proposition 4.4 has been reduced to the proof of a lower bound $\mathrm{gap}^{+,n} = \Omega(L^{-5/2-\varepsilon})$, uniformly in $n \leqslant c_o \log L$. Note that the case $n = 0$ does not appear in (4.33). Indeed, in that case the corresponding equilibrium is concentrated on the single "all plus" element $\mathcal{S}^{0,+}$. We start with the simplest case, i.e. $n = 1$, which can be analyzed by rather standard arguments.

Lemma 4.6. *There exists a constant $c = c(\lambda) > 0$, such that*

$$\mathrm{gap}^{+,1} \geqslant c\,L^{-5/2}\,. \tag{4.34}$$

Proof. When $n = 1$, the process restricted to $\mathcal{S}^{+,1}$ consists of a nearest neighbor random walk on $E_L \setminus (\{-L\} \cup \{L\})$, the sites in $\{-L+2, \dots, L-2\}$ with the same parity as L. Letting $\sigma(x)$ denote the element of $\mathcal{S}^{+,1}$ with unique crossing at x, the corresponding jump rates are given by

$$c(x, x \pm 2) = \sum_{y \in O_L} \theta_y(\sigma(x)) \mathbf{1}(\sigma(x)^y = \sigma(x \pm 2))\,. \tag{4.35}$$

These rates are of order 1 (except at $x = -L+2$ where $c(-L+2, -L) = 0$ and at $x = L-2$ where $c(L-2, L) = 0$) since there is a uniformly positive probability for the polymer conditioned to have signs $\sigma(x)$ of having a 0 at $x \pm 2$. Moreover, the reversible invariant probability measure $\rho(x)$ for this chain is given by

$$\rho(x) = \pi^o(\eta_{x-1}, \eta_x, \eta_{x+1} = (1, 0, -1) \,|\, \mathrm{sign}(\eta) \in \mathcal{S}^{+,1})\,. \tag{4.36}$$

Recalling (4.4) and Lemma 2.2, the event $\mathrm{sign}(\eta) \in \mathcal{S}^{+,1}$ has uniformly positive probability at equilibrium, and $\rho(x) \asymp \rho_0(x)$, where ρ_0 is the probability

$$\rho_0(x) = \frac{(L+x)^{-3/2}(L-x)^{-3/2}}{z_L}\,, \quad x \in E_L \setminus (\{-L\} \cup \{L\})\,, \tag{4.37}$$

where z_L is the normalizing constant. Since the rates (4.35) are of order 1, a standard comparison argument shows that it is sufficient to prove the bound $\mathrm{gap}_0^{+,1} \geqslant c\,L^{-5/2}$ for the gap of the chain with reversible measure ρ_0 given by (4.37) with rates defined by $c_0(x, x+2) = 1$ and $c_0(x, x-2) = \rho_0(x-2)/\rho_0(x)$. The latter process has the Dirichlet form

$$\mathcal{E}_0(f, f) = \sum_{x=-L+2}^{L-4} \rho_0(x)[f(x+2) - f(x)]^2\,. \tag{4.38}$$

On the other hand, writing $f(y) - f(x) = \sum_{j=x}^{y-2}[f(j+2) - f(j)]$, $x < y$, and using Schwarz' inequality, the variance $\mathrm{Var}_{\rho_0}(f)$ can be bounded above as follows:

$$\begin{aligned}\mathrm{Var}_{\rho_0}(f) &= \frac{1}{2}\sum_{x,y} \rho_0(x)\rho_0(y)[f(y) - f(x)]^2 \\ &\leqslant 2L \sum_{j=-L+2}^{L-4} \rho_0(j)[f(j+2) - f(j)]^2 \sum_{x \leqslant j} \sum_{y > j} \frac{\rho_0(x)\rho_0(y)}{\rho_0(j)}\,.\end{aligned}$$

From (4.37) one has

$$\max_j \sum_{x \leqslant j} \sum_{y>j} \frac{\rho_0(x)\rho_0(y)}{\rho_0(j)} = O(L^{\frac{3}{2}}) .$$

It follows that $\mathrm{gap}_0^{+,1} \geqslant c\, L^{-5/2}$, which implies (4.34). □

Remark 4.7. The bound of Lemma 4.6 is optimal in the sense that

$$\mathrm{gap}^{+,1} \leqslant c^{-1}\, L^{-5/2} . \tag{4.39}$$

This can be seen by taking a test function $\varphi(x)$ in the variational principle for $\mathrm{gap}^{+,1}$, of the form $\varphi(x) = g(x/L)$, where $g : [-1,1] \mapsto \mathbb{R}$ is given by $g(s) = -1$ if $s < -1/2$, $g(s) = 1$ if $s > 1/2$, and is linear between $-1/2$ and $1/2$. With this choice one has that the variance of φ w.r.t. ρ defined by (4.36) is of order 1, while the Dirichlet form - given by (4.38) with ρ instead of ρ_0 - is bounded above by the probability of $x \in [-L/2, L/2]$ times L^{-2}, i.e. $L^{-5/2}$. This implies (4.39). We point out that the results in (4.34) and (4.39) could have been obtained by appealing to a known criterium for birth and death chains; see [17].

We turn to the proof of a lower bound on $\mathrm{gap}^{+,n}$ for $n > 1$. Recall the definition (4.32) of the measure $\nu_{n,o}$. We introduce a further dynamics. We view the ξ_i as particle positions. Each particle $i = 1, \dots, n$ has an independent Poisson clock with parameter 1. When particle i rings, we freeze all positions ξ_k, $k \neq i$, and update ξ_i with the new position ξ_i' sampled from the conditional distribution $\nu_n^i := \nu_{n,o}(\cdot \,|\, \xi_k,\, k \neq i)$, where $\nu_{n,o}$ is given in (4.32). The Dirichlet form of this process is given by

$$\mathcal{E}_{\mathrm{eq}}^{n,o}(f,f) = \sum_{i=1}^{n} \nu_{n,o}\left[\mathrm{Var}_{\nu_n^i}(f)\right] , \tag{4.40}$$

where $\mathrm{Var}_{\nu_n^i}(f) = \nu_n^i(f^2) - \nu_n^i(f)^2$ is the variance conditioned on the values of ξ_k, $k \neq i$. Let $\mathrm{gap}_{\mathrm{eq}}^{n,o}$ denote the associated spectral gap:

$$\mathrm{gap}_{\mathrm{eq}}^{n,o} = \inf_f \frac{\mathcal{E}_{\mathrm{eq}}^{n,o}(f,f)}{\mathrm{Var}_{\nu_{n,o}}(f)} , \tag{4.41}$$

where the infimum ranges over all functions of the crossing positions $\xi_1, \dots, \xi_n$. The next estimate allows one to reduce the proof of Proposition 4.4 to the proof of a lower bound $\mathrm{gap}_{\mathrm{eq}}^{n,o} = \Omega(L^{-\varepsilon})$, uniformly in $n \leqslant c_o \log L$.

Lemma 4.8. *There exists a constant $c = c(\lambda) > 0$, such that for all $n \geqslant 1$.*

$$\mathrm{gap}^{+,n} \geqslant c\, L^{-5/2} \mathrm{gap}_{\mathrm{eq}}^{n,o} . \tag{4.42}$$

Proof. For $n = 1$ this coincides with the result of Lemma 4.6, since in this case $\nu_{1,o}[\mathrm{Var}_{\nu_1^1}(f)]$ coincides with $\mathrm{Var}_{\nu_{1,o}}(f)$. The general case follows from the observation that Lemma 4.6 can be applied with the segment $\{\xi_{i-1}, \dots, \xi_{i+1}\}$ replacing the usual $\{-L, \dots, L\}$, to obtain

$$\mathrm{Var}_{\nu_n^i}(f) \leqslant \frac{1}{c}\,(\xi_{i+1} - \xi_{i-1})^{5/2} \mathcal{E}^i(f,f) \leqslant \frac{1}{c_1}\, L^{5/2} \mathcal{E}^i(f,f) ,$$

where $c_1 = c/2^{5/2}$, and $\mathcal{E}^i(f,f)$ denotes the Dirichlet form of the random walk corresponding to the rates (4.35), with the segment $\{\xi_{i-1}, \dots, \xi_{i+1}\}$ in place of $\{-L, \dots, L\}$. Taking the $\nu_{n,o}$-expectation, and summing over $i = 1, \dots, n$ one obtains the estimate

$$\mathcal{E}_{\mathrm{eq}}^{n,o}(f,f) \leqslant \frac{1}{c_1}\, L^{5/2} \sum_{i=1}^{n} \nu_{n,o}\left[\mathcal{E}^i(f,f)\right] .$$

The desired conclusion follows from the observation that $\sum_{i=1}^n \nu_{n,o}\left[\mathcal{E}^i(f,f)\right] = \mathcal{E}^{+,n}(f,f)$, where $\mathcal{E}^{+,n}(f,f)$ is the Dirichlet form of the process restricted to $\mathcal{S}^{+,n}$ with spectral gap given by $\mathrm{gap}^{+,n}$. □

4.4. **Lower bound on** $\mathrm{gap}_{\mathrm{eq}}^{n,o}$**.** To complete the proof we need the lower bound $\mathrm{gap}_{\mathrm{eq}}^{n,o} = \Omega(L^{-\varepsilon})$. We first remove the restriction $N(\eta) \leqslant c_o \log L$ in the definition of the measure $\nu_{n,o}$. Namely, introduce the probability measure

$$\nu_n(x_1,\dots,x_n) = \frac{\rho_+(x_1+L)\rho_+(x_2-x_1)\cdots\rho_+(x_n-x_{n-1})\rho_+(L-x_n)}{Z_{n,L}}\,, \tag{4.43}$$

defined by the kernel

$$\rho_+(j) = \frac{2^{-j}Z_j^+}{z_+}\,, \qquad Z_j^+ = \sum_{\eta\in\Omega_j^+}\lambda^{N(\eta)}\,, \tag{4.44}$$

where

$$\Omega_j^+ = \{\eta \geqslant 0:\ \eta_0 = \eta_j = 0\,,\ \eta_{x+1} = \eta_x \pm 1\,,\ x = 0,\dots,j-1\}\,.$$

Note that (4.43) coincides with (4.32) except for the removal of the constraint $N(\eta) \leqslant c_o \log L$ in the definition of the kernel ρ_+. As in (4.4) one can check that ρ_+ is equal to $\rho_{+,o}$ up to $O(L^{-p})$ corrections for some large $p > 0$ and therefore $\nu_n(x_1,\dots,x_n) = \nu_{n,o}(x_1,\dots,x_n)(1+O(L^{-p}))$, uniformly. Proceeding as in (4.40) and (4.41) we define

$$\mathcal{E}_{\mathrm{eq}}^n(f,f) = \sum_{i=1}^n \nu_n\left[\mathrm{Var}_{\nu_n^i}(f)\right]\,, \tag{4.45}$$

and $\mathrm{gap}_{\mathrm{eq}}^n$, the spectral gap associated to the measure ν_n and the Dirichlet form (4.45). From the previous observations we see that $\mathrm{Var}_{\nu_n}(f) \leqslant c\,\mathrm{Var}_{\nu_{n,o}}(f)$ and $\mathcal{E}_{\mathrm{eq}}^{n,o}(f,f) \leqslant c\,\mathcal{E}_{\mathrm{eq}}^n(f,f)$, for some constant $c > 0$, for every function f. Therefore $\mathrm{gap}_{\mathrm{eq}}^n \leqslant c^2\,\mathrm{gap}_{\mathrm{eq}}^{n,o}$. For later purposes it is important to keep track of the dependence on L in our notation, and therefore we write $\mathrm{gap}_{\mathrm{eq}}^n(L)$ below. The desired bound $\mathrm{gap}_{\mathrm{eq}}^{n,o} = \Omega(L^{-\varepsilon})$ follows from the comparison mentioned above and

Proposition 4.9. *For any $\varepsilon > 0$, there exists $L_0(\varepsilon) > 0$ such that for $L \geqslant L_0$ and $n \leqslant c_o \log L$*

$$\mathrm{gap}_{\mathrm{eq}}^n(L) \geqslant L^{-\varepsilon}\,. \tag{4.46}$$

As a preliminary step towards the proof of Proposition 4.9, we establish the following lemma.

Lemma 4.10. *There exists $c > 0$ such that for any n, uniformly in $L \geqslant n$:*

$$\mathrm{gap}_{\mathrm{eq}}^n(L) \geqslant c\,e^{-n/c}\,.$$

The crucial point of the above estimate is that it does not depend on the size of the system L, but only on the number of particles n. Note that Lemma 4.10 gives the required lower bound (4.46) if we take $n \leqslant \varepsilon_1 \log L$ with ε_1 suitably small. The case of larger n will be considered afterwards.

Proof of Lemma 4.10. Every particle has an independent Poisson(1) clock. When particle i rings we update its position according to the equilibrium measure conditioned on the neighboring positions ξ_{i-1} and ξ_{i+1}, that is the probability for the i-th particle to be in position $\xi_i' = \xi_{i-1}+k$, $k = 2,4,\dots,\xi_{i+1}-\xi_{i-1}-2$, is given by

$$\frac{\rho_+(k)\rho_+(\xi_{i+1}-\xi_{i-1}-k)}{Z}\,, \qquad \text{where} \quad Z = \sum_{k=2}^{\xi_{i+1}-\xi_{i-1}-2}\rho_+(k)\rho_+(\xi_{i+1}-\xi_{i-1}-k)\,. \tag{4.47}$$

An important property of (4.47) is that the probability α of the event $\xi_i' = \xi_{i-1} + 2$ (or $\xi_i' = \xi_{i+1} - 2$) is positive, uniformly in the length $\xi_{i+1} - \xi_{i-1}$. The idea is to use this property to prove that with probability at least $\frac{1}{2}\alpha^n$, uniformly in the initial configuration, the process hits the minimal configuration

$$\xi_1 = -L + 2, \xi_2 = -L + 4, \dots, \xi_n = -L + 2n\,, \tag{4.48}$$

before time n^2. Once this result is available, one concludes with a simple coupling argument. Indeed, using an independent coupling, the total variation distance between two evolutions at time $t = m \times n^2$, with arbitrary initial conditions, is bounded above by $(1-\alpha^{2n}/4)^m \leqslant e^{-m\alpha^{2n}/4}$. Thus, the mixing time of this chain is at most $8n^2\alpha^{-2n}$. Using e.g. (2.8) we obtain the lemma.

To prove the above claim, we use the notation $(\tau_m, \ell_m)_{m \geqslant 1}$ for the sequence of updating marks, that is the m-th update occurs at time τ_m and it concerns the ℓ_m-th particle, where ℓ_m is a number in $\{1, \cdots, n\}$. Consider the event E that there exist integers $1 \leqslant \phi(1) < \phi(2) < \cdots < \phi(n)$ such that $\tau_{\phi(n)} \leqslant n^2$, $\ell_{\phi(i)} = i$, and such that for every $i = 1, \cdots, n$ and $\phi(i-1) < j \leqslant \phi(i)$, one has $\ell_j \neq \ell_{\phi(i-1)}$ (where $\phi(0) = 0$ and $\ell_0 = 0$). In words, E is the event that within time n^2 there has been a sequence of n updates at times $\tau_{\phi(1)}, \dots, \tau_{\phi(n)}$, such that the update at time $\tau_{\phi(i)}$ concerned the i-th particle and such that the i-th particle is not touched again before time $\tau_{\phi(i+1)}$. Conditioned on the event E one has a probability of at least α^n of hitting the configuration described in (4.48). Indeed, at time $\tau_{\phi(1)}$ we set $\xi_1 = 2$ with probability α, at time $\tau_{\phi(2)}$ we set $\xi_2 = 4$ with probability α, and so on. Therefore, to prove the lemma it remains to show that the event E has probability at least $1/2$. This can be easily seen as follows. Consider the event F that a sequence of integers $1 \leqslant \psi(1) < \psi(2) < \cdots < \psi(n)$ exists such that $\tau_{\psi(n)} \leqslant n^2$, and $\ell_{\psi(i)} = i$, for every $i = 1, \cdots, n$. Let $\tau^{(1)}$ denote the first time particle 1 is updated and define recursively $\tau^{(i)}$ as the first time after $\tau^{(i-1)}$ when particle i is updated. Clearly, $F = \{\tau^{(n)} \leqslant n^2\}$. Using $\mathbb{E}[\tau^{(n)}] = n$, by Markov's inequality the probability of F is at least $1 - n/n^2 \geqslant 1/2$ for $n \geqslant 2$. On the other hand $E = F$, since one can define the sequence ϕ from ψ by choosing $\phi(n) = \psi(n)$, and for $1 \leqslant i \leqslant n-1$,

$$\phi(n-i) = \max\{j < \psi(n-i+1) \,:\, \ell_j = n-i\}.$$

This ends the proof of Lemma 4.10. □

Proof of Proposition 4.9. The proof of Proposition 4.9 is based on a block-dynamics argument that allows one to reduce to $n \leqslant \varepsilon_1 \log L$ particles, in which case the result will follow from Lemma 4.10. Fix an integer $K \leqslant n$. A block is a collection of particles with adjacent labels, and our particles will be partitioned into $\Delta := \lceil n/K \rceil$ non-overlapping blocks, in such a way that the first $\Delta - 1$ blocks contain exactly K particles each, and the last block contains at most K particles. For the sake of simplicity, since it does not change any of our estimates, we will suppose that all blocks have exactly K particles, i.e. $n = K\Delta$. With this notation, the configuration of the i-th block can be described by the variables

$$(\xi_{K(i-1)+1}, \dots, \xi_{Ki})\,, \qquad i = 1, \dots, \Delta\,. \tag{4.49}$$

As usual, the ξ_i are interpreted as particle positions or crossing positions. To define the block-dynamics, we consider independent Poisson(1) clocks on each block, when one of them rings we put all crossings of the relative block simultaneously at equilibrium conditioned on the position of all crossings belonging to the other blocks. That is, if B_i denotes the i-th block (4.49), and Var_{B_i} is the variance with respect to

$$\nu_n(\cdot \,|\, B_j,\ j \neq i)\,,$$

then the Dirichlet form of the block-dynamics is given by

$$\mathcal{E}^n_{\rm bl}(f,f) = \sum_{i=1}^{\Delta} \nu_n[{\rm Var}_{B_i}(f)]\,, \tag{4.50}$$

for any function f of the particle positions. Call $\mathrm{gap}^n_{\rm bl}$ the associated spectral gap.

The gap of the original dynamics for a single block B_i is given by $\mathrm{gap}^K_{\rm eq}(\ell)$, where $\ell = \xi_{iK+1} - \xi_{(i-1)K}$ is the length of the portion of the system occupied by the K particles in the i-th block:

$$\mathrm{gap}^K_{\rm eq}(\ell)\ {\rm Var}_{B_i}(f) \leqslant \sum_{r=1}^{k} \nu_n\left[{\rm Var}_{\nu^r_n}(f)\,|\,B_j,\ j\neq i\right]\,,$$

where ν^r_n is the equilibrium distribution of the r-th particle in the block B_i, conditioned on the positions of the remaining particles, cf. (4.45). Using this estimate in (4.50), taking the ν_n-expectation and summing over i one obtains

$$\mathrm{gap}^n_{\rm eq}(L) \geqslant \mathrm{gap}^n_{\rm bl} \times \min_{\ell} \mathrm{gap}^K_{\rm eq}(\ell)\,. \tag{4.51}$$

Note that this is a special case of a well known estimate that controls the gap of the original dynamics in terms of the gap of the block-dynamics (see e.g. [16]). From Lemma 4.10 we know that $\mathrm{gap}^K_{\rm eq}(\ell) \geqslant c\,e^{-K/c}$ uniformly in the length ℓ. Therefore, Proposition 4.9 follows directly from Lemma 4.11 below by taking $K = \varepsilon_1 \log L$, $K \leqslant n \leqslant c_o \log L$, with ε_1 sufficiently small (depending on ε) and $\Delta = n/K$ (the case $n \leqslant \varepsilon_1 \log L$ being contained in Lemma 4.10).

Lemma 4.11. *For all $\Delta \geqslant 1$, there exists $c(\Delta) > 0$ such that for all L and $\Delta \leqslant n \leqslant c_o \log L$,*

$$\mathrm{gap}^n_{\rm bl} \geqslant c(\Delta)\, K^{-4\Delta}\,, \qquad K = n/\Delta\,. \tag{4.52}$$

Before proving Lemma 4.11, we need to establish some preliminary facts. Consider $\nu^{2L}_n := \nu_n$, the equilibrium measure (4.43) of the system of n particles over the segment $\{-L,\dots,L\}$. Passing to the increment variables $\zeta_i := \xi_i - \xi_{i-1}$, $i = 1,\dots,n+1$ one writes ν^{2L}_n as the conditional probability $\rho_+^{\otimes(n+1)}\big(\,\cdot\,\mid \sum_{i=1}^{n+1}\zeta_i = 2L\big)$.

Lemma 4.12. *For all $n = o(L^{1/6})$:*

$$\nu^{2L}_n(\zeta_1 \geqslant 2L - L^{1/3}) = \frac{1}{n+1}(1+o(1))\,.$$

Proof of Lemma 4.12. Observe that

$$\nu^{2L}_n(\zeta_1 \geqslant 2L - L^{1/3}) = \frac{\sum_{i\,\geqslant\, 2L-L^{1/3}} \rho_+(i)\rho_+^{\otimes(n)}(\sum_{i=1}^n \zeta_i = 2L-i)}{\rho_+^{\otimes(n+1)}(\sum_{i=1}^{n+1}\zeta_i = 2L)}\,.$$

Recall that

$$\rho_+(j) \sim c_+ j^{-3/2}, \tag{4.53}$$

for some known constant $c_+ > 0$; see (2.23). In particular,

$$\rho_+(i) = \rho_+(2L)\left(1 + O\big(L^{-2/3}\big)\right)\,, \qquad i \in \{2L - L^{1/3},\cdots,2L\}\,. \tag{4.54}$$

Also, since $2L \gg (n+1)^2$, using [8, Theorem A], one has

$$\rho_+^{\otimes(n+1)}\Big(\sum_{i=1}^{n+1}\zeta_i = 2L\Big) = (n+1)(1+o(1))\,\rho_+(2L)\,. \tag{4.55}$$

From (4.54) and (4.55),

$$(n+1)\nu_n^{2L}(\zeta_1 \geqslant 2L - L^{1/3}) = (1+o(1))\,\rho_+^{\otimes(n)}\Big(\sum_{i=1}^n \zeta_i \in [0, L^{1/3})\Big)\,.$$

It remains to show that $\rho_+^{\otimes(n)}\big(\sum_{i=1}^n \zeta_i \in [0, L^{1/3})\big) = 1 + o(1)$. However, this is an immediate consequence of the assumption $n = o(L^{1/6})$ and well known estimates for heavy tailed distributions of the form (4.53); see e.g. [8]. □

Proof of Lemma 4.11. As a corollary of Lemma 4.12, one has that for L large enough, uniformly in $2L \geqslant 2\ell \geqslant L^{1/3}$ and in $n = o(\ell^{1/6})$,

$$\nu_n^{2\ell}(\zeta_1 > 2\ell - L^{1/3}) \geqslant \nu_n^{2\ell}(\zeta_1 > 2\ell - \ell^{1/3}) \geqslant \frac{1}{2(n+1)}\,. \tag{4.56}$$

We shall use this observation together with a coupling argument. Start the dynamics at two arbitrary initial configurations ξ, ξ' of n particles, and call $\xi(t), \xi'(t)$ the corresponding states of the dynamics at time t. It will be shown that for a suitable coupling $\mathbb{P}$:

$$\mathbb{P}(\xi(1) = \xi'(1)) \geqslant c(\Delta)(5(K+1))^{-4\Delta}\,. \tag{4.57}$$

Once this estimate is available the conclusion follows easily. Indeed, (4.57) implies that at time T the total variation distance between $\xi(T)$ and $\xi'(T)$, is bounded above by

$$(1 - c(\Delta)(5(K+1))^{-4\Delta})^{\lfloor T \rfloor} \leqslant e^{-c(\Delta)(5(K+1))^{-4\Delta}\lfloor T \rfloor}\,,$$

and therefore the mixing time of the chain is bounded by $2c(\Delta)^{-1}(5(K+1))^{4\Delta}$. Using (2.8) we obtain the same bound for the inverse spectral gap. After adjusting the constant $c(\Delta)$, this proves Lemma 4.11.

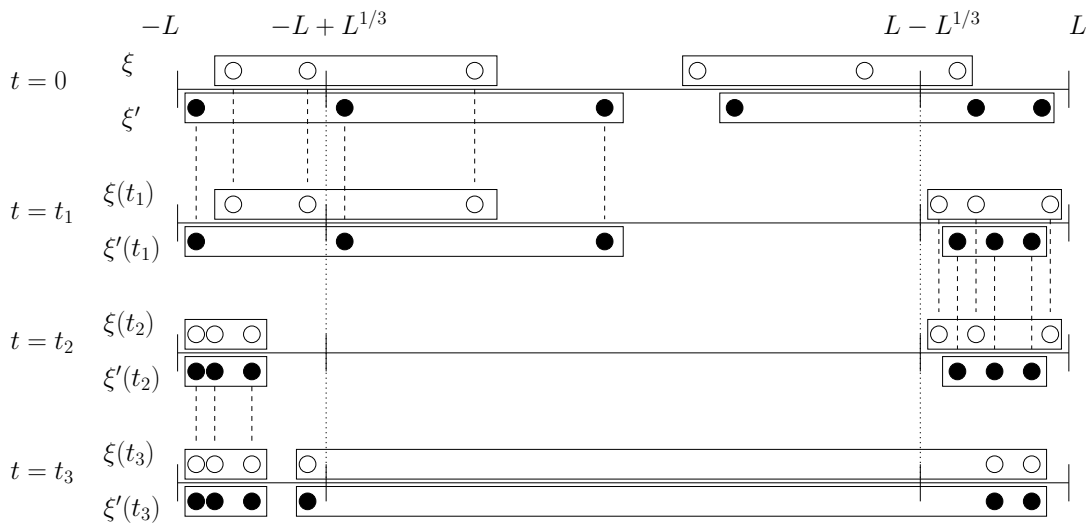

FIGURE 3. Illustration of the coupling used in the proof of Lemma 4.11 in the case $\Delta = 2$, $n = 6$ and $K = 3$. At time t_1, all particles in the second block are placed to the right of $L - L^{1/3}$. At time t_2, all particles in the first block are matched to the left of $-L + L^{1/3}$. At time t_3, all particles in the second block are matched.

To prove (4.57) we proceed as follows. We have Δ independent Poisson(1) clocks, one at each block, and we note $(t_m, c_m)_{m \geqslant 1}$ the sequence of update marks: for any $m \geqslant 1$, t_m is the time of the m-th update and $c_m \in \{1, \cdots, \Delta\}$ is the label of the block to be updated at time t_m. Consider the event E that within time $t = 1$ a sequence of updates $t_1, \ldots, t_{\Delta-1}, t_\Delta, t_{\Delta+1}, \ldots, t_{2\Delta-1}$ has occurred, such that $c_m = \Delta - m + 1$ for $m = 1, \ldots \Delta$, and $c_{\Delta+i} = i + 1$ for $i = 1, \ldots, \Delta - 1$. Clearly, E has a positive probability $c(\Delta) > 0$. On the event E we define a coupling of the two evolutions by using the same marks (t_m, c_m), i.e. at time t_m we update block c_m in both configurations. We refer to Figure 3 for a representation of the case $\Delta = 2$. At time t_1 we have to update the particle positions in the block B_Δ. The corresponding equilibrium is of the form $\nu_K^{2\ell}$, with $2\ell = L - \xi_{K(\Delta-1)}$. Then, the new particle positions will satisfy

$$\xi_{K(\Delta-1)+1} \geqslant L - L^{1/3}\,, \qquad \text{with probability at least } \frac{1}{2(K+1)}\,. \tag{4.58}$$

Indeed, either $\xi_{K(\Delta-1)} \geqslant L - L^{1/3}$ and the requirement is automatically satisfied because of $\xi_{K(\Delta-1)+1} \geqslant \xi_{K(\Delta-1)}$, or $\xi_{K(\Delta-1)} < L - L^{1/3}$, in which case $2\ell \geqslant L^{1/3}$ and the claim follows from (4.56). Therefore, using an independent coupling, one has a probability at least $(4(K+1)^2)^{-1}$ for the event

$$\min\{\xi_{K(\Delta-1)+1}(t_1), \xi'_{K(\Delta-1)+1}(t_1)\} \geqslant L - L^{1/3}\,.$$

Then, one updates the block $B_{\Delta-1}$ at time t_2, and so on until one updates the block B_2 at time $t_{\Delta-1}$. Iterating the argument given above, one has a probability of at least $(4(K+1)^2)^{-\Delta+1}$ for the event

$$\min\{\xi_{K+1}(t_{\Delta-1}), \xi'_{K+1}(t_{\Delta-1})\} \geqslant L - (\Delta-1)L^{1/3}\,. \tag{4.59}$$

Next, at time t_Δ, one updates the block B_1. Let us show that on the event (4.59) there is a coupling $\mathbb{P}$ of the two equilibria on block B_1 (conditioned on the value of $\xi_{K+1}(t_{\Delta-1})$ and $\xi'_{K+1}(t_{\Delta-1})$ respectively) such that the event

$$M_1 := \{\xi_i(t_\Delta) = \xi'_i(t_\Delta) \leqslant -L + L^{1/3}\,,\ i = 1, \ldots, K\}\,,$$

has probability

$$\mathbb{P}(M_1) \geqslant (5(K+1)^2)^{-1}\,. \tag{4.60}$$

Clearly, it suffices to show that $\xi_K(t_\Delta) = \xi'_K(t_\Delta) \leqslant -L + L^{1/3}$ with at least that probability (all other particles $\xi_1(t_\Delta), \ldots, \xi_{K-1}(t_\Delta)$ are then automatically matched using the diagonal coupling since B_1 is the first block and $\xi_0 = \xi'_0 = -L$ is fixed). Setting $2\ell = \xi_{K+1}(t_{\Delta-1}) + L$ and $2\ell' = \xi'_{K+1}(t_{\Delta-1}) + L$, we need to couple the measures $\nu_K^{2\ell}(\xi_K = i)$, $\nu_K^{2\ell'}(\xi'_K = i)$. Consider first the problem of coupling $\nu_1(i) := \nu_K^{2\ell}(\xi_K = L + i \,|\, \xi_K \leqslant -L + L^{1/3})$ and $\nu_2(i) := \nu_K^{2\ell'}(\xi'_K = L + i \,|\, \xi'_K \leqslant -L + L^{1/3})$. We have

$$\|\nu_1 - \nu_2\| = O(L^{-2/3})\,. \tag{4.61}$$

Indeed, first note that, by (4.59), one has

$$2L \geqslant 2\ell\,,\ 2\ell' \geqslant 2L - (\Delta-1)L^{1/3}\,. \tag{4.62}$$

With the notation $\rho_+^{\otimes(K)}(i) := \rho_+^{\otimes(K)}(\sum_{j=1}^K \zeta_j = i)$ one has

$$\begin{aligned}\|\nu_1 - \nu_2\| &= \frac{1}{2}\sum_{i=1}^{L^{1/3}} |\nu_1(i) - \nu_2(i)| \\ &= \frac{1}{2}\sum_{i=1}^{L^{1/3}} \Big| \frac{\rho_+^{\otimes(K)}(i)\rho_+(2\ell - i)}{\sum_{m=1}^{L^{1/3}} \rho_+^{\otimes K}(m)\rho_+(2\ell - m)} - \frac{\rho_+^{\otimes(K)}(i)\rho_+(2\ell' - i)}{\sum_{m=1}^{L^{1/3}} \rho_+^{\otimes K}(m)\rho_+(2\ell' - m)} \Big| .\end{aligned}$$

Using (4.54) and (4.62) one has $\rho_+(2\ell' - i) = \rho_+(2\ell - j)(1 + O(L^{-2/3}))$ uniformly in $0 \leqslant i, j \leqslant L^{1/3}$. Therefore (4.61) follows from

$$\|\nu_1 - \nu_2\| = O(L^{-2/3}) \times \sum_{i=1}^{L^{1/3}} \frac{\rho_+^{\otimes(K)}(i)\rho_+(2\ell - i)}{\sum_{m=1}^{L^{1/3}} \rho_+^{\otimes K}(m)\rho_+(2\ell - m)} = O(L^{-2/3}) .$$

We turn to the proof of (4.60). We define the coupling of $\nu_K^{2\ell}(\xi_K = i)$, $\nu_K^{2\ell'}(\xi'_K = i)$ as follows. Flip two independent coins with head probability $p = \nu_K^{2\ell}(\xi_K \leqslant -L + L^{1/3})$ and $p' = \nu_K^{2\ell'}(\xi'_K \leqslant -L + L^{1/3})$. If both coins end up being head, then sample the pair ξ_K, ξ'_K according to the maximal coupling of ν_1, ν_2 defined by $\|\nu_1 - \nu_2\|$. If the p coin is head and the p' coin is tail then sample independently ξ_K according to $\nu_K^{2\ell}(\cdot \,|\, \xi_K \leqslant -L + L^{1/3})$ and ξ'_K according to $\nu_K^{2\ell'}(\cdot \,|\, \xi'_K > -L + L^{1/3})$. Similarly, if the p coin is tail and the p' coin is head then sample independently ξ'_K according to $\nu_K^{2\ell'}(\cdot \,|\, \xi'_K \leqslant -L + L^{1/3})$ and ξ_K according to $\nu_K^{2\ell}(\cdot \,|\, \xi_K > -L + L^{1/3})$. Finally, if both coins are tail then sample independently ξ_K, ξ'_K according to $\nu_K^{2\ell}(\cdot \,|\, \xi_K > -L + L^{1/3})$, $\nu_K^{2\ell'}(\cdot \,|\, \xi'_K > -L + L^{1/3})$. Using this coupling, we obtain that

$$\mathbb{P}(M_1) \geqslant p\, p' \,(1 - \|\nu_1 - \nu_2\|) \geqslant (4(K+1)^2)^{-1}(1 + o(1)) ,$$

where we have used the fact that $\min\{p, p'\} \geqslant 1/2(K+1)$ by (4.56). This ends the proof of (4.60).

Repeating the same argument leading to (4.60) one shows that on the event M_1, the event

$$M_2 := \{\xi_{K+i}(t_{\Delta+1}) = \xi'_{K+i}(t_{\Delta+1}) \leqslant -L + 2L^{1/3} ,\ i = 1, \dots, K\} ,$$

has probability at least $(5(K+1)^2)^{-1}$. Thus, iterating, one concludes that

$$\mathbb{P}(\xi(1) = \xi'(1)) \geqslant \mathbb{P}(E)\,(4(K+1)^2)^{-\Delta+1}\,(5(K+1)^2)^{-\Delta} \geqslant c(\Delta)(5(K+1))^{-4\Delta} .$$

This ends the proof of (4.57). □

4.5. **Proof of Theorem 1.2.** Concerning the bound $T_{\mathrm{rel}}^{\mathcal{S}} \leqslant L^{5/2+\varepsilon}$, we note that it follows from the same arguments used for the proof of Proposition 4.4. In fact, the situation is simpler here due to the absence of constraints on the number of crossings. We omit the details.

Concerning the lower bound, we can actually prove that for some constant $c > 0$, one has

$$T_{\mathrm{rel}}^{\mathcal{S}} \geqslant c\, L^{5/2}\,(\log L)^{-1} . \tag{4.63}$$

We use an argument similar to that in (4.39). Recall that

$$T_{\mathrm{rel}}^{\mathcal{S}} \geqslant \frac{\mathrm{Var}_\nu(f)}{\mathcal{D}(f,f)} \tag{4.64}$$

for any $f : \mathcal{S} \mapsto \mathbb{R}$, where ν is defined by (1.7) and

$$\mathcal{D}(f,f) = \sum_{x \in O_L} \nu\,[\mathrm{Var}_{\nu_x}(f)] ,$$

ν_x denoting the conditional probability $\nu(\cdot \,|\, \sigma_y\,,\ y \neq x)$. Let ζ denote the number of +'s, i.e.

$$\zeta(\sigma) = \sum_{x \in O_L} \mathbf{1}(\sigma_x = +1)\,, \qquad \sigma \in \mathcal{S}\,.$$

Note that $\zeta \in \{0, \dots, L\}$. Define the function $f(\sigma) = g(\zeta(\sigma)/L)$, where $g : [0,1] \mapsto [-1,1]$ is given by $g(s) = -1$ for $s < 1/4$, $g(s) = 1$ for $s > 3/4$, and by the linear interpolation for $s \in [1/4, 3/4]$. Since $\zeta = 0$ and $\zeta = L$ have both positive probability uniformly in L, one has $\mathrm{Var}(f) \geqslant c$ for some constant $c > 0$. Let us estimate the Dirichlet form of f. We have

$$\mathcal{D}(f,f) \leqslant c\, L^{-2} \sum_{\sigma \in \mathcal{S}} \nu(\sigma) \sum_{x \in O_L} p_x(\sigma) \mathbf{1}(L/4 \leqslant \zeta \leqslant 3L/4)\,,$$

for some constant c, where we write $p_x(\sigma)$ for the probability of a flip at x in σ. Recall that the sum over x of the probabilities $p_x(\sigma)$ between consecutive crossings give a contribution of order 1; this follows easily from Lemma 2.2. That yields $\sum_{x \in O_L} p_x(\sigma) \leqslant c\,\chi$ for a suitable constant c, where χ is the number of crossings (4.1). Therefore, adjusting the constant c:

$$\mathcal{D}(f,f) \leqslant c\, L^{-2}\, \nu\left(\chi;\ L/4 \leqslant \zeta \leqslant 3L/4\right)\,.$$

However,

$$\nu\left(\chi;\ L/4 \leqslant \zeta \leqslant 3L/4\right) \leqslant \nu\left(\chi\,;\ \chi > c\,\log L\right) + c\,(\log L)\,\nu\left(L/4 \leqslant \zeta \leqslant 3L/4\right)\,.$$

From Lemma 2.2 we deduce that $\nu\left(\chi\,;\ \chi > c\,\log L\right) = O(L^{-p})$ for some large p, provided c is large. On the other hand $L/4 \leqslant \zeta \leqslant 3L/4$ implies that there exists a crossing in some position $x \in \{-L + L/8, \dots, L - L/8\}$. From the estimates of Lemma 2.2 this last event has probability $O(L^{-1/2})$, so that

$$\nu\left(L/4 \leqslant \zeta \leqslant 3L/4\right) = O(L^{-1/2})\,. \tag{4.65}$$

In conclusion, adjusting the constant c, one has $\mathcal{D}(f,f) \leqslant c\, L^{-5/2}\, \log L$. This ends the proof of (4.63). □

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