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Control of McKean-Vlasov Dynamics versus Mean Field Games*

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1 Introduction

The problem studied in this paper involves stochastic differential games with a finite number of players. We compare two methods of investigations which offer, in the asymptotic regime of large populations (i.e when the number of players $N \rightarrow \infty$), a structure which is simple enough to be amenable to actual solutions, both from the theoretical and numerical points of view.

In order to derive tractable solutions, we assume that all the players are similar in their behavior, and that each individual on his own, can hardly influence the outcome of the game. We further strengthen the symmetry of the problem by assuming that the interaction between the players is of mean field type in the sense that whenever an individual player has to make a decision, he or she sees only averages of functions of the private states of the other players. These games are symmetric in the sense that all the players are identical, but they are not anonymous (see for example [?]) or with weak interaction in the sense of [?]. In the large population limit, a given player should feel the presence of the other players through the statistical distribution of the private states of the other players, and should determine his optimal strategy by optimizing his appropriately modified objective criterion taking the limit $N \rightarrow \infty$ into account. The search for an approximate Nash equilibrium of the game in this asymptotic regime is essentially dictated by the Mean Field Game (MFG for short) proposition of Lasry and Lions. See for example [?, ?, ?, ?, ?, ?].

However, without the optimization part, and when all the players use the same distributed feedback strategy (which is eventually what transpires in the MFG approach), the large population regime is reminiscent of Mark Kac's propagation of chaos theory put on rigorous mathematical ground by Mc Kean and Vlasov (MKV for short). See for example Sznitmann's

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beautiful mathematical treatment [?]. Indeed, according to these works, one expects that in the limit $N \rightarrow \infty$, the private states of the individual players evolve independently of each other, each of them satisfying a specific stochastic differential equation with coefficients depending upon the statistical distribution of the private state in question. Having each player optimize his own objective function under the constraints of this new private state dynamics amounts to the stochastic control of McKean-Vlasov dynamics, mathematical problem not understood in general. See nevertheless [?]. Coming back to our original problem we may wonder if the optimization over the feedback strategies in this new control problem lead to one form of approximate equilibrium for the original N player game? In any case, if such a problem can be solved, the issue is to understand how an optimal feedback strategy for such a problem compares with the result of the MFG analysis.

The main thrust of this paper is to investigate the similarities and the differences between the MFG approach and the search for an optimal strategy for MKV dynamics. In Section 4 we give a pedagogical introduction to these two very related (and sometimes confused) problems with the specific goal to explain the differences between their nature and solutions. As both these problems lead to the analysis of Forward Backward Stochastic Differential Equations (FBSDEs) where the marginal distributions of the solutions appear in the coefficients, we devote Section 5 to the construction of solutions to these problems which have not been studied in the literature. The remainder part of the paper is devoted to the special class of Linear Quadratic (LQ) stochastic games. We implement the MFG approach and analyze the optimal control of MKV dynamics in sections 6 and 7¹. We provide explicit examples in Section 8. While bearing some similarity to the contents of this section, the results of [?] on some linear quadratic MFGs are different in the sense that they concern some infinite horizon stationary cases without any attempt to compare the results of the MFG approach to the control of the corresponding Mc Kean Vlasov dynamics. We conclude in Section 9 where we revisit the original example of CO₂ emissions regulation which is introduction in Section 2 as a motivation.

2 Motivation from GHG Emissions Regulation

For the sake of motivation, we present a toy model of the simplest form of Green House Gas (GHG) emissions regulation. Since our goal is to motivate the analysis of the mathematical problems discussed later in the paper, we shall not worry about the lack of realism of this model.

A set $\{1, \dots, N\}$ of firms compete in an economy where green house gas emissions are regulated. For each firm i , we denote by E_t^i the cumulative emissions, up to time t , of firm i , and we assume that their dynamics satisfy

$$dE_t^i = (b_t^i - \alpha_t^i)dt + \sigma_t^i dW_t, \quad E_0^i = 0. \quad (1)$$

where assumptions on the individual emission rates b_t^i and the volatilities σ_t^i can be chosen as in [?] for the sake of definiteness. The process $(\alpha_t^i)_{0 \leq t \leq T}$ represents the abatement rate of firm i , and it can be viewed as the control the firm can exert on its emissions output. Note

¹After completion of this work, we were made aware of the appearance on the web of a very recent technical report by A. Bensoussan, K. C. J. Sung, S. C. P. Yam, and S. P. Yung entitled *Linear Quadratic Mean Field Games*. In this independent work, the authors present a study of linear quadratic mean field games in relation to the control of McKean-Vlasov dynamics very much in the spirit of what we do in Section 6 and Section 7

that in the case $\alpha_t^i \equiv 0$, the process $(E_t^i)_{0 \leq t \leq T}$ gives the cumulative emissions of firm i in the absence of regulation, situation which is called *Business as Usual*, BAU in short. At the start $t = 0$ of the regulation implementation period, each firm i is allocated a number Λ_i of permits (also called certificates of emission, or allowances). The cap

$$\Lambda^{(N)} = \sum_{i=1}^N \Lambda_i \quad (2)$$

can be viewed as the emissions target set by the regulator for the period $[0, T]$. If at the end of the regulation period $[0, T]$, the aggregate emissions in the economy exceed the cap, i.e. if:

$$\sum_{i=1}^N E_T^i > \Lambda^{(N)}, \quad (3)$$

then each firm i has to offset its emissions E_T^i (expressed in CO₂ ton equivalent) by redeeming one permit per ton, or by paying a penalty λ for each ton not covered by a certificate. In other words, firm i has to pay

$$\lambda_i = \lambda(E_T^i - \Lambda_i)^+ \mathbf{1}_{[\Lambda^{(N)}, \infty)} \left(\sum_{j=1}^N E_T^j \right) \quad (4)$$

where we use the notation $x^+ = \max(x, 0)$ for the positive part of the real number x , and $\mathbf{1}_A$ for the indicator function of the set A . The penalty λ is currently equal to 100 euros in the European Union Emissions Trading Scheme (EU ETS).

Remark 1. *In the case of cap-and-trade regulations, a market on which emissions certificates can be bought and sold is also created as part of the regulation. In order to avoid penalties, firms whose productions are likely to lead to cumulative emissions in excess of their initial allocations Λ_i may engage in buying allowances from firms which expect to meet demand with less emissions than their own initial allocations. For the sake of simplicity, the present discussion concentrates on the cap part, and disregards the trade part of the regulation.*

In search for an optimal behavior, each firm needs to solve the following optimization problem. If we assume that the abatement costs for firm i are given by a function $c^i : \mathbb{R} \rightarrow \mathbb{R}$ which is C^1 and strictly convex, is normalized to satisfy $c^i(0) = \min c^i$, and satisfies Inada-like conditions, ($c^i(x) = \beta|x|^{1+\alpha}$ for some $\beta > 0$ and $\alpha > 0$ is an example of such a cost function), and if its abatement strategy is α^i , its terminal wealth is given by

$$W_T^i = x^i - \int_0^T c^i(\alpha_t^i) dt - \lambda(E_T^i - \Lambda_i)^+ \mathbf{1}_{[\Lambda^{(N)}, \infty)} \left(\sum_{j=1}^N E_T^j \right).$$

Recall that in our simple model, firms do not trade allowances. Now if we view the cumulative emissions E_t^i as the private state of firm i , and if we assume that each firm tries to maximize its expected terminal wealth, or equivalently minimize the objective function

$$J^i(\alpha) = \mathbb{E} \left[\int_0^T c^i(\alpha_t^i) dt + \lambda(E_T^i - \Lambda_i)^+ \mathbf{1}_{[\Lambda, \infty)} \left(\sum_{j=1}^N E_T^j \right) \right] \quad (5)$$

where we set $\alpha_t = (\alpha_t^1, \dots, \alpha_t^N)$, then we have formulated our emission regulation model as a stochastic differential game. Notice that in this particular example, the equations (1) giving the dynamics of the private states E_t^i are decoupled. However, equations (5) giving the expected costs to minimize are coupled in a very specific way, namely through the average of the values of all the private states. This is a particular form of interaction of *mean field* type which we define rigorously in the next subsection.

3 Stochastic Differential Game with Mean Field Interactions

Motivated by the simple example of emissions regulation discussed above, we single out a class of stochastic differential games where the interaction between the players is given in terms of functions of average characteristics of the private states and actions of the individual players.

In our formulation, the state of the system (which is controlled by the actions of the individual players) is given at each time t by a vector $X_t = (X_t^1, \dots, X_t^N)$ whose N components X_t^i can be interpreted as the *private states* of the individual players. A typical example capturing the kind of symmetry which we would like to include in our models is given by models in which the dynamics of the private states are given by coupled stochastic differential equations of the form

$$dX_t^i = \frac{1}{N} \sum_{j=1}^N \tilde{b}(t, X_t^i, X_t^j, \alpha_t^i) dt + \sigma dW_t^i \quad (6)$$

where \tilde{b} is a function of time, the values of two private states, and the control of one player. For the sake of simplicity, we assume that each process $(X_t^i)_{0 \leq t \leq T}$ is univariate. Otherwise, the notations become more involved while the results remain essentially the same. The present discussion can accommodate models where the volatility σ is a function with the same structure as \tilde{b} . We refrain from considering this level of generality to keep the notations to a reasonable level. We use the notation $\alpha_t = (\alpha_t^1, \dots, \alpha_t^N)$ for the players strategies. Notice that the dynamics (6) can be rewritten in the form:

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, \alpha_t^i) dt + \sigma dW_t^i \quad (7)$$

if the function b of time, a private state, a probability distribution on private states, and a control, is defined by

$$b(t, x, \mu, \alpha) = \int \tilde{b}(t, x, x', \alpha) d\mu(x') \quad (8)$$

and the measure $\bar{\mu}_t^N$ is defined as the empirical distribution of the private states, i.e.

$$\bar{\mu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}. \quad (9)$$

Interactions given by functions of the form (8) will be called *linear* or *of order 1*. We could imagine that the drift of (6) giving the interaction between the private states is of the form

$$\frac{1}{N^2} \sum_{j,k=1}^N \tilde{b}(t, X_t^i, X_t^j, X_t^k, \alpha_t^i)$$

which could be rewritten, still with $\mu = \bar{\mu}_t^N$, in the form

$$b(t, x, \mu, \alpha) = \int \tilde{b}(t, x, x', x'', \alpha) d\mu(x') d\mu(x''). \quad (10)$$

Interactions of this form will be called *quadratic* or *of order 2*. Clearly, one can extend this definition to interactions of all orders, and more generally, we will say that the interaction is *fully nonlinear* if it is given by a drift of the form $b(t, X_t^i, \bar{\mu}_t^N, \alpha_t)$ for a general function b defined on $[0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times A$ where $\mathcal{P}(\mathbb{R})$ denotes the set of probabilities on the real line and A the space in which at each time $t \in [0, T]$, the controls can be chosen by the individual players. In general, we say that the game involves interactions of the *mean field* type if the coefficients of the stochastic differential equation giving the dynamics of a private state depend upon the other private states exclusively through the empirical distribution of these private states – in other words if the interaction is fully nonlinear in the sense just defined – and if the running and terminal cost functions have the same structure.

Remark 2. *As in the case of the toy model discussed earlier, the dependence upon the empirical distribution $\bar{\mu}_t^N$ of the private states can degenerate to a dependence upon some moments of this distribution. To be more specific, we can have*

$$b(t, x, \mu, \alpha) = \int \tilde{b}(t, x, \langle \varphi, \mu \rangle, \alpha) \quad (11)$$

for some scalar function φ of the private states, where we use the adjointity notation

$$\langle \varphi, \mu \rangle = \int \varphi(x') d\mu(x')$$

for the integral of a function with respect to a measure. In such a case, we shall say that the interaction is scalar.

To summarize the problem at hand, the game consists in minimizing simultaneously costs of the form

$$J^i(\alpha) = \mathbb{E} \left[\int_0^T f(t, X_t^i, \bar{\mu}_t^N, \alpha_t^i) dt + g(X_T, \bar{\mu}_T^N) \right] \quad (12)$$

under constraints of the form

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, \alpha_t^i) dt + \sigma dW_t^i. \quad (13)$$

Note that for symmetry reasons, we chose the running and terminal cost functions f and g to be the same for all the players. Our goal is to search for equilibriums of such a stochastic differential game.

3.1 The Single Player Problem in the Large Population Regime $N \rightarrow \infty$

Given that the problem is intractable in most cases (see nevertheless the discussion of the linear quadratic case in Section ?? below), we try to identify realistic models for which approximate optimal strategies and Nash equilibria can be identified and computed.

For the sake of definiteness, we restrict ourselves to equilibriums given by Markovian strategies in closed loop feedback form

$$\alpha_t = (\phi^1(t, X_t), \dots, \phi^N(t, X_t))$$

for some deterministic functions ϕ^1, \dots, ϕ^N of time and the state of the system. Further, we assume that the strategies are of distributed type in the sense that the function ϕ^i giving the strategy of player i depends upon the state X_t of the system only through the private state X_t^i of player i . In other words, we request that:

$$\alpha_t^i = \phi^i(t, X_t^i), \quad i = 1, \dots, N.$$

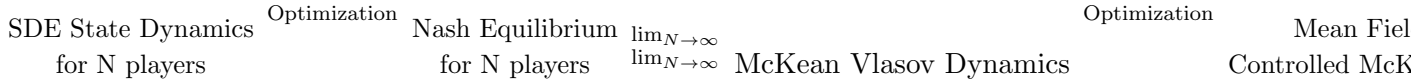
Moreover, given the symmetry of the set-up, we restrict our search for equilibriums to situations in which all the players use the same feedback strategy function, i.e.

$$\phi^1(t, \cdot) = \dots = \phi^N(t, \cdot) = \phi(t, \cdot), \quad 0 \leq t \leq T.$$

for some common deterministic function ϕ . The search for an equilibrium for a game with a fixed finite number of players involves an *optimization*. The large population regime involves taking a *limit*, and it would be interesting to understand if the order in which these operations are performed influences the outcome of the analysis.

3.2 Natural Question

The thrust of the paper is to give the rationale for the four arrows of the following diagram, explain the mathematics behind these arrows, and settle the question of the commutativity of the diagram.



It is important to emphasize one more time what we mean with the limit $N \rightarrow \infty$. We want to identify properties of the limit which, when re-injected into the game with finitely many players, give an approximate solution to the problem we are unable to solve directly for the stochastic game with N players.

4 Mean Field Games (MFG) versus Control of MKV Dynamics

4.1 Optimizing First

If we first search for a Nash equilibrium when the number N of players is finite and fixed, each player i assumes that the other players have already chosen their strategies, say $\alpha_t^{1*} = \phi^{1*}(t, X_t^1), \dots, \alpha_t^{i-1*} = \phi^{i-1*}(t, X_t^{i-1}), \alpha_t^{i+1*} = \phi^{i+1*}(t, X_t^{i+1}), \dots, \alpha_t^{N*} = \phi^{N*}(t, X_t^N)$, and under this assumption, solves the optimization problem:

$$\phi^{i*} = \arg \inf_{\phi} \mathbb{E} \left[\int_0^T f(t, X_t^i, \bar{\mu}_t^N, \phi(t, X_t^i)) dt + g(X_T, \bar{\mu}_T^N) \right] \quad (14)$$

when the dynamics of his own private state is controlled by the feedback control $\alpha_t^i = \phi(t, X_t^i)$ while the dynamics of the private states of the other players are controlled by the feedbacks $\alpha_t^j = \phi^{j*}(t, X_t^j)$ for $j = 1, \dots, i-1, i+1, \dots, N$. So when writing an equation for a critical point (typically a *first order condition*), we apply an infinitesimal perturbation to ϕ without perturbing any of the other ϕ^{j*} for $j \neq i$ and look for conditions under which this deviation for

player i does not make him better off when the other players do not deviate from the strategies they use. A perturbation of ϕ could in principle affect the dynamics of all the private states by changing the empirical distribution of these private states. However, especially when the number of players N is large, a form of the law of large numbers should make this empirical measure $\bar{\mu}_t^N$ quite insensitive to small perturbations of ϕ . So for all practical purposes, the optimization problem (14) can be solved as (or at least its solution can be approximated by) a standard stochastic control problem once the the family $(\bar{\mu}_t^N)_{0 \leq t \leq T}$ of probability measures is fixed.

So in this approach, the search for an approximate Nash equilibrium is based on the following strategy: first one fixes a family $(\mu_t)_{0 \leq t \leq T}$ of probability measures, next, one solves the standard stochastic control problem (parameterized by the choice of the family $(\mu_t)_{0 \leq t \leq T}$):

$$\phi^* = \arg \inf_{\phi} \mathbb{E} \left[\int_0^T f(t, X_t, \mu_t, \phi(X_t)) dt + g(X_T, \mu_T) \right] \quad (15)$$

subject to the dynamic constraint

$$dX_t = b(t, X_t, \mu_t, \phi(X_t)) dt + \sigma d\tilde{W}_t \quad (16)$$

for some Wiener process $(\tilde{W}_t)_{0 \leq t \leq T}$, and finally, once a solution ϕ is found for each choice of the family $(\mu_t)_{0 \leq t \leq T}$, then the family of measures is determined (typically by a fixed point argument) so that at each time t , the statistical distribution of the solution X_t of (16) is exactly μ_t .

The role of the limit $N \rightarrow \infty$ (large number of players) is to guarantee the stability of the empirical measure $\bar{\mu}_t^N$ when a single player perturbs his strategy while the other players keep theirs unchanged, and the fact that this stable distribution has to be the common statistical distribution of all the private states X_t^i . Performing the optimization over ϕ when the family $(\mu_t)_{0 \leq t \leq T}$ of probability measures is kept fixed is the proper way to implement the notion of Nash equilibrium whereby each player is not better off if he deviates from his strategy while all the other players keep theirs untouched, as implied by the lack of change in $(\mu_t)_{0 \leq t \leq T}$ and hence $(\bar{\mu}_t^N)_{0 \leq t \leq T}$.

Summary of the procedure:

1. Fix a deterministic function $[0, T] \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R})$;
2. Solve the standard stochastic control problem

$$\phi^* = \arg \inf_{\phi} \mathbb{E} \left[\int_0^T f(t, X_t, \mu_t, \phi(t, X_t)) dt + g(X_T, \mu_T) \right] \quad (17)$$

subject to

$$dX_t = b(t, X_t, \mu_t, \phi(t, X_t)) dt + \sigma dW_t;$$

3. Determine the function $[0, T] \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R})$ so that

$$\forall t \in [0, T], \quad \mathcal{L}(X_t) = \mu_t.$$

Once these three steps have been taken successfully, the feedback function ϕ^* provides strategies $\alpha_t^{j*} = \phi^*(t, X_t^j)$ that, if used by all the players $j = 1, \dots, N$, form an approximate Nash equilibrium for the game.

This is the philosophy of the Mean Field Game (MFG) approach to the search for an approximate Nash equilibrium when the interaction is of mean field type.

Assuming that the family $\underline{\mu} = (\mu_t)_{0 \leq t \leq T}$ of probability distributions has been frozen, the Hamiltonian of the stochastic control problem is

$$H^\mu(t, x, y, \alpha) := yb(t, x, \mu_t, \alpha) + f(t, x, \mu_t, \alpha). \quad (18)$$

Recall that we are limiting ourselves to the case of constant and identical volatilities σ for the sake of simplicity. In all the examples considered in this paper, there exists a regular function $\hat{\alpha}$ satisfying:

$$\hat{\alpha} = \hat{\alpha}^\mu : [0, T] \times \mathbb{R} \times \mathbb{R} \ni (t, x, y) \mapsto \hat{\alpha}(t, x, y) \in \arg \inf_{\alpha \in A} H^\mu(t, x, y, \alpha). \quad (19)$$

We denote by $\mathcal{H}^\mu(t, x, y)$ this infimum:

$$\mathcal{H}^\mu(t, x, y) = \inf_{\alpha \in A} H^\mu(t, x, y, \alpha). \quad (20)$$

In this paper, we solve stochastic control problems using the probabilistic approach based on the Pontryagin minimum principle and solving the adjoint forward-backward stochastic differential equations. For the sake of completeness, we review the Lasry-Lions' approach using the Hamilton-Jacobi-Bellman (HJB for short) equation leading to the solution of a forward-backward system of nonlinear Partial Differential Equations (PDEs for short).

HJB / PDE approach

Since $\underline{\mu}$ is frozen, the stochastic control problem is Markovian and we can introduce the HJB value function:

$$v(t, x) = \inf_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[\int_t^T f(s, X_s, \mu_s, \alpha_s) ds + g(X_T, \mu_T) | X_t = x \right]. \quad (21)$$

where \mathcal{A}_t denotes the set of admissible controls over the interval $[t, T]$. We expect that the HJB value function v is the solution in some sense (most likely in the viscosity sense only) of the Hamilton-Jacobi-Bellman (HJB) equation (see [?])

$$\partial_t v + \frac{\sigma^2}{2} \partial_{xx}^2 v + \mathcal{H}^\mu(t, x, \partial_x v(t, x)) = 0 \quad (22)$$

with terminal condition $v(T, x) = g(x, \mu_T)$.

In the end, we would like $\underline{\mu} = (\mu_t)_{0 \leq t \leq T}$ to be the flow of marginal distributions of the optimally controlled private state. Clearly, this sounds like a circular argument at this stage since this stochastic differential equation actually depends upon $\underline{\mu}$, and it seems that only a fixed point argument can resolve such a quandary. In any case, the flow of statistical distributions should satisfy Kolmogorov's equation. In other words, if we use the notation

$$\beta(t, x) = b(t, x, \mu_t, \phi(t, x))$$

where ϕ is the optimal feedback control (think of $\phi(t, x) = \hat{\alpha}^\mu(t, x, \sigma^{-1} \partial_x v(t, x))$), then the flow $(\nu_t)_{0 \leq t \leq T}$ of measures should be given by $\nu_t = \mathcal{L}(X_t)$ and satisfy Kolmogorov's equation

$$\partial_t \nu - \frac{\sigma^2}{2} \partial_{xx}^2 \nu - \operatorname{div}(\beta(t, x) \nu) = 0 \quad (23)$$

with initial condition $\nu_0 = \mu_0$. This PDE can be given a rigorous meaning in the sense of distributions. When ν_t has a smooth density, integration by part can be used to turn the formal PDE (23) for ν into a classical PDE for the density of ν .

Setting $\nu_t = \mathcal{L}(X_t) = \mu_t$ when $(X_t)_{0 \leq t \leq T}$ is the diffusion optimally controlled by the feedback function ϕ gives a system of coupled nonlinear forward-backward PDEs (22)-(23) called the MFG PDE system. See [?, ?, ?, ?, ?, ?].

Pontryagin Principle/ FBSDE approach

Given the frozen folw of measures $\underline{\mu}$, the stochastic optimization problem is a standard stochastic control problem and as such, its solution can be approached via the stochastic Pontryagin principle. For each open-loop adapted control $\underline{\alpha} = (\alpha_t)_{0 \leq t \leq T}$, we denote by $\underline{X}^\alpha = (X_t^\alpha)_{0 \leq t \leq T}$ the associated state, any solution $(Y_t, Z_t)_{0 \leq t \leq T}$ of the BSDE

$$dY_t = -\partial_x H^\mu(t, X_t, \mu_t, \alpha_t)dt + Z_t dW_t, \quad (24)$$

is called a set of adjoint processes and the BSDE (24) is called the adjoint equation. It turns out that when the Hamiltonian H^μ is convex with respect to the variables (x, α) and the terminal cost g is convex with respect to the variable x , the forward component of any solution to the forward-backward stochastic system (FBSDE)

$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}^\mu(t, X_t, p_t))dt + \sigma dW_t, \\ dY_t = -\partial_x H^\mu(t, X_t, \mu_t, \hat{\alpha}^\mu(t, X_t, Y_t))dt + Z_t dW_t, \end{cases}$$

with the right initial condition for X_0 and terminal condition $Y_T = \partial_x g(X_T, \mu_T)$, is an optimally controlled path for (17). In particular, if we want to include point 3. of the MFG approach as outlined earlier, namely if we want to enforce $\mu_t = \mathcal{L}(X_t)$, the optimal state dynamics in the MFG approach is given by the forward component of any solution to the stochastic forward-backward system

$$\begin{aligned} dX_t &= b(t, X_t, \mathcal{L}(X_t), \hat{\alpha}^{\mathcal{L}(X)}(t, X_t, p_t))dt + \sigma dW_t, \\ dY_t &= -\partial_x H^{\mathcal{L}(X)}(t, X_t, \mathcal{L}(X_t), \hat{\alpha}^{\mathcal{L}(X)}(t, X_t, Y_t))dt + Y_t dW_t, \end{aligned} \quad (25)$$

with the right initial condition for X_0 and with the terminal condition $Y_T = \partial_x g(X_T, \mathcal{L}(X_T))$. This FBSDE is of McKean-Vlasov type as formally introduced in the next subsection.

4.2 Taking the Limit $N \rightarrow \infty$ First

If we now suppose that the feedback function ϕ is fixed, at least temporarily, the theory of propagation of chaos states that if we consider the solution $X_t^N = (X_t^{N,1}, \dots, X_t^{N,N})$ of the system of N stochastic differential equations (13) with $\alpha_t^i = \phi(t, X_t^{N,i})$, then in the limit $N \rightarrow \infty$, for any fixed integer k , the joint distribution of the k -dimensional process $\{(X_t^{N,1}, \dots, X_t^{N,k})\}_{0 \leq t \leq T}$ converges to a product distribution (in other words the k processes $(X_t^{N,i})_{0 \leq t \leq T}$ for $i = 1, \dots, k$ become independent in the limit) and the distribution of each single marginal process converges toward the distribution of the unique solution $\underline{\xi} = (\xi_t)_{0 \leq t \leq T}$ of the McKean-Vlasov evolution equation

$$d\xi_t = b(t, \xi_t, \mathcal{L}(\xi_t), \phi(t, \xi_t))dt + \sigma d\tilde{W}_t \quad (26)$$

where $(\tilde{W}_t)_{0 \leq t \leq T}$ is a standard Wiener process. So if the common feedback control function ϕ is fixed, in the limit $N \rightarrow \infty$, the private states of the players become independent of each other, and for each given i , the distribution of the private state process $(X_t^{N,i})_{0 \leq t \leq T}$ evolving according to (13) converges toward the distribution of the solution of (26). So if we optimize after taking the limit $N \rightarrow \infty$, i.e. assuming that the limit has already been taken, the objective of each player becomes the minimization of the functional

$$J(\phi) = \mathbb{E} \left[\int_0^T f(t, \xi_t, \mathcal{L}(\xi_t), \phi(t, \xi_t)) dt + g(\xi_T, \mathcal{L}(\xi_T)) \right] \quad (27)$$

over a class of admissible feedback controls ϕ . Minimization of (27) over ϕ under the dynamical constraint (26) is a form of optimal stochastic control problem where the controls are in closed loop feedback form. More generally, such a problem can be stated as in (17) for open loop control $\underline{\alpha} = (\alpha_t)_{0 \leq t \leq T}$ adapted to any specific information structure.

$$\underline{\alpha}^* = \arg \inf_{\phi} \mathbb{E} \left[\int_0^T f(t, \xi_t, \mathcal{L}(\xi_t), \alpha_t) dt + g(\xi_T, \mathcal{L}(\xi_T)) \right] \quad (28)$$

subject to

$$d\xi_t = b(t, \xi_t, \mathcal{L}(\xi_t), \alpha_t) dt + \sigma dW_t; .nonumber \quad (29)$$

Naturally, we call this problem the *optimal control of the stochastic McKean-Vlasov dynamics*.

Standard techniques from stochastic control of Markovian systems cannot be used for this type of stochastic differential equations and a solution to this problem is not known in general even though solutions based on appropriate analogs of the Pontryagin minimum principle have been sought for by several authors. We first review the results of [?] which are the only published ones which we know of. They concern scalar interactions only for which the dependence upon the probability measure of the drift and cost functions are of the form

$$\begin{aligned} b(t, x, \mu, \alpha) &= b(t, x, \langle \psi, \mu \rangle, \alpha) \\ f(t, x, \mu, \alpha) &= f(t, x, \langle \gamma, \mu \rangle, \alpha) \\ g(x, \mu) &= g(x, \langle \zeta, \mu \rangle) \end{aligned}$$

for some convex functions ψ , γ , and ζ . Here and throughout the paper we use the adjointity notation $\langle \varphi, \mu \rangle$ to denote the integral of the function φ with respect to the measure μ . Notice that we use the same notations b , f and g for functions where the variable μ , which was a measure, is replaced by a numeric variable. As we explained earlier, this setting could be sufficient for the analysis of most of the linear quadratic models we consider in sections 6 and 7. We review the more general results [?] of the first two named authors in the following section. The results of [?] also require that we also assume that

- (A1) g is convex in (x, x') ;
- (A2) the Hamiltonian H is convex in (x, x'_1, x'_2, α) ;
- (A3) the partial derivatives $\partial_{x'} b$, $\partial_{x'} f$ and $\partial_{x'} g$ are non-negative.

where the Hamiltonian function H is defined as:

$$H(t, x, x'_1, x'_2, y, \alpha) = yb(t, x, x'_1, \alpha) + f(t, x, x'_2, \alpha) \quad (30)$$

The major difference with the classical case is in the form of the adjoint equation. Given a control process $\underline{\alpha} = (\alpha_t)_{0 \leq t \leq T}$ and a process $\underline{x}_t = (\xi_t)_{0 \leq t \leq T}$ satisfying

$$d\xi_t = b(t, \xi_t, \mathcal{L}(\xi_t), \alpha_t)dt + \sigma dW_t \quad (31)$$

a pair of processes $\underline{y} = (y_t)_{0 \leq t \leq T}$ and $\underline{z} = (z_t)_{0 \leq t \leq T}$ satisfying

$$\begin{cases} dy_t &= -[\partial_x b(t, X_t, \mathbb{E}\{\psi(\xi_t)\}, \underline{\alpha}_t)y_t + \partial_x f(t, \xi_t, \mathbb{E}\{\gamma(\xi_t)\}, \underline{\alpha}_t)]dt + z_t d\tilde{W}_t \\ &\quad - [\mathbb{E}\{\partial_{x'} b(t, \xi_t, \mathbb{E}\{\psi(\xi_t)\}, \underline{\alpha}_t)y_t\} \partial_x \psi(\xi_t) + \mathbb{E}\{\partial_{x'} f(t, \xi_t, \mathbb{E}\{\gamma(\xi_t)\}, \underline{\alpha}_t)\} \partial_x \gamma(\xi_t)]dt \\ y_T &= \partial_x g(\xi_T, \mathbb{E}\{\zeta(\xi_T)\}) + \mathbb{E}\{\partial_{x'} g(\xi_T, \mathbb{E}\{\zeta(X_T)\})\} \partial_x \zeta(\xi_T) \end{cases} \quad (32)$$

is called a pair of adjoint processes and this BSDE of the McKean-Vlasov type (as its coefficients depend upon the distribution of the solution) is called the adjoint equation. Under these condition, the main result of [?] states that:

Theorem 4.1. *If ψ , γ , and ζ are convex and (A1) - (A3) are satisfied, if $\underline{\xi} = \xi$ satisfies (31) for some control process $\underline{\alpha} = (\alpha_t)_{0 \leq t \leq T}$, if $\underline{y} = (y_t)_{0 \leq t \leq T}$ and $\underline{z} = (z_t)_{0 \leq t \leq T}$ is a pair of adjoint processes and if*

$$\tilde{H}(t, \xi_t, y_t, \alpha_t) = \inf_{\alpha \in A} \tilde{H}(t, \xi_t, y_t, \alpha) \quad (33)$$

for all $t \in [0, T]$ almost surely, then the control $\underline{\alpha} = (\alpha_t)_{0 \leq t \leq T}$ is optimal.

The Hamiltonian H appearing in (33) is defined for any random variable ξ as:

$$\tilde{H}(t, \xi, y, \alpha) = yb(t, \xi, \mathbb{E}\{\psi(\xi)\}, \alpha) + f(t, \xi, \mathbb{E}\{\gamma(\xi)\}, \alpha) \quad (34)$$

4.3 First Discussion of the Differences

The general form of MKV dynamics is given by the solution of non-standard stochastic differential equations in which the distribution of the solution appears in the coefficients:

$$d\xi_t = b(t, \xi_t, \mathcal{L}(\xi_t), \alpha_t)dt + \sigma dW_t, \quad \xi_0 = x, \quad (35)$$

where σ is a positive constant. For a given admissible control $\underline{\alpha} = (\alpha_t)_{0 \leq t \leq T}$ we write ξ^α for the unique solution to (35), which exists under the usual growth and Lipchitz conditions on the function b , see [?] for example. The problem is to optimally control this process so as to minimize the expectation:

$$J_{MKV}(\underline{\alpha}) = \mathbb{E} \left[\int_0^T f(t, \xi_t^\alpha, \mathcal{L}(\xi_t^\alpha), \alpha_t)dt + g(\xi_T^\alpha, \mathcal{L}(\xi_T^\alpha)) \right]. \quad (36)$$

On the other hand, the dynamics arising from the MFG approach are given by the solution of standard stochastic differential equation:

$$dX_t = b(t, X_t, \mu_t, a_t)dt + \sigma dW_t, \quad X_0 = x, \quad (37)$$

where $\underline{\mu} = (\mu_t)_{0 \leq t \leq T}$ is a deterministic function with values in the space of probability measures, which can be understood as a (temporary) proxy or candidate for the anticipated statistical distribution of the random variables X_t . The expectation that has to be minimized is now:

$$J_{MFG}(a) = \mathbb{E} \left[\int_0^T f(t, X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T) \right], \quad (38)$$

and an equilibrium takes place if for an optimal control $\underline{\alpha}$, the anticipated distribution μ_t actually coincides with $\mathcal{L}(X_t^{\underline{\alpha}})$ for every $t \in [0, T]$. While very similar, the two problems differ by a very important point: the timing of the optimization.

In the MKV control problem (35)-(36), at each time t , the statistical distribution μ_t is matched to the distribution of the state, and once we have $\mu_t = \mathcal{L}(X_t)$, then do we optimize the objective function. On the other hand, for the MFG problem (37)-(38), the optimization is performed for each fixed family $\underline{\mu} = (\mu_t)_{0 \leq t \leq T}$ of probability distributions, and once the optimization is performed, one matches the distributions μ_t to the laws of the optimally controlled process.

5 Solvability of General MKV Forward-Backward Stochastic Equations

We argued that optimal paths to both MFG and MKV control problems appear as realizations of the forward component of the solution of systems of forward-backward stochastic equations of MKV type, though the corresponding forward-backward systems are different. On one hand, the forward-backward system of the MFG approach appears as a standard forward-backward system derived from the classical form of the stochastic Pontryagin principle, a nonlinear term of MKV type appearing as a result of the fixed point argument of the third step of the MFG approach. On the other hand, as we saw in the special case of scalar interactions studied in [?] when we introduced the adjoint equation in this set-up, the forward-backward system associated with the MKV control problem involves additional terms coming from the need to optimize with respect to the interaction terms.

In this section, we address the solvability of general stochastic forward-backward equations of MKV type. Due to the heavy technical nature of the results and their proofs, we only give sketches of proof, all the arguments being detailed in the forthcoming work [?] by the two first named authors.

Forward-backward systems such as (21) and (32) as special cases of more general fully coupled forward and backward stochastic differential equations involving the marginal distributions of the forward solutions. Changing the notation ever so slightly in order to accommodate the FBSDEs appearing in both analyses, we consider equations of the form:

$$\begin{aligned} dX_t &= B(t, X_t, Y_t, \mathcal{L}(X_t))dt + \sigma dW_t \\ dY_t &= -F(t, X_t, Y_t, \mathcal{L}(X_t))dt + Z_t dW_t, \quad 0 \leq t \leq T, \end{aligned} \tag{39}$$

with $Y_T = G(X_T, \mathcal{L}(X_T))$ as terminal condition. For simplicity, X and Y are both one dimensional, but our strategy also applies in the higher-dimensional setting.

Because of the coupled structure of the system, solvability is a hard question to tackle: fully coupled forward-backward systems are instances of stochastic two-point-boundary-value problem for which both existence and uniqueness are known to fail under standard Cauchy-Lipschitz condition. When the coefficients do not depend on the marginal distributions of the solutions, the forward and backward equations may be decoupled by taking advantage of the noise. Indeed, when the noise is non-degenerate, it has a decoupling effect by regularizing the underlying FBSDE value function. We refer to [?] for a review of this strategy. Recall that by FBSDE value function, we mean the function u giving Y_t as a function of X_t , say $u(t, X_t)$.

In general, it satisfies the quasilinear PDE

$$\partial_t u(t, x) + \frac{1}{2} \sigma^2 \partial_{xx}^2 u(t, x) + B(t, x, u(t, x)) \partial_x u(t, x) + F(t, x, u(t, x)) = 0, \quad (40)$$

for $t \in [0, T]$ and $x \in \mathbb{R}$, with $u(T, x) = G(x)$. Within the theory of optimal control, when the forward-backward system derives from the stochastic Pontryagin principle, the FBSDE value function u appears as the derivative of the HJB value function v which solves the HJB equation.

When the solution interacts with itself, e.g. with its own distribution, the standard Markov structure breaks down. Indeed, the marginal distribution of the process X at time t is needed to compute the transitions towards the future values of the state of the system. Basically, the Markov property must be considered in a larger space, namely the Cartesian product of the state space of the forward process, which is \mathbb{R} in the present situation, with the space of probability measures on the state space, which is infinite dimensional if the state space is not finite. The PDE approach is not sufficient anymore: the underlying heat kernel derived from the noise still has a smoothing property in the finite-dimensional component, but does not have any regularizing effect in the infinite-dimensional direction.

In the forthcoming paper [?], the solvability of the equation is tackled by a compactness argument and the Schauder fixed point theorem. Given an element μ of the space $E = \mathcal{P}_1(C([0, T]))$ of probability measures on the space of $C([0, T])$ of real valued continuous functions on $[0, T]$, we consider the system

$$\begin{aligned} dX_t &= B(t, X_t, Y_t, \mu_t) dt + \sigma dW_t \\ dY_t &= -F(t, X_t, Y_t, \mu_t) dt + Z_t dW_t, \quad 0 \leq t \leq T, \end{aligned} \quad (41)$$

with $Y_T = G(X_T, \mu_T)$ as terminal condition. Here $\underline{\mu} = (\mu_t)_{0 \leq t \leq T}$ is the flow of marginal distributions where μ_t denotes the image (pulled forward) of μ under the t -th coordinate map $C([0, T]) \ni w \mapsto w(t) \in \mathbb{R}$. If this system admits a unique solution, we can consider as output the measure $\mathcal{L}(\underline{X})$. The following result is proven in [?]:

Theorem 5.1. *Assume $\sigma > 0$ and that B , F and G are bounded Lipschitz continuous with respect to the space variable for the Euclidean distance, and with respect to the McKean Vlasov component for the Wasserstein metric, uniformly in the other variables. Then, the mapping $\Phi : E \ni \underline{\mu} \mapsto \mathcal{L}(\underline{X}) \in E$ has a fixed point, where $E = \mathcal{P}_1(C([0, T], \mathbb{R}))$.*

Recall that the Wasserstein metric between two probability measures η and η' on \mathbb{R}^N , $N \geq 1$ is the square root of the infimum of $\int_{\mathbb{R}^{2N}} |z - z'|^2 d\pi(z, z')$ over all the probability measures π on \mathbb{R}^{2N} admitting η and η' as marginals.

Notice that given any μ in E , the forward-backward system is standard, so that unique solvability holds, see e.g. Delarue [?]. The proof consists in showing that Φ leaves a bounded closed convex subset $\Gamma \subset E$ stable and that the restriction of Φ to Γ is continuous and has a relatively compact range, E being endowed with the topology of weak convergence of measures.. A complete proof is given in [?].

The application of Theorem 5.1 to the solvability of the adjoint equations in (25) and (32) is not so straightforward. Indeed, the coefficients B and F therein derive from a convex Hamiltonian structure, which means that they stand for the derivatives with respect to the state space parameter (and also possibly with respect to the measure parameter) of the

underlying Hamiltonian function. In practice, the Hamiltonian function is expected to grow at infinity at a rate which could be quadratic with respect to the state space parameter and the control variable, so that boundedness of the coefficients B and F is expected to fail. Similarly, the terminal cost G in (25) and (32) may have quadratic growth, so that the terminal condition in the adjoint equations is not bounded in practice. As a consequence, refinements of Theorem 5.1 are necessary for our specific purposes.

Here is a first straightforward refined statement of this kind: when the coefficient B grows at most linearly in (x, y) and $\mathcal{L}(X)$, and F and G are bounded, existence of a solution still holds true. When we say that B is at most of linear growth with respect to $(x, y, \mathcal{L}(X))$, we mean that

$$|B(t, x, y, \mathcal{L}(X))| \leq C \left[1 + |x| + |y| + \left(\int_{\mathbb{R}^N} |x'|^2 d\mu(x') \right)^{1/2} \right], \quad \mu = \mathcal{L}(X).$$

Solvability follows from a compactness argument again. Approximating the drift B by a sequence of bounded drifts $(B_n)_{n \geq 1}$, we can prove that, for the corresponding solutions $((X_t^n, Y_t^n = u^n(t, X_t^n))_{0 \leq t \leq T})_{n \geq 1}$, the distributions $(\mathcal{L}(\underline{X}^n))_{n \geq 1}$ are tight and that the functions $(u^n)_{n \geq 1}$ are equicontinuous on compact subsets of $[0, T] \times \mathbb{R}^N$. The proof follows from two key observations: first, since f and g are bounded, the functions $(u^n)_{n \geq 1}$ are uniformly bounded, so that the processes $(\underline{X}^n)_{n \geq 1}$ are tight; second, by smoothing properties of heat kernels, the functions $(u^n)_{n \geq 1}$ are locally uniformly continuous, see [?]. Extracting a convergent subsequence, we can pass to the limit by using stability of forward-backward stochastic differential equations.

Here is another straightforward refinement: when G is bounded, and F is bounded with respect to all the parameters but y , and has a linear growth with respect to y , the solvability still holds. Indeed, a standard maximum principle for BSDEs says that the process Y is bounded, the bound depending upon the bound of G and the growth of F only, so that F may be seen as a bounded driver.

5.1 Solvability of the MFG Adjoint Equations

For our specific purposes, the main point is to allow F and G to grow at most linearly. The strategy developed in [?] consists in approximating the Hamiltonian structure of the MFG/MKV control problem by a sequence of convex Hamiltonian structures with bounded derivatives. Such an approximation argument relies on the following lemma:

Lemma 5.2. *If $d \geq 1$, given any continuous c -strongly convex function $\Phi : (x, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ with $c > 0$, there exists a non-decreasing sequence $(\Phi^n)_{n \geq 1}$ of smooth convex functions which converge toward Φ uniformly on compacts, such that $\partial_x \Phi^n$ is bounded on the whole space for each $n \geq 1$, and Φ^n is c -strongly convex in α for each $n \geq 1$.*

Sketch of the Proof. We use the Legendre transform and define Φ^n as a mollification of $\Phi^{n,0}(x, \alpha) = \sup_{|y| \leq n} \inf_{z \in \mathbb{R}^d} [\langle y, x - z \rangle + \Phi(z, \alpha)]$. The desired properties then follow from standard argument in convex analysis. \square

We can now state the following result for the MFG approach.

Theorem 5.3. *Let us assume that*

(i) *The cost function f has non-negative values and is twice-continuously differentiable with respect to x and α , with Lipschitz-continuous derivatives with respect to x , α and μ ,*

the Lipschitz property holding for the Euclidean norm in x and α and for the Wasserstein distance in μ . Moreover, f is convex in (x, α) for (t, μ) fixed in the sense that there exists a constant $\lambda > 0$ such that:

$$f(t, x', \mu, \alpha') - f(t, x, \mu, \alpha) - \langle (x' - x, \alpha' - \alpha), \nabla_{(x, \alpha)} f(x, \mu, \alpha) \rangle \geq \lambda |\alpha' - \alpha|^2,$$

for all $t \in [0, T]$, $x, x', \alpha, \alpha' \in \mathbb{R}$ and $\mu \in \mathcal{P}_1(\mathbb{R})$;

(ii) The terminal cost g is non-negative twice-continuously differentiable with respect to x , with Lipschitz-continuous derivatives in x and μ , the Lipschitz property holding for the Euclidean norm in x and for the Wasserstein distance in μ . Moreover, g is convex in x when μ is fixed;

(iii) b is affine in x and α in the sense that $b(t, x, \mu, \alpha) = b_0(t, \mu) + b_1(t)x + b_2(t)\alpha$ where b_0, b_1 and b_2 are bounded and b_0 is Lipschitz-continuous with respect to μ for the Wasserstein distance.

Then, the adjoint equation (25) has at least one solution.

We emphasize that the mean-field linear-quadratic models investigated later in the paper do not satisfy the boundedness condition required for b_0 in (iii). In the proof, the boundedness condition ensures that the fixed point measures of the approximating MFG problems are tight. In the linear-quadratic framework, the tightness condition can be expressed differently, as the problem may be reformulated directly, by investigating solvability of associated Riccati equations, see Section ?? below.

Sketch of Proof. Condition (i) says that $\partial_{\alpha, \alpha}^2 f \geq \lambda$, so that $\partial_{\alpha} f$ is strictly increasing with respect to α . Recalling (18) and (43), $\hat{\alpha}(t, x, p, \mu)$ which is the unique solution of the equation $b_2(t)p + \partial_{\alpha} f(t, x, \mu, \hat{\alpha}) = 0$, inherits the regularity properties of the coefficients and is Lipschitz continuous with respect to x, p and μ , the Lipschitz continuity with respect to μ being with respect to the Wasserstein distance. We are looking for the conclusion of Theorem 5.1 with $B(t, x, p, \mu) = b(t, x, \hat{\alpha}(t, x, p, \mu))$, $F(t, x, p, \mu) = \partial_x f(t, x, \mu, \hat{\alpha}(t, x, p, \mu))$ and $G(x, \mu) = \partial_x g(x, \mu)$.

Approximating f and g by two non-decreasing sequences $(f^n)_{n \geq 1}$ and $(g^n)_{n \geq 1}$, as given by Lemma 5.2, we can find, for each $n \geq 1$, a solution (X^n, Y^n) to the adjoint equations associated with the approximate Hamiltonian

$$H^{n, \mu}(t, x, y, \alpha) = b(t, x, \mu, \alpha)y + f^n(t, x, \mu, y, \alpha), \quad t \in [0, T], \quad x, y, \alpha \in \mathbb{R}, \quad \mu \in \mathcal{P}_1(\mathbb{R}),$$

and thus with $B^n(t, x, p, \mu) = b(t, x, \hat{\alpha}^n(t, x, p, \mu))$, $F^n(t, x, p, \mu) = \partial_x f^n(t, x, \mu, \hat{\alpha}^n(t, x, p, \mu))$ and $G^n(x, \mu) = \partial_x g^n(x, \mu)$, where

$$\hat{\alpha}^n(t, x, p, \mu) = \operatorname{argmin}_{\alpha} H^{n, \mu}(t, x, p, \alpha).$$

Clearly, $\hat{\alpha}^n(t, x, p, \mu)$ converges towards $\hat{\alpha}(t, x, p, \mu)$ as $n \rightarrow +\infty$.

For any $n \geq 1$, we denote by \underline{X}^n the process controlled by $(\hat{\alpha}_t^n = \hat{\alpha}^n(t, X_t^n, Y_t^n, \mathcal{L}(X_t^n)))_{0 \leq t \leq T}$ and u^n the FBSDE value function such that $Y_t^n = u^n(t, X_t^n)$, for any $t \in [0, T]$. Our goal is to establish tightness of the processes $(\underline{X}^n)_{n \geq 1}$ and relative compactness of the functions $(u^n)_{n \geq 1}$ for the topology of uniform convergence on compact subsets of $[0, T] \times \mathbb{R}^d$. The crucial step is to prove that the growth of the functions $(u^n)_{n \geq 1}$ can be controlled, uniformly in $n \geq 1$.

In order to do so, we compare the behavior of \underline{X}^n to the behavior of the process \underline{U}^n controlled by $\underline{\beta}^n$ defined by:

$$\beta_s^n = \begin{cases} \hat{\alpha}_s^n, & 0 \leq s \leq t, \\ 0, & t < s \leq T. \end{cases}$$

Repeating the proof of the stochastic maximum principle and using the strict convexity of the approximate Hamiltonian H^n with respect to the parameter α , we bound the conditional cost

$$\begin{aligned} & \mathbb{E} \left[g^n(X_T^n, \mathcal{L}(X_T^n)) + \int_t^T [\lambda |\hat{\alpha}_s^n|^2 + f^n(s, X_s^n, \mathcal{L}(X_s^n), \hat{\alpha}_s^n)] ds | \mathcal{F}_t \right] \\ & \leq \mathbb{E} \left[g^n(U_T^n, \mathcal{L}(U_T^n)) + \int_t^T f^n(s, U_s^n, \mathcal{L}(X_s^n), 0) ds | \mathcal{F}_t \right]. \end{aligned}$$

The key observation is to note that, by boundedness of b_0 , we have:

$$\mathbb{E} \left[\sup_{t \leq s \leq T} |U_s^n|^2 | \mathcal{F}_t \right] \leq C(1 + |X_t^n|^2),$$

for some constant C independent of n . Modifying the constant C if necessary, we deduce that

$$\mathbb{E} \left[\int_t^T |\hat{\alpha}_s^n|^2 | \mathcal{F}_t \right] \leq C(1 + |X_t^n|^2),$$

from which

$$\mathbb{E} \left[\sup_{t \leq s \leq T} |X_s^n|^2 | \mathcal{F}_t \right] \leq C(1 + |X_t^n|^2)$$

easily follows. Standard BSDE arguments imply that $|u^n(t, x)| \leq C(1 + |x|)$, and in turn, tightness of the family $(\underline{X}^n)_{n \geq 1}$. Uniform continuity of the family $(u^n)_{n \geq 1}$ on compact subsets follows from the smoothing effect of heat kernels, see [?].

Let $(u, \underline{\mu})$ be the limit of a convergent subsequence which we still denote $(u^n, \mathcal{L}(\underline{X}^n))_{n \geq 1}$. The smoothing effect of heat kernels can also be used to prove that the functions $(u^n)_{n \geq 1}$ are uniformly locally Lipschitz-continuous in space, so that u itself is locally Lipschitz-continuous in space. Since u grows at most linearly, the SDE

$$dX_t = B(t, X_t, u(t, X_t), \mu_t) dt + \sigma d\tilde{W}_t,$$

is strongly solvable. We can also solve the backward equation

$$dY_t = -F(t, X_t, Y_t, \mu_t) dt + Z_t d\tilde{W}_t, \quad Y_T = G(X_T, \mu_T).$$

By stability of forward and backward equations separately, it is plain to see that \underline{X} is the limit of the $(\underline{X}^n)_{n \geq 1}$ and that \underline{Y} is the limit of the $((u^n(t, X_t^n))_{0 \leq t \leq T})_{n \geq 1}$. Passing to the limit in the approximating forward-backward systems, we conclude that $(\underline{X}, \underline{Y})$ solves the adjoint equation in (25). \square

5.2 Solvability of the adjoint Equations of the MKV Control Problem

We claim:

Theorem 5.4. *In the framework of Theorem 4.1, assume that γ and ζ are twice-differentiable convex functions, with bounded second-order derivatives and that b , f and g are twice-differentiable functions with bounded second-order derivatives that satisfy (A1) - (A3). Assume in addition that b is linear with respect to x , x' and α , namely $b(t, x, x', \alpha) = b_0(t) + b_1(t)x + b_2(t)x' + b_3(t)\alpha$, where b_1 , b_2 and b_3 are bounded functions. Assume similarly that $\psi(\xi) = \xi$ so that $\langle \psi, \mu \rangle$ stands for the expectation of μ . Assume finally f is λ -convex with respect to α , for some $\lambda > 0$, so that H in (30) is also λ -convex with respect to α .*

Then, the forward-backward system (32), with $\alpha_t = \hat{\alpha}(t, \xi_t, y_t, \mathcal{L}(\xi_t))$, is uniquely solvable, where $\hat{\alpha}(t, \xi, y, \mu) = \operatorname{argmin}_{\alpha} H(t, \xi, \langle \psi, \mu \rangle, \langle \gamma, \mu \rangle, y, \alpha)$.

Sketch of Proof. As in the proof of Theorem 5.3, we first notice that the function $\hat{\alpha}$ is well-defined and Lipschitz-continuous w.r.t to ξ , y and μ . Still as in the proof of Theorem 5.3, we consider non-decreasing sequences of convex functions $(f^n)_{n \geq 1}$, $(g^n)_{n \geq 1}$, $(\gamma^n)_{n \geq 1}$ and $(\zeta^n)_{n \geq 1}$, with bounded first- and second-order derivatives, converging towards f , g , γ and ζ respectively as $n \rightarrow +\infty$. For each $n \geq 1$, we define the Hamiltonian H^n by replacing f by f^n in (30), and we consider the forward-backward system (32) by replacing (f, g, γ, ζ) therein by $(f^n, g^n, \gamma^n, \zeta^n)$ and α_t by $\hat{\alpha}_t^n = \hat{\alpha}^n(t, \xi_t, y_t, \mathcal{L}(\xi_t))$ the argument of the minimum of $H^n(t, \xi, \langle \psi, \mu \rangle, \langle \gamma, \mu \rangle, y, \alpha)$: it is the forward-backward system associated with the Hamiltonian H^n . We denote by (X^n, Y^n) its solution.

By the stochastic Pontryagin principle, the process $(\hat{\alpha}_t^n)_{0 \leq t \leq T}$ minimizes the cost function (27), with f , g , γ and ζ therein replaced by f^n , g^n , γ^n and ζ^n . In particular, $J^n(\hat{\alpha}^n) \leq J^n(\hat{\alpha}^{n+p})$, for $p \geq 0$. Actually, by λ -convexity of f^n with respect to α , we can even write:

$$J^n(\hat{\alpha}^n) + \lambda \mathbb{E} \int_0^T |\hat{\alpha}_s^n - \hat{\alpha}_s^{n+p}|^2 ds \leq J^n(\hat{\alpha}^{n+p}).$$

Recalling the proof of Lemma 5.2, f^n and g^n inherits the monotonicity property of f and g with respect to x' , and for any admissible control $\underline{\beta}$, $J^n(\underline{\beta}) \leq J^{n+p}(\underline{\beta})$. Therefore,

$$J^n(\hat{\alpha}^n) + \lambda \mathbb{E} \int_0^T |\hat{\alpha}_s^n - \hat{\alpha}_s^{n+p}|^2 ds \leq J^{n+p}(\hat{\alpha}^{n+p}).$$

In particular, the sequence $(J^n(\hat{\alpha}^n))_{n \geq 1}$ is non-decreasing. Thus, it has a limit since it is bounded by $\sup_{n \geq 1} (J^n(0)) < \infty$. We deduce that $(\hat{\alpha}^n)_{n \geq 1}$ is a Cauchy sequence. By stability of standard McKean Vlasov SDEs, we deduce that that

$$\lim_{n \rightarrow +\infty} \sup_{p \geq 1} \mathbb{E} \left[\sup_{0 \leq s \leq T} |X_s^{n+p} - X_s^n|^2 \right] = 0.$$

This proves that the processes $(\underline{X}^n)_{n \geq 1}$ converge for the norm $\mathbb{E}[\sup_{0 \leq s \leq T} |\cdot|^2]^{1/2}$, towards a continuous adapted process \underline{X} . By standard results of stability for BSDEs, we then deduce that the same holds for the processes $(\underline{Y}^n)_{n \geq 1}$, that is

$$\lim_{n \rightarrow +\infty} \sup_{p \geq 1} \mathbb{E} \left[\sup_{0 \leq s \leq T} |Y_s^{n+p} - Y_s^n|^2 \right] = 0.$$

One then deduce the existence of the limit \underline{Y} , and the fact that $(\underline{X}, \underline{Y})$ satisfies the forward-backward system (32). Uniqueness easily follows from the stochastic Pontryagin principle. \square

Remark 3. *Uniqueness in Theorem 5.4 says that, for any $t \in [0, T]$, Y_t is a function of X_t . This also follows from a standard change of filtration, as usually done in the theory of BSDEs. In particular, the optimal control $\hat{\alpha}$ is a feedback control, as expected.*

6 Mean Field Linear Quadratic (LQ) Games

This section is devoted to a class of models for which we can push the analysis further, to the point of deriving explicit formulas in some cases. In the general case of d' -dimensional private states, with d' possibly greater than 1, imposing mean field interaction to a linear quadratic model of a stochastic differential games forces the dynamics of the private states to be of the form

$$dX_t^i = [A_t X_t^i + \bar{A}_t \bar{X}_t + b_t \alpha_t^i + \beta_t] dt + \sigma dW_t^i, \quad i = 1, \dots, N$$

with individual costs:

$$J^i(\alpha) = \mathbb{E} \left\{ \int_0^T \left[\frac{1}{2} \alpha_t^{i\dagger} N_t \alpha_t^i + \|M_t X_t^i + \tilde{M}_t \bar{X}_t\|^2 \right] dt + \frac{1}{2} \|Q X_T + \tilde{Q} \bar{X}_T\|^2 \right\}$$

where \bar{X}_t denotes the average of the X_t^i . In the notation used in this paper for the drift and cost functions, this means

$$\begin{aligned} b(t, x, \mu, \alpha) &= A_t x + \bar{A}_t \bar{\mu} + B_t \alpha + \beta_t \\ f(t, x, \mu, \alpha) &= \frac{1}{2} \alpha_t^\dagger N_t \alpha_t \\ g(x, \mu) &= \frac{1}{2} \|Qx + \bar{Q}\bar{\mu}\|^2 \end{aligned}$$

where the $d' \times d'$ matrices A_t , \bar{A}_t , M_t and \bar{M}_t , the $d' \times k'$ matrix B_t , the $k' \times k'$ matrix N_t , and the $d' \times 1$ vector β_t are deterministic continuous functions of $t \in [0, T]$, and the $d' \times d'$ matrices Q , and \bar{Q} are deterministic and symmetric. Recall that we use the notation $\bar{\mu}$ for the mean of measure μ , i.e. $\bar{\mu} = \int x d\mu(x)$.

6.1 Solving the N -player Game

Under specific assumptions on the matrix coefficients (in particular B_t and N_t being symmetric and invertible), existence of open loop Nash equilibriums for Linear Quadratic (LQ for short) stochastic differential games of the type considered in this section can be proven using the stochastic Pontryagin principle approach. See for example [?]. We refrain from giving more specific in the general set-up as we are about to re-derive these results in the particular case of mean field models which we consider below.

As in our introductory discussion, we assume that $d' = 1$ for the sake of simplicity. In other words, individual private states are one dimensional. This assumption is not essential, its goal is only to keep the notation to a reasonable level of complexity. See nevertheless Remark 4 below. Moreover, we also assume that $k' = 1$ so that the actions of the individual players are also one-dimensional. Using lower cases for scalars where we used upper cases for matrices, the dynamics of the private states rewrite:

$$dx_t^i = [a_t x_t^i + \bar{a}_t \bar{x}_t + b_t \alpha_t^i + \beta_t] dt + \sigma dW_t^i, \quad i = 1, \dots, N.$$

Similarly, the individual costs become:

$$J^i(\alpha) = \mathbb{E} \left\{ \int_0^T \left[\frac{n_t}{2} (\alpha_t^i)^2 + (m_t x_t + \bar{m}_t \bar{x}_t)^2 \right] dt + \frac{1}{2} (q x_T + \bar{q} \bar{x}_T)^2 \right\}$$

We will assume throughout that $b_t > 0$ and $n_t > 0$.

6.2 Implementing the MFG Approach

As explained earlier, the first step of the MFG approach is to fix a flow $\underline{\mu} = (\mu_t)_{0 \leq t \leq T}$ of probability measures in lieu of the empirical measures of the players' private states, and solve the resulting control problem for one single player. Since the empirical measure of the players' private states enters only the state equations and the cost functions through its mean, it is easier to choose a real valued deterministic function $(\bar{\mu}_t)_{0 \leq t \leq T}$ (which we should denote $\bar{\underline{\mu}}$ in

order to be consistent with the convention used so far in our notation system) as a proxy for the mean of the empirical distribution of the private states. Then the individual stochastic control problem is to minimize

$$J(a) = \mathbb{E} \left\{ \int_0^T \left[\frac{n_t}{2} \alpha_t^2 + (m_t x_t + \bar{m}_t \bar{\mu}_t)^2 \right] dt + \frac{1}{2} (q x_T + \bar{q} \bar{\mu}_T)^2 \right\}$$

subject to the dynamical constraint

$$dx_t = [a_t x_t + \bar{a}_t \bar{\mu}_t + b_t \alpha_t + \beta_t] dt + \sigma dW_t,$$

Given that the deterministic function $t \mapsto \bar{\mu}_t$ is assumed fixed, the Hamiltonian of the stochastic control problem is equal to

$$H^{\bar{\mu}}(t, x, y, \alpha) = y[a_t x + \bar{a}_t \bar{\mu}_t + b_t \alpha + \beta_t] + \frac{1}{2} (m_t x + \bar{m}_t \bar{\mu}_t)^2 + \frac{1}{2} n_t \alpha^2. \quad (42)$$

The optimal control $\hat{\alpha}$ minimizing the Hamiltonian is given by:

$$\hat{\alpha} = -\frac{b_t}{n_t} y \quad (43)$$

and the minimum Hamiltonian by

$$\mathcal{H}^{\bar{\mu}}(t, x, y) = \inf_{\alpha \in \mathcal{A}} H^{\bar{\mu}}(t, x, y, \alpha) = (a_t x + \bar{a}_t \bar{\mu}_t + \beta_t) y + \frac{1}{2} (m_t x + \bar{m}_t \bar{\mu}_t)^2 - \frac{1}{2} \frac{b_t^2}{n_t} y^2$$

HJB Approach

As explained earlier, we state the MFG system of coupled nonlinear PDEs for the sake of completeness only since we eventually use the stochastic Pontryagin principle to solve this optimal control problem. In the particular case at hand, the classical forward-backward MFG system reads

$$\partial_t v(t, x) = \frac{\sigma^2}{2} \partial_{xx}^2 v(t, x) + (a_t x + \bar{a}_t \bar{\mu}_t + \beta_t) \partial_x v(t, x) - \frac{1}{2} \frac{b_t^2}{n_t} \partial_x v(t, x)^2 + \frac{1}{2} (m_t x + \bar{m}_t \bar{\mu}_t)^2 \quad (44)$$

for the HJB equation, and

$$\partial_t \nu_t = \frac{\sigma^2}{2} \partial_{xx}^2 \nu_t - \operatorname{div}([a_t x_t + \bar{a}_t \bar{\mu}_t - \frac{b_t^2}{n_t} \partial_x v(t, x) + \beta_t] \nu_t) \quad (45)$$

for the Kolmogorov's equation. Recall the expression (43) for the optimal control and the fact that the adjoint variable y is related to the state variable x by the FBSDE value function which is the derivative of the HJB value function. In any case, this Kolmogorov's equation needs to be solved simultaneously under the fixed point condition $\bar{\mu}_t = \int x d\nu_t(x)$ for all $t \in [0, T]$.

Pontryagin Minimum Principle Approach

In the present situation, the classical adjoint equation reads

$$dy_t = -[a_t y_t + m_t(m_t x_t + \bar{m}_t \bar{\mu}_t)]dt + z_t dW_t, \quad y_T = q(qx_T + \bar{q}\bar{\mu}_T). \quad (46)$$

Notice that because of the special form of the coefficients, this BSDE does not involve the control $\hat{\alpha}_t$ explicitly. With our choice of $\hat{\alpha}_t$, the forward dynamics of the state become:

$$dx_t = [a_t x_t + \bar{a}_t \bar{\mu}_t - \frac{b_t^2}{n_t} y_t + \beta_t]dt + \sigma dW_t. \quad (47)$$

and the problem reduces to the proof of the existence of a solution to the standard FBSDE (47) - (46) and the analysis of the properties of such a solution. In particular, we will need to check that this solution is amenable to the construction of a fixed point for $\bar{\mu}$. Notice that, once this FBSDE is solved (43) will provide us with an open loop optimal control for the problem. For the remainder of the proof, we streamline the notation and rewrite the FBSDE as

$$\begin{cases} dx_t &= [a_t x_t + b_t y_t + c_t]dt + \sigma dW_t, & x_0 = x \\ dy_t &= [m_t x_t - a_t y_t + d_t]dt + z_t dW_t, & y_T = qx_T + r \end{cases} \quad (48)$$

where in order to better emphasize the linear nature of the FBSDE we set:

- $a_t = a_t, \quad b_t = -b_t^2/n_t, \quad c_t = \beta_t + \bar{a}_t \bar{\mu}_t;$
- $m_t = -m_t^2, \quad d_t = -m_t \bar{m}_t \bar{\mu}_t;$
- $q = q^2, \quad r = q\bar{q}\bar{\mu}_T.$

Notice the abuse of notation: the $a_t, b_t, c_t, m_t, d_t, q$ and r appearing in the left hand sides are new variables and parameters while the quantities in the right hand sides are the original coefficients of our LQ-game. Linear FBSDEs of the form (48) have been studied in [?, ?], but for the sake of completeness we construct a solution from scratch. Because of the linearity of the system (48), we expect the FBSDE value function to be affine in the space variable, so we search for deterministic functions η_t and χ_t such that

$$y_t = \eta_t x_t + \chi_t, \quad 0 \leq t \leq T. \quad (49)$$

Computing dy_t from this ansatz using the expression of dx_t from (48) we get

$$dy_t = [(\dot{\eta}_t + a_t \eta_t + b_t \eta_t^2)x_t + \dot{\chi}_t + b_t \eta_t \chi_t + c_t \eta_t]dt + \sigma \eta_t dW_t$$

and identifying term by term with the expression of dy_t given in (48) we get:

$$\begin{cases} \dot{\eta}_t = -b_t \eta_t^2 - 2a_t \eta_t + m_t, & \eta_T = q \\ \dot{\chi}_t + (a_t + b_t \eta_t)\chi_t = d_t - c_t \eta_t, & \chi_T = r \\ z_t = r \eta_t. \end{cases} \quad (50)$$

The first equation is a Ricatti equation. Its solution is obtained by solving the second order linear equation

$$-b_t \ddot{\theta}_t - [\dot{b}_t - 2a_t b_t] \dot{\theta}_t + m_t b_t^2 \theta_t = 0,$$

with terminal conditions $\theta_T = 1$ and $\dot{\theta}_T = b_T q$. In terms of the original coefficients, this second order ordinary differential equation rewrites:

$$\frac{b_t^2}{n_t} \ddot{\theta}_t - \left[\frac{\dot{b}_t^2}{n_t} - 2a_t \frac{b_t^2}{n_t} \right] \dot{\theta}_t - m_t^2 \frac{b_t^4}{n_t^2} \theta_t = 0,$$

with terminal conditions $\theta_T = 1$ and $\dot{\theta}_T = -qb_T^2/n_T$. We need a solution $t \mapsto \theta_t$ which does not vanish, so an extra assumption guaranteeing the strict positivity of the discriminant of this equation needs to be imposed on the coefficients. Once such a positive solution is found, the solution η_t of the Ricatti equation is obtained by setting

$$\eta_t = -\frac{n_t \dot{\theta}_t}{b_t^2 \theta_t}.$$

Remark 4. *While we claimed that restricting ourselves to the one dimensional case $d' = 1$ was not restrictive, but it is only fair to mention that the solution of matrix valued Ricatti's equations is more involved and requires extra assumptions, and the assumptions needed to guarantee the invertibility of θ_t are rather technical.*

In any case, once η_t is computed, we plug its value in the third equation to determine z_t , and in the second equation which can then be solved by:

$$\chi_t = r e^{\int_t^T [a_u + b_u \eta_u] du} - \int_t^T [d_s - c_s \eta_s] e^{\int_t^s [a_u + b_u \eta_u] du} ds,$$

or in terms of the original coefficients

$$\chi_t = q \bar{q} \bar{\mu}_T e^{\int_t^T [a_u - \eta_u b_u^2 / n_u] du} + \int_t^T [m_s \bar{m}_s \bar{\mu}_s + (\beta_s + \bar{a}_s \bar{\mu}_s) \eta_s] e^{\int_t^s [a_u - \eta_u b_u^2 / n_u] du} ds. \quad (51)$$

Now that the deterministic functions η_t and χ_t are computed, we rewrite the forward stochastic differential equation for the dynamics of the state using the ansatz (67)

$$dx_t = [(a_t + b_t \eta_t)x_t + b_t \chi_t + c_t] dt + \sigma dW_t, \quad x_0 = x.$$

Such a stochastic differential equation is solved explicitly

$$x_t = x e^{\int_0^t (a_u + b_u \eta_u) du} + \int_0^t (b_s \chi_s + c_s) e^{\int_s^t (a_u + b_u \eta_u) du} ds + \sigma \int_0^t e^{\int_s^t (a_u + b_u \eta_u) du} dW_s. \quad (52)$$

We see here that things can go wrong if $a_u + b_u \eta_u$ is not integrable near $u = 0$. Even without going back to the original coefficients, we notice that x_t is a Gaussian process of the Ornstein-Uhlenbeck type. Its expectation is

$$\mathbb{E}\{x_t\} = x e^{\int_0^t (a_u + b_u \eta_u) du} + \int_0^t (b_s \chi_s + c_s) e^{\int_s^t (a_u + b_u \eta_u) du} ds. \quad (53)$$

Notice that the function $t \mapsto \bar{\mu}_t$ which we started from is present, though hidden, in the right hand side. A solution of the MFG approach, is obtained if we can determine this function so that the expectation $\mathbb{E}\{x_t\}$ given by (53) above is exactly $\bar{\mu}_t$. In other words, we need to find a fixed point to the mapping U which takes the function $\bar{\mu} : [0, T] \ni t \mapsto \bar{\mu}_t$ into the function

$\bar{\nu} = U(\bar{\mu}) : [0, T] \ni t \mapsto \bar{\nu}_t$ where $\bar{\nu}_t$ is defined as the right hand side of (53). We introduce some temporary notation in order to express U in a more manageable way, and prove the existence of a fixed point. First we notice that the solution $(\eta_t)_{0 \leq t \leq T}$ of the Riccati equation in (50) does not depend upon $\bar{\mu} = (\bar{\mu}_t)_{0 \leq t \leq T}$, so that $\alpha_t = a_t + b_t \eta_t$ is also independent of $\bar{\mu}$. Looking closely at formula (51) giving the value of χ_t , we notice that it can be written in the form

$$\chi_t = q\bar{\mu}_T e^{\int_t^T \alpha_u du} + \int_t^T \gamma_s \bar{\mu}_s e^{\int_t^s \alpha_u du} ds + \delta_t$$

where the constant q and the function α , γ and δ do not depend upon $\bar{\mu}$.

Consequently, $\bar{\mu}_t$ is of the form

$$\bar{\mu}_t = \alpha_t x + \beta_t + q\gamma_t \bar{x}_T + \int_0^t \delta_s \bar{x}_s ds + \int_0^t \eta_s \int_s^T \bar{x}_u \zeta_u du ds$$

for some nice functions α , β , γ , δ , η , ζ , but it is not clear to me what will make such a function a contraction – short of assuming T small enough !

A Simple Example

For the sake of illustration, we consider the frequently used example where the drift b reduces to the control, namely $b(t, x, \mu, \alpha) = \alpha$, so that $a_t = \bar{a}_t = \beta_t = 0$, $b_t = 1$ and the state equation reads

$$dx_t = \alpha_t dt + \sigma dW_t.$$

We also assume that the running cost is simply the square of the control, i.e. $n_t = 1$ and $m_t = \bar{m}_t = 0$. Using the notations and the results above, we see that the FBSDE of the MFG approach has the simple form

$$\begin{cases} dx_t = -y_t dt + \sigma dW_t \\ dy_t = z_t dW_t \end{cases} \quad (54)$$

which we solve by postulating $y_t = \eta_t x_t + \chi_t$, and solving for the two deterministic functions η and χ . We find:

$$\eta_t = \frac{q}{1 + q(T - t)}, \quad \chi_t = \frac{r}{1 + q(T - t)},$$

and plugging these expressions into (52) we get

$$x_t = x \frac{1 + q(T - t)}{1 + qT} - \frac{rt}{1 + qT} + \sigma [1 + q(T - t)] \int_0^t \frac{dW_s}{1 + q(T - s)}. \quad (55)$$

Notice further that the optimal control α_t and the adjoint process y_t satisfy

$$-\alpha_t = y_t = \frac{q}{1 + q(T - t)} x_t + \frac{r}{1 + q(T - t)}$$

and that the only quantity depending upon the fixed mean function $t \mapsto \bar{\mu}_t$ is the constant $r = q\bar{q}\bar{\mu}_T$ which depends only upon the mean state at the end of the time interval. This makes the search for a fixed point very simple and one easily check that if

$$\bar{\mu}_T = \frac{x}{1 + q(q + \bar{q})T}$$

then the mean at time T of the random variable x_T given by (55) is $\bar{\mu}_T$.

6.3 More Tractable Examples

Motivated by the specific form of our introductory example of emission regulation, we generalize the simple example discussed above by keeping the same forms for the state dynamics and the running cost, but considering more general terminal cost functions $g(x, \mu)$. From the specific form of Kolmogorov's equation

$$\bar{\mu}_t = \mathbb{E}\{X_t\} = x + \int_0^t \int \partial_x v(t, x, \bar{\mu}_s) \nu_s(dx) ds,$$

where v is the solution to (22) for $(\bar{\mu}_t)_{t \in [0, T]}$ frozen. The HJB equation being:

$$\partial_t v + \frac{\sigma^2}{2} \partial_{xx}^2 v + \frac{|\partial_x v|^2}{2} = 0, \quad v|_{t=T}(x) = g(x, \bar{\mu}_T),$$

a Hopf-Cole change of variable reducing this nonlinear HJB equation to a classical linear heat equation shows that we can get solutions in closed form when g is linear or quadratic in its first variable.

Linear terminal cost.

In this case the terminal cost is of the form $g(x, x') = x\gamma(x')$ and the fixed point equation for the mean is

$$\bar{\mu}_t = x + t \gamma(\bar{\mu}_t).$$

Consequently, existence of a solution amounts to the existence of fixed points for the transformation $x' \mapsto x + t \gamma(x')$, for a given x and for every $t \in [0, T]$. The invariant distributions of this dynamical system are described by the Poisson's equation

$$\dot{\theta}(t) = \gamma(\theta(T)), \quad t \in (0, T], \quad \theta(0) = x.$$

Quadratic terminal cost.

In this case, $g(x, x') = x^2 \gamma(x')$ and we have to consider the existence of a fixed point for the function

$$(\theta_s)_{s \in (0, t]} \mapsto x + \gamma(\theta(T)) \int_0^t \theta(s) ds, \quad \forall t \in (0, T],$$

or equivalently the existence of a solution to the following initial value problem:

$$\dot{\theta}(t) = \gamma(\theta(T))\theta(t), \quad t \in (0, T], \quad \theta(0) = x.$$

Remark 5. *We did not consider the case of a constant terminal cost $g(x, x') = \gamma(x')$ because the solution is trivial one ($\bar{\mu}_t \equiv x$).*

7 Control of Mean Field LQ McKean Vlasov Dynamics

As before, we assume $d' = 1$ and $k' = 1$. The problem of the optimal control of the Mean Field Linear Quadratic McKean-Vlasov dynamics consists in the minimization of the functional

$$J(\alpha) = \mathbb{E} \left[\int_0^T \left[\frac{1}{2} (m_t \xi_t + \bar{m}_t \mathbb{E}\{\xi_t\})^2 + \frac{1}{2} n_t \alpha_t^2 \right] dt + \frac{1}{2} (q \xi_T + \bar{q} \mathbb{E}\{\xi_T\})^2 \right] \quad (56)$$

over all the admissible control processes $\alpha = (\alpha_t)_{0 \leq t \leq T}$ under the constraint

$$d\xi_t = [a_t \xi_t + \bar{a}_t \mathbb{E}\{\xi_t\} + b_t \alpha_t + \beta_t] dt + \sigma dW_t. \quad (57)$$

In order to be admissible, a control must be a progressively measurable process taking values in a fixed subset A of an Euclidean space and satisfy an integrability condition to be specified. So in the notations of the Pontryagin principle stated earlier in Theorem 4.1, we have

- $\psi(x) = \gamma(x) = \zeta(x) = x$
- $b(t, x, x', a) = a_t x + \bar{a}_t x' + b_t a + \beta_t$
- $f(t, x, x', a) = \frac{1}{2}(m_t x + \bar{m}_t x')^2 + \frac{1}{2} n_t a^2$
- $g(x, x') = \frac{1}{2}(q x + \bar{q} x')^2$

so that the adjoint equation becomes

$$\begin{aligned} dp_t &= -[a_t p_t + m_t(m_t \xi_t + \bar{m}_t \mathbb{E}\{\xi_t\})] dt + q_t dW_t \\ &\quad - [\mathbb{E}\{\bar{a}_t p_t\} + \mathbb{E}\{\bar{m}_t(m_t \xi_t + \bar{m}_t \mathbb{E}\{\xi_t\})\}] dt, \quad p_T = (q + \bar{q})(q \xi_T + \bar{q} \mathbb{E}\{\xi_T\}). \end{aligned} \quad (58)$$

Notice that because of the special form of the coefficients, this BSDE does not involve the control α_t explicitly. The control appears only indirectly, through the state ξ_t and its mean $\mathbb{E}\{\xi_t\}$. In order to apply the Pontryagin minimum principle, we guess the candidate for the optimal control by minimizing the Hamiltonian:

$$\hat{\alpha} = \arg \inf_{\alpha \in A} H(t, x, p, \alpha) \quad (59)$$

$$= \arg \inf_{a \in A} p[a_t x + \bar{a}_t x' + b_t a + \beta_t] + \frac{1}{2}(m_t x + \bar{m}_t x')^2 + \frac{1}{2} n_t a^2 \quad (60)$$

and the first order condition gives:

$$p b_t + n_t \alpha = 0$$

so that we choose

$$\alpha_t = -\frac{b_t}{n_t} p_t \quad (61)$$

as candidate for the optimal control. With this choice, the dynamics of the state become:

$$d\xi_t = [a_t \xi_t + \bar{a}_t \mathbb{E}\{\xi_t\} - \frac{b_t^2}{n_t} p_t + \beta_t] dt + \sigma dW_t. \quad (62)$$

and the problem reduces to the proof of the existence of a solution to the FBSDE (62) - (58) of the McKean-Vlasov type, and the analysis of the properties of such a solution.

In order to make the adjoint equation (58) more tractable, we assume (as in the case of the Mean Field approach) that all the coefficients $a_t, \bar{a}_t, b_t, \beta_t, m_t, \bar{m}_t, n_t$ are deterministic continuous functions of $t \in [0, T]$, and that q , and \bar{q} are deterministic constants. In this case, the FBSDE (62) - (58) of McKean-Vlasov type reads:

$$\begin{aligned} d\xi_t &= [a_t \xi_t + \bar{a}_t \mathbb{E}\{\xi_t\} - \frac{b_t^2}{n_t} p_t + \beta_t] dt + \sigma dW_t, \quad \xi_0 = \xi \\ dp_t &= -[\tilde{a}_t p_t + \tilde{m}_t(\tilde{m}_t \xi_t + \tilde{\bar{m}}_t \mathbb{E}\{\xi_t\})] dt + q_t dW_t \\ &\quad - [\mathbb{E}\{\bar{a}_t p_t\} + \mathbb{E}\{\bar{m}_t(m_t \xi_t + \bar{m}_t \mathbb{E}\{\xi_t\})\}] dt, \quad p_T = (q + \bar{q})(q \xi_T + \bar{q} \mathbb{E}\{\xi_T\}) \end{aligned} \quad (63)$$

Because of these equations are linear, we can solve first for the functions $t \mapsto \mathbb{E}\{\xi_t\}$ and $t \mapsto \mathbb{E}\{p_t\}$. Indeed, taking expectations on both sides of (66) and rewriting the resulting system using the notation $\bar{\xi}_t$ and \bar{p}_t for the expectations $\mathbb{E}\{\xi_t\}$ and $\mathbb{E}\{p_t\}$ respectively, we obtain:

$$\begin{cases} d\bar{\xi}_t &= [(a_t + \bar{a}_t)\bar{\xi}_t - \frac{b_t^2}{n_t}\bar{p}_t + \beta_t]dt, & \bar{\xi}_0 = \xi \\ d\bar{p}_t &= -[(a_t + \bar{a}_t)\bar{p}_t + (m_t + \bar{m}_t)\bar{\xi}_t]dt, & \bar{p}_T = (q + \bar{q})^2\bar{\xi}_T. \end{cases} \quad (64)$$

This is a typical linear system of Ordinary Differential Equations (ODEs) for the function $\bar{\xi}_t$ and \bar{p}_t with mixed initial/terminal conditions, $\bar{\xi}_0 = x$, $\bar{p}_T = (q + \bar{q})^2\bar{\xi}_T$, so there is absolutely no guarantee that a solution exists. In order to decouple the initial and terminal boundary conditions, we search a solution in the form $\bar{p}_t = \eta_t\bar{\xi}_t + \chi_t$ for two deterministic functions η_t and χ_t to be found. Simple arithmetic shows that the system (64) is equivalent to

$$\begin{cases} \dot{\eta}_t &= \frac{b_t^2}{n_t}\eta_t^2 - 2(a_t + \bar{a}_t)\eta_t - (m_t + \bar{m}_t)^2, & \eta_T = (q + \bar{q})^2 \\ \dot{\chi}_t &= -[\frac{b_t^2}{n_t} - (a_t + \bar{a}_t)]\chi_t = -\beta_t\eta_t, & \chi_T = 0. \end{cases} \quad (65)$$

The first equation is a Ricatti equation whose solution is obtained by solving the second order linear equation

$$\frac{b_t^2}{n_t}\ddot{\theta}_t - [\frac{\dot{b}_t^2}{n_t} - 2(a_t + \bar{a}_t)\frac{b_t^2}{n_t}]\dot{\theta}_t - (m_t + \bar{m}_t)^2\frac{b_t^4}{n_t^2}\theta_t = 0$$

with terminal conditions $\theta_T = 1$ and $\dot{\theta}_T = (q + \bar{q})^2\frac{b_T^2}{n_T}$ and setting

$$\eta_t = -\frac{n_t}{b_t^2}\frac{\dot{\theta}_t}{\theta_t}.$$

The same remarks about the positivity of θ_t apply here as well. Once η_t is computed, we plug its value in the second equation which can then be solved by:

$$\chi_t = \int_t^T \beta_s \eta_s e^{\int_t^s [a_u + \bar{a}_u - b_u^2/n_u] du} ds.$$

Now that the deterministic functions η_t and χ_t are computed, we solve for $\bar{\xi}_t$ and \bar{p}_t by plugging our ansatz $\bar{p}_t = \eta_t\bar{\xi}_t + \chi_t$ into the first equation of (64). We get:

$$d\bar{\xi}_t = [(a_t + \bar{a}_t - \frac{b_t^2}{n_t}\eta_t)\bar{\xi}_t + \beta_t - \frac{b_t^2}{n_t}\chi_t]dt, \quad \bar{\xi}_0 = \xi$$

whose solution is

$$\bar{\xi}_t = e^{\int_0^t (a_u + \bar{a}_u - \eta_u b_u^2/n_u) du} \xi + \int_0^t [\beta_s - \frac{b_s^2}{n_s}\chi_s] e^{\int_s^t (a_u + \bar{a}_u - \eta_u b_u^2/n_u) du} ds,$$

which we in turn, plug in the ansatz $\bar{p}_t = \eta_t\bar{\xi}_t + \chi_t$ to get the value of \bar{p}_t .

We can now replace the expectations $\mathbb{E}\{\xi_t\}$ and $\mathbb{E}\{p_t\}$ appearing in the FBSDE (66) of McKean-Vlasov type by the deterministic functions $\bar{\xi}_t$ and \bar{p}_t obtained solving the forward backward system of ODEs, and solve the linear FBSDE

$$\begin{cases} d\xi_t &= [a_t\xi_t + b_t p_t + c_t]dt + \sigma dW_t, & \xi_0 = \xi \\ dp_t &= [m_t\xi_t - a_t p_t + d_t]dt + q_t dW_t, & p_T = q\xi_T + r \end{cases} \quad (66)$$

where for the purpose of this part of the proof, and in order to lighten the notation, we use the same abuse of notation we used in the case of the MFG approach by setting:

- $a_t = a_t$, $b_t = -b_t^2/n_t$, $c_t = \beta_t + \bar{a}_t\bar{\xi}_t$;
- $m_t = -m^2$, $d_t = -\bar{m}_t(2m_t + \bar{m}_t)\bar{\xi}_t - \bar{a}_t\bar{p}_t$;
- $q = q(q + \bar{q})$, $r = \bar{q}(q + \bar{q})\bar{\xi}_T$

We are now dealing with a Markovian linear FBSDE, so we suspect that the FBSDE value function is affine and we search for deterministic functions η_t and χ_t such that

$$p_t = \eta_t \xi_t + \chi_t, \quad 0 \leq t \leq T. \quad (67)$$

As in the case of the search for the means $\bar{\xi}_t$ and \bar{p}_t , we look for the necessary consequences of such an ansatz, and we try to derive the final form of the solution of the FBSDE from there. Computing dp_t from the ansatz using the expression of $d\xi_t$ from (66) and identifying the result with the expression of dp_t given in (66) we get:

$$\begin{cases} \dot{\eta}_t = -b_t \eta_t^2 - 2a_t \eta_t + m_t, & \eta_T = q \\ \dot{\chi}_t + (a_t + b_t \eta_t) \chi_t = d_t - c_t \eta_t, & \chi_T = r \\ q_t = r \eta_t. \end{cases} \quad (68)$$

We proceed as before, first finding η_t (which gives automatically q_t by the third equation of (68)) by solving the Ricatti equation, and then, plugging the resulting η_t into the second equation, we solve for χ_t . As before, once η_t and χ_t are determined, we rewrite the forward stochastic differential equation for the dynamics of the state using the ansatz

$$d\xi_t = [(a_t + b_t \eta_t) \xi_t + b_t \chi_t + c_t] dt + \sigma dW_t, \quad \xi_0 = \xi$$

which is solved explicitly

$$\xi_t = e^{\int_0^t (a_u + b_u \eta_u) du} \xi + \int_0^t (b_s \chi_s + c_s) e^{\int_s^t (a_u + b_u \eta_u) du} ds. \quad (69)$$

Notice that (69) shows that the optimally controlled state is still Gaussian despite the nonlinearity due to the McKean-Vlasov nature of the dynamics. While the expectation $\mathbb{E}\{\xi_t\} = \bar{\xi}_t$ was already computed, expression (69) can be used to compute the variance of ξ_t as a function of time. Because of the linearity of the ansatz and the fact that η_t and χ_t are deterministic, the adjoint process $(p_t)_{0 \leq t \leq T}$ is also Gaussian.

Remark 6. *Using again the form of the ansatz, we see that the optimal control α_t which was originally identified as an open loop control in (61), is in fact in closed loop feedback form since it can be rewritten as*

$$\alpha_t = -\frac{b_t}{n_t} \eta_t \xi_t - \frac{b_t}{n_t} \chi_t \quad (70)$$

via the feedback function $\phi(t, \xi) = -b_t(\eta_t \xi + \chi_t)/n_t$ which incidently shows that the optimal control is also a Gaussian process.

8 Comparison of Mean Field and McKean-Vlasov Approximations

In this section we provide simple examples confirming that the problems are different by showing that they have different solutions. Note that we shall see that they can coincide in

some special cases. We first concentrate on the simple case:

$$b(t, x, \mu, \alpha) = \alpha, \quad f(t, x, \mu, \alpha) = \frac{\alpha^2}{2}$$

which we already considered earlier. We saw that in both cases, the optimal control is simply the negative of the adjoint process, the difference between the two cases being that the FBSDEs giving the optimally controlled state and the adjoint process are different. While the analysis of the previous sections was given for a very specific form of the quadratic terminal cost, we discuss now several related forms of the function g for which we can still conclude.

The case $g(x, x') = xx'$. Here we only compare the end-of-the-period means. We know from section ?? that the solution to the MFG problem is such that $\bar{\mu}_T = x + \bar{\mu}_T T$, so that we have:

$$\bar{\mu}_T = \frac{c}{1 - T}. \quad (71)$$

On the other hand, taking expectations on both sides of the FBSDE we get that:

$$\begin{aligned} \bar{p}_t &= -2\mathbb{E}[X_T], & t \in [0, T], \\ \bar{x}_t &= -\bar{p}_t, & \bar{x}_0 = x, \end{aligned}$$

from which we see that

$$\mathbb{E}[x_T] = \frac{c}{1 - 2T}. \quad (72)$$

(71) and (72) clearly show that the terminal means of the optimally controlled state are different in the MKV control problem and the MFG problem.

The case $g(x, x') = xx'^2$. In this example we point out that we can also use system (??)-(??) to find necessary conditions to the MFG problem. First we take $(\bar{\mu}_t)_{t \in [0, T]}$ as given, and then we write the system it must satisfy:

$$\begin{aligned} \bar{p}_t^{mfg} &= -\mathbb{E}[X_T]^2, & t \in [0, T], \\ \mathbb{E}[X_t] &= -\bar{p}_t^{mfg}, & \mathbb{E}[X_0] = x. \end{aligned}$$

At the equilibrium, we must have $\bar{\mu}_T = \mathbb{E}[X_T]$, then

$$T \bar{\mu}_T^2 - \bar{\mu}_T + x = 0. \quad (73)$$

On the other hand, we write the averaged FBSDE after taking expectations for the MKV control problem, we get:

$$3T \bar{x}_T^2 - \bar{x}_T + x = 0. \quad (74)$$

Existence of a solution and uniqueness depend upon the parameters T and x . Let us have a quick look at these issues more in details. Let $S_{mfg} = \{(T, x) \in \mathbb{R}_+^2; 1 - 4Tx \geq 0\}$ and S_{mkv} be defined in the same way. They represent the sets of solutions to both problems. It is easy to see that $S_{mkv} \subset S_{mfg}$. Moreover, there exists a continuum of pairs (T, x) such that there is uniqueness in the MKV control problem while there are two solutions to the MFG problems. In particular, if $T = 1$, $x = 1/12$, then we have

$$\bar{x}_T = \frac{1}{6}, \quad \text{and} \quad \bar{\mu}_T = \frac{3 \pm \sqrt{6}}{6}.$$

Additive running cost. It is obvious from the averaged FBSDE after taking expectations that the solutions \bar{x} and $\bar{\mu}$ do not coincide when the coefficients of the model have the form:

- $b(t, x, x', a) = b_1(x) + b_2(x') + b_3(a)$;
- $f(t, x, x', a) = f_1(t, x) + f_2(t, x') + a^2/2$;
- $g(x, x') = g_1(x,') + g_2(x, x')$.

The case $f(t, x, x', a) = a^2/2 + xx'$. In this example we take $g = 0$ and $b(t, x, x', a) = a$. Writing the necessary the averaged FBSDE after taking expectations for both problems, we get:

$$\begin{aligned} \frac{d}{dt} \bar{p}_t^{mfg} &= -\bar{\mu}_t, \quad \bar{p}_T^{mfg} = 0 & \frac{d}{dt} \bar{p}_t^{mkv} &= -2\bar{x}_t, \quad \bar{p}_T^{mkv} = 0, \\ \frac{d}{dt} \bar{\mu}_t &= -\bar{p}_t^{mfg}, \quad \bar{\mu}_0 = c & \frac{d}{dt} \bar{x}_t &= -\bar{p}_t^{mkv}, \quad \bar{x}_0 = x. \end{aligned}$$

These are classical systems with exponential solutions. After computations, we finally have the following solutions:

$$\bar{x}_t = x \left(\frac{3}{2} - \frac{e^{\sqrt{2}(T-t)} + e^{-\sqrt{2}(T-t)}}{e^{\sqrt{2}T} + e^{-\sqrt{2}T}} \right), \quad \text{and} \quad \bar{\mu}_t = x \left(2 - \frac{e^{(T-t)} + e^{-(T-t)}}{e^T + e^{-T}} \right).$$

The linear-quadratic case with zero terminal cost. This classical case corresponds to $b(t, x, x', a) = a$, $f(t, x, x', a) = a^2/2 + (x - x')^2/2$. As we have already investigated the impact of the terminal cost, we take $g = 0$ here. Here it is easy to see that the trivial solution ($\bar{p} = 0$) solves both problems.

9 Back to the Carbon Emissions Regulation Example

We revisit the emissions regulation example presented in Section 2, switching from the original notation of the emissions literature to the notation used in this paper. The private states $X_t^i = E_t^i$ are one-dimensional and the drift of their dynamics and the running cost function are of the form:

$$b(t, x, \mu, \alpha) = -\alpha, \quad \text{and} \quad f(t, x, \mu, \alpha) = c(\alpha)$$

(for the sake of simplicity we assume that the BAU drift b_t^i is zero) while the function giving the terminal cost is the function g defined on $\mathbb{R} \times \mathcal{P}_1(\mathbb{R})$ by

$$g(x, \mu) = \lambda(x - \Lambda_N)^+ \mathbf{1}_{\{\bar{\mu} > \Lambda_N\}}$$

where $\Lambda_N = \Lambda/N$ stands for the cap per firm. The quantity Λ_N is the relevant form of the cap as it makes sense in the limit $N \rightarrow \infty$ of a large number of firms treated similarly by the regulator.

9.1 The Mean Field Game Approximation

As explained earlier, approximate Nash equilibrium provided by the MFG approach are given by the solutions of a forward-backward system of nonlinear PDEs which, in the present set-up, if we assume quadratic abatement costs $c(\xi) = \xi^2$, reads:

$$\begin{aligned} \text{(Kolmogorov)} \quad & \partial_t m - \frac{\sigma^2}{2} m'' + (-v' m)' = 0, \quad m(0, \cdot) = m_0(\cdot), \\ \text{(HJB)} \quad & \partial_t v + \frac{\sigma^2}{2} v'' - \frac{1}{2} (v')^2 = 0, \quad v(T, \cdot) = v_T(\cdot, m(T)). \end{aligned}$$

where the function $m(t, \cdot)$ stands for the density of the measure $\mu_t = \mathcal{L}(X_t)$ so that:

$$\bar{m}_t = \bar{\mu}_t = \int x m(t, x) dx$$

and v for the HJB value function of the stochastic control problem when the family $\{\mu_t\}_{0 \leq t \leq T}$ is frozen. These PDEs are simple enough to be solved by a Hopf-Cole change of variables:

$$\begin{aligned} v &= -\sigma^2 \log u \quad \leftrightarrow \quad u = \exp\left(-\frac{v}{\sigma^2}\right) \\ m &= u\psi \quad \leftrightarrow \quad \psi = \frac{m}{u}. \end{aligned}$$

Then the pair (u, ψ) solves the simple forward-backward system:

$$\begin{aligned} \partial_t \psi - \frac{\sigma^2}{2} \psi'' &= 0, & \psi(0, \cdot) &= \psi_0(u(0, \cdot)), \\ \partial_t u + \frac{\sigma^2}{2} u'' &= 0, & u(T, \cdot) &= u_T(\cdot, \psi(T, \cdot)), \end{aligned}$$

consisting of two fundamental heat equations coupled only at the end points of the time interval. Here

$$\psi_0(f(y)) = \frac{m_0(y)}{f(y)} \quad \text{and} \quad \psi_T(y, f(y)) = \exp\left(-\frac{\lambda y}{\sigma^2} \mathbb{1}\left\{\int y e^{-\lambda y/\sigma^2} f(y) dy > \Lambda_N\right\}\right).$$

Though MFG problems cannot be solved analytically in general, here the specific and simple form of the mean-field interaction term v_T (depending only upon the global distribution's first moment) leads to an easy computation of the solutions by allowing us to take full advantage of the adjoint structure of the terminal cost. We need to distinguish three cases:

- *The Business As Usual (BAU) Solution.* If we assume that the cap is not reached, then $u(T, y) = 1$ for all $y \in \mathbb{R}$, which implies that for all $(t, y) \in [0, T] \times \mathbb{R}$, we have $v(t, y) = 0$, and consequently the optimal abatement strategy is to do nothing, i.e. $a_t = 0$. This corresponds to BAU. So in this case $m(t, y) = \varphi_{0, \sigma^2 t} \star m_0(y)$, where $\varphi_{0, \sigma^2 t}$ is the density of the Gaussian distribution with mean 0 and variance $2\sigma^2 t$. Then one should check that the mean of $\varphi_{0, \sigma^2 T} \star m_0(y)$ is lower than Λ in order to conclude that this is a Nash equilibrium approximation. This happens whenever $\bar{y}_0 < \Lambda$.
- *The Abatement Solution.* Suppose now that the cap is exceeded. In this case the terminal condition is linear: $v_T(y) = \lambda y$. An easy computation shows that $v(t, y) = -\sigma^2 \log(\varphi_{0, \sigma^2(T-t)}(y))$, $\phi(t, x) = \lambda$ (abatement strategy) and $m(t, x) = \varphi_{\lambda t, \sigma^2 t} \star m_0(x)$. Now one should check that we eventually have the $\bar{m}_T > \Lambda_N$ which, since $\bar{m}_T = \bar{m}_0 - \lambda T$, is equivalent to the condition $\bar{m}_0 > \Lambda_N + \lambda T$.
- *Non-Existence.* Finally, if \bar{m}_0 belongs to the interval $[\Lambda_N, \Lambda_N + \lambda T]$, it is easy to see that there is no solution (i.e. no MFG approximate Nash equilibrium). Indeed, in the reasonable (from the point of view of the choices of Λ , λ , and T) cap-and-trade schemes, the cap is expected to be reached so that the players have an abatement strategy, and it is often observed that the cap is in fact not reached at the end of the period. Now, from the MFG theory viewpoint, the intuition is as follows. An individual firm with negligible impact on the overall emission can emit whatever it wants without impacting significantly the global emissions. However, as soon as this becomes everybody's strategy, the cap is reached.

Let us emphasize that we could also have used the result of Section 6 (linear terminal cost case) to directly get the existence condition for the specific $g = u_T$.