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Remarks on curve classes on rationally connected varieties

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To Joe Harris, on his 60th birthday

1 Introduction

Let $X$ be a smooth complex projective variety. Define

$$Z^{2i}(X) = \frac{\text{Hdg}^{2i}(X, \mathbb{Z})}{H^{2i}(X, \mathbb{Z})_{\text{alg}}}.$$  (1)

where $\text{Hdg}^{2i}(X, \mathbb{Z})$ is the space of integral Hodge classes on $X$ and $H^{2i}(X, \mathbb{Z})_{\text{alg}}$ is the subgroup of $H^{2i}(X, \mathbb{Z})$ generated by classes of codimension $i$ closed algebraic subsets of $X$.

These groups measure the defect of the Hodge conjecture for integral Hodge classes, hence they are trivial for $i = 0, 1$ and $n = \dim X$, but in general they can be nonzero by [1]. Furthermore they are torsion if the Hodge conjecture for rational Hodge classes on $X$ of degree $2i$ holds. In addition to the previously mentioned case, this happens when $i = n - 1, n = \dim X$, due to the Lefschetz theorem on $(1, 1)$-classes and the hard Lefschetz isomorphism (cf. [23]). We will call classes in $\text{Hdg}^{2n-2}(X, \mathbb{Z})$ “curve classes”, as they are also degree 2 homology classes.

Note that the Kollár counterexamples (cf. [14]) to the integral Hodge conjecture already exist for curve classes (that is degree 4 cohomology classes in this case) on projective threefolds, unlike the Atiyah-Hirzebruch examples which work for degree 4 integral Hodge classes in higher dimension.

It is remarked in [21], [23] that the two groups

$$Z^4(X), \ Z^{2n-2}(X), \ n := \dim X$$

are birational invariants. (For threefolds, this is the same group, but not in higher dimension.) The nontriviality of these birational invariants for rationally connected varieties is asked in [23]. Still more interesting is the nontriviality of these invariants for unirational varieties, having in mind the Lüroth problem (cf. [3], [2], [4]).

Concerning the group $Z^4(X)$, Colliot-Thélène and the author proved in [8], building on the work of Colliot-Thélène and Ojanguren [5], that it can be nonzero for unirational varieties starting from dimension 6. What happens in dimensions 5 and 4 is unknown (the four dimensional case being particularly challenging in our mind), but in dimension 3, there is the following result proved in [22]:

**Theorem 1.1.** (Voisin 2006) Let $X$ be a smooth projective threefold which is either uniruled or Calabi-Yau. Then the group $Z^4(X)$ is equal to 0.

This result, and in particular the Calabi-Yau case, implies that the group $Z^6(X)$ is also 0 for a Fano fourfold $X$ which admits a smooth anticanonical divisor. Indeed, a smooth anticanonical divisor $j : Y \hookrightarrow X$ is a Calabi-Yau threefold, so that we have $Z^4(Y) = 0$ by Theorem 1.1 above. As $H^2(Y, \mathcal{O}_Y)$, every class in $H^2(Y, \mathbb{Z})$ is a Hodge class, and it follows
that $H^4(Y,\mathbb{Z}) = H^4(Y,\mathbb{Z})_{\text{alg}}$. As the Gysin map $j_* : H^4(Y,\mathbb{Z}) \to H^0(X,\mathbb{Z})$ is surjective by the Lefschetz theorem on hyperplane sections, it follows that $H^0(X,\mathbb{Z}) = H^0(X,\mathbb{Z})_{\text{alg}}$, and thus $Z^n(X) = 0$.

In the paper [11], it was proved more generally that if $X$ is any Fano fourfold, the group $Z^n(X)$ is trivial. Similarly, if $X$ is a Fano fivefold on index 2, the group $Z^n(X)$ is trivial. These results have been generalized to higher dimensional Fano manifolds of index $n - 3$ and dimension $\geq 8$ by Enrica Floris [9] who proves the following result:

**Theorem 1.2.** Let $X$ be a Fano manifold over $\mathbb{C}$ of dimension $n \geq 8$ and index $n - 3$. then the group $Z^{2n-2}(X)$ is equal to 0. Equivalently, any integral cohomology class of degree $2n - 2$ on $X$ is algebraic.

The purpose of this note is to provide a number of evidences for the vanishing of the group $Z^{2n-2}(X)$, for any rationally connected variety over $\mathbb{C}$. Note that in this case, since $H^2(X,\mathcal{O}_X) = 0$, the Hodge structure on $H^2(X,\mathbb{Q})$ is trivial, and so is the Hodge structure on $H^2(X,\mathbb{Q})$, so that $Z^{2n-2}(X) = H^{2n-2}(X,\mathbb{Z})/H^{2n-2}(X,\mathbb{Z})_{\text{alg}}$. We will first prove the following two results.

**Proposition 1.3.** The group $Z^{2n-2}(X)$ is locally deformation invariant for rationally connected manifolds $X$.

Let us explain the meaning of the statement. Consider a smooth projective morphism $\pi : X \to B$ between connected quasi-projective complex varieties, with $n$ dimensional fibers. Recall from [15] that if one fiber $X_b := \pi^{-1}(b)$ is rationally connected, so is every fiber. Let us endow everything with the usual topology. Then the sheaf $R^{2n-2}\pi_*\mathbb{Z}$ is locally constant on $B$. On any Euclidean open set $U \subset B$ where this local system is trivial, the group $Z^{2n-2}(X_b), b \in U$ is the finite quotient of the constant group $H^{2n-2}(X_b,\mathbb{Z})$ by its subgroup $H^{2n-2}(X_b,\mathbb{Z})_{\text{alg}}$. To say that $Z^{2n-2}(X_b)$ is locally constant means that on open sets $U$ as above, the subgroup $H^{2n-2}(X_b,\mathbb{Z})_{\text{alg}}$ of the constant group $H^{2n-2}(X_b,\mathbb{Z})$ does not depend on $b$.

It follows from the above result that the vanishing of the group $Z^{2n-2}(X)$ for $X$ a rationally connected manifold reduces to the similar statement for $X$ defined over a number field.

Let us now define an $l$-adic analogue $Z^{2n-2}(X)_l$ of the group $Z^{2n-2}(X)$ (cf. [6], [7]). Let $X$ be a smooth projective variety defined over a field $K$ which in the sequel will be either a finite field or a number field. Let $\overline{K}$ be an algebraic closure of $K$. Any cycle $Z \in CH^i(X_{\overline{K}})$ is defined over a finite extension of $K$. Let $l$ be a prime integer different from $p = \text{char } K$ if $K$ is finite. It follows that the cycle class

$$cl(Z) \in H^{2i}_cl(X_{\overline{K}},\mathbb{Q}_l(s))$$

is invariant under a finite index subgroup of Gal$(\overline{K}/K)$.

Classes satisfying this property are called Tate classes. The Tate conjecture for finite fields asserts the following:

**Conjecture 1.4.** (cf. [18] for a recent account) Let $X$ be smooth and projective over a finite field $K$. The cycle class map gives for any $s$ a surjection

$$cl : CH^{2i}(X_{\overline{K}}) \otimes \mathbb{Q}_l \to H^{2i}(X_{\overline{K}},\mathbb{Q}_l(s))_{\text{Tate}}.$$ 

Note that the cycle class defined on $CH^i(X_{\overline{K}})$ takes in fact values in $H^{2i}(X_{\overline{K}},\mathbb{Z}_l(s))$, and more precisely in the subgroup $H^{2i}(X_{\overline{K}},\mathbb{Z}_l(s))_{\text{Tate}}$ of classes invariant under a finite index subgroup of Gal$(\overline{K}/K)$. We thus get for each $i$ a morphism

$$cl^i : CH^{2i}(X_{\overline{K}}) \otimes \mathbb{Z}_l \to H^{2i}(X_{\overline{K}},\mathbb{Z}_l(i))_{\text{Tate}}.$$ 

We can thus introduce the following variant of the groups $Z^{2i}(X)$:

$$Z^{2i}_{cl}(X)_l := H^{2i}_cl(X_{\overline{K}},\mathbb{Z}_l(i))_{\text{Tate}}/\text{Im } cl^i.$$
An argument similar to the one used for the proof of Proposition 1.3 will lead to the following result:

**Proposition 1.5.** Let $X$ be a smooth rationally connected variety defined over a number field $K$, with ring of integers $\mathcal{O}_K$. Assume given a projective model $\mathcal{X}$ of $X$ over $\text{Spec} \mathcal{O}_K$. Fix a prime integer $l$. Then except for finitely many $p \in \text{Spec} \mathcal{O}_K$, the group $Z_{\text{et}}^{2n-2}(X)_{l}$ is isomorphic to the group $Z_{\text{et}}^{2n-2}(X_{p})$.

In the course of the paper, we will also consider variants $Z_{\text{rat}}^{2n-2}(X)$, resp. $Z_{\text{et, rat}}^{2n-2}(X)_{l}$ of the groups $Z_{\text{et}}^{2n-2}(X)$, resp. $Z_{\text{et}}^{2n-2}(X)_{l}$, obtained by taking the quotient of the group of integral Hodge classes (resp. integral $l$-adic Tate classes) by the subgroup generated by classes of rational curves. This variant is suggested by Kollár’s paper (cf. [16, Question 3, (1)]). By the same arguments, these groups are also deformation and specialization invariants for rationally connected varieties.

Our last result is conditional but it strongly suggests the vanishing of the group $Z^{2n-2}(X)$ for $X$ a smooth rationally connected variety over $\mathbb{C}$. Indeed, we will prove using the main result of [19] and the two propositions above the following consequence of Theorem 1.5:

**Theorem 1.6.** Assume Tate’s conjecture 1.4 holds for degree 2 Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety $X$ over $\mathbb{C}$.

**Thanks.** I thank the organizers of the beautiful conference “A celebration of algebraic geometry” for inviting me there. I also thank Jean-Louis Colliot-Thélène, Olivier Debarre and János Kollár for useful discussions.

It is a pleasure to dedicate this note to Joe Harris, whose influence on the subject of rational curves on algebraic varieties (among other topics!) is invaluable.

## 2 Deformation and specialization invariance

**Proof of Proposition 1.3.** We first observe that, due to the fact that relative Hilbert schemes parameterizing curves in the fibers of $B$ are a countable union of varieties which are projective over $B$, given a simply connected open set $U \subset B$ (in the classical topology of $B$), and a class $\alpha \in \Gamma(U, R^{2n-2}_t \mathbb{Z})$ such that $\alpha_t$ is algebraic for $t \in V$, where $V$ is a smaller nonempty open set $V \subset U$, then $\alpha_t$ is algebraic for any $t \in U$.

To prove the deformation invariance, we just need using the above observation to prove the following:

**Lemma 2.1.** Let $t \in U \subset B$, and let $C \subset X_t$ be a curve and let $[C] \in H^{2n-2}(X_t, \mathbb{Z})$ be its cohomology class. Then the class $[C]_{s}$ is algebraic for $s$ in a neighborhood of $t$ in $U$.

**Proof of Lemma 2.1.** By results of [15], there are rational curves $R_i \subset X_t$ with ample normal bundle which meet $C$ transversally at distinct points, and with arbitrary tangent directions at these points. We can choose an arbitrarily large number $D$ of such curves with generically chosen tangent directions at the attachment points. We then know by [10, §2.1] that the curve $C' = C \cup_{i \in D} R_i$ is smoothable in $X_t$ to a smooth unobstructed curve $C'' \subset X_t$, that is $H^{1}(C'', N_{C''/X_t}) = 0$. This curve $C''$ then deforms with $X_t$ (cf. [12], [13, II.1]) in the sense that the morphism from the deformation of the pair $(C'', X_t)$ to $B$ is smooth, and in particular open. So there is a neighborhood of $V$ of $t$ in $U$ such that for $s \in V$, there is a curve $C''_s \subset X_s$ which is a deformation of $C'' \subset X_t$. The class $[C''_s] = [C'']_s$ is thus algebraic on $X_s$. On the other hand, we have

$$[C''] = [C'] = [C] + \sum_i [R_i].$$

As the $R_i$’s are rational curves with positive normal bundle, they are also unobstructed, so that the classes $[R_i]_s$ also are algebraic on $X_s$ for $s$ in a neighborhood of $t$ in $U$. Thus
Lemma 2.5. The following lemma: smooth $i$ $\mathbb{P}$ is surjective. Since $\text{dim} C$ $\in \mathbb{C},$ the condition that $X_\sigma$ is smooth of dimension $n$ follows from the fact that the incidence set $(C, X)$ is smooth along $n.$

Theorem 2.4. 1) Assume that the sheaves $\mathcal{E} \otimes \mathcal{I}_C$ are generated by global sections for $i = 1, \ldots, k.$ Then if $X_\sigma$ is smooth rationally connected for general $\sigma,$ the group $Z_\sigma^{2n-2}(X_\sigma)$ vanishes for any $\sigma$ such that $X_\sigma$ is smooth of dimension $n.$

2) Under the same assumptions as in 1), assume the curves $C_i \subset X$ are rational. Then if $X_\sigma$ is smooth rationally connected for general $\sigma,$ the group $Z_\sigma^{2n-2}(X_\sigma)$ vanishes for any $\sigma$ such that $X_\sigma$ is smooth of dimension $n.$

Proof. 1) Let $j_\sigma : X_\sigma \to X$ be the inclusion map. Since $n \geq 3$ and $\mathcal{E}$ is ample, by Sommese’s theorem [20], the Gysin map $j_{\sigma*} : H^{2n-2}(X_\sigma, Z) \to H^{2n+2r-2}(X, Z)$ is an isomorphism. It follows that the group $H^{2n-2}(X_\sigma, Z)$ is a constant group. In order to show that $Z^{2n-2}(X_\sigma)$ is trivial, it suffices to show that the classes $(j_{\sigma*})^{-1}([C_i])$ are algebraic on $X_\sigma$ since they generate $H^{2n-2}(X_\sigma, Z).$ Since the $X_\sigma$’s are rationally connected, Theorem 1.3 tells us that it suffices to show that for each $i,$ there exists a $\sigma(i)$ such that $X_{\sigma(i)}$ is smooth $n$-dimensional and that the class $(j_{\sigma(i)*})^{-1}([C_i])$ is algebraic on $X_{\sigma(i)}.$

It clearly suffices to exhibit one smooth $X_{\sigma(i)}$ containing $C_i,$ which follows from the following lemma:

Lemma 2.5. Let $X$ be a variety of dimension $n + r$ with $n \geq 2,$ $C \subset X$ be a smooth curve, $\mathcal{E}$ be a rank $r$ vector bundle on $X$ such that $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global section. Then for a generic $\sigma \in H^0(X, \mathcal{E} \otimes \mathcal{I}_C),$ the zero set $X_\sigma$ is smooth of dimension $n.$

Proof. The fact that $X_\sigma$ is smooth of dimension $n$ away from $C$ is standard and follows from the fact that the incidence set $(\sigma, x) \in \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C)) \times (X \setminus C), \sigma(x) = 0$ is smooth of dimension $n + N,$ where $N := \text{dim} \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C)).$ It thus suffices to check the smoothness along $C$ for generic $\sigma.$

This is checked by observing that since $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global sections, its restriction $\mathcal{E} \otimes N^*_C/X$ is also generated by global sections. This implies that for each point $c \in C,$ the condition that $X_\sigma$ is singular at $c$ defines a codimension $n$ closed algebraic subset $P_c$ of $P := \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C)),$ determined by the condition that $d\sigma c : N^*_C/X, c \to \mathcal{E},$ is not surjective. Since $\text{dim} C = 1,$ the union of the $P_c$’s cannot be equal to $P$ if $n \geq 2.$

This concludes the proof of 1) and the proof of 2) works exactly in the same way.
Let us finish this section with the proof of Proposition 1.5.

**Proof of Proposition 1.5.** Let \( p \in \text{Spec} \mathcal{O}_K \), with residue field \( k(p) \). Assume \( X_p \) is smooth. For \( l \) prime to char \( k(p) \), the (adequately constructed) specialization map

\[
H_{ct}^{2n-2}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l(n-1)) \to H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1))
\]

is then an isomorphism (cf. [17, Chapter VI, §4]).

Observe also that since \( X_{\overline{\mathbb{Q}}} \) is rationally connected, the rational étale cohomology group \( H_{ct}^{2n-2}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(n-1)) \) is generated over \( \mathbb{Q}_l \) by curve classes. Hence the same is true for \( H_{ct}^{2n-2}(X_p, \mathbb{Q}_l(n-1)) \). Thus the whole cohomology groups

\[
H_{ct}^{2n-2}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l(n-1)), \quad H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1))
\]

consist of Tate classes, and (2) gives an isomorphism

\[
H_{ct}^{2n-2}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l(n-1))_{\text{Tate}} \to H_{ct}^{2n-2}(X_p, \mathbb{Z}_l(n-1))_{\text{Tate}}.
\]

In order to prove Proposition 1.5, it thus suffices to prove the following:

**Lemma 2.6.** 1) For almost every \( p \in \text{Spec} \mathcal{O}_K \), the fiber \( X_{\overline{p}} \) is smooth and separably rationally connected.

2) If \( X_{\overline{p}} \) is smooth and separably rationally connected, for any curve \( C_{\overline{p}} \subset X_{\overline{p}} \), the inverse image \( [C_{\overline{p}}]_{X_{\overline{p}}} \in H_{ct}^{2n-2}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l(n-1)) \) of the class \([C_{\overline{p}}]\) in \( H_{ct}^{2n-2}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l(n-1)) \) via the isomorphism (3) is the class of a 1-cycle on \( X_{\overline{p}} \).

**Proof.** 1) When the fiber \( X_p \) is smooth, the separable rational connectedness of \( X_p \) is equivalent to the existence of a smooth rational curve \( C_{\overline{p}} \cong \mathbb{P}^1_{k(p)} \) together with a morphism \( \phi : C_{\overline{p}} \to X_{\overline{p}} \) such that the vector bundle \( \phi^* T_{X_{\overline{p}}} \) on \( \mathbb{P}^1_{k(p)} \) is a direct sum \( \oplus_i \mathcal{O}_{\mathbb{P}^1_{k(p)}}(a_i) \) where all \( a_i \) are positive. Equivalently

\[
H^1(\mathbb{P}^1_{k(p)}, \phi^* T_{X_{\overline{p}}}(-2)) = 0.
\]

The smooth projective variety \( X_{\overline{p}} \) being rationally connected in characteristic 0, it is separably rationally connected, hence there exists a finite extension \( K' \) of \( K \), a curve \( C \) and a morphism \( \phi : C \to X \) defined over \( K' \), such that \( C \cong \mathbb{P}^1_{K'} \) and \( H^1(\mathbb{P}^1_{K'}, \phi^* T_{X_{K'}}(-2)) = 0 \).

We choose a model

\[
\Phi : C \cong \mathbb{P}^1_{K'} \to X'
\]

of \( C \) and \( \phi \) defined over a Zariski open set of \( \text{Spec} \mathcal{O}_{K'} \). By upper-semi-continuity of cohomology, the vanishing (4) remains true after restriction to almost every closed point \( p \in \text{Spec} \mathcal{O}_{K'} \), which proves 1).

2) The proof is identical to the proof of Proposition 1.3: we just have to show that the curve \( C_{\overline{p}} \subset X_{\overline{p}} \) is algebraically equivalent in \( X_{\overline{p}} \) to a difference \( C_{\overline{p}} - \sum_i R_i \), where each curve \( R_i \) resp. \( R_i \) (they are in fact defined over a finite extension \( k(p) \) of \( k(p) \)), lifts to a curve \( C'_{\overline{p}} \) resp. \( R_i \) in \( X_{K'} \), for some finite extension \( K' \) of \( K \).

Assuming the curves \( C_{\overline{p}} \), \( R_i \) are smooth, the existence of such a lifting is granted by the condition \( H^1(C_{\overline{p}}, N_{C_{\overline{p}}/X_{\overline{p}}}) = 0, \) resp. \( H^1(R_i, N_{R_i/X_{\overline{p}}}) = 0 \).

Starting from \( C \subset X_{\overline{p}} \) where \( X_{\overline{p}} \) is separably rationally connected over \( \overline{p} \), we obtain such curves \( C_{\overline{p}}, R_i \) as in the previous proof, applying [10, §2.1].

The proof of Proposition 1.5 is finished.

Again, this proof leads as well to the proof of the specialization invariance of the \( l \)-adic analogues \( Z_{ct, rat}^{2n-2}(X) \) of the groups \( Z^{2n-2}_{ct, rat}(X) \) introduced in Remark 2.2.

**Variant 2.7.** Let \( X \) be a smooth rationally connected variety defined over a number field \( K \), with ring of integers \( \mathcal{O}_K \). Assume given a projective model \( X \) of \( X \) over \( \text{Spec} \mathcal{O}_K \). Fix a prime integer \( l \). Then for any \( p \in \text{Spec} \mathcal{O}_K \) such that \( X_{\overline{p}} \) is smooth separably connected, the group \( Z_{ct, rat}^{2n-2}(X)_l \) is isomorphic to the group \( Z_{ct, rat}^{2n-2}(X_p)_l \).
3 Consequence of a result of Chad Schoen

In [19], Chad Schoen proves the following theorem:

**Theorem 3.1.** Let $X$ be a smooth projective variety of dimension $n$ defined over a finite field $k$ of characteristic $p$. Assume that the Tate conjecture holds for degree 2 Tate classes on smooth projective surfaces defined over a finite extension of $k$. Then the étale cycle class map:

$$cl : CH^{n-1}(X_{\bar{k}}) \otimes \mathbb{Z}_l \rightarrow H^{2n-2}(X_{\bar{k}}, \mathbb{Z}_l(n-1))_{Tate}$$

is surjective, that is $Z^{2n-2}_{et}(X)_l = 0$.

In other words, the Tate conjecture 1.4 for degree 2 rational Tate classes implies that the groups $Z^{2n-2}_{et}(X)_l$ should be trivial for all smooth projective varieties defined over finite fields. This is of course very different from the situation over $\mathbb{C}$ where the groups $Z^{2n-2}(X)$ are known to be possibly nonzero.

**Remark 3.2.** There is a similarity between the proof of Theorem 3.1 and the proof of Theorem 1.1. Schoen proves that given an integral Tate class $\alpha$ on $X$ (defined over a finite field), there exist a smooth complete intersection surface $S \subset X$ and an integral Tate class $\beta$ on $S$ such that $j_{S*}\beta = \alpha$ where $j_S$ is the inclusion of $S$ in $X$. The result then follows from the fact that if the Tate conjecture holds for degree 2 rational Tate classes on $S$, it holds for degree 2 integral Tate classes on $S$.

I prove that for $X$ a uniruled or Calabi-Yau, and for $\beta \in Hdg^4(X, \mathbb{Z})$ there exists surfaces $S_i$ such that there exists a rational Tate class $\beta_i$ in $Hdg^2(S_i, \mathbb{Z})$ such that $\alpha = \sum_i j_{S_i*}\beta_i$. The result then follows from the Lefschetz theorem on $(1,1)$-classes applied to the $\beta_i$.

We refer to [7] for some comments on and other applications of Schoen’s theorem, and conclude this note with the proof of the following theorem (cf. Theorem 1.6 of the introduction).

**Theorem 3.3.** Assume Tate’s conjecture 1.4 holds for degree 2 Tate classes on smooth projective varieties defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety $X$ over $\mathbb{C}$.

**Proof.** We first recall that for a smooth rationally connected variety $X$, the group $Z^{2n-2}(X)$ is equal to the quotient $H^{2n-2}(X, \mathbb{Z})/H^{2n-2}(X, \mathbb{Z})_{alg}$, due to the fact that the Hodge structure on $H^{2n-2}(X, \mathbb{Q})$ is trivial. In fact, we have more precisely

$$H^{2n-2}(X, \mathbb{Q}) = H^{2n-2}(X, \mathbb{Q})_{alg}$$

by hard Lefschetz theorem and the fact that

$$H^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{alg}$$

by the Lefschetz theorem on $(1,1)$-classes.

Next, in order to prove that $Z^{2n-2}(X)$ is trivial, it suffices to prove that for each $l$, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/(\text{Im } cl) \otimes \mathbb{Z}_l$ is trivial.

We apply Proposition 1.3 which tells us that for any connected $\mathbb{C}$, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ is locally deformation invariant for families of smooth rationally connected varieties. Note that our smooth projective rationally connected variety $X$ is the fiber $X_t$ of a smooth projective morphism $\phi : \mathcal{X} \rightarrow B$ defined over a number field, where $\mathcal{X}$ and $B$ are quasiprojective, geometrically connected and defined over a number field. By local deformation invariance, the vanishing of $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ is equivalent to the vanishing of $Z^{2n-2}(X_{t'}) \otimes \mathbb{Z}_l$ for any point $t' \in B(\mathbb{C})$. Taking for $t'$ a point of $B$ defined over a number field, $X_{t'}$ is defined over a number field. Hence it suffices to prove the vanishing of $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ for $X$ rationally connected defined over a number field $L$. 


We have
\[ Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/(\text{Im } cl) \otimes \mathbb{Z}_l, \]
and by the Artin comparison theorem (cf. [17, Chapter III, §3]), this is equal to
\[ \frac{H^{2n-2}_c(X, \mathbb{Z}_l(n-1))}{(\text{Im } cl) \otimes \mathbb{Z}_l} = Z^{2n-2}_c(X)_l \]
since \( H^{2n-2}_c(X, \mathbb{Z}_l(n-1)) \) consists of Tate classes. Hence it suffices to prove that for \( X \) rationally connected defined over a number field and for any \( l \), the group \( Z^{2n-2}_c(X)_l \) is trivial.

We now apply Proposition 1.5 to \( X \) and its reduction \( X_p \) for almost every closed point \( p \in \text{Spec } \mathcal{O}_L \). It follows that the vanishing of \( Z^{2n-2}_c(X)_l \) is implied by the vanishing of \( Z^{2n-2}_c(X_p)_l \). According to Schoen’s theorem 3.1, the last vanishing is implied by the Tate conjecture for degree 2 Tate classes on smooth projective surfaces.

Remark 3.4. This argument does not say anything on the groups \( Z^{2n-2}_{rat}(X) \), since there is no control on the 1-cycles representing given degree 2 \( n-2 \) Tate classes on varieties defined over finite fields. Similarly, Theorem 1.1 does not say anything on \( Z^4_{rat}(X) \) for \( X \) a rationally connected threefold.

References


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