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► **To cite this version:**

Ahmed El Ghini, Mohamed El Machkouri. Kernel deconvolution estimation for random fields. 2011.
hal-00655770

HAL Id: hal-00655770

<https://hal.science/hal-00655770>

Preprint submitted on 2 Jan 2012

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KERNEL DECONVOLUTION ESTIMATION FOR RANDOM FIELDS

Ahmed EL GHINI and Mohamed EL MACHKOURI

Abstract

In this work, we establish the asymptotic normality of the deconvolution kernel density estimator in the context of strongly mixing random fields. Only minimal conditions on the bandwidth parameter are required and a simple criterion on the strong mixing coefficients is provided. Our approach is based on the Lindeberg's method rather than on Bernstein's technique and coupling arguments widely used in previous works on nonparametric estimation for spatial processes. We deal also with nonmixing random fields which can be written as a (nonlinear) functional of i.i.d. random fields by considering the physical dependence measure coefficients introduced by Wu [20].

AMS Subject Classifications (2000): 62G05, 62G07, 60G60.

Key words and phrases: Central limit theorem, deconvolution kernel density estimator, strongly mixing random fields, nonmixing random fields, physical dependence measure.

Short title: Deconvolution kernel density estimator for random fields.

1 Introduction and main results

Let $X = (X_i)_{i \in \mathbb{Z}^d}$ be a stationary real random field defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We observe the random field X on a region Λ_n , $n \in \mathbb{N}^*$, but the observations are contaminated with noise such as measurement errors. In fact, we observe only the random field $Y = (Y_i)_{i \in \mathbb{Z}^d}$ defined for any i in \mathbb{Z}^d by $Y_i = X_i + \theta_i$ where the error variables $(\theta_i)_{i \in \mathbb{Z}^d}$ are identically distributed and independent of X . We denote by f_Y , f_X and f_θ the marginal density of Y , X and θ respectively and we have $f_Y = f_X \star f_\theta$. We observe a sample of Y and we want to estimate f_X using the deconvolution kernel approach introduced by Stefanski and Carroll [18]. Previous key results on deconvolution kernel density estimators for time series are Fan [7], [8] and Masry [12], [13]. For strongly mixing random fields indexed by the lattice \mathbb{Z}^d , Li [10] obtained a central limit theorem for the deconvolution kernel density estimator using the so-called Bernstein's small and large blocks technique and coupling arguments initiated by Tran [19]. Note that the extension of asymptotic result for time series to the spatial

setting is not trivial because of difficulties coming from spatial ordering. The purpose of this work is to put on light a new approach for the asymptotic normality of kernel density estimators. In fact, we are going to apply the Lindeberg's method (see [11]) in order to improve the result by Li [10] in several directions. This new approach was recently applied successfully in El Machkouri and Stoica [5] and El Machkouri [4] for the Nadaraya-Watson estimator and the Parzen-Rosenblatt estimator respectively in the setting of random fields.

For any finite subset B of \mathbb{Z}^d , denote $|B|$ the number of elements in B and ∂B its boundary defined by $\partial B = \{i \in B; \exists j \notin B |i - j| = 1\}$ where $|s| = \max_{1 \leq k \leq d} |s_k|$ for any $s = (s_1, \dots, s_d)$ in \mathbb{Z}^d . In the sequel, we assume that we observe $(X_i)_{i \in \mathbb{Z}^d}$ on a sequence $(\Lambda_n)_{n \geq 1}$ of finite subsets of \mathbb{Z}^d which satisfies

$$\lim_{n \rightarrow \infty} |\Lambda_n| = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{|\partial \Lambda_n|}{|\Lambda_n|} = 0. \quad (1)$$

Given two σ -algebras \mathcal{U} and \mathcal{V} of \mathcal{F} , we recall the α -mixing coefficient introduced by Rosenblatt [16] defined by $\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, A \in \mathcal{U}, B \in \mathcal{V}\}$. For any τ in $\mathbb{N}^* \cup \{\infty\}$ and any positive integer n , we consider the mixing coefficient $\alpha_{1,\tau}(n)$ defined by

$$\alpha_{1,\tau}(n) = \sup \{ \alpha(\sigma(X_k), \mathcal{F}_B), k \in \mathbb{Z}^d, |B| \leq \tau, \rho(B, \{k\}) \geq n \},$$

where $\mathcal{F}_B = \sigma(X_i; i \in B)$ and the distance ρ is defined for any subsets B_1 and B_2 of \mathbb{Z}^d by $\rho(B_1, B_2) = \min\{|i - j|, i \in B_1, j \in B_2\}$. We say that the random field $(X_i)_{i \in \mathbb{Z}^d}$ is strongly mixing if $\lim_{n \rightarrow \infty} \alpha_{1,\tau}(n) = 0$ for some τ in $\mathbb{N}^* \cup \{\infty\}$.

Let $(b_n)_{n \geq 1}$ be a sequence of positive numbers going to zero as n goes to infinity. The deconvolving kernel density estimator for f_X is defined for any x in \mathbb{R} by

$$\hat{f}_n(x) = \frac{1}{|\Lambda_n| b_n} \sum_{i \in \Lambda_n} g_n \left(\frac{x - Y_i}{b_n} \right) \quad (2)$$

where for any z in \mathbb{R} ,

$$g_n(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} \frac{\phi_K(t)}{\phi_\theta(t/b_n)} dt.$$

The kernel density estimator \hat{f}_n defined by (2) can be written for any x in \mathbb{R} as

$$\hat{f}_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \hat{\phi}_n(t) \frac{\phi_K(tb_n)}{\phi_\theta(t)} dt \quad (3)$$

where

$$\hat{\phi}_n(t) = \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} e^{itY_i}.$$

We consider the following assumptions:

(A1) The marginal probability distribution of each Y_k is absolutely continuous with continuous positive density function f_Y .

(A2) There exists $\kappa > 0$ such that $\sup_{(x,y) \in \mathbb{R}^2} f_{i|j}(y|x) \leq \kappa$ where $f_{i|j}$ is the conditional density function of Y_i given Y_j for any i and j in \mathbb{Z}^d .

(A3) There exists $\beta > 0$ and $B > 0$ such that $|t|^\beta |\phi_\theta(t)| \xrightarrow[t \rightarrow +\infty]{} B$.

(A4) The characteristic function ϕ_K of the kernel K vanishes outside $[-1, 1]$.

(A5) The bandwidth b_n converges to zero and $|\Lambda_n|b_n$ goes to infinity.

The following result establishes the asymptotic bias of the estimator \hat{f}_n .

Proposition 1 (Li [10], 2008) *If ϕ_K is continuous then, for any real x ,*

$$\mathbb{E}(\hat{f}_n(x)) \xrightarrow[n \rightarrow +\infty]{} f(x).$$

Now, we investigate the asymptotic variance of the estimator \hat{f}_n .

Proposition 2 *Assume that $\sum_{m \geq 1} m^{2d-1} \alpha_{1,1}(m) < \infty$. For any x in \mathbb{R} , we have*

$$\lim_{n \rightarrow \infty} |\Lambda_n| b_n^{2\beta+1} \mathbb{V}(\hat{f}_n(x)) = \frac{f_Y(x)}{B^2} \int_{\mathbb{R}} |t|^{2\beta} |\phi_K(t)|^2 dt := \sigma^2(x). \quad (4)$$

Our main result is the following.

Theorem 1 *Assume that Assumptions (A1), ..., (A5) hold and*

$$\sum_{m=1}^{+\infty} m^{2d-1} \alpha_{1,\infty}(m) < \infty. \quad (5)$$

Then for any positive integer k and any distinct points x_1, \dots, x_k in \mathbb{R} ,

$$(|\Lambda_n| b_n^{2\beta+1})^{1/2} \begin{pmatrix} \hat{f}_n(x_1) - \mathbb{E}\hat{f}_n(x_1) \\ \vdots \\ \hat{f}_n(x_k) - \mathbb{E}\hat{f}_n(x_k) \end{pmatrix} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, V) \quad (6)$$

where V is a diagonal matrix with diagonal elements $v_{ii} = \frac{f_Y(x_i)}{B^2} \int_{\mathbb{R}} |t|^{2\beta} |\phi_K(t)|^2 dt$.

Remark 1. Theorem 1 improves Theorem 4.1 and Theorem 4.2 in [10] in three directions: the regions Λ_n where the random field is observed are not reduced to rectangular ones, the assumption (A5) on the bandwidth b_n is minimal and the mixing condition

(5) does not depend on the bandwidth parameter b_n .

Since the mixing property is often unverifiable and might be too restrictive, it is important to provide limit theorems for nonmixing and possibly nonlinear spatial processes. So, in the sequel, we consider that $(X_i)_{i \in \mathbb{Z}^d}$ is a field of identically distributed real random variables with a marginal density f such that

$$X_i = F(\varepsilon_{i-s}; s \in \mathbb{Z}^d), \quad i \in \mathbb{Z}^d, \quad (7)$$

where $(\varepsilon_j)_{j \in \mathbb{Z}^d}$ are i.i.d. random variables and F is a measurable function defined on $\mathbb{R}^{\mathbb{Z}^d}$. In the one-dimensional case ($d = 1$), the class (7) includes linear as well as many widely used nonlinear time series models as special cases. More importantly, it provides a very general framework for asymptotic theory for statistics of stationary time series (see [20] and the review paper [21]). Let $(\varepsilon'_j)_{j \in \mathbb{Z}^d}$ be an i.i.d. copy of $(\varepsilon_j)_{j \in \mathbb{Z}^d}$ and consider for any positive integer n the coupled version X_i^* of X_i defined by $X_i^* = F(\varepsilon_{i-s}^*; s \in \mathbb{Z}^d)$ where $\varepsilon_j^* = \varepsilon_j \mathbb{1}_{\{j \neq 0\}} + \varepsilon'_0 \mathbb{1}_{\{j=0\}}$ for any j in \mathbb{Z}^d . In other words, we obtain X_i^* from X_i by just replacing ε_0 by its copy ε'_0 . Following Wu [20], we introduce appropriate dependence measures: let i in \mathbb{Z}^d and $p > 0$ be fixed. If X_i belongs to \mathbb{L}_p (that is, $\mathbb{E}|X_i|^p$ is finite), we define the physical dependence measure $\delta_{i,p} = \|X_i - X_i^*\|_p$ where $\|\cdot\|_p$ is the usual \mathbb{L}^p -norm and we say that the random field $(X_i)_{i \in \mathbb{Z}^d}$ is p -stable if $\sum_{i \in \mathbb{Z}^d} \delta_{i,p} < \infty$. For $d \geq 2$, the reader should keep in mind the following two examples already given in [6] :

Linear random fields: Let $(\varepsilon_i)_{i \in \mathbb{Z}^d}$ be i.i.d random variables with ε_i in \mathbb{L}^p , $p \geq 2$. The linear random field X defined for any i in \mathbb{Z}^d by

$$X_i = \sum_{s \in \mathbb{Z}^d} a_s \varepsilon_{i-s}$$

with $(a_s)_{s \in \mathbb{Z}^d}$ in $\mathbb{R}^{\mathbb{Z}^d}$ such that $\sum_{i \in \mathbb{Z}^d} a_i^2 < \infty$ is of the form (7) with a linear functional g . For any i in \mathbb{Z}^d , $\delta_{i,p} = \|a_i\| \|\varepsilon_0 - \varepsilon'_0\|_p$. So, X is p -stable if $\sum_{i \in \mathbb{Z}^d} |a_i| < \infty$. Clearly, if H is a Lipschitz continuous function, under the above condition, the subordinated process $Y_i = H(X_i)$ is also p -stable since $\delta_{i,p} = O(|a_i|)$.

Volterra field : Another class of nonlinear random field is the Volterra process which plays an important role in the nonlinear system theory (Casti [2], Rugh [17]): consider the second order Volterra process

$$X_i = \sum_{s_1, s_2 \in \mathbb{Z}^d} a_{s_1, s_2} \varepsilon_{i-s_1} \varepsilon_{i-s_2},$$

where a_{s_1, s_2} are real coefficients with $a_{s_1, s_2} = 0$ if $s_1 = s_2$ and $(\varepsilon_i)_{i \in \mathbb{Z}^d}$ are i.i.d. random variables with ε_i in \mathbb{L}^p , $p \geq 2$. Let

$$A_i = \sum_{s_1, s_2 \in \mathbb{Z}^d} (a_{s_1, i}^2 + a_{i, s_2}^2) \quad \text{and} \quad B_i = \sum_{s_1, s_2 \in \mathbb{Z}^d} (|a_{s_1, i}|^p + |a_{i, s_2}|^p).$$

By the Rosenthal inequality, there exists a constant $C_p > 0$ such that

$$\delta_{i,p} = \|X_i - X_i^*\|_p \leq C_p A_i^{1/2} \|\varepsilon_0\|_2 \|\varepsilon_0\|_p + C_p B_i^{1/p} \|\varepsilon_0\|_p^2.$$

Theorem 2 *Let $(X_i)_{i \in \mathbb{Z}^d}$ be defined by the relation (7) and assume that (A1), ..., (A5) hold. If (5) is replaced by the condition*

$$\sum_{i \in \mathbb{Z}^d} |i|^{\frac{5d}{2}} \delta_i < \infty \tag{8}$$

then the conclusion of Theorem 1 still holds.

2 Proofs

Throughout this section, the symbol κ will denote a generic positive constant which the value is not important and for any $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$, we denote $|i| = \max_{1 \leq k \leq d} |i_k|$. Recall also that for any finite subset B of \mathbb{Z}^d , we denote $|B|$ the number of elements in B . Let $\tau \in \mathbb{N}^* \cup \{\infty\}$ be fixed and consider the sequence $(m_{n,\tau})_{n \geq 1}$ defined by

$$m_{n,\tau} = \max \left\{ v_n, \left\lceil \left(\frac{1}{b_n} \sum_{|i| > v_n} |i|^d \alpha_{1,\tau}(|i|) \right)^{\frac{1}{d}} \right\rceil + 1 \right\} \tag{9}$$

where $v_n = \lceil b_n^{-\frac{1}{2d}} \rceil$ and $\lceil \cdot \rceil$ denotes the integer part function. The following technical lemma is a spatial version of a result by Bosq, Merlevède and Peligrad ([1], pages 88-89).

Lemma 1 *If $\tau \in \mathbb{N}^* \cup \{\infty\}$ and $\sum_{m \geq 1} m^{2d-1} \alpha_{1,\tau}(m) < \infty$ then*

$$m_{n,\tau} \rightarrow \infty, \quad m_{n,\tau}^d b_n \rightarrow 0 \quad \text{and} \quad \frac{1}{m_{n,\tau}^d b_n} \sum_{|i| > m_{n,\tau}} |i|^d \alpha_{1,\tau}(|i|) \rightarrow 0.$$

Proof of Proposition 2. Let z be fixed in \mathbb{R} . For any i in \mathbb{Z}^d , we denote

$$Z_i(z) = \frac{1}{b_n} \left(g_n \left(\frac{z - Y_i}{b_n} \right) - \mathbb{E} g_n \left(\frac{z - Y_i}{b_n} \right) \right).$$

The proof of the following lemma is done in the appendix.

Lemma 2 For any z in \mathbb{R} ,

$$b_n^{2\beta+1}\mathbb{E}(Z_0^2(z)) \xrightarrow{n \rightarrow \infty} \sigma^2(z) := \frac{f_Y(z)}{B^2} \int_{\mathbb{R}} |u|^{2\beta} |\phi_K(u)|^2 du \quad (10)$$

and $\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} |Z_0(s)Z_i(t)| = O(b_n^{-2\beta})$ for any s and t in \mathbb{R} .

Let x in \mathbb{R} be fixed. We have

$$|\Lambda_n| b_n^{2\beta+1} \mathbb{V}(\hat{f}_n(x)) = b_n^{2\beta+1} \mathbb{E}(Z_0^2(x)) + \frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \text{Cov}(Z_i(x), Z_j(x)). \quad (11)$$

Since $(Z_i)_{i \in \mathbb{Z}^d}$ is stationary, we have

$$\begin{aligned} \frac{b_n^{2\beta+1}}{|\Lambda_n|} \left| \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \text{Cov}(Z_i(x), Z_j(x)) \right| &\leq b_n^{2\beta+1} \sum_{i \in \mathbb{Z}^d \setminus \{0\}} |\mathbb{E}(Z_0(x)Z_i(x))| \\ &\leq b_n^{2\beta+1} \left(m_{n,1}^d \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} |Z_0(x)Z_i(x)| + \sum_{|i| > m_{n,1}} |\mathbb{E}(Z_0(x)Z_i(x))| \right). \end{aligned}$$

By Rio's covariance inequality (cf. [15], Theorem 1.1), we know that $|\mathbb{E}(Z_0(x)Z_i(x))| \leq \kappa \|Z_0(x)\|_{\infty}^2 \alpha_{1,1}(|i|)$. Since $\|Z_0(x)\|_{\infty} \leq \kappa b_n^{-\beta-1}$ and $\tau \in \mathbb{N}^* \cup \{\infty\}$, we obtain

$$\frac{b_n^{2\beta+1}}{|\Lambda_n|} \left| \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \text{Cov}(Z_i(x), Z_j(x)) \right| \leq \kappa \left(m_{n,1}^d b_n^{2\beta+1} \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} |Z_0(x)Z_i(x)| + \frac{1}{m_{n,1}^d b_n} \sum_{|i| > m_{n,1}} |i|^d \alpha_{1,1}(|i|) \right).$$

Applying Lemma 1 and the second part of Lemma 2, we derive

$$\lim_{n \rightarrow \infty} \frac{b_n^{2\beta+1}}{|\Lambda_n|} \left| \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \text{Cov}(Z_i(x), Z_j(x)) \right| = 0. \quad (12)$$

Combining (10), (11) and (12), we obtain (4). The proof of Proposition 2 is complete.

Proof of Theorem 1. Without loss of generality, we consider only the case $k = 2$ and we refer to x_1 and x_2 as x and y ($x \neq y$). Let λ_1 and λ_2 be two constants such that $\lambda_1^2 + \lambda_2^2 = 1$ and denote

$$S_n = \lambda_1 (|\Lambda_n| b_n^{2\beta+1})^{1/2} (\hat{f}_n(x) - \mathbb{E} \hat{f}_n(x)) + \lambda_2 (|\Lambda_n| b_n^{2\beta+1})^{1/2} (\hat{f}_n(y) - \mathbb{E} \hat{f}_n(y)) = \sum_{i \in \Lambda_n} \frac{b_n^{\beta+\frac{1}{2}} \Delta_i}{|\Lambda_n|^{1/2}}$$

where $\Delta_i = \lambda_1 Z_i(x) + \lambda_2 Z_i(y)$ and for any z in \mathbb{R} ,

$$Z_i(z) = \frac{1}{b_n} \left(g_n \left(\frac{z - Y_i}{b_n} \right) - \mathbb{E} g_n \left(\frac{z - Y_i}{b_n} \right) \right).$$

We consider the notation

$$\eta = \frac{\lambda_1^2 f_Y(x) + \lambda_2^2 f_Y(y)}{B^2} \int_{\mathbb{R}} |t|^{2\beta} |\phi_K(t)|^2 dt. \quad (13)$$

The proof of the following technical result is postponed to the annex.

Lemma 3 $b_n^{2\beta+1} \mathbb{E}(\Delta_0^2)$ converges to η as n goes to infinity and $\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\Delta_0 \Delta_i| = O(b_n^{-2\beta})$.

In order to prove the convergence in distribution of S_n to $\sqrt{\eta} \tau_0$ where $\tau_0 \sim \mathcal{N}(0, 1)$, we follow the Lindeberg's method used in the proof of the central limit theorem for stationary random fields by Dedecker [3]. Let φ be a one to one map from $[1, \kappa] \cap \mathbb{N}^*$ to a finite subset of \mathbb{Z}^d and $(\xi_i)_{i \in \mathbb{Z}^d}$ a real random field. For all integers k in $[1, \kappa]$, we denote

$$S_{\varphi(k)}(\xi) = \sum_{i=1}^k \xi_{\varphi(i)} \quad \text{and} \quad S_{\varphi(k)}^c(\xi) = \sum_{i=k}^{\kappa} \xi_{\varphi(i)}$$

with the convention $S_{\varphi(0)}(\xi) = S_{\varphi(\kappa+1)}^c(\xi) = 0$. To describe the set Λ_n , we define the one to one map φ from $[1, |\Lambda_n|] \cap \mathbb{N}^*$ to Λ_n by: φ is the unique function such that $\varphi(k) <_{\text{lex}} \varphi(l)$ for $1 \leq k < l \leq |\Lambda_n|$. From now on, we consider a field $(\tau_i)_{i \in \mathbb{Z}^d}$ of i.i.d. random variables independent of $(\Delta_i)_{i \in \mathbb{Z}^d}$ such that τ_0 has the standard normal law $\mathcal{N}(0, 1)$. We introduce the fields Γ and γ defined for any i in \mathbb{Z}^d by

$$\Gamma_i = \frac{b_n^{\beta+\frac{1}{2}} \Delta_i}{|\Lambda_n|^{1/2}} \quad \text{and} \quad \gamma_i = \frac{\tau_i \sqrt{\eta}}{|\Lambda_n|^{1/2}}$$

where η is defined by (13). Let h be any function from \mathbb{R} to \mathbb{R} . For $0 \leq k \leq l \leq |\Lambda_n|+1$, we introduce $h_{k,l}(\Gamma) = h(S_{\varphi(k)}(\Gamma) + S_{\varphi(l)}^c(\gamma))$. With the above convention we have that $h_{k,|\Lambda_n|+1}(\Gamma) = h(S_{\varphi(k)}(\Gamma))$ and also $h_{0,l}(\Gamma) = h(S_{\varphi(l)}^c(\gamma))$. In the sequel, we will often write $h_{k,l}$ instead of $h_{k,l}(\Gamma)$. We denote by $B_1^4(\mathbb{R})$ the unit ball of $C_b^4(\mathbb{R})$: h belongs to $B_1^4(\mathbb{R})$ if and only if it belongs to $C^4(\mathbb{R})$ and satisfies $\max_{0 \leq i \leq 4} \|h^{(i)}\|_{\infty} \leq 1$. It suffices to prove that for all h in $B_1^4(\mathbb{R})$,

$$\mathbb{E} \left(h \left(S_{\varphi(|\Lambda_n|)}(\Gamma) \right) \right) \xrightarrow{n \rightarrow +\infty} \mathbb{E} \left(h \left(\tau_0 \sqrt{\eta} \right) \right).$$

We use Lindeberg's decomposition:

$$\mathbb{E} \left(h \left(S_{\varphi(|\Lambda_n|)}(\Gamma) \right) - h \left(\tau_0 \sqrt{\eta} \right) \right) = \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(h_{k,k+1} - h_{k-1,k} \right).$$

Now,

$$h_{k,k+1} - h_{k-1,k} = h_{k,k+1} - h_{k-1,k+1} + h_{k-1,k+1} - h_{k-1,k}.$$

Applying Taylor's formula we get that:

$$h_{k,k+1} - h_{k-1,k+1} = \Gamma_{\varphi(k)} h'_{k-1,k+1} + \frac{1}{2} \Gamma_{\varphi(k)}^2 h''_{k-1,k+1} + R_k$$

and

$$h_{k-1,k+1} - h_{k-1,k} = -\gamma_{\varphi(k)} h'_{k-1,k+1} - \frac{1}{2} \gamma_{\varphi(k)}^2 h''_{k-1,k+1} + r_k$$

where $|R_k| \leq \Gamma_{\varphi(k)}^2 (1 \wedge |\Gamma_{\varphi(k)}|)$ and $|r_k| \leq \gamma_{\varphi(k)}^2 (1 \wedge |\gamma_{\varphi(k)}|)$. Since $(\Gamma, \tau_i)_{i \neq \varphi(k)}$ is independent of $\tau_{\varphi(k)}$, it follows that

$$\mathbb{E} \left(\gamma_{\varphi(k)} h'_{k-1,k+1} \right) = 0 \quad \text{and} \quad \mathbb{E} \left(\gamma_{\varphi(k)}^2 h''_{k-1,k+1} \right) = \mathbb{E} \left(\frac{\eta}{|\Lambda_n|} h''_{k-1,k+1} \right)$$

Hence, we obtain

$$\begin{aligned} \mathbb{E} \left(h \left(S_{\varphi(|\Lambda_n|)}(\Gamma) \right) - h \left(\tau_0 \sqrt{\eta} \right) \right) &= \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1} \right) \\ &\quad + \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\left(\Gamma_{\varphi(k)}^2 - \frac{\eta}{|\Lambda_n|} \right) \frac{h''_{k-1,k+1}}{2} \right) \\ &\quad + \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(R_k + r_k \right). \end{aligned}$$

Let $1 \leq k \leq |\Lambda_n|$ be fixed. Noting that $|\Delta_0|$ is bounded by $\kappa b_n^{-\beta-1}$ and applying the first part of Lemma 3, we derive

$$\mathbb{E} |R_k| \leq \frac{b_n^{3\beta+\frac{3}{2}} \mathbb{E} |\Delta_0|^3}{|\Lambda_n|^{3/2}} = O \left(\frac{1}{(|\Lambda_n|^3 b_n)^{1/2}} \right)$$

and

$$\mathbb{E} |r_k| \leq \frac{\mathbb{E} |\gamma_0|^3}{|\Lambda_n|^{3/2}} \leq \frac{\eta^{3/2} \mathbb{E} |\tau_0|^3}{|\Lambda_n|^{3/2}} = O \left(\frac{1}{|\Lambda_n|^{3/2}} \right).$$

Consequently, we obtain

$$\sum_{k=1}^{|\Lambda_n|} \mathbb{E} (|R_k| + |r_k|) = O \left(\frac{1}{(|\Lambda_n| b_n)^{1/2}} + \frac{1}{|\Lambda_n|^{1/2}} \right) = o(1).$$

Now, it is sufficient to show

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \left(\mathbb{E}(\Gamma_{\varphi(k)} h'_{k-1,k+1}) + \mathbb{E} \left(\left(\Gamma_{\varphi(k)}^2 - \frac{\eta}{|\Lambda_n|} \right) \frac{h''_{k-1,k+1}}{2} \right) \right) = 0. \quad (14)$$

First, we focus on $\sum_{k=1}^{|\Lambda_n|} \mathbb{E}(\Gamma_{\varphi(k)} h'_{k-1,k+1})$. On the lattice \mathbb{Z}^d we define the lexicographic order as follows: if $i = (i_1, \dots, i_d)$ and $j = (j_1, \dots, j_d)$ are distinct elements of \mathbb{Z}^d , the notation $i <_{\text{lex}} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, \dots, d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \leq q < p$. Let the sets $\{V_i^M; i \in \mathbb{Z}^d, M \in \mathbb{N}^*\}$ be defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d; j <_{\text{lex}} i\} \text{ and for } M \geq 2, V_i^M = V_i^1 \cap \{j \in \mathbb{Z}^d; |i - j| \geq M\}.$$

For any subset L of \mathbb{Z}^d define $\mathcal{F}_L = \sigma(\Delta_i; i \in L)$ and set

$$\mathbb{E}_M(\Delta_i) = \mathbb{E}(\Delta_i | \mathcal{F}_{V_i^M}), \quad M \in \mathbb{N}^*.$$

For all M in \mathbb{N}^* and all integer k in $[1, |\Lambda_n|]$, we define

$$\mathbb{E}_k^M = \varphi([1, k] \cap \mathbb{N}^*) \cap V_{\varphi(k)}^M \quad \text{and} \quad S_{\varphi(k)}^M(\Gamma) = \sum_{i \in \mathbb{E}_k^M} \Gamma_i.$$

For any function Ψ from \mathbb{R} to \mathbb{R} , we define $\Psi_{k-1,l}^M = \Psi(S_{\varphi(k)}^M(\Gamma) + S_{\varphi(l)}^c(\gamma))$. We are going to apply this notation to the successive derivatives of the function h . Our aim is to show that

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1} - \Gamma_{\varphi(k)} \left(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{m_n, \infty}(\Gamma) \right) h''_{k-1,k+1} \right) = 0.$$

where $(m_n)_{n \geq 1}$ is the sequence defined by (9). First, we use the decomposition

$$\Gamma_{\varphi(k)} h'_{k-1,k+1} = \Gamma_{\varphi(k)} h'_{k-1,k+1}{}^{m_n, \infty} + \Gamma_{\varphi(k)} \left(h'_{k-1,k+1} - h'_{k-1,k+1}{}^{m_n, \infty} \right).$$

We consider a one to one map ψ from $[1, |\mathbb{E}_k^{m_n, \infty}|] \cap \mathbb{N}^*$ to $\mathbb{E}_k^{m_n, \infty}$ and such that $|\psi(i) - \varphi(k)| \leq |\psi(i-1) - \varphi(k)|$. This choice of ψ ensures that $S_{\psi(i)}(\Gamma)$ and $S_{\psi(i-1)}(\Gamma)$ are $\mathcal{F}_{V_{\varphi(k)}^{|\psi(i)-\varphi(k)|}}$ -measurable. The fact that γ is independent of Γ imply that

$$\mathbb{E} \left(\Gamma_{\varphi(k)} h' \left(S_{\varphi(k+1)}^c(\gamma) \right) \right) = 0.$$

Therefore

$$\left| \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1}{}^{m_n, \infty} \right) \right| = \left| \sum_{i=1}^{|\mathbb{E}_k^{m_n, \infty}|} \mathbb{E} \left(\Gamma_{\varphi(k)} (\Theta_i - \Theta_{i-1}) \right) \right| \quad (15)$$

where $\Theta_i = h' \left(S_{\psi(i)}(\Gamma) + S_{\varphi(k+1)}^c(\gamma) \right)$. Since $S_{\psi(i)}(\Gamma)$ and $S_{\psi(i-1)}(\Gamma)$ are $\mathcal{F}_{V_{\varphi(k)}^{|\psi(i)-\varphi(k)|}}$ -measurable, we can take the conditional expectation of $\Gamma_{\varphi(k)}$ with respect to $\mathcal{F}_{V_{\varphi(k)}^{|\psi(i)-\varphi(k)|}}$ in the right hand side of (15). On the other hand the function h' is 1-Lipschitz, hence $|\Theta_i - \Theta_{i-1}| \leq |\Gamma_{\psi(i)}|$. Consequently,

$$|\mathbb{E}(\Gamma_{\varphi(k)}(\Theta_i - \Theta_{i-1}))| \leq \mathbb{E}|\Gamma_{\psi(i)}\mathbb{E}_{|\psi(i)-\varphi(k)}(\Gamma_{\varphi(k)})|$$

and

$$\left| \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1}{}^{m_n,\infty} \right) \right| \leq \sum_{i=1}^{|\mathbb{E}_k^{m_n,\infty}|} \mathbb{E}|\Gamma_{\psi(i)}\mathbb{E}_{|\psi(i)-\varphi(k)}(\Gamma_{\varphi(k)})|.$$

Hence,

$$\begin{aligned} \left| \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1}{}^{m_n,\infty} \right) \right| &\leq \frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} \sum_{i=1}^{|\mathbb{E}_k^{m_n,\infty}|} \mathbb{E}|\Delta_{\psi(i)}\mathbb{E}_{|\psi(i)-\varphi(k)}(\Delta_{\varphi(k)})| \\ &\leq b_n^{2\beta+1} \sum_{|j| \geq m_n,\infty} \|\Delta_j \mathbb{E}_{|j}(\Delta_0)\|_1. \end{aligned}$$

For any j in \mathbb{Z}^d , we have

$$\|\Delta_j \mathbb{E}_{|j}(\Delta_0)\|_1 = \text{Cov} \left(|\Delta_j| \left(\mathbb{1}_{\mathbb{E}_{|j}(\Delta_0) \geq 0} - \mathbb{1}_{\mathbb{E}_{|j}(\Delta_0) < 0} \right), \Delta_0 \right).$$

So, applying Rio's covariance inequality (cf. [15], Theorem 1.1), we obtain

$$\|\Delta_j \mathbb{E}_{|j}(\Delta_0)\|_1 \leq 4 \int_0^{\alpha_{1,\infty}(|j|)} Q_{\Delta_0}^2(u) du$$

where Q_{Δ_0} is defined by $Q_{\Delta_0}(u) = \inf\{t \geq 0; \mathbb{P}(|\Delta_0| > t) \leq u\}$ for any u in $[0, 1]$. Since $|\Delta_0|$ is bounded by $\kappa b_n^{-\beta-1}$, we have

$$Q_{\Delta_0}(u) \leq \kappa b_n^{-\beta-1} \quad \text{and} \quad \|\Delta_j \mathbb{E}_{|j}(\Delta_0)\|_1 \leq \kappa b_n^{-2\beta-2} \alpha_{1,\infty}(|j|).$$

Finally, we derive

$$\left| \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1}{}^{m_n,\infty} \right) \right| \leq \frac{\kappa}{b_n} \sum_{|j| \geq m_n,\infty} \alpha_{1,\infty}(|j|) \leq \frac{\kappa}{m_{n,\infty}^d b_n} \sum_{|j| \geq m_n,\infty} |j|^d \alpha_{1,\infty}(|j|)$$

and by Lemma 1, we obtain $\lim_{n \rightarrow +\infty} \left| \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1}{}^{m_n,\infty} \right) \right| = 0$. Applying again Taylor's formula, it remains to consider

$$\Gamma_{\varphi(k)}(h'_{k-1,k+1} - h'_{k-1,k+1}{}^{m_n,\infty}) = \Gamma_{\varphi(k)}(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{m_n,\infty}(\Gamma))h''_{k-1,k+1} + R'_k,$$

where $|R'_k| \leq 2|\Gamma_{\varphi(k)}(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{m_{n,\infty}}(\Gamma))(1 \wedge |S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{m_{n,\infty}}(\Gamma)|)$. Denoting $W_n = \{-m_{n,\infty} + 1, \dots, m_{n,\infty} - 1\}^d$ and $W_n^* = W_n \setminus \{0\}$, it follows that

$$\begin{aligned} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| &\leq 2b_n^{2\beta+1} \mathbb{E} \left(|\Delta_0| \left(\sum_{i \in W_n} |\Delta_i| \right) \left(1 \wedge \frac{b_n^{\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\Delta_i| \right) \right) \\ &= 2b_n^{2\beta+1} \mathbb{E} \left(\left(\Delta_0^2 + \sum_{i \in W_n^*} |\Delta_0 \Delta_i| \right) \left(1 \wedge \frac{b_n^{\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\Delta_i| \right) \right) \\ &\leq \frac{2b_n^{3\beta+\frac{3}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} \mathbb{E}(\Delta_0^2 |\Delta_i|) + 2b_n^{2\beta+1} \sum_{i \in W_n^*} \mathbb{E}|\Delta_0 \Delta_i|. \end{aligned}$$

Since $|\bar{\Delta}_0|$ is bounded by $\kappa b_n^{-\beta-1}$, we derive

$$\begin{aligned} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| &\leq \frac{\kappa b_n^{2\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} \mathbb{E}(|\Delta_0 \Delta_i|) + 2b_n^{2\beta+1} \sum_{i \in W_n^*} \mathbb{E}|\Delta_0 \Delta_i| \\ &= \frac{\kappa b_n^{2\beta+1} \mathbb{E}(\Delta_0^2)}{(|\Lambda_n| b_n)^{1/2}} + \left(\frac{\kappa b_n^{2\beta+1}}{(|\Lambda_n| b_n)^{1/2}} + 2b_n^{2\beta+1} \right) \sum_{i \in W_n^*} \mathbb{E}(|\Delta_0 \Delta_i|) \\ &\leq \kappa \left(\frac{1}{(|\Lambda_n| b_n)^{1/2}} + \left(\frac{b_n^{2\beta+1}}{(|\Lambda_n| b_n)^{1/2}} + b_n^{2\beta+1} \right) m_{n,\infty}^d b_n^{-2\beta} \right) \quad (\text{by Lemma 3}) \\ &= \kappa \left(\frac{1}{(|\Lambda_n| b_n)^{1/2}} + \left(\frac{1}{(|\Lambda_n| b_n)^{1/2}} + 1 \right) m_{n,\infty}^d b_n \right) \\ &= o(1) \quad (\text{by Lemma 1 and Assumption (A5)}). \end{aligned}$$

So, we have shown that

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1} - \Gamma_{\varphi(k)}(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{m_{n,\infty}}(\Gamma)) h''_{k-1,k+1} \right) = 0.$$

In order to obtain (14) it remains to control

$$F_0 = \mathbb{E} \left(\sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1} \left(\frac{\Gamma_{\varphi(k)}^2}{2} + \Gamma_{\varphi(k)} \left(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{m_{n,\infty}}(\Gamma) \right) - \frac{\eta}{2|\Lambda_n|} \right) \right).$$

Let μ be the law of the stationary real random field $(\Delta_i)_{i \in \mathbb{Z}^d}$ and consider the projection π_0 from $\mathbb{R}^{\mathbb{Z}^d}$ to \mathbb{R} defined by $\pi_0(\omega) = \omega_0$ and the family of translation operators $(T^i)_{i \in \mathbb{Z}^d}$ from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$ defined by $(T^i(\omega))_k = \omega_{k+i}$ for any $i \in \mathbb{Z}^d$ and any ω in $\mathbb{R}^{\mathbb{Z}^d}$. Denote by \mathcal{B} the Borel σ -algebra of \mathbb{R} . The random field $(\pi_0 \circ T^i)_{i \in \mathbb{Z}^d}$ defined on the probability space $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mu)$ is stationary with the same law as $(\Delta_i)_{i \in \mathbb{Z}^d}$,

hence, without loss of generality, one can suppose that $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mu)$ and $\Delta_i = \pi_0 \circ T^i$. Recall that ρ is the metric defined for any finite subsets B_1 and B_2 of \mathbb{Z}^d by $\rho(B_1, B_2) = \min\{|i - j|; i \in B_1, j \in B_2\}$ and $|i - j| = \max_{1 \leq k \leq d} |i_k - j_k|$ for any $i = (i_1, \dots, i_d)$ and $j = (j_1, \dots, j_d)$ in \mathbb{Z}^d . We consider the following sets:

$$\Lambda_n^{m_{n,\infty}} = \{i \in \Lambda_n; \rho(\{i\}, \partial\Lambda_n) \geq m_{n,\infty}\} \quad \text{and} \quad \Gamma_n^{m_{n,\infty}} = \{1 \leq k \leq |\Lambda_n|; \varphi(k) \in \Lambda_n^{m_{n,\infty}}\},$$

and the function Ψ from $\mathbb{R}^{\mathbb{Z}^d}$ to \mathbb{R} such that

$$\Psi(\Delta) = \Delta_0^2 + \sum_{i \in V_0^1 \cap W_n} 2\Delta_0 \Delta_i \quad \text{where} \quad W_n = \{-m_{n,\infty} + 1, \dots, m_{n,\infty} - 1\}^d.$$

For $1 \leq k \leq |\Lambda_n|$, we set $D_k^{(n)} = \eta - b_n^{2\beta+1} (\Psi \circ T^{\varphi(k)}(\Delta))$. By definition of Ψ and of the set $\Gamma_n^{m_{n,\infty}}$, we have for any k in $\Gamma_n^{m_{n,\infty}}$

$$\Psi \circ T^{\varphi(k)}(\Delta) = \Delta_{\varphi(k)}^2 + 2\Delta_{\varphi(k)}(S_{\varphi(k-1)}(\Delta) - S_{\varphi(k)}^{m_{n,\infty}}(\Delta)).$$

Therefore for k in $\Gamma_n^{m_{n,\infty}}$

$$\frac{D_k^{(n)}}{|\Lambda_n|} = \frac{\eta}{|\Lambda_n|} - \Gamma_{\varphi(k)}^2 - 2\Gamma_{\varphi(k)}(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{m_{n,\infty}}(\Gamma)).$$

Using (1), we know that $\lim_{n \rightarrow +\infty} |\Lambda_n|^{-1} |\Gamma_n^{m_{n,\infty}}| = 1$. So, it remains to consider

$$F_1 = \left| \mathbb{E} \left(\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} D_k^{(n)} \right) \right|.$$

Applying Lemma 3 and Lemma 1, we obtain

$$\begin{aligned} F_1 &\leq \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} (\Delta_{\varphi(k)}^2 - \mathbb{E}(\Delta_0^2)) \right) \right| + |\eta - b_n^{2\beta+1} \mathbb{E}(\Delta_0^2)| + 2b_n^{2\beta+1} \sum_{j \in V_0^1 \cap W_n} \mathbb{E}|\Delta_0 \Delta_j| \\ &\leq \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} (\Delta_{\varphi(k)}^2 - \mathbb{E}(\Delta_0^2)) \right) \right| + o(1). \end{aligned}$$

So, it suffices to prove that

$$F_2 = \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} (\Delta_{\varphi(k)}^2 - \mathbb{E}(\Delta_0^2)) \right) \right|$$

goes to zero as n goes to infinity. Let $C > 0$ be fixed. We have $F_2 \leq F_2' + F_2''$ where

$$F_2' = \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h_{k-1,k+1}'' (\Delta_{\varphi(k)}^2 - \mathbb{E}_C (\Delta_{\varphi(k)}^2)) \right) \right|$$

and

$$F_2'' = \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h_{k-1,k+1}'' (\mathbb{E}_C (\Delta_{\varphi(k)}^2) - \mathbb{E}(\Delta_0^2)) \right) \right|$$

where we recall the notation $\mathbb{E}_C (\Delta_{\varphi(k)}^2) = \mathbb{E} (\Delta_{\varphi(k)}^2 | \mathcal{F}_{V_{\varphi(k)}^C})$. The following result is a Serfling type inequality which can be found in [14].

Lemma 4 *Let \mathcal{U} and \mathcal{V} be two σ -algebras and let X be a random variable measurable with respect to \mathcal{U} . If $1 \leq p \leq r \leq \infty$ then*

$$\|\mathbb{E}(X|\mathcal{V}) - \mathbb{E}(X)\|_p \leq 2(2^{1/p} + 1) (\alpha(\mathcal{U}, \mathcal{V}))^{\frac{1}{p} - \frac{1}{r}} \|X\|_r.$$

Applying Lemma 4 and keeping in mind that $|\Delta_0|$ is bounded by $\kappa b_n^{-\beta-1}$, we derive

$$F_2'' \leq b_n^{2\beta+1} \|\mathbb{E}_C (\Delta_0^2) - \mathbb{E}(\Delta_0^2)\|_1 \leq \kappa b_n^{-1} \alpha_{1,\infty}(C)$$

In the other part,

$$F_2' \leq \frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} (J_k^1(C) + J_k^2(C))$$

where

$$J_k^1(C) = \left| \mathbb{E} \left(h_{k-1,k+1}'' \circ T^{-\varphi(k)} (\Delta_0^2 - \mathbb{E}_C (\Delta_0^2)) \right) \right| = 0$$

since $h_{k-1,k+1}'' \circ T^{-\varphi(k)}$ is $\mathcal{F}_{V_0^C}$ -measurable and

$$\begin{aligned} b_n^{2\beta+1} J_k^2(C) &= b_n^{2\beta+1} \left| \mathbb{E} \left(\left(h_{k-1,k+1}'' \circ T^{-\varphi(k)} - h_{k-1,k+1}'' \circ T^{-\varphi(k)} \right) (\Delta_0^2 - \mathbb{E}_C (\Delta_0^2)) \right) \right| \\ &\leq b_n^{2\beta+1} \mathbb{E} \left(\left(2 \wedge \sum_{|i|<C} \frac{b_n^{\beta+\frac{1}{2}} |\Delta_i|}{|\Lambda_n|^{1/2}} \right) \Delta_0^2 \right) \\ &\leq \frac{\kappa b_n^{2\beta+1} \mathbb{E}(\Delta_0^2)}{(|\Lambda_n| b_n)^{1/2}} + \frac{\kappa b_n^{2\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{\substack{|i|<C \\ i \neq 0}} \mathbb{E} |\Delta_0 \Delta_i| \quad \text{since } |\Delta_0| \leq \kappa b_n^{-\beta-1} \text{ a.s.} \\ &\leq \kappa \left(\frac{1}{(|\Lambda_n| b_n)^{1/2}} + \frac{C^d \sqrt{b_n}}{|\Lambda_n|^{1/2}} \right) \quad (\text{by Lemma 3}) \end{aligned}$$

So, putting $C = b_n^{\frac{-1}{2d-1}}$ and keeping in mind that $\sum_{m \geq 0} m^{2d-1} \alpha_{1,\infty}(m) < +\infty$, we derive

$$F_2 = O(C^{2d-1} \alpha_{1,\infty}(C)) + O\left(\frac{1 + b_n^{\frac{d-1}{2d-1}}}{(|\Lambda_n| b_n)^{1/2}}\right) = o(1).$$

The proof of Theorem 1 is complete.

2.1 Proof of Theorem 2

The proof follows the same lines as in the proof of Theorem 1. We consider the sequence $(m_n)_{n \geq 1}$ defined by

$$m_n = \max \left\{ v_n, \left[\left(\frac{1}{b_n^3} \sum_{|i| > v_n} |i|^{\frac{5d}{2}} \delta_i \right)^{\frac{1}{3d}} \right] + 1 \right\} \quad (16)$$

where $v_n = \lfloor b_n^{\frac{-1}{2d}} \rfloor$ and $[.]$ denotes the integer part function. As in the proof of Theorem 1, the sequence $(m_n)_{n \geq 1}$ satisfies the following lemma which the proof is left to the reader.

Lemma 5 *If $\sum_{i \in \mathbb{Z}^d} |i|^{\frac{5d}{2}} \delta_i < +\infty$ holds then*

$$m_n \rightarrow \infty, \quad m_n^d b_n \rightarrow 0 \quad \text{and} \quad \frac{1}{(m_n^d b_n)^{3/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_i \rightarrow 0.$$

For any z in \mathbb{R} , we denote

$$G_n(z, i) = g_n \left(\frac{z - Y_i}{b_n} \right) \quad \text{and} \quad \bar{G}_n(z, i) = \mathbb{E}(G_n(z, i) | \mathcal{F}_{n,i}) \quad (17)$$

where $\mathcal{F}_{n,i} = \sigma(\varepsilon_{i-s}; |s| \leq m_n)$. Denoting $M_n = 2m_n + 1$, $(\bar{G}_n(z, i))_{i \in \mathbb{Z}^d}$ is an M_n -dependent random field (i.e. $\bar{G}_n(z, i)$ and $\bar{G}_n(z, j)$ are independent as soon as $|i - j| \geq M_n$). For any x in \mathbb{R} and any integer $n \geq 1$, we denote

$$\bar{f}_n(x) = \frac{1}{|\Lambda_n| b_n} \sum_{i \in \Lambda_n} \bar{G}_n(x, i).$$

The proof of the following lemma is done in the appendix.

Lemma 6 *For any $p \geq 2$, any x in \mathbb{R} , any positive integer n and any $(a_i)_{i \in \mathbb{Z}^d}$ in $\mathbb{R}^{\mathbb{Z}^d}$,*

$$\left\| \sum_{i \in \Lambda_n} a_i (G_n(x, i) - \bar{G}_n(x, i)) \right\|_p \leq \frac{\kappa m_n^d}{b_n^{1+\beta}} \left(p \sum_{i \in \Lambda_n} a_i^2 \right)^{1/2} \sum_{|i| > m_n} \delta_{i,p}.$$

Let $x \neq y$ be fixed and let λ_1 and λ_2 be two constants such that $\lambda_1^2 + \lambda_2^2 = 1$. We have

$$\begin{aligned} \lambda_1(|\Lambda_n|b_n^{2\beta+1})^{1/2}(\hat{f}_n(x) - \mathbb{E}\hat{f}_n(x)) + \lambda_2(|\Lambda_n|b_n^{2\beta+1})^{1/2}(\hat{f}_n(y) - \mathbb{E}\hat{f}_n(y)) &= \sum_{i \in \Lambda_n} \frac{b_n^{\beta+\frac{1}{2}}\Delta_i}{|\Lambda_n|^{1/2}} \\ \lambda_1(|\Lambda_n|b_n^{2\beta+1})^{1/2}(\bar{f}_n(x) - \mathbb{E}\bar{f}_n(x)) + \lambda_2(|\Lambda_n|b_n^{2\beta+1})^{1/2}(\bar{f}_n(y) - \mathbb{E}\bar{f}_n(y)) &= \sum_{i \in \Lambda_n} \frac{b_n^{\beta+\frac{1}{2}}\bar{\Delta}_i}{|\Lambda_n|^{1/2}} \end{aligned}$$

where

$$\Delta_i = \lambda_1 Z_i(x) + \lambda_2 Z_i(y) \quad \text{and} \quad \bar{\Delta}_i = \lambda_1 \bar{Z}_i(x) + \lambda_2 \bar{Z}_i(y)$$

and for any z in \mathbb{R} ,

$$Z_i(z) = \frac{1}{b_n} (\mathbf{G}_n(z, i) - \mathbb{E}\mathbf{G}_n(z, i)) \quad \text{and} \quad \bar{Z}_i(z) = \frac{1}{b_n} (\bar{\mathbf{G}}_n(z, i) - \mathbb{E}\bar{\mathbf{G}}_n(z, i))$$

where $\mathbf{G}_n(z, i)$ and $\bar{\mathbf{G}}_n(z, i)$ are defined by (17). Applying Lemma 5 and Lemma 6, we know that

$$\frac{b_n^{\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \left\| \sum_{i \in \Lambda_n} (\Delta_i - \bar{\Delta}_i) \right\|_2 \leq \frac{\kappa(|\lambda_1| + |\lambda_2|)}{(m_n^d b_n)^{3/2}} \sum_{|i| > m_n} |i|^{\frac{5d}{2}} \delta_i = o(1). \quad (18)$$

So, it suffices to prove the asymptotic normality of $\left(b_n^{\beta+\frac{1}{2}} |\Lambda_n|^{-1/2} \sum_{i \in \Lambda_n} \bar{\Delta}_i \right)_{n \geq 1}$. Let η be defined by (13). The proof of the following lemma is also postponed to the appendix.

Lemma 7 $b_n^{2\beta+1} \mathbb{E}(\bar{\Delta}_0^2)$ converges to η as n goes to infinity. Moreover, if (8) holds then $\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\bar{\Delta}_0 \bar{\Delta}_i| = o(m_n^{-d} b_n^{-2\beta-1})$.

As in the proof of Theorem 1, in order to describe the set Λ_n , we consider the one to one map φ from $[1, |\Lambda_n|] \cap \mathbb{N}^*$ to Λ_n by: φ is the unique function such that $\varphi(k) <_{\text{lex}} \varphi(l)$ for $1 \leq k < l \leq |\Lambda_n|$ where $<_{\text{lex}}$ denotes the lexicographic order on \mathbb{Z}^d and we consider a field $(\tau_i)_{i \in \mathbb{Z}^d}$ of i.i.d. random variables independent of $(\bar{\Delta}_i)_{i \in \mathbb{Z}^d}$ such that τ_0 has the standard normal law $\mathcal{N}(0, 1)$. We introduce the fields Γ and γ defined for any i in \mathbb{Z}^d by

$$\Gamma_i = \frac{b_n^{\beta+\frac{1}{2}} \bar{\Delta}_i}{|\Lambda_n|^{1/2}} \quad \text{and} \quad \gamma_i = \frac{\tau_i \sqrt{\eta}}{|\Lambda_n|^{1/2}}.$$

Note that Γ is an M_n -dependent random field where $M_n = 2m_n + 1$ and m_n is defined by (16). Keeping in mind the notations introduced in the proof of Theorem 1, it suffices to prove that for any function h in $B_1^4(\mathbb{R})$,

$$\mathbb{E} \left(h \left(S_{\varphi(|\Lambda_n|)}(\Gamma) \right) \right) \xrightarrow{n \rightarrow +\infty} \mathbb{E} \left(h \left(\tau_0 \sqrt{\eta} \right) \right).$$

Applying Lindeberg's decomposition, we have

$$\mathbb{E} \left(h \left(S_{\varphi(|\Lambda_n|)}(\Gamma) \right) - h \left(\tau_0 \sqrt{\eta} \right) \right) = \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(h_{k,k+1} - h_{k-1,k} \right).$$

Now,

$$h_{k,k+1} - h_{k-1,k} = h_{k,k+1} - h_{k-1,k+1} + h_{k-1,k+1} - h_{k-1,k}.$$

By Taylor's formula we get

$$h_{k,k+1} - h_{k-1,k+1} = \Gamma_{\varphi(k)} h'_{k-1,k+1} + \frac{1}{2} \Gamma_{\varphi(k)}^2 h''_{k-1,k+1} + R_k$$

and

$$h_{k-1,k+1} - h_{k-1,k} = -\gamma_{\varphi(k)} h'_{k-1,k+1} - \frac{1}{2} \gamma_{\varphi(k)}^2 h''_{k-1,k+1} + r_k$$

where $|R_k| \leq \Gamma_{\varphi(k)}^2 (1 \wedge |\Gamma_{\varphi(k)}|)$ and $|r_k| \leq \gamma_{\varphi(k)}^2 (1 \wedge |\gamma_{\varphi(k)}|)$. Since $(\Gamma, \tau_i)_{i \neq \varphi(k)}$ is independent of $\tau_{\varphi(k)}$, it follows that

$$\mathbb{E} \left(\gamma_{\varphi(k)} h'_{k-1,k+1} \right) = 0 \quad \text{and} \quad \mathbb{E} \left(\gamma_{\varphi(k)}^2 h''_{k-1,k+1} \right) = \mathbb{E} \left(\frac{\eta}{|\Lambda_n|} h''_{k-1,k+1} \right)$$

Hence, we obtain

$$\begin{aligned} \mathbb{E} \left(h \left(S_{\varphi(|\Lambda_n|)}(\Gamma) \right) - h \left(\tau_0 \sqrt{\eta} \right) \right) &= \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1} \right) \\ &\quad + \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\left(\Gamma_{\varphi(k)}^2 - \frac{\eta}{|\Lambda_n|} \right) \frac{h''_{k-1,k+1}}{2} \right) \\ &\quad + \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(R_k + r_k \right). \end{aligned}$$

Let $1 \leq k \leq |\Lambda_n|$ be fixed. Noting that $|\overline{\Delta}_0|$ is bounded by $\kappa b_n^{-\beta-1}$ and applying Lemma 7, we derive

$$\mathbb{E} |R_k| \leq \frac{b_n^{3\beta+\frac{3}{2}} \mathbb{E} |\overline{\Delta}_0|^3}{|\Lambda_n|^{3/2}} = O \left(\frac{1}{(|\Lambda_n| b_n)^{1/2}} \right)$$

and

$$\mathbb{E} |r_k| \leq \frac{\mathbb{E} |\gamma_0|^3}{|\Lambda_n|^{3/2}} \leq \frac{\eta^{3/2} \mathbb{E} |\tau_0|^3}{|\Lambda_n|^{3/2}} = O \left(\frac{1}{|\Lambda_n|^{3/2}} \right).$$

Consequently, we obtain

$$\sum_{k=1}^{|\Lambda_n|} \mathbb{E} (|R_k| + |r_k|) = O \left(\frac{1}{(|\Lambda_n| b_n)^{1/2}} + \frac{1}{|\Lambda_n|^{1/2}} \right) = o(1).$$

Now, it is sufficient to show

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \left(\mathbb{E}(\Gamma_{\varphi(k)} h'_{k-1,k+1}) + \mathbb{E} \left(\left(\Gamma_{\varphi(k)}^2 - \frac{\eta}{|\Lambda_n|} \right) \frac{h''_{k-1,k+1}}{2} \right) \right) = 0. \quad (19)$$

We focus on $\sum_{k=1}^{|\Lambda_n|} \mathbb{E}(\Gamma_{\varphi(k)} h'_{k-1,k+1})$. Recall that the sets $\{V_i^k; i \in \mathbb{Z}^d, k \in \mathbb{N}^*\}$ are defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d; j <_{\text{lex}} i\} \text{ and for } k \geq 2, V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d; |i - j| \geq k\}.$$

For all n in \mathbb{N}^* and all integer k in $[1, |\Lambda_n|]$, we define

$$E_k^{M_n} = \varphi([1, k] \cap \mathbb{N}^*) \cap V_{\varphi(k)}^{M_n} \quad \text{and} \quad S_{\varphi(k)}^{M_n}(\Gamma) = \sum_{i \in E_k^{M_n}} \Gamma_i$$

where $M_n = 2m_n + 1$. For any function Ψ from \mathbb{R} to \mathbb{R} , we define $\Psi_{k-1,l}^{M_n} = \Psi(S_{\varphi(k)}^{M_n}(\Gamma) + S_{\varphi(l)}^c(\gamma))$. Our aim is to show that

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1} - \Gamma_{\varphi(k)} \left(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{M_n}(\Gamma) \right) h''_{k-1,k+1} \right) = 0. \quad (20)$$

First, we note that

$$\Gamma_{\varphi(k)} h'_{k-1,k+1} = \Gamma_{\varphi(k)} h'_{k-1,k+1}^{M_n} + \Gamma_{\varphi(k)} \left(h'_{k-1,k+1} - h'_{k-1,k+1}^{M_n} \right).$$

Applying again Taylor's formula,

$$\Gamma_{\varphi(k)} \left(h'_{k-1,k+1} - h'_{k-1,k+1}^{M_n} \right) = \Gamma_{\varphi(k)} \left(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{M_n}(\Gamma) \right) h''_{k-1,k+1} + R'_k,$$

where

$$|R'_k| \leq 2 |\Gamma_{\varphi(k)} \left(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{M_n}(\Gamma) \right)| (1 \wedge |S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{M_n}(\Gamma)|).$$

Since $(\Gamma_i)_{i \in \mathbb{Z}^d}$ is M_n -dependent, we have $\mathbb{E} \left(\Gamma_{\varphi(k)} h'_{k-1,k+1}^{M_n} \right) = 0$ and consequently (20) holds if and only if $\lim_{n \rightarrow +\infty} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| = 0$. In fact, denoting $W_n = \{-M_n + 1, \dots, M_n - 1\}^d$ and $W_n^* = W_n \setminus \{0\}$, it follows that

$$\begin{aligned} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| &\leq 2b_n^{2\beta+1} \mathbb{E} \left(|\bar{\Delta}_0| \left(\sum_{i \in W_n} |\bar{\Delta}_i| \right) \left(1 \wedge \frac{b_n^{\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\bar{\Delta}_i| \right) \right) \\ &= 2b_n^{2\beta+1} \mathbb{E} \left(\left(\bar{\Delta}_0^2 + \sum_{i \in W_n^*} |\bar{\Delta}_0 \bar{\Delta}_i| \right) \left(1 \wedge \frac{b_n^{\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} |\bar{\Delta}_i| \right) \right) \\ &\leq \frac{2b_n^{3\beta+\frac{3}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} \mathbb{E}(\bar{\Delta}_0^2 |\bar{\Delta}_i|) + 2b_n^{2\beta+1} \sum_{i \in W_n^*} \mathbb{E}|\bar{\Delta}_0 \bar{\Delta}_i|. \end{aligned}$$

Since $|\overline{\Delta}_0|$ is bounded by $\kappa b_n^{-\beta-1}$, we derive

$$\begin{aligned} \sum_{k=1}^{|\Lambda_n|} \mathbb{E}|R'_k| &\leq \frac{\kappa b_n^{2\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{i \in W_n} \mathbb{E}(|\overline{\Delta}_0 \overline{\Delta}_i|) + 2b_n^{2\beta+1} \sum_{i \in W_n^*} \mathbb{E}|\overline{\Delta}_0 \overline{\Delta}_i| \\ &\leq \frac{\kappa b_n^{2\beta+1} \mathbb{E}(\overline{\Delta}_0^2)}{(|\Lambda_n| b_n)^{1/2}} + \kappa b_n^{2\beta+1} \left(1 + \frac{1}{(|\Lambda_n| b_n)^{1/2}}\right) m_n^d \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}(|\overline{\Delta}_0 \overline{\Delta}_i|) \\ &= o(1) \quad (\text{by Lemma 7 and Assumption (A5)}). \end{aligned}$$

In order to obtain (19) it remains to control

$$F_0 = \mathbb{E} \left(\sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} \left(\frac{\Gamma_{\varphi(k)}^2}{2} + \Gamma_{\varphi(k)} \left(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{M_n}(\Gamma) \right) - \frac{\eta}{2|\Lambda_n|} \right) \right).$$

Denote by \mathcal{B} the Borel σ -algebra of \mathbb{R} . Without loss of generality, one can suppose that $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d}, \mu)$ and $\overline{\Delta}_k = \pi_0 \circ T^k$ where μ is the law of the stationary real random field $(\overline{\Delta}_k)_{k \in \mathbb{Z}^d}$, π_0 is the projection from $\mathbb{R}^{\mathbb{Z}^d}$ to \mathbb{R} defined by $\pi_0(\omega) = \omega_0$ and $(T^k)_{k \in \mathbb{Z}^d}$ is the family of translation operators from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$ defined by $(T^k(\omega))_i = \omega_{i+k}$ for any $k \in \mathbb{Z}^d$ and any ω in $\mathbb{R}^{\mathbb{Z}^d}$. Recall also the metric ρ defined for any finite subsets B_1 and B_2 of \mathbb{Z}^d by $\rho(B_1, B_2) = \min\{|i-j|; i \in B_1, j \in B_2\}$ and $|i-j| = \max_{1 \leq k \leq d} |i_k - j_k|$ for any $i = (i_1, \dots, i_d)$ and $j = (j_1, \dots, j_d)$ in \mathbb{Z}^d . We consider the following sets:

$$\Lambda_n^{M_n} = \{i \in \Lambda_n; \rho(\{i\}, \partial\Lambda_n) \geq M_n\} \quad \text{and} \quad \mathbb{I}_n^{M_n} = \{1 \leq k \leq |\Lambda_n|; \varphi(k) \in \Lambda_n^{M_n}\},$$

and the function Ψ from $\mathbb{R}^{\mathbb{Z}^d}$ to \mathbb{R} such that

$$\Psi(\overline{\Delta}) = \overline{\Delta}_0^2 + \sum_{i \in V_0^1 \cap W_n} 2\overline{\Delta}_0 \overline{\Delta}_i.$$

For $1 \leq k \leq |\Lambda_n|$, we set $D_k^{(n)} = \eta - b_n^{2\beta+1} \Psi \circ T^{\varphi(k)}(\overline{\Delta})$. By definition of Ψ and of the set $\mathbb{I}_n^{M_n}$, we have for any k in $\mathbb{I}_n^{M_n}$, $\Psi \circ T^{\varphi(k)}(\overline{\Delta}) = \overline{\Delta}_{\varphi(k)}^2 + 2\overline{\Delta}_{\varphi(k)}(S_{\varphi(k-1)}(\overline{\Delta}) - S_{\varphi(k)}^{M_n}(\overline{\Delta}))$. Therefore for k in $\mathbb{I}_n^{M_n}$,

$$\frac{D_k^{(n)}}{|\Lambda_n|} = \frac{\eta}{|\Lambda_n|} - \Gamma_{\varphi(k)}^2 - 2\Gamma_{\varphi(k)}(S_{\varphi(k-1)}(\Gamma) - S_{\varphi(k)}^{M_n}(\Gamma)).$$

Since (1) holds, we have $\lim_{n \rightarrow +\infty} |\Lambda_n|^{-1} |\mathbb{I}_n^{M_n}| = 1$. So, it remains to consider

$$F_1 = \left| \mathbb{E} \left(\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1, k+1} D_k^{(n)} \right) \right|.$$

Applying Lemma 7, we have

$$\begin{aligned}
F_1 &\leq \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1}(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2)) \right) \right| + |\eta - b_n^{2\beta+1} \mathbb{E}(\overline{\Delta}_0^2)| + 2b_n^{2\beta+1} \sum_{j \in V_0^1 \cap W_n} \mathbb{E}|\overline{\Delta}_0 \overline{\Delta}_j| \\
&\leq \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1}(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2)) \right) \right| + o(1).
\end{aligned}$$

So, it suffices to prove that

$$F_2 = \left| \mathbb{E} \left(\frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} h''_{k-1,k+1}(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2)) \right) \right|$$

goes to zero as n goes to infinity. In fact, $F_2 \leq \frac{b_n^{2\beta+1}}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} (J_k^1(n) + J_k^2(n))$ where $J_k^1(n) = \left| \mathbb{E} \left(h''_{k-1,k+1}(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2)) \right) \right| = 0$ since $h''_{k-1,k+1}$ is $\sigma(\overline{\Delta}_i; i \in V_{\varphi(k)}^{M_n})$ -measurable and the conditional expectation of $\overline{\Delta}_{\varphi(k)}^2$ with respect to $\sigma(\overline{\Delta}_i; i \in V_{\varphi(k)}^{M_n})$ is null (cf. $(\overline{\Delta}_i)_{i \in \mathbb{Z}^d}$ is an M_n -dependent random field).

$$\begin{aligned}
b_n^{2\beta+1} J_k^2(n) &= b_n^{2\beta+1} \left| \mathbb{E} \left(\left(h''_{k-1,k+1} - h''_{k-1,k+1}^{M_n} \right) \left(\overline{\Delta}_{\varphi(k)}^2 - \mathbb{E}(\overline{\Delta}_0^2) \right) \right) \right| \\
&\leq b_n^{2\beta+1} \mathbb{E} \left(\left(2 \wedge \sum_{|i| < M_n} \frac{b_n^{\beta+\frac{1}{2}} |\overline{\Delta}_i|}{|\Lambda_n|^{1/2}} \right) \overline{\Delta}_0^2 \right) \\
&\leq \frac{\kappa b_n^{2\beta+1} \mathbb{E}(\overline{\Delta}_0^2)}{(|\Lambda_n| b_n)^{1/2}} + \frac{\kappa b_n^{2\beta+\frac{1}{2}}}{|\Lambda_n|^{1/2}} \sum_{\substack{|i| < M_n \\ i \neq 0}} \mathbb{E}|\overline{\Delta}_0 \overline{\Delta}_i| \quad \text{since } |\overline{\Delta}_0| \leq \kappa b_n^{-\beta-1} \text{ a.s.} \\
&\leq \frac{\kappa \left(b_n^{2\beta+1} \mathbb{E}(\overline{\Delta}_0^2) + b_n^{2\beta+1} m_n^d \sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}(|\overline{\Delta}_0 \overline{\Delta}_i|) \right)}{(|\Lambda_n| b_n)^{1/2}} \\
&= o(1) \quad (\text{by Lemma 7 and Assumption (A5)}).
\end{aligned}$$

The proof of Theorem 2 is complete.

3 Appendix

Proof of Lemma 1. Let τ be fixed in $\mathbb{N}^* \cup \{\infty\}$ and let $m_{n,\tau}$ be defined by equation (1). Since b_n goes to zero and $m_{n,\tau} \geq v_n = \lfloor b_n^{-\frac{1}{2d}} \rfloor$ where $\lfloor \cdot \rfloor$ is the integer part function, we obtain $m_{n,\tau}$ goes to infinity as n goes to infinity. For any positive integer m , we

consider

$$r(m) = \sum_{|i|>m} |i|^d \alpha_{1,\tau}(|i|).$$

Since $\sum_{m \geq 1} m^{2d-1} \alpha_{1,\tau}(m) < \infty$, we have $r(m)$ converges to zero as m goes to infinity. Moreover,

$$m_{n,\tau}^d b_n \leq \max \left\{ \sqrt{b_n}, \kappa \left(\sqrt{r(v_n)} + b_n \right) \right\} \xrightarrow{n \rightarrow +\infty} 0$$

We have also

$$m_{n,\tau}^d \geq \frac{1}{b_n} \sqrt{r(v_n)} \geq \frac{1}{b_n} \sqrt{r(m_{n,\tau})} \quad \text{since } v_n \leq m_{n,\tau}.$$

Finally, we obtain

$$\frac{1}{m_{n,\tau}^d b_n} \sum_{|i|>m_{n,\tau}} |i|^d \alpha_{1,\tau}(|i|) \leq \sqrt{r(m_{n,\tau})} \xrightarrow{n \rightarrow +\infty} 0.$$

The proof of Lemma 1 is complete.

Proof of Lemma 2. Using Assumption **(A3)**, for any t in \mathbb{R} , there exists a sequence $(r_n(t))_{n \geq 1}$ going to 1 as n goes to infinity such that $\phi_\theta(t/b_n) = Br_n(t)b_n^\beta/|t|^\beta$. Consequently,

$$\begin{aligned} \int_{\mathbb{R}} g_n^2(u) du &= \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} e^{-itu} \frac{\phi_K(t)}{\phi_\theta(t/b_n)} dt \right)^2 du \\ &= \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} e^{-itu} \frac{|t|^\beta \phi_K(t)}{Br_n(t)b_n^\beta} dt \right)^2 du \\ &= \frac{1}{B^2 b_n^{2\beta}} \int_{\mathbb{R}} \frac{|t|^{2\beta} \phi_K^2(t)}{r_n^2(t)} dt \quad (\text{By Plancherel's theorem}) \end{aligned}$$

So, we obtain

$$\lim_{n \rightarrow \infty} b_n^{2\beta} \int_{\mathbb{R}} g_n^2(u) du = \frac{1}{B^2} \int_{\mathbb{R}} |t|^{2\beta} \phi_K^2(t) dt \quad (21)$$

Similarly,

$$\lim_{n \rightarrow \infty} b_n^\beta \int_{\mathbb{R}} |g_n(u)| du = \frac{1}{B} \int_{\mathbb{R}} |t|^\beta |\phi_K(t)| dt \quad (22)$$

For any z in \mathbb{R} and any i in \mathbb{Z}^d , we recall that $G_n(z, i) := g_n\left(\frac{z-Y_i}{b_n}\right)$. Using (22) and keeping in mind Assumption **(A1)**, we have

$$b_n^{\beta-1} \mathbb{E} |G_n(z, i)| = b_n^\beta \int_{\mathbb{R}} |g_n(u)| f_Y(z - ub_n) du \xrightarrow{n \rightarrow +\infty} \frac{f_Y(z)}{B} \int_{\mathbb{R}} |t|^\beta |\phi_K(t)| dt. \quad (23)$$

Similarly, using (21), we derive

$$b_n^{2\beta-1} \mathbb{E} G_n^2(z, i) = b_n^{2\beta} \int_{\mathbb{R}} g_n^2(u) f_Y(z - ub_n) du \xrightarrow{n \rightarrow +\infty} \frac{f_Y(z)}{B^2} \int_{\mathbb{R}} |t|^{2\beta} \phi_K^2(t) dt. \quad (24)$$

In the other part, for any s in \mathbb{R} ,

$$b_n^{2\beta+1} \mathbb{E}(Z_0^2(s)) = b_n^{2\beta-1} (\mathbb{E} G_n^2(s, 0) - (\mathbb{E} G_n(s, 0))^2). \quad (25)$$

Combining (23), (24) and (25), we derive for any s in \mathbb{R} ,

$$\lim_{n \rightarrow \infty} b_n^{2\beta+1} \mathbb{E}(Z_0^2(s)) = \frac{f_Y(s)}{B^2} \int_{\mathbb{R}} |t|^{2\beta} \phi_K^2(t) dt. \quad (26)$$

Now, let s and t be fixed in \mathbb{R} and let i be fixed in $\mathbb{Z}^d \setminus \{0\}$. Then,

$$b_n^2 \mathbb{E}|Z_0(s)Z_i(t)| \leq \mathbb{E}|G_n(s, 0)G_n(t, i)| + 3\mathbb{E}|G_n(s, 0)| \mathbb{E}|G_n(t, 0)| \quad (27)$$

and

$$\begin{aligned} \mathbb{E}|G_n(s, 0)G_n(t, i)| &\leq b_n^2 \iint |g_n(u)g_n(v)| f_Y(s - ub_n) |f_{i|0}(s - ub_n, t - vb_n) - f_Y(t - vb_n)| dudv \\ &\quad + b_n^2 \int |g_n(u)| f_Y(s - ub_n) du \int |g_n(v)| f_Y(t - vb_n) dv \end{aligned}$$

where $f_{i|0}$ is the conditional density of Y_i given Y_0 . Using (23) and Assumption **(A2)**, we obtain

$$\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|G_n(s, 0)G_n(t, i)| \leq \kappa b_n^{2-2\beta}. \quad (28)$$

Combining (23), (27) and (28), we derive

$$\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|Z_0(s)Z_i(t)| = O(b_n^{-2\beta})$$

The proof of Lemma 2 is complete.

Proof of Lemma 3. For any z in \mathbb{R} and any i in \mathbb{Z}^d , recall that $G_n(z, i) = g_n\left(\frac{z - Y_i}{b_n}\right)$. We have

$$\mathbb{E}(\Delta_0^2) = \lambda_1^2 \mathbb{E}(Z_0^2(x)) + \lambda_2^2 \mathbb{E}(Z_0^2(y)) + 2\lambda_1 \lambda_2 \mathbb{E}(Z_0(x)Z_0(y)) \quad (29)$$

and

$$b_n^2 \mathbb{E}(Z_0(x)Z_0(y)) = \mathbb{E}(G_n(x, 0)G_n(y, 0)) - \mathbb{E}(G_n(x, 0)) \mathbb{E}(G_n(y, 0)). \quad (30)$$

Moreover,

$$\mathbb{E}(G_n(x, 0)G_n(y, 0)) = b_n \int_{\mathbb{R}} g_n(u)g_n\left(u + \frac{y-x}{b_n}\right) f_Y(x - ub_n) du.$$

Keeping in mind Assumption **(A4)** and $x \neq y$, we have

$$\left| g_n \left(u + \frac{y-x}{b_n} \right) \right| \leq \frac{1}{2\pi |u + (y-x)/b_n|} \int_{\mathbb{R}} \left| \frac{\Phi_K \left(\frac{t}{u+(y-x)/b_n} \right)}{\Phi_\theta \left(\frac{t}{y-x+ub_n} \right)} \right| dt = O(b_n).$$

So, using (23), we derive

$$|\mathbb{E}(G_n(x, 0)G_n(y, 0))| \leq \kappa b_n^2 \int_{\mathbb{R}} |g_n(u)| f_Y(x - ub_n) du = O(b_n^{2-\beta}). \quad (31)$$

Combining (23), (30) and (31) and applying Lemma 1, we derive

$$|\mathbb{E}(Z_0(x)Z_0(y))| = O(b_n^{-2\beta}). \quad (32)$$

Combining (26), (29) and (32), we obtain $\lim_{n \rightarrow +\infty} b_n^{2\beta+1} \mathbb{E}(\Delta_0^2) = \eta$. Let i in $\mathbb{Z}^d \setminus \{0\}$ be fixed. Noting that

$$\mathbb{E}|\Delta_0 \Delta_i| \leq \lambda_1^2 \mathbb{E}|Z_0(x)Z_i(x)| + 2\lambda_1 \lambda_2 \mathbb{E}|Z_0(x)Z_i(y)| + \lambda_2^2 \mathbb{E}|Z_0(y)Z_i(y)|$$

and applying the second part of Lemma 2, we obtain $\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\Delta_0 \Delta_i| = O(b_n^{-2\beta})$. The proof of Lemma 3 is complete.

Proof of Lemma 6. We follow the proof of Proposition 1 in [6]. For any i in \mathbb{Z}^d and any x in \mathbb{R} , we denote $R_i = G_n(x, i) - \overline{G}_n(x, i)$. Since there exists a measurable function H such that $R_i = H(\varepsilon_{i-s}; s \in \mathbb{Z}^d)$, we are able to define the physical dependence measure coefficients $(\delta_{i,p}^{(n)})_{i \in \mathbb{Z}^d}$ associated to the random field $(R_i)_{i \in \mathbb{Z}^d}$. We recall that $\delta_{i,p}^{(n)} = \|R_i - R_i^*\|_p$ where $R_i^* = H(\varepsilon_{i-s}^*; s \in \mathbb{Z}^d)$ and $\varepsilon_j^* = \varepsilon_j \mathbf{1}_{\{j \neq 0\}} + \varepsilon'_0 \mathbf{1}_{\{j=0\}}$ for any j in \mathbb{Z}^d . In other words, we obtain R_i^* from R_i by just replacing ε_0 by its copy ε'_0 . Let $\tau : \mathbb{Z} \rightarrow \mathbb{Z}^d$ be a bijection. For any $l \in \mathbb{Z}$, for any $i \in \mathbb{Z}^d$, we denote

$$P_l R_i = \mathbb{E}(R_i | \mathcal{F}_l) - \mathbb{E}(R_i | \mathcal{F}_{l-1}) \quad (33)$$

where $\mathcal{F}_l = \sigma(\varepsilon_{\tau(s)}; s \leq l)$.

Lemma 8 *For any l in \mathbb{Z} and any i in \mathbb{Z}^d , we have $\|P_l R_i\|_p \leq \delta_{i-\tau(l), p}^{(n)}$.*

Proof of Lemma 8. Let l in \mathbb{Z} and i in \mathbb{Z}^d be fixed.

$$\|P_l R_i\|_p = \|\mathbb{E}(R_i | \mathcal{F}_l) - \mathbb{E}(R_i | \mathcal{F}_{l-1})\|_p = \|\mathbb{E}(R_0 | T^i \mathcal{F}_l) - \mathbb{E}(R_0 | T^i \mathcal{F}_{l-1})\|_p$$

where $T^i \mathcal{F}_l = \sigma(\varepsilon_{\tau(s)-i}; s \leq l)$.

$$\begin{aligned}
\|P_l R_i\|_p &= \left\| \mathbb{E} \left(H((\varepsilon_{-s})_{s \in \mathbb{Z}^d}) | T^i \mathcal{F}_l \right) - \mathbb{E} \left(H((\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{i-\tau(l)\}}; \varepsilon'_{\tau(l)-i}) | T^i \mathcal{F}_l \right) \right\|_p \\
&\leq \left\| H((\varepsilon_{-s})_{s \in \mathbb{Z}^d}) - H((\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{i-\tau(l)\}}; \varepsilon'_{\tau(l)-i}) \right\|_p \\
&= \left\| H((\varepsilon_{i-\tau(l)-s})_{s \in \mathbb{Z}^d}) - H((\varepsilon_{i-\tau(l)-s})_{s \in \mathbb{Z}^d \setminus \{i-\tau(l)\}}; \varepsilon'_0) \right\|_p \\
&= \|R_{i-\tau(l)} - R_{i-\tau(l)}^*\|_p \\
&= \delta_{i-\tau(l), p}^{(n)}.
\end{aligned}$$

The proof of Lemma 8 is complete.

For all i in \mathbb{Z}^d , $R_i = \sum_{l \in \mathbb{Z}} P_l R_i$. Consequently, $\|\sum_{i \in \Lambda_n} a_i R_i\|_p = \|\sum_{l \in \mathbb{Z}} \sum_{i \in \Lambda_n} a_i P_l R_i\|_p$. Applying the Burkholder inequality (cf. [9], page ??) for the martingale difference sequence $(\sum_{i \in \Lambda_n} a_i P_l R_i)_{l \in \mathbb{Z}}$, we obtain

$$\left\| \sum_{i \in \Lambda_n} a_i R_i \right\|_p \leq \left(2p \sum_{l \in \mathbb{Z}} \left\| \sum_{i \in \Lambda_n} a_i P_l R_i \right\|_p^2 \right)^{\frac{1}{2}} \leq \left(2p \sum_{l \in \mathbb{Z}} \left(\sum_{i \in \Lambda_n} |a_i| \|P_l R_i\|_p \right)^2 \right)^{\frac{1}{2}}$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i \in \Lambda_n} |a_i| \|P_l R_i\|_p \right)^2 \leq \sum_{i \in \Lambda_n} a_i^2 \|P_l R_i\|_p \times \sum_{i \in \Lambda_n} \|P_l R_i\|_p$$

and by Lemma 8, $\sum_{i \in \mathbb{Z}^d} \|P_l R_i\|_p \leq \sum_{j \in \mathbb{Z}^d} \delta_{j,p}^{(n)}$. So, we derive

$$\left\| \sum_{i \in \Lambda_n} a_i R_i \right\|_p \leq \left(2p \sum_{j \in \mathbb{Z}^d} \delta_{j,p}^{(n)} \sum_{i \in \Lambda_n} a_i^2 \sum_{l \in \mathbb{Z}} \|P_l R_i\|_p \right)^{\frac{1}{2}}.$$

Applying again Lemma 8, we have $\sum_{l \in \mathbb{Z}} \|P_l R_i\|_p \leq \sum_{j \in \mathbb{Z}^d} \delta_{j,p}^{(n)}$ for any i in \mathbb{Z}^d and finally, we derive

$$\left\| \sum_{i \in \Lambda_n} a_i R_i \right\|_p \leq \left(2p \sum_{i \in \Lambda_n} a_i^2 \right)^{\frac{1}{2}} \sum_{i \in \mathbb{Z}^d} \delta_{i,p}^{(n)}.$$

Since $\overline{G}_n^*(x, i) = \mathbb{E}(G_n^*(x, i) | \mathcal{F}_{n,i}^*)$ where $\mathcal{F}_{n,i}^* = \sigma(\varepsilon_{i-s}^*; |s| \leq m_n)$ and $(G_n(x, i) - \overline{G}_n(x, i))^* = G_n^*(x, i) - \overline{G}_n^*(x, i)$, we derive $\delta_{i,p}^{(n)} \leq 2\|G_n(x, i) - G_n^*(x, i)\|_p$. Moreover, for any s and t in \mathbb{R} , $|g_n(s) - g_n(t)| \leq \kappa b_n^{-\beta} |s - t|$. So, we obtain

$$\delta_{i,p}^{(n)} \leq \kappa b_n^{-1-\beta} \delta_{i,p} \tag{34}$$

where $\delta_{i,p} = \|X_i - X_i^*\|_p$.

Lemma 9 For any $p \geq 2$, any positive integer n and any x in \mathbb{R} ,

$$\|G_n(x, 0) - \overline{G}_n(x, 0)\|_p \leq \frac{\kappa\sqrt{p}}{b_n^{1+\beta}} \sum_{|j|>m_n} \delta_{j,p}.$$

Proof of Lemma 9. We consider the sequence $(B_n)_{n \geq 0}$ of finite subsets of \mathbb{Z}^d defined by $B_0 = \{(0, \dots, 0)\}$ and for any n in \mathbb{N}^* , $B_n = \{i \in \mathbb{Z}^d; |i| = n\}$. The cardinality of the set B_n is $|B_n| = 2d(2n+1)^{d-1}$ for $n \geq 1$. Let $\tau: \mathbb{N}^* \rightarrow \mathbb{Z}^d$ be the bijection defined by $\tau(1) = (0, \dots, 0)$ and

- for any n in \mathbb{N}^* , if $l \in]a_{n-1}, a_n]$ then $\tau(l) \in B_n$,
- for any n in \mathbb{N}^* , if $(p, q) \in]a_{n-1}, a_n]^2$ and $p < q$ then $\tau(p) <_{\text{lex}} \tau(q)$

where $a_n = \sum_{j=0}^n |B_j|$ goes to infinity as n goes to infinity. Let $(m_n)_{n \geq 1}$ be the sequence of positive integers defined by (16). For any n in \mathbb{N}^* , we recall that $\mathcal{F}_{n,0} = \sigma(\varepsilon_{-s}; |s| \leq m_n)$ and we consider also the σ -algebra $\mathcal{G}_n := \sigma(\varepsilon_{\tau(p)}; 1 \leq p \leq n)$. By the definition of the bijection τ , for any n in \mathbb{N} , $1 \leq p \leq a_n$ if and only if $|\tau(p)| \leq n$. So, we have $\mathcal{G}_{a_{m_n}} = \mathcal{F}_{n,0}$. Consequently, $G_n(x, 0) - \overline{G}_n(x, 0) = \sum_{l>a_{m_n}} D_l$ with $D_l = \mathbb{E}(G_n(x, 0)|\mathcal{G}_l) - \mathbb{E}(G_n(x, 0)|\mathcal{G}_{l-1})$ for any l in \mathbb{Z} . Let $p \geq 2$ be fixed. Since $(D_l)_{l \in \mathbb{Z}}$ is a martingale-difference sequence, applying Burkholder's inequality (cf. [9], page ??), we derive

$$\|G_n(x, 0) - \overline{G}_n(x, 0)\|_p \leq \left(2p \sum_{l>a_{m_n}} \|D_l\|_p^2 \right)^{1/2}.$$

Denoting $G'_n(x, 0) = g_n \left(b_n^{-1} \left(x - F \left((\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{-\tau(l)\}}; \varepsilon'_{\tau(l)} \right) - \theta_0 \right) \right)$, we obtain

$$\|D_l\|_p = \|\mathbb{E}(G_n(x, 0)|\mathcal{G}_l) - \mathbb{E}(G'_n(x, 0)|\mathcal{G}_l)\|_p \leq \|G_n(x, 0) - G'_n(x, 0)\|_p.$$

Keeping in mind that $|g_n(s) - g_n(t)| \leq \kappa b_n^{-\beta} |s - t|$ for any s and t in \mathbb{R} , we derive

$$\begin{aligned} \|D_l\|_p &\leq \kappa b_n^{-1-\beta} \left\| F \left((\varepsilon_{-s})_{s \in \mathbb{Z}^d} \right) - F \left((\varepsilon_{-s})_{s \in \mathbb{Z}^d \setminus \{-\tau(l)\}}; \varepsilon'_{\tau(l)} \right) \right\|_p \\ &= \kappa b_n^{-1-\beta} \left\| F \left((\varepsilon_{-\tau(l)-s})_{s \in \mathbb{Z}^d} \right) - F \left((\varepsilon_{-\tau(l)-s})_{s \in \mathbb{Z}^d \setminus \{-\tau(l)\}}; \varepsilon'_0 \right) \right\|_p \\ &= \kappa b_n^{-1-\beta} \|X_{-\tau(l)} - X_{-\tau(l)}^*\|_p = \kappa b_n^{-1-\beta} \delta_{-\tau(l), p} \end{aligned}$$

and finally

$$\|G_n(x, 0) - \overline{G}_n(x, 0)\|_p \leq \kappa b_n^{-1-\beta} \left(2p \sum_{l>a_{m_n}} \delta_{-\tau(l), p}^2 \right)^{1/2} \leq \kappa b_n^{-1-\beta} \sqrt{p} \sum_{|j|>m_n} \delta_{j,p}.$$

The proof of Lemma 9 is complete.

Noting that $\delta_{i,p}^{(n)} \leq 2\|G_n(x, 0) - \bar{G}_n(x, 0)\|_p$ and applying Lemma 9, we derive

$$\delta_{i,p}^{(n)} \leq \frac{\kappa\sqrt{p}}{b_n^{1+\beta}} \sum_{|j|>m_n} \delta_{j,p}. \quad (35)$$

Combining (34) and (35), we obtain

$$\sum_{i \in \mathbb{Z}^d} \delta_{i,p}^{(n)} \leq \kappa \left(\frac{m_n^d \sqrt{p}}{b_n^{1+\beta}} \sum_{|j|>m_n} \delta_{j,p} + \frac{1}{b_n^{1+\beta}} \sum_{|j|>m_n} \delta_{j,p} \right).$$

The proof of Lemma 6 is complete.

Proof of Lemma 7. Let s and t be fixed in \mathbb{R} . We have

$$\begin{aligned} |\mathbb{E}(\bar{G}_n(s, 0)\bar{G}_n(t, 0)) - \mathbb{E}(G_n(s, 0)G_n(t, 0))| &\leq \|G_n(s, 0)\|_2 \|G_n(t, 0) - \bar{G}_n(t, 0)\|_2 \\ &\quad + \|\bar{G}_n(t, 0)\|_2 \|G_n(s, 0) - \bar{G}_n(s, 0)\|_2. \end{aligned}$$

Using (24) and applying Lemma 9, we derive

$$|\mathbb{E}(\bar{G}_n(s, 0)\bar{G}_n(t, 0)) - \mathbb{E}(G_n(s, 0)G_n(t, 0))| \leq \frac{\kappa}{b_n^{2\beta+\frac{1}{2}}} \sum_{|j|>m_n} \delta_j.$$

Since $b_n^2 |\mathbb{E}(Z_0(s)Z_0(t)) - \mathbb{E}(\bar{Z}_0(s)\bar{Z}_0(t))| = |\mathbb{E}(G_n(s, 0)G_n(t, 0)) - \mathbb{E}(\bar{G}_n(s, 0)\bar{G}_n(t, 0))|$, we obtain

$$m_n^d b_n^{2\beta+1} |\mathbb{E}(Z_0(s)Z_0(t)) - \mathbb{E}(\bar{Z}_0(s)\bar{Z}_0(t))| \leq \frac{\kappa}{(m_n^d b_n)^{3/2}} \sum_{|j|>m_n} |j|^{\frac{5d}{2}} \delta_j. \quad (36)$$

Combining (26), (36) and Lemma 5, for any z in \mathbb{R} , we obtain

$$b_n^{2\beta+1} \mathbb{E}(\bar{Z}_0^2(z)) \xrightarrow{n \rightarrow +\infty} \sigma^2(z). \quad (37)$$

Similarly, combining (32), (36) and Lemma 5, we derive

$$|\mathbb{E}(\bar{Z}_0(x)\bar{Z}_0(y))| = o(m_n^{-d} b_n^{-2\beta-1}). \quad (38)$$

In the other part, we have

$$\mathbb{E}(\bar{\Delta}_0^2) = \lambda_1^2 \mathbb{E}(\bar{Z}_0^2(x)) + \lambda_2^2 \mathbb{E}(\bar{Z}_0^2(y)) + 2\lambda_1 \lambda_2 \mathbb{E}(\bar{Z}_0(x)\bar{Z}_0(y)). \quad (39)$$

Combining (37), (38) and (39), we obtain the convergence of $b_n^{2\beta+1}\mathbb{E}(\overline{\Delta}_0^2)$ to η as n goes to infinity.

Let $i \neq 0$ be fixed in \mathbb{Z}^d and let s and t be fixed in \mathbb{R} . We have

$$m_n^d b_n^{2\beta+1} \mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| \leq m_n^d b_n^{2\beta-1} (\mathbb{E}|\overline{G}_n(s,0)\overline{G}_n(t,i)| + 3\mathbb{E}|G_n(s,0)|\mathbb{E}|G_n(t,0)|). \quad (40)$$

Since $||\alpha| - |\beta|| \leq |\alpha - \beta|$ for any (α, β) in \mathbb{R}^2 and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\mathbb{E}|\overline{G}_n(s,0)\overline{G}_n(t,i)| - \mathbb{E}|G_n(s,0)G_n(t,i)|| &\leq \|\overline{G}_n(s,0)\|_2 \|\overline{G}_n(t,0) - G_n(t,0)\|_2 \\ &\quad + \|G_n(t,0)\|_2 \|\overline{G}_n(s,0) - G_n(s,0)\|_2 \end{aligned}$$

Using again (24) and applying Lemma 9, we derive

$$m_n^d b_n^{2\beta-1} |\mathbb{E}|\overline{G}_n(s,0)\overline{G}_n(t,i)| - \mathbb{E}|G_n(s,0)G_n(t,i)|| \leq \frac{\kappa}{(m_n^d b_n)^{3/2}} \sum_{|j|>m_n} |j|^{\frac{5d}{2}} \delta_j. \quad (41)$$

Combining (28), (41) and Lemma 5, we obtain

$$\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\overline{G}_n(s,0)\overline{G}_n(t,i)| = o(m_n^{-d} b_n^{-2\beta+1}). \quad (42)$$

Moreover, using (23) and applying again Lemma 5, we have

$$\mathbb{E}|G_n(s,0)|\mathbb{E}|G_n(t,0)| = o(m_n^{-d} b_n^{-2\beta+1}). \quad (43)$$

Combining (40), (42) and (43), we derive

$$\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\overline{Z}_0(s)\overline{Z}_i(t)| = o(m_n^{-d} b_n^{-2\beta-1}).$$

Since $\mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| \leq \lambda_1^2 \mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(x)| + 2\lambda_1\lambda_2 \mathbb{E}|\overline{Z}_0(x)\overline{Z}_i(y)| + \lambda_2^2 \mathbb{E}|\overline{Z}_0(y)\overline{Z}_i(y)|$, we have also $\sup_{i \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E}|\overline{\Delta}_0\overline{\Delta}_i| = o(m_n^{-d} b_n^{-2\beta-1})$. The proof of Lemma 7 is complete.

References

- [1] D. Bosq, Merlevède F., and M. Peligrad. Asymptotic normality for density kernel estimators in discrete and continuous time. *J. Multivariate Anal.*, 68:78–95, 1999.
- [2] John L. Casti. *Nonlinear system theory*, volume 175 of *Mathematics in Science and Engineering*. Academic Press Inc., Orlando, FL, 1985.

- [3] J. Dedecker. A central limit theorem for stationary random fields. *Probab. Theory Relat. Fields*, 110:397–426, 1998.
- [4] M. El Machkouri. Asymptotic normality for the parzen-rosenblatt density estimator for strongly mixing random fields. *Statistical Inference for Stochastic Processes*, 14(1):73–84, 2011.
- [5] M. El Machkouri and R. Stoica. Asymptotic normality of kernel estimates in a regression model for random fields. *J. Nonparametr. Stat.*, 22(8):955–971, 2010.
- [6] M. El Machkouri, D. Volný, and W. B. Wu. A central limit theorem for stationary random fields. arXiv:1109.0838v1, 2011.
- [7] J. Fan. Asymptotic normality for deconvolution kernel density estimators. *Sankhya Ser. A*, 53(1):97–110, 1991.
- [8] J. Fan. On the optimal rates of convergence for nonparametric deconvolution problems. *Ann. Statist.*, 19(3):1257–1272, 1991.
- [9] P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic Press, New York, 1980.
- [10] Z. Li. Asymptotic normality for deconvolution kernel density estimators from random fields. arXiv:0810.3121v1, 2008.
- [11] J. W. Lindeberg. Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift*, 15:211–225, 1922.
- [12] E. Masry. Asymptotic normality for deconvolution estimators of multivariate densities of stationary processes. *J. Multivariate Anal.*, 44(1):47–68, 1993.
- [13] E. Masry. Strong consistency and rates for deconvolution of multivariate densities of stationary processes. *Stochastic Process. Appl.*, 47(1):53–74, 1993.
- [14] D. L. McLeish. A maximal inequality and dependent strong laws. *Ann. Probab.*, 3(5):829–839, 1975.
- [15] E. Rio. Covariance inequalities for strongly mixing processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 29(4):587–597, 1993.

- [16] M. Rosenblatt. A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. USA*, 42:43–47, 1956.
- [17] W. J. Rugh. *Nonlinear system theory*. Johns Hopkins Series in Information Sciences and Systems. Johns Hopkins University Press, Baltimore, Md., 1981.
- [18] L. Stefanski and R. J. Carroll. Deconvoluting kernel density estimators. *Statistics*, 21(2):169–184, 1990.
- [19] L.T. Tran. Kernel density estimation on random fields. *J. Multivariate Anal.*, 34:37–53, 1990.
- [20] W. B. Wu. Nonlinear system theory: another look at dependence. *Proc. Natl. Acad. Sci. USA*, 102(40):14150–14154 (electronic), 2005.
- [21] W. B. Wu. Asymptotic theory for stationary processes. *Statistics and Its Interface*, 0:1–20, 2011.