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Conley index condition for asymptotic stability[☆]

Emmanuel Moulay*

*Xlim (UMR-CNRS 6172) – Département SIC
Université de Poitiers – Bât. SP2MI
Bvd Marie et Pierre Curie – BP 30179
86962 Futuroscope Chasseneuil Cedex – France*

Qing Hui**

*Department of Mechanical Engineering
Texas Tech University
Lubbock, TX 79409-1021 – USA*

Abstract

In this paper, we use Conley index theory to develop necessary conditions for stability of equilibrium and periodic solutions of nonlinear continuous-time systems. The Conley index is a topological generalization of Morse theory which has been developed to analyze dynamical systems using topological methods. In particular, the Conley index of an invariant set with respect to a dynamical system is defined as the relative homology of an index pair for the invariant set. The Conley index can then be used to examine the structure of the system invariant set as well as the system dynamics within the invariant set, including system stability properties. Efficient numerical algorithms using homology theory have been developed in the literature to compute the Conley index and can be used to deduce the stability properties of nonlinear dynamical systems.

Keywords: Asymptotic Lyapunov stability, Conley index.

1. Introduction

One of the most basic issues in system theory is stability of dynamical systems. The most complete contribution to stability analysis of nonlinear dynamical systems was introduced in the late nineteenth century by A. M. Lyapunov in his seminal work in 1892 entitled *The General Problem of the Stability of Motion* [1, 2]. Lyapunov's results which include the direct and indirect methods,

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* (33) 5 49 49 68 55

** (806) 742-3563 238

Email addresses: emmanuel.moulay@univ-poitiers.fr (Emmanuel Moulay), qing.hui@ttu.edu (Qing Hui)

along with the Krasovskii-LaSalle invariance principle [3, 4], provide a powerful framework for analyzing the stability of equilibrium and periodic solutions of nonlinear dynamical systems. Lyapunov’s direct method for examining the stability of an equilibrium state of a dynamical system requires the construction of a positive-definite function of the system states (Lyapunov function) for which its time rate of change due to perturbations in a neighborhood of the system’s equilibrium is always negative or zero. Stability of periodic solutions of a dynamical system can also be addressed by constructing a Lyapunov-like function satisfying the Krasovskii-LaSalle invariance principle.

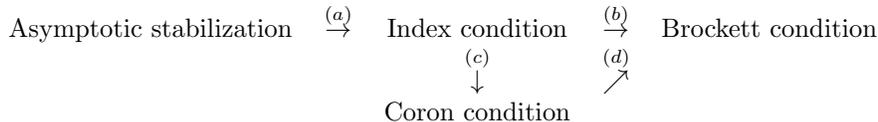
Alternatively, in the case where the trajectory of a dynamical system can be relatively easily integrated, Poincaré’s theorem [5] provides a powerful tool in analyzing the stability properties of periodic orbits and limit cycles. Specifically, Poincaré’s theorem provides necessary and sufficient conditions for stability of periodic orbits based on the stability properties of a fixed point of a discrete-time dynamical system constructed from a Poincaré return map. However, in many applications, especially for high-dimensional nonlinear systems, system trajectories cannot be relatively easily integrated and the construction of a Lyapunov function for establishing stability properties of a dynamical system can be a daunting task.

In this paper, we use Conley index theory [6, 7, 8] to develop necessary conditions for stability of equilibrium and periodic solutions of nonlinear continuous-time systems. The Conley index is a topological generalization of Morse theory which has been developed to analyze dynamical systems using topological methods. In particular, the Conley index of an invariant set \mathcal{K} with respect to a dynamical system is defined as the relative homology of an index pair for \mathcal{K} . The Conley index can then be used to examine the structure of the system invariant set as well as the system dynamics within the invariant set, including system stability properties. Specifically, Conley index theory is based on isolating neighborhoods of the system state space which enclose components of chain-recurrent sets that can be used to detect connecting orbits between the components of these sets. The method generates a simplicial complex which can be used to provide an understanding of the behavior of sets of trajectories rather than individual orbits of dynamical systems. Efficient numerical algorithms using homology theory have been developed to compute the Conley index [9] and can be used to deduce the stability properties of nonlinear dynamical systems.

There are still several topological necessary conditions for asymptotic stability and stabilization for an equilibrium of differential equations. The oldest one is the *Brockett condition* given in [10, Theorem 1] for smooth systems. This condition is also true for continuous systems [11] and even for a class of discontinuous ones [12]. Then, another condition named *index condition* is addressed in [13, Theorem 52.1]. This condition uses the degree theory which is surveyed in [14]. It is true for finite dimensional systems by using the Brouwer degree and for infinite dimensional systems by using the Leray-Schauder degree (see for instance [15] and [16, Theorem 2.2]). This condition has been extended to discontinuous systems in [17]. Finally, the last necessary condition for stabilization is the *Coron condition* given in [18]. It is based on homology (or

homotopy) groups. For an introduction to algebraic topology, the reader may refer to [19, 20]. Recently, the Coron condition has been extended for the stabilization of a codimension one submanifold by using the Poincaré duality in [21].

The relations between these conditions are as follow:



With some additional assumptions, a converse of (a) is presented in [16, Theorem 2.2]. A converse of (c) is given in [22, Proposition A.1] and [23, Theorem 11.8] for dynamical stabilization of analytic systems. Point (b) can be seen for instance in the proof of the Brockett condition given by Sontag in [24, Theorem 22] or by Coron in [23, Theorem 11.4]. They use the fact that the degree is a homotopy invariant as long as the homotopy satisfies a boundary condition (see [24, Theorem 22] for more details). Moreover, in the following paper [25], the author presents a version of the Brockett and index conditions dedicated to global attractors. Homotopy theory has also been used for a topological obstruction to simultaneous stabilization in [26]. The index condition is dedicated to equilibrium points and is based on the degree theory. It was first addressed for asymptotic stability in [13] and can be easily used for stabilization.

The main goal of this paper is to use the more general Conley index theory in order to give a new index condition for compact invariant sets, in particular for equilibria and periodic orbit. A recent topological study of compact invariant sets has been addressed in [27]. The paper is organized as follow. After some preliminaries on Conley index in section 2, several Conley index conditions are given in section 3 for asymptotic stability of an equilibrium point and stable periodic orbits. Finally, a conclusion is addressed in Section 4.

2. Preliminaries on Conley index

In this paper, we consider nonlinear continuous-time dynamical systems of the form:

$$\dot{x}(t) = f(x(t)), \quad x \in \mathcal{D}, \quad t \in \mathbb{R}_{\geq 0}, \quad (1)$$

where \mathcal{D} is an open subset of \mathbb{R}^n and f a C^1 -vector field. In this paper, we assume that all solutions to (1) are bounded over their maximal interval of existence, and hence, by the Peano-Cauchy theorem [28, pp. 16, 17] can be extended to infinity. System (1) generates a *semiflow* denoted by $\varphi(t, x)$ for all $x \in \mathcal{D}$ and $t \in \mathbb{R}_{\geq 0}$, that is a continuous map satisfying:

1. $\varphi(0, x) = x$;
2. $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$.

For each $t \in \mathbb{R}_{\geq 0}$, we let φ_t denote the map $x \mapsto \varphi(t, x)$.

$\mathcal{K} \subset \mathbb{R}^n$ is *invariant* if $f(\mathcal{K}) = \mathcal{K}$. Let $\mathcal{N} \subset \mathbb{R}^n$, the *maximal invariant subset* of \mathcal{N} with respect to the flow φ is denoted by:

$$\text{Inv}(\mathcal{N}) = \{x \in \mathcal{N} : \varphi(\mathbb{R}, x) \subseteq \mathcal{N}\}$$

$\overline{\mathcal{N}}$ denotes the closure of \mathcal{N} and $\overset{\circ}{\mathcal{N}}$ the interior of \mathcal{N} . For given sets \mathcal{A} and \mathcal{B} , $\mathcal{A} \cong \mathcal{B}$ denotes that \mathcal{A} is isomorphic to \mathcal{B} . For $x \in \mathbb{R}^n$, $\mathcal{B}_\varepsilon(x)$ denotes the ball of radius ε centered at x .

Let p be an *equilibrium point* of f , i.e. $f(p) = 0$, and let $Df(p)$ denote the Jacobian matrix of f at p . If the matrix $Df(p)$ has no eigenvalues with zero real parts then p is called an *hyperbolic equilibrium point*.

We recall the definitions of the index pair and Conley index given in [6, 8].

Definition 1. *An isolating neighborhood is a compact set \mathcal{N} such that $\text{Inv}(\mathcal{N}, s) \subset \overset{\circ}{\mathcal{N}}$. \mathcal{K} is an isolated invariant set if $\mathcal{K} = \text{Inv}(\mathcal{N})$ for some isolating neighborhood \mathcal{N} .*

Isolating neighborhoods are robust in the sense that if \mathcal{N} is an isolating neighborhood for a flow φ , then \mathcal{N} is also an isolating neighborhood for all nearby flows ψ . Isolating neighborhoods can be easily found for a typical flow of a dynamical system. The Conley index [6] provides a method for obtaining information about the dynamics of $\text{Inv}(\mathcal{N})$ from information on the isolating neighborhood \mathcal{N} . In particular, it can be shown that if \mathcal{N} and \mathcal{N}' are isolating neighborhoods such that $\text{Inv}(\mathcal{N}) = \text{Inv}(\mathcal{N}')$, then the Conley index of \mathcal{N} is the same as the Conley index of \mathcal{N}' . Thus, one can consider the Conley index as an index of isolated invariant sets.

Definition 2. *Let \mathcal{K} be an isolated invariant set. A pair of compact sets $(\mathcal{N}, \mathcal{L})$, where $\mathcal{L} \subset \mathcal{N}$, is called an index pair for \mathcal{K} if:*

1. $\mathcal{K} = \text{Inv}(\overline{\mathcal{N} \setminus \mathcal{L}})$ and $\mathcal{N} \setminus \mathcal{L}$ is a neighborhood of \mathcal{K} ;
2. \mathcal{L} is positively invariant in \mathcal{N} ; that is given $x \in \mathcal{L}$ and $\varphi([0, t], x) \subset \mathcal{N}$, then $\varphi([0, t], x) \subset \mathcal{L}$;
3. \mathcal{L} is an exit set for \mathcal{N} ; that is given $x \in \mathcal{N}$ and $t_1 > 0$ such that $\varphi(t_1, x) \notin \mathcal{N}$, then there exists $t_0 \in [0, t_1]$ such that $\varphi([0, t_0], x) \subset \mathcal{N}$ and $\varphi(t_0, x) \in \mathcal{L}$.

Definition 3. *Let $(\mathcal{N}, \mathcal{L})$ be an index pair and let $\mathcal{K} = \text{Inv}(\overline{\mathcal{N} \setminus \mathcal{L}})$. Then the Conley index of \mathcal{K} , denoted by $CH_*(\mathcal{K})$, is the relative homology of \mathcal{N} modulo \mathcal{L} , that is:*

$$CH_*(\mathcal{K}) \triangleq H_*(\mathcal{N}, \mathcal{L}). \quad (2)$$

The notion of relative homology is presented in [20, p. 115]. In Conley's original definition [6], the index was defined to be the homotopy type of the particular topological space. Since working with homotopy types of spaces is

very difficult (and virtually impossible to compute), the homological definition given above is much more useful. In light of Definition 3, several important properties of the Conley index can be summarized as follows. First, knowledge of the invariant set \mathcal{K} is not necessary in order to compute its Conley index. In addition, if $(\mathcal{N}, \mathcal{L})$ is an index pair for the flow φ , then $\overline{\mathcal{N} \setminus \mathcal{L}}$ is an isolating neighborhood for φ , and hence, for all flows sufficiently close to φ . Hence, if ψ denotes a flow sufficiently close to φ , then the Conley index of \mathcal{K} is isomorphic to the Conley index of $\mathcal{K}_\psi \triangleq \text{Inv}(\overline{\mathcal{N} \setminus \mathcal{L}}, \psi)$. Finally, given an isolated invariant set \mathcal{K} , its Conley index is independent of the index pair used to compute it.

Next, we give several key theorems concerning the Conley index. For the proofs of these results, see [7].

Theorem 4. *If \mathcal{K} is an isolated invariant set, then there exists an index pair for \mathcal{K} , and hence, the Conley index of \mathcal{K} exists.*

Theorem 5. *Let $(\mathcal{N}, \mathcal{L})$ and $(\mathcal{N}', \mathcal{L}')$ be index pairs for an isolated invariant set \mathcal{K} . Then*

$$H_*(\mathcal{N}, \mathcal{L}) \cong H_*(\mathcal{N}', \mathcal{L}'), \quad (3)$$

and hence, the Conley index is well defined.

Note that in the case where $\mathcal{N} = \emptyset$ is an isolated invariant set (vacuously), (\emptyset, \emptyset) is an index pair for \emptyset . In this case, $CH_*(\emptyset) \cong 0$. Even though this result is trivial, its converse leads to the following fundamental result known as the *Wazewski property* [7]. The following result establishes the simplest possible existence result which can be obtained using the Conley index.

Theorem 6 (Wazewski property). *If \mathcal{N} is an isolating neighborhood such that $CH_*(\mathcal{N}) \not\cong 0$, then $\text{Inv}(\mathcal{N}) \neq \emptyset$.*

Theorem 6 states that if the Conley index of an isolating neighborhood is nontrivial, then the invariant set is not an empty set. It is important to note that Theorem 6 gives no information on the structure of the invariant set \mathcal{S} .

The reader can find more details about the Conley index theory in [8].

3. Conley index condition

3.1. The case of stable equilibria

Let us recall the definition of asymptotic stability which is a local property.

Definition 7. *Suppose that $p \in \mathcal{D}$ is an equilibrium point of f , p is Lyapunov stable if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(p)$ then $\varphi(t, x_0) \in \mathcal{B}_\varepsilon(p)$ for all $t \geq 0$. p is asymptotically stable if it is Lyapunov stable and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(p)$ then $\|\varphi(t, x_0) - p\| \rightarrow 0$ as $t \rightarrow \infty$.*

Theorem 8 ([8]). *Let p be a hyperbolic fixed point with an unstable manifold of dimension n . Then*

$$CH_i(p) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Theorem 8 can be used to develop stability results for continuous-time nonlinear dynamical systems. In particular, if we can compute the Conley index of a hyperbolic equilibrium point, then we can determine its stability from its Conley index. This is quite powerful for two reasons. First, the Conley index provides a way for determining the stability properties of dynamical systems on a manifold or a polyhedron, which is usually difficult to deduce by using standard methods. Second, the eigenvalues of high dimensional Jacobian matrices are not easy to obtain numerically. However, the Conley index can be efficiently computed using homology theory. Specifically, *computational homology* has been developed in the literature [9] to give a robust algorithm for computing the Conley index. Hence, merging stability theory with Conley index theory can provide an alternative approach for determining the stability properties of nonlinear dynamical systems. The following result, first given in [29] and whose proof is derived from the one of Theorem 8, provides a necessary condition for asymptotic stability of an equilibrium point by using the Conley index.

Theorem 9 (Conley index condition for equilibria). *Let us consider the system (1). If p is asymptotically stable then*

$$CH_i(p) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that p is an hyperbolic equilibrium point. If $g = Df(p)$ denotes the linearization of f at point p , then the Hartman-Grobman's theorem given in [30] implies that there exists a neighborhood U of p and a homeomorphism $h : U \rightarrow \mathbb{R}^n$ such that

$$f|_U = h^{-1} \circ g \circ h$$

that is, in a neighborhood U of p , f is topologically conjugate to its linearization. Thus, the flow of the system (1) in U is topologically equivalent to the flow in a neighborhood of the origin of

$$\dot{y} = Df(p) y. \quad (5)$$

A linear change of variables transforms (5) into

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (6)$$

where A is a $m \times m$ matrix for which the real parts of all its eigenvalues are less than zero and B a $k \times k$ matrix for which the real parts of all its eigenvalues are

greater than zero with $m+k=n$. As $([-1, 1]^m \times [-1, 1]^k, [-1, 1]^m \times \partial([-1, 1]^k))$ is an index pair for the origin of the system (6), it implies that for all $i \in \mathbb{N}$,

$$CH_i(0_{(6)}) = H_i([-1, 1]^m \times [-1, 1]^k, [-1, 1]^m \times \partial([-1, 1]^k)) = \begin{cases} \mathbb{Z} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} .$$

By using the continuation homotopy property of the Conley index recalled in [8, Theorem 3.10] and [31, Theorem 1.10], we have that $CH_i(0_{(6)}) = CH_i(0_{(5)}) = CH_i(p)$. As p is asymptotically stable, the Lyapunov's indirect method, given for instance in [32, Theorem 4.7], implies that $k=0$ and the result follows from the previous calculus. ■

If there exists $i \geq 1$ such that $CH_i(p) \cong \mathbb{Z}$ then p cannot be asymptotically stable. The Conley index can be efficiently computed by using the computational Homology [9]. The condition of asymptotic stability given by Theorem 9 is only necessary. The following corollary can be used to compute the Conley index of an hyperbolic equilibrium.

Corollary 10. *Let $\{\lambda_j\}_{1 \leq j \leq n}$ denote the eigenvalues of $Df(p)$. If there exist $(m, k) \in \mathbb{N}^2$ such that*

1. $m+k=n$,
2. for all $1 \leq q \leq m$ and $1 \leq r \leq k$ we have $Re(\lambda_q) < 0$ and $Re(\lambda_r) > 0$,

then for all $i \in \mathbb{N}$,

$$CH_i(p) \cong \begin{cases} \mathbb{Z} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} .$$

3.2. The case of stable periodic orbits

Next, we use the Conley index to develop necessary conditions for asymptotic stability of periodic orbits. To begin, we introduce the notions of Lyapunov and asymptotic stability of a periodic orbit of the nonlinear dynamical system (1). For this definition $\text{dist}(p, \mathcal{M})$ denotes the smallest distance from a point p to any point in the set \mathcal{M} , that is, $\text{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} \|p - x\|$. Furthermore, the following definition of periodic solutions and periodic orbits of (1) is needed.

Definition 11. *A solution $\varphi(t, x_0)$ of (1) is periodic if there exists a finite time $T > 0$ such that $\varphi(t+T, x_0) = \varphi(t, x_0)$ for all $t \geq 0$. The minimal $T > 0$ for which the solution $\varphi(t, x_0)$ of (1) is periodic is called the period. A set $\mathcal{O} \subset \mathcal{D}$ is a periodic orbit of (1) if $\mathcal{O} = \{x \in \mathcal{D} : x = \varphi(t, x_0), 0 \leq t \leq T\}$ for some periodic solution $\varphi(t, x_0)$ of (1).*

Definition 12. *A periodic orbit \mathcal{O} of (1) is Lyapunov stable if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\text{dist}(x_0, \mathcal{O}) < \delta$, then $\text{dist}(\varphi(t, x_0), \mathcal{O}) < \varepsilon$, $t \geq 0$. A periodic orbit \mathcal{O} of (1) is asymptotically stable if it is Lyapunov stable and there exists $\delta > 0$ such that if $\text{dist}(x_0, \mathcal{O}) < \delta$, then $\text{dist}(\varphi(t, x_0), \mathcal{O}) \rightarrow 0$ as $t \rightarrow \infty$.*

The next result gives a sufficient condition for determining the existence of periodic orbits via the Conley index. Recall that $\Xi \subset \mathcal{X}$ is a *Poincaré section* for φ in \mathcal{N} if Ξ is a local section, $\Xi \cap \mathcal{N}$ is closed, and for every $x \in \mathcal{N}$, $\varphi((0, \infty), x) \cap \Xi \neq \emptyset$.

Theorem 13 ([33]). *Assume that \mathcal{N} is an isolating neighborhood for the semiflow φ defined on a metric space \mathcal{X} which admits a Poincaré section Ξ . If $\text{Inv}(\mathcal{N})$ has the Conley index of a hyperbolic periodic orbit, then $\text{Inv}(\mathcal{N})$ contains a periodic orbit.*

To proceed, we assume that for the point $p \in \mathcal{D}$, the dynamical system (1) has a periodic solution $\varphi(t, p), t \geq 0$, with period $T > 0$ that generates the periodic orbit

$$\mathcal{O} := \{x \in \mathcal{D} : x = \varphi(t, p), 0 \leq t \leq T\}.$$

Note that \mathcal{O} is a compact invariant set. Furthermore, we assume that there exists a continuously differentiable function $\mathcal{X} : \mathcal{D} \rightarrow \mathbb{R}$ such that the $(n - 1)$ -dimensional hyperplane defined by

$$\mathcal{H} := \{x \in \mathcal{D} : \mathcal{X}(x) = 0\}$$

contains the point $x = p$ and $D\mathcal{X}(p) \neq 0$. In addition, we assume that the hyperplane \mathcal{H} is not tangent to the periodic orbit \mathcal{O} at $x = p$, that is, $D\mathcal{X}(p)f(p) \neq 0$. Next, define the local section $\mathcal{S} \subset \mathcal{H}$ such that $p \in \mathcal{S}$, $D\mathcal{X}(x) \neq 0, x \in \mathcal{S}$, and all trajectories of (1) starting in \mathcal{S} are not tangent to \mathcal{H} , that is, $\mathcal{X}'(x)f(x) \neq 0, x \in \mathcal{S}$. Note that a trajectory $\varphi(t, p)$ will intersect \mathcal{S} at p in T seconds. Furthermore, let

$$\mathcal{U} := \{x \in \mathcal{S} : \exists \tau(x) > 0; \varphi(\tau(x), x) \in \mathcal{S} \text{ and } \varphi(t, x) \notin \mathcal{S}, 0 < t < \tau(x)\}. \quad (7)$$

Finally, define the Poincaré return map $P : \mathcal{U} \rightarrow \mathcal{S}$ by

$$P(x) := \varphi(\tau(x), x), \quad x \in \mathcal{U}. \quad (8)$$

Next, define

$$\mathcal{D}_1 := \{x \in \mathcal{D} : \exists \tau(x) > 0; \varphi(\tau(x), x) \in \mathcal{S}\}.$$

The existence of \mathcal{D}_1 is guaranteed by continuous dependence of solutions of (1) on initial data; moreover, for every $x \in \mathcal{O}$ there exists $\delta = \delta(x) > 0$ such that $\mathcal{B}_\delta(x) \subset \mathcal{D}_1$, and hence, \mathcal{O} is a subset of $\mathring{\mathcal{D}}_1$. Similarly, define

$$\begin{aligned} \mathcal{O}_\alpha &:= \{x \in \mathcal{D}_1 : \varphi(\tau(x), x) \in \mathcal{S}_\alpha\}, \\ \mathcal{U}_\alpha &:= \{x \in \mathcal{S}_\alpha : \varphi(\tau(x), x) \in \mathcal{S}_\alpha\}, \end{aligned}$$

where $\mathcal{S}_\alpha := \mathcal{B}_\alpha(p) \cap \mathcal{S}$, $\alpha > 0$, and $\mathcal{O} \subset \mathring{\mathcal{O}}_\alpha \subseteq \mathcal{D}_1$. The function $\tau : \mathcal{D}_1 \rightarrow \overline{\mathbb{R}}_{\geq 0}$ defines the time required for the trajectory $\varphi(t, x), x \in \mathcal{D}_1$, to return to the local section \mathcal{S} . Note that $\tau(x) > 0, x \in \mathcal{U}$. The following lemma shows that $\tau(\cdot)$ is continuous on $\mathcal{D}_1 \setminus \mathcal{H}$.

Lemma 14. *Consider the nonlinear dynamical system (1). Assume that the point $p \in \mathcal{D}_1$ generates the periodic orbit $\mathcal{O}_1 := \{x \in \mathcal{D}_1 : x = \varphi(t, p), 0 \leq t \leq T\}$, where $\varphi(t, p), t \geq 0$, is the periodic solution with period $T \equiv \tau(p)$. Then the function $\tau : \mathcal{D}_1 \rightarrow \overline{\mathbb{R}}_+$ is continuous on $\mathcal{D}_1 \setminus \mathcal{H}$.*

Proof. Let $\varepsilon > 0$ and $x \in \mathcal{D}_1 \setminus \mathcal{H}$. Note that $x^* := \varphi(\tau(x), x) \in \mathcal{S}$, and hence, $D\mathcal{X}(x^*) \neq 0$ and $D\mathcal{X}(x^*)f(x^*) \neq 0$. Now, since $\varphi(\cdot, x)$ is continuous in t , $[0, t_1]$ is a compact interval, and \mathcal{S} is closed. It follows from the definition of $\tau(\cdot)$ that for any $t_1 \in (0, \tau(x))$,

$$\sigma(t_1) := \inf_{0 \leq t \leq t_1} \text{dist}(\varphi(t, x), \mathcal{S}) > 0. \quad (9)$$

Next, for $\hat{\varepsilon} > 0$ sufficiently small, define $x_2 := \varphi(t_2, x)$ and $t_2 := \tau(x) + \frac{\hat{\varepsilon}}{2}$. Since $D\mathcal{X}(x^*) \neq 0$ and $D\mathcal{X}(x^*)f(x^*) \neq 0$, it follows that $\text{dist}(x_2, \mathcal{S}) > 0$. Now, define $t_1 := \tau(x) - \frac{\hat{\varepsilon}}{2}$ and $x_1 := \varphi(t_1, x)$. Then it follows from the continuous dependence of solutions to (1) in time and initial data that there exists $\delta > 0$ such that for all $y \in \mathcal{B}_\delta(x)$,

$$\sup_{0 \leq t \leq t_2} \|s(t, y) - s(t, x)\| < \min\{\text{dist}(x_2, \mathcal{S}), \sigma(t_1)\}.$$

Hence, for all $y \in \mathcal{B}_\delta(x)$, it follows that $t_1 < \tau(y) < t_2$. Now, taking $\hat{\varepsilon} < \varepsilon$, it follows that $|\tau(y) - \tau(x)| < \varepsilon$, establishing the continuity of $\tau(\cdot)$ at $x \in \mathcal{D}_1 \setminus \mathcal{H}$.

■

Finally, define the discrete-time dynamical system given by

$$z(k+1) = P(z(k)), \quad z(0) \in \mathcal{U}, \quad k \in \overline{\mathbb{N}}. \quad (10)$$

Clearly $x = p$ is a fixed point of (10) since $T = \tau(p)$ and hence $p = P(p)$. Since Poincaré's theorem provides necessary and sufficient conditions for stability of periodic orbits based on the stability properties of a fixed point of the discrete-time dynamical system (10), stability notions of discrete-time systems are required. Next, we give two key necessary results for developing Poincaré's theorem in terms of the Conley index theory.

Theorem 15 ([34]). *Consider the nonlinear dynamical system (1) with the Poincaré map defined by (8). Assume that the point $p \in \mathcal{D}$ generates the periodic orbit $\mathcal{O} := \{x \in \mathcal{D} : x = \varphi(t, p), 0 \leq t \leq T\}$, where $\varphi(t, p), t \geq 0$, is the periodic solution with period $T \equiv \tau(p)$. Then the following statements hold:*

- i) $p \in \mathcal{D}$ is a Lyapunov stable fixed point of (10) if and only if the periodic orbit \mathcal{O} generated by p is Lyapunov stable.*
- ii) $p \in \mathcal{D}$ is an asymptotically stable fixed point of (10) if and only if the periodic orbit \mathcal{O} generated by p is asymptotically stable.*

Theorem 15 is a restatement of the classical Poincaré theorem, presenting necessary and sufficient conditions for Lyapunov and asymptotic stability of a periodic orbit of the nonlinear dynamical system (1) based on the stability

properties of a fixed point of the n -dimensional discrete-time dynamical system (10) involving the Poincaré map (8).

Next, we present a classical corollary to Poincaré's theorem that allows us to analyze the stability of periodic orbits by replacing the n th-order nonlinear dynamical system by an $(n - 1)$ th-order discrete-time system. To present this result assume, without loss of generality, that $\frac{\partial \mathcal{X}(x)}{\partial x_n} \neq 0, x \in \mathcal{S}_\alpha$, where $x = [x_1, \dots, x_n]^T$ and $\alpha > 0$ is sufficiently small. Then it follows from the implicit function theorem [32] that $x_n = g(x_1, \dots, x_{n-1})$, where $g(\cdot)$ is a continuously differentiable function at $x_r := [x_1, \dots, x_{n-1}]^T$ such that $[x_r^T, g(x_r)]^T \in \mathcal{S}_\alpha$. Note that in this case $P : \mathcal{U}_\alpha \rightarrow \mathcal{S}_\alpha$ in (10) is given by $P(x) := [P_1(x), \dots, P_n(x)]^T$, where

$$P_n(x_r, g(x_r)) = g(P_1(x_r, g(x_r)), \dots, P_{n-1}(x_r, g(x_r))). \quad (11)$$

Hence, we can reduce the n -dimensional discrete-time system (10) to the $(n - 1)$ -dimensional discrete-time system given by

$$z_r(k + 1) = \mathcal{P}_r(z_r(k)), \quad k \in \overline{\mathbb{N}}, \quad (12)$$

where $z_r \in \mathbb{R}^{n-1}$, $[z_r^T(\cdot), g(z_r(\cdot))]^T \in \mathcal{S}_\alpha$, and

$$\mathcal{P}_r(x_r) := \begin{bmatrix} P_1(x_r, g(x_r)) \\ \vdots \\ P_{n-1}(x_r, g(x_r)) \end{bmatrix}. \quad (13)$$

Note that it follows from (11) and (13) that $p := [p_r^T, g(p_r)]^T \in \mathcal{S}_\alpha$ is a fixed point of (10) if and only if p_r is a fixed point of (12). To present the following result define

$$\begin{aligned} \mathcal{S}_{r\alpha} &:= \{x_r \in \mathbb{R}^{n-1} : [x_r^T, g(x_r)]^T \in \mathcal{S}_\alpha\}, \\ \mathcal{U}_{r\alpha} &:= \{x_r \in \mathcal{S}_{r\alpha} : [x_r^T, g(x_r)]^T \in \mathcal{U}_\alpha\}. \end{aligned}$$

Corollary 16 ([34]). *Consider the nonlinear dynamical system (1) with the Poincaré return map defined by (8). Assume that $\frac{\partial \mathcal{X}(x)}{\partial x_n} \neq 0, x \in \mathcal{S}_\alpha$, and the point $p \in \mathcal{S}_\alpha$ generates the periodic orbit $\mathcal{O} := \{x \in \mathcal{D} : x = \varphi(t, p), 0 \leq t \leq T\}$, where $\varphi(t, p), t \geq 0$, is the periodic solution with the period $T = \tau(p)$ such that $\varphi(\tau(p), p) = p$. Then for $p = [p_r^T, g(p_r)]^T \in \mathcal{S}_\alpha$, p_r is an asymptotically stable fixed point of (12) if and only if the periodic orbit \mathcal{O} is asymptotically stable.*

The next theorem presents a necessary condition for asymptotic stability of periodic orbits in terms of the Conley index.

Theorem 17 (Conley index condition for periodic orbits). *Consider the nonlinear dynamical system (1) with the Poincaré map defined by (8) and let $x_0 = [x_{r0}^T, g(x_{r0})]^T \in \mathcal{S}_\alpha$ be a hyperbolic fixed point. Assume that $\frac{\partial \mathcal{X}(x)}{\partial x_n} \neq 0, x \in \mathcal{S}_\alpha$. If the periodic orbit $\mathcal{O} = \{x \in \mathcal{D} : x = \varphi(t, x_0), 0 \leq t \leq T\}$ generated by $x_0 \in \mathcal{S}_\alpha$ is asymptotically stable, then*

$$CH_i(x_0) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$CH_i(x_{r0}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using similar arguments as in the proof of Theorem 9, it follows that if x_0 is asymptotically stable for (10), then

$$CH_i(x_0) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 15, x_0 is asymptotically stable for (10) if and only if \mathcal{O} is asymptotically stable. Now, the conclusion follows immediately by combining the above results together.

The second conclusion can be proved in a similar way by noting Corollary 16.

■

3.3. Remarks about stable sets

The main goal of this section is to introduce some existing results about asymptotically stable sets, linked with the Conley index theory.

Definition 18. Let $\mathcal{K} \subset \mathcal{M}$ be an invariant set for System (1). The set \mathcal{K} is asymptotically stable if the following two conditions hold:

1. \mathcal{K} is Lyapunov stable, that is, for every open neighborhood $\mathcal{V} \subseteq \mathcal{M}$ of \mathcal{K} , there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{M}$ of \mathcal{K} such that $\varphi_t(\mathcal{U}) \subseteq \mathcal{V}$ for every $t \geq 0$.
2. \mathcal{K} is attractive, that is, there exists an open neighborhood $\mathcal{W} \subseteq \mathcal{M}$ of \mathcal{K} such that, for every $x \in \mathcal{W}$ and every open neighborhood $\mathcal{U} \subseteq \mathcal{M}$ of \mathcal{K} , there exists $T \geq 0$ such that $\varphi(t, x) \in \mathcal{U}$ for all $t > T$.

The domain of attraction of an asymptotically stable set $\mathcal{K} \subseteq \mathcal{M}$ is the set \mathcal{A} of points x such that, for every open neighborhood \mathcal{U} of \mathcal{K} , there exists $T > 0$ such that $\varphi(t, x) \in \mathcal{U}$ for all $t \geq T$. It is well known that the domain of attraction \mathcal{A} of a compact asymptotically stable set is open and invariant, that is, $\varphi_t(\mathcal{A}) = \mathcal{A}$ for all $t \geq 0$.

Theorem 19 ([8]). Let \mathcal{K} be a hyperbolic invariant set that is diffeomorphic to a circle. Assume that \mathcal{K} has an oriented unstable manifold of dimension $n + 1$. Then

$$CH_i(\mathcal{K}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, n + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Definition 20 ([35]). A smooth function $V : \mathcal{U} \rightarrow \mathbb{R}$ defined on an open neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of \mathcal{K} is said to be a Lyapunov function if it satisfies

- i) $V(x) \geq 0$ for all $x \in \mathcal{U}$ and $V(x) = 0$ if and only if $x \in \mathcal{K}$.
- ii) $\frac{d}{dt}V(x(t)) \leq 0$ on $\mathcal{U} \setminus \mathcal{K}$ for every $x(0) \in \mathcal{U} \setminus \mathcal{K}$.

iii) V tends to a constant (possibly infinite) value on the boundary $\partial\mathcal{U}$.

Lemma 21. *Assume \mathcal{K} is compact and asymptotically stable. Then there exists a Lyapunov function $V(\cdot)$ such that for all $c > 0$ small enough,*

$$CH_{n-1}(V^{-1}(c)) \cong \mathbb{Z}. \quad (15)$$

Proof. See Proposition 2.1 of [35]. ■

4. Conclusion

The aim of this paper is to use the Conley index theory in order to provide necessary conditions for asymptotic stability of compact sets. Precise necessary conditions are given for equilibria and periodic orbits. The more general case of stable sets is still a challenge for future work, in particular the case of closed sets, not necessarily bounded.

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