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CALOGERO-MOSER VERSUS KAZHDAN-LUSZTIG CELLS

CÉDRIC BONNAFÉ AND RAPHAËL ROUQUIER

1. INTRODUCTION

In [KaLu], Kazhdan and Lusztig developed a combinatorial theory associated with Coxeter groups. They defined in particular partitions of the group in left and two-sided cells. For Weyl groups, these have a representation theoretic interpretation in terms of primitive ideals, and they play a key role in Lusztig's description of unipotent characters for finite groups of Lie type [Lu3]. Lusztig generalized this theory to Hecke algebras of Coxeter groups with unequal parameters [Lu2, Lu4].

We propose a definition of left cells and two-sided cells for complex reflection groups, based on ramification theory for Calogero-Moser spaces. These spaces have been defined via rational Cherednik algebras by Etingof and Ginzburg [EtGi]. We conjecture that these coincide with Kazhdan-Lusztig cells, for real reflection groups. Counterparts of families of irreducible characters have been studied by Gordon and Martino [GoMa], and we provide here a version of left cell representations. The Calogero-Moser cells are studied in detail in [BoRou].

2. CALOGERO-MOSER SPACES AND CELLS

2.1. Rational Cherednik algebras at $t = 0$. Let us recall some constructions and results from [EtGi]. Let V be a finite-dimensional complex vector space and W a finite subgroup of $GL(V)$. Let \mathcal{S} be the set of reflections of W , *i.e.*, elements g such that $\ker(g-1)$ is a hyperplane. We assume that W is a reflection group, *i.e.*, it is generated by \mathcal{S} .

We denote by \mathcal{S}/\sim the quotient of \mathcal{S} by the conjugacy action of W and we let $\{\underline{c}_s\}_{s \in \mathcal{S}/\sim}$ be a set of indeterminates. We put $A = \mathbf{C}[\mathbf{C}^{\mathcal{S}/\sim}] = \mathbf{C}[\{\underline{c}_s\}_{s \in \mathcal{S}/\sim}]$. Given $s \in \mathcal{S}$, let $v_s \in V$ (resp. $\alpha_s \in V^*$) be an eigenvector for s associated to the non-trivial eigenvalue.

The 0-rational Cherednik algebra \mathbf{H} is the quotient of $A \otimes T(V \oplus V^*) \rtimes W$ by the relations

$$[x, x'] = [\xi, \xi'] = 0$$

$$[\xi, x] = \sum_{s \in \mathcal{S}} \underline{c}_s \frac{\langle x, \alpha_s \rangle \cdot \langle v_s, \xi \rangle}{\langle v_s, \alpha_s \rangle} \quad \text{for } x, x' \in V^* \text{ and } \xi, \xi' \in V.$$

We put $Q = Z(\mathbf{H})$ and $P = A \otimes S(V^*)^W \otimes S(V)^W \subset Q$. The ring Q is normal. It is a free P -module of rank $|W|$.

2.2. Galois closure. Let $K = \text{Frac}(P)$ and $L = \text{Frac}(Q)$. Let M be a Galois closure of the extension L/K and R the integral closure of Q in M . Let $G = \text{Gal}(M/K)$ and $H = \text{Gal}(M/L)$. Let $\mathcal{P} = \text{Spec } P = \mathbf{A}_{\mathbf{C}}^{S/\sim} \times V/W \times V^*/W$, $\mathcal{Q} = \text{Spec } Q$ the Calogero-Moser space, and $\mathcal{R} = \text{Spec } R$.

We denote by $\pi : \mathcal{R} \rightarrow \mathcal{Q}$ the quotient by H , and by $\Upsilon : \mathcal{Q} \rightarrow \mathcal{P}$ and $\phi : \mathcal{P} \rightarrow \mathbf{A}_{\mathbf{C}}^{S/\sim}$ the canonical maps. We put $p = \Upsilon\pi : \mathcal{R} \rightarrow \mathcal{P}$ the quotient by G .

2.3. Ramification. Let $\mathfrak{r} \in \mathcal{R}$ be a prime ideal of R . We denote by $D(\mathfrak{r}) \subset G$ its decomposition group and by $I(\mathfrak{r}) \subset D(\mathfrak{r})$ its inertia group.

We have a decomposition into irreducible components

$$\mathcal{R} \times_{\mathcal{P}} \mathcal{Q} = \bigcup_{g \in G/H} \mathcal{O}_g, \text{ where } \mathcal{O}_g = \{(x, \pi(g^{-1}(x))) \mid x \in \mathcal{R}\}$$

inducing a decomposition into irreducible components

$$V(\mathfrak{r}) \times_{\mathcal{P}} \mathcal{Q} = \coprod_{g \in I(\mathfrak{r}) \backslash G/H} \mathcal{O}_g(\mathfrak{r}), \text{ where } \mathcal{O}_g(\mathfrak{r}) = \{(x, \pi(g^{-1}g'(x))) \mid x \in V(\mathfrak{r}), g' \in I(\mathfrak{r})\}.$$

2.4. Undeformed case. Let $\mathfrak{p}_0 = \phi^{-1}(0) = \sum_{s \in S/\sim} P_{\mathfrak{C}_s}$. We have $P/\mathfrak{p}_0 = \mathbf{C}[V \oplus V^*]^{W \times W}$, $Q/\mathfrak{p}_0Q = \mathbf{C}[V \oplus V^*]^{\Delta W}$, where $\Delta(W) = \{(w, w) \mid w \in W\} \subset W \times W$. A Galois closure of the extension of $\mathbf{C}(\mathfrak{p}_0Q) = \mathbf{C}(V \oplus V^*)^{\Delta W}$ over $\mathbf{C}(\mathfrak{p}_0) = \mathbf{C}(V \oplus V^*)^{W \times W}$ is $\mathbf{C}(V \oplus V^*)^{\Delta Z(W)}$.

Let $\mathfrak{r}_0 \in \mathcal{R}$ above \mathfrak{p}_0 . Since \mathfrak{p}_0Q is prime, we have $G = D(\mathfrak{r}_0)H = HD(\mathfrak{r}_0)$ and $I(\mathfrak{r}_0) = 1$. Fix an isomorphism $\iota : \mathbf{C}(\mathfrak{r}_0) \xrightarrow{\sim} \mathbf{C}(V \oplus V^*)^{\Delta Z(W)}$ extending the canonical isomorphism of $\mathbf{C}(\mathfrak{p}_0Q)$ with $\mathbf{C}(V \oplus V^*)^{\Delta W}$.

The application ι induces an isomorphism $D(\mathfrak{r}_0) \xrightarrow{\sim} (W \times W)/\Delta Z(W)$, that restricts to an isomorphism $D(\mathfrak{r}_0) \cap H \xrightarrow{\sim} \Delta W/\Delta Z(W)$. This provides a bijection $G/H \xrightarrow{\sim} (W \times W)/\Delta W$. Composing with the inverse of the bijection $W \xrightarrow{\sim} (W \times W)/\Delta W$, $w \mapsto (w, 1)$, we obtain a bijection $G/H \xrightarrow{\sim} W$.

From now on, we identify the sets G/H and W through this bijection. Note that this bijection depends on the choices of \mathfrak{r}_0 and of ι . Since M is the Galois closure of L/K , we have $\bigcap_{g \in G} H^g = 1$, hence the left action of G on W induces an injection $G \subset \mathfrak{S}(W)$.

2.5. Calogero-Moser cells.

Definition 2.1. Let $\mathfrak{r} \in \mathcal{R}$. The \mathfrak{r} -cells of W are the orbits of $I(\mathfrak{r})$ in its action on W .

Let $c \in \mathbf{A}_{\mathbf{C}}^{S/\sim}$. Choose $\mathfrak{r}_c \in \mathcal{R}$ with $\overline{p(\mathfrak{r}_c)} = \bar{c} \times 0 \times 0$. The \mathfrak{r}_c -cells are called the *two-sided Calogero-Moser c -cells* of W . Choose now $\mathfrak{r}_c^{\text{left}} \in \mathcal{R}$ contained in \mathfrak{r}_c with $\overline{p(\mathfrak{r}_c^{\text{left}})} = \bar{c} \times V/W \times 0 \in \mathcal{P}$. The $\mathfrak{r}_c^{\text{left}}$ -cells are called the *left Calogero-Moser c -cells* of W . We have $I(\mathfrak{r}_c^{\text{left}}) \subset I(\mathfrak{r}_c)$. Consequently, every left cell is contained in a unique two-sided cell.

The map sending $w \in W$ to $\pi(w^{-1}(\mathfrak{r}_c))$ induces a bijection from the set of two-sided cells to $\Upsilon^{-1}(c \times 0 \times 0)$.

2.6. Families and cell multiplicities. Let E be an irreducible representation of $\mathbf{C}[W]$. We extend it to a representation of $S(V) \rtimes W$ by letting V act by 0. Let

$$\Delta(E) = e \cdot \text{Ind}_{S(V) \rtimes W}^{\mathbf{H}}(A \otimes_{\mathbf{C}} E), \text{ where } e = \frac{1}{|W|} \sum_{w \in W} w,$$

be the spherical Verma module associated with E . It is a Q -module.

Let $c \in \mathbf{A}_{\mathbf{C}}^{S/\sim}$ and let $\Delta^{\text{left}}(E) = (R/\mathfrak{r}_c^{\text{left}}) \otimes_P \Delta(E)$.

Definition 2.2. Given Γ a left cell, we define the cell multiplicity $m_{\Gamma}(E)$ of E as the multiplicity of $\Delta^{\text{left}}(E)$ at the component $\mathcal{O}_{\Gamma}(\mathfrak{r}_c^{\text{left}})$.

Note that $\sum_{\Gamma} m_{\Gamma}(E) \cdot [\mathcal{O}_{\Gamma}(\mathfrak{r}_c^{\text{left}})]$ is the support cycle of $\Delta^{\text{left}}(E)$.

There is a unique two-sided cell Λ containing all left cells Γ such that $m_{\Gamma}(E) \neq 0$. Its image in \mathcal{Q} is the unique $\mathfrak{q} \in \Upsilon^{-1}(c \times 0 \times 0)$ such that $(Q/\mathfrak{q}) \otimes_Q \Delta(E) \neq 0$. The corresponding map $\text{Irr}(W) \rightarrow \Upsilon^{-1}(c \times 0 \times 0)$ is surjective, and its fibers are the *Calogero-Moser families* of $\text{Irr}(W)$, as defined by Gordon [Go1].

2.7. Dimension 1. Let V be a one-dimensional complex vector space, let $d \geq 2$ and let W be the group of d -th roots of unity acting on V . Let $\zeta = \exp(2i\pi/d)$, let $s = \zeta \in W$ and $\underline{c}_i = \underline{c}_{s^i}$ for $1 \leq i \leq d-1$. We have $A = \mathbf{C}[\underline{c}_1, \dots, \underline{c}_{d-1}]$ and

$$\mathbf{H} = A\langle x, \xi, s \mid sxs^{-1} = \zeta^{-1}x, s\xi s^{-1} = \zeta\xi \text{ and } [\xi, x] = \sum_{i=1}^{d-1} \underline{c}_i s^i \rangle.$$

Let $\text{eu} = \xi x - \sum_{i=1}^{d-1} (1 - \zeta^i)^{-1} \underline{c}_i s^i$. We have $P = A[x^d, \xi^d]$ and $Q = A[x^d, \xi^d, \text{eu}]$. Define $\underline{\kappa}_1, \dots, \underline{\kappa}_d = \underline{\kappa}_0$ by $\underline{\kappa}_1 + \dots + \underline{\kappa}_d = 0$ and $\sum_{i=1}^{d-1} \underline{c}_i s^i = \sum_{i=0}^{d-1} (\underline{\kappa}_i - \underline{\kappa}_{i+1}) \varepsilon_i$, where $\varepsilon_i = \frac{1}{d} \sum_{j=0}^{d-1} \zeta^{ij} s^j$. We have $A = \mathbf{C}[\underline{\kappa}_1, \dots, \underline{\kappa}_d] / (\underline{\kappa}_1 + \dots + \underline{\kappa}_d)$.

The normalization of the Galois closure is described as follows. There is an isomorphism of A -algebras

$$A[X, Y, Z] / (XY - \prod_{i=1}^d (Z - \underline{\kappa}_i)) \xrightarrow{\sim} Q, \quad X \mapsto x^d, \quad Y \mapsto \xi^d \text{ and } Z \mapsto \text{eu}.$$

We have an isomorphism of A -algebras

$$A[X, Y, \lambda_1, \dots, \lambda_d] / (e_1(\lambda) = e_1(\underline{\kappa}), \dots, e_{d-1}(\lambda) = e_{d-1}(\underline{\kappa}), e_d(\lambda) = e_d(\underline{\kappa}) + (-1)^{d+1} XY) \xrightarrow{\sim} R$$

where $Z = \lambda_d$ and where e_i denotes the i -th elementary symmetric function. We have $G = \mathfrak{S}_d$, acting by permuting the λ_i 's, and $H = \mathfrak{S}_{d-1}$.

Let $\mathfrak{p}_0 = (\underline{\kappa}_1, \dots, \underline{\kappa}_d) \in \text{Spec } P$ and $\mathfrak{r}_0 = (\underline{\kappa}_1, \dots, \underline{\kappa}_d, \lambda_1 - \zeta \lambda_d, \dots, \lambda_{d-1} - \zeta^{d-1} \lambda_d) \in \text{Spec } R$. We have $D(\mathfrak{r}_0) = \langle (1, 2, \dots, d) \rangle \subset \mathfrak{S}_d$ and $\mathbf{C}(\mathfrak{r}_0) = \mathbf{C}(X, Y, \lambda_d = \sqrt[d]{XY}) = \mathbf{C}(X, Y, Z = \sqrt[d]{XY})$. The composite bijection $D(\mathfrak{r}_0) \xrightarrow{\sim} G/H \xrightarrow{\sim} W$ is an isomorphism of groups given by $(1, \dots, d) \mapsto s$.

Fix $c \in \mathbf{C}^{d-1}$ and let $\kappa_1, \dots, \kappa_d \in \mathbf{C}$ corresponding to c . Consider $\mathfrak{r} = \mathfrak{r}_c$ or $\mathfrak{r}_c^{\text{left}}$ as in §2.5. Then $I(\mathfrak{r})$ is the subgroup of \mathfrak{S}_d stabilizing $(\kappa_1, \dots, \kappa_d)$. The left c -cells coincide with the two-sided c -cells and two elements s^i and s^j are in the same cell if and only if $\kappa_i = \kappa_j$. Finally, the multiplicity $m_{\Gamma}(\det^j)$ is 1 if $s^j \in \Gamma$ and 0 otherwise.

3. COXETER GROUPS

3.1. Kazhdan-Lusztig cells. Following Kazhdan-Lusztig [KaLu] and Lusztig [Lu2, Lu4], let us recall the construction of cells.

We assume here V is the complexification of a real vector space $V_{\mathbf{R}}$ acted on by W . We choose a connected component C of $V_{\mathbf{R}} - \bigcup_{s \in S} \ker(s - 1)$ and we denote by S the set of $s \in S$ such that $\ker(s - 1) \cap \bar{C}$ has codimension 1 in \bar{C} . This makes (W, S) into a Coxeter group, and we denote by l the length function.

Let Γ be a totally ordered free abelian group and let $L : W \rightarrow \Gamma$ be a weight function, *i.e.*, a function such that $L(ww') = L(w) + L(w')$ if $l(ww') = l(w) + l(w')$. We denote by v^γ the element of the group algebra $\mathbf{Z}[\Gamma]$ corresponding to $\gamma \in \Gamma$.

We denote by H the Hecke algebra of W : this is the $\mathbf{Z}[\Gamma]$ -algebra generated by elements T_s with $s \in S$ subject to the relations

$$(T_s - v^{L(s)})(T_s + v^{-L(s)}) = 0 \text{ and } \underbrace{T_s T_t T_s \cdots}_{m_{st} \text{ terms}} = \underbrace{T_t T_s T_t \cdots}_{m_{st} \text{ terms}} \text{ for } s, t \in S \text{ with } m_{st} \neq \infty$$

where m_{st} is the order of st . Given $w \in W$, we put $T_w = T_{s_1} \cdots T_{s_n}$, where $w = s_1 \cdots s_n$ is a reduced decomposition.

Let i be the ring involution of H given by $i(v^\gamma) = v^{-\gamma}$ for $\gamma \in \Gamma$ and $i(T_s) = T_s^{-1}$. We denote by $\{C_w\}_{w \in W}$ the Kazhdan-Lusztig basis of H . It is uniquely defined by the properties that $i(C_w) = C_w$ and $C_w - T_w \in \bigoplus_{w' \in W} \mathbf{Z}[\Gamma_{<0}]T_{w'}$.

We introduce the partial order \prec_L on W . It is the transitive closure of the relation given by $w' \prec_L w$ if there is $s \in S$ such that the coefficient of $C_{w'}$ in the decomposition of $C_s C_w$ in the Kazhdan-Lusztig basis is non-zero. We define $w \sim_L w'$ to be the corresponding equivalence relation: $w \sim_L w'$ if and only if $w \prec_L w'$ and $w' \prec_L w$. The equivalence classes are the left cells. We define \prec_{LR} as the partial order generated by $w \prec_{LR} w'$ if $w \prec_L w'$ or $w^{-1} \prec_L w'^{-1}$. As above, we define an associated equivalence relation \sim_{LR} . Its equivalence classes are the two-sided cells.

When $\Gamma = \mathbf{Z}$, $L = l$, and W is a Weyl group, a definition of left cells based on primitive ideals in enveloping algebras was proposed by Joseph [Jo]: let \mathfrak{g} be a complex semi-simple Lie algebra with Weyl group W . Let ρ be the half-sum of the positive roots. Given $w \in W$, let I_w be the annihilator in $U(\mathfrak{g})$ of the simple module with highest weight $-w(\rho) - \rho$. Then, w and w' are in the same left cell if and only if $I_w = I_{w'}$.

3.2. Representations and families. Let Γ be a left cell. Let $W_{\leq \Gamma}$ (resp. $W_{< \Gamma}$) be the set of $w \in W$ such that there is $w' \in \Gamma$ with $w \prec_L w'$ (resp. $w \prec_L w'$ and $w \notin \Gamma$). The left cell representation of W over \mathbf{C} associated with Γ [KaLu, Lu4] is the unique representation, up to isomorphism, that deforms into the left H -module

$$\left(\bigoplus_{w \in W_{\leq \Gamma}} \mathbf{Z}[\Gamma]C_w \right) / \left(\bigoplus_{w \in W_{< \Gamma}} \mathbf{Z}[\Gamma]C_w \right).$$

Lusztig [Lu1, Lu4] has defined the set of constructible characters of W inductively as the smallest set of characters with the following properties: it contains the trivial character, it is stable under tensoring by the sign representation and it is stable under J -induction from a

parabolic subgroup. Lusztig's families are the equivalence classes of irreducible characters of W for the relation generated by $\chi \sim \chi'$ if χ and χ' occur in the same constructible character. Lusztig has determined constructible characters and families for all W and all parameters.

Lusztig has shown for equal parameters, and conjectured in general, that the set of left cell characters coincides with the set of constructible characters.

3.3. A conjecture. Let $c \in \mathbf{R}^{S/\sim}$. Let Γ be the subgroup of \mathbf{R} generated by \mathbf{Z} and $\{c_s\}_{s \in S}$. We endow it with the natural order on \mathbf{R} . Let $L : W \rightarrow \Gamma$ be the weight function determined by $L(s) = c_s$ if $s \in S$.

The following conjecture is due to Gordon and Martino [GoMa]. A similar conjecture has been proposed independently by the second author¹. It is known to hold for types A_n , B_n , D_n and $I_2(n)$ [Go2, GoMa, Be, Ma1, Ma2].

Conjecture 3.1. *The Calogero-Moser families of irreducible characters of W coincide with the Lusztig families.*

We propose now a conjecture involving partitions of elements of W , via ramification. The part dealing with left cell characters could be stated in a weaker way, using Q and not R , and thus not needing the choice of prime ideals, by involving constructible characters.

Conjecture 3.2. *There is a choice of $\mathfrak{r}_c^{\text{left}} \subset \mathfrak{r}_c$ such that*

- *the Calogero-Moser two-sided cells (resp. left cells) coincide with the Kazhdan-Lusztig two-sided cells (resp. left cells)*
- *the representations $\sum_{E \in \text{Irr}(W)} m_\Gamma(E)E$, where Γ is a Calogero-Moser left cell, coincide with the left cell representations of Kazhdan-Lusztig.*

Various particular cases and general results supporting Conjecture 3.2 are provided in [BoRou]. In particular, the conjecture holds for $W = B_2$, for all choices of parameters.

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