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► **To cite this version:**

Wolfgang Lucha, Dmitri Melikhov, Silvano Simula. Decay constants of heavy pseudoscalar mesons from QCD sum rules. *Journal of Physics G: Nuclear and Particle Physics*, 2011, 38 (10), pp.105002. 10.1088/0954-3899/38/10/105002 . hal-00655647

HAL Id: hal-00655647

<https://hal.science/hal-00655647>

Submitted on 1 Jan 2012

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Decay constants of heavy pseudoscalar mesons from QCD sum rules

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(Dated: June 30, 2011)

We revisit the sum-rule extraction of the decay constants of the D , D_s , B , and B_s mesons from the two-point correlator of heavy–light pseudoscalar currents. We use the operator product expansion of this correlator expressed in terms of the $\overline{\text{MS}}$ heavy-quark mass, for which the perturbative expansion exhibits a reasonable convergence. Our main emphasis is laid on the control over the uncertainties in the decay constants, related both to the input QCD parameters and to the limited intrinsic accuracy of the method of QCD sum-rules. This becomes possible due to the application of our procedure of extracting hadron observables that involves as novel feature dual thresholds depending on the Borel parameter. For charmed mesons, we find the decay constants $f_D = (206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{sys})})$ MeV and $f_{D_s} = (245.3 \pm 15.7_{(\text{OPE})} \pm 4.5_{(\text{sys})})$ MeV. For beauty mesons, the decay constants turn out to be extremely sensitive to the precise value of $\overline{m}_b(\overline{m}_b)$. By requiring our sum-rule estimate to match the average of the lattice results for f_B , a very accurate value $\overline{m}_b(\overline{m}_b) = 4.245 \pm 0.025$ GeV is extracted, leading to $f_B = (193.4 \pm 12.3_{(\text{OPE})} \pm 4.3_{(\text{sys})})$ MeV and $f_{B_s} = (232.5 \pm 18.6_{(\text{OPE})} \pm 2.4_{(\text{sys})})$ MeV.

PACS numbers: 11.55.Hx, 12.38.Lg, 03.65.Ge

1. INTRODUCTION

The extraction of the ground-state decay constant within the method of QCD sum rules [1] is a complicated problem:

First, one should derive a reliable operator product expansion (OPE) for the Borel-transformed (vulgo “Borelized”) correlator $\Pi(\tau)$. The OPE in QCD is a double expansion: a perturbative expansion in powers of the strong coupling α_s and an expansion in powers of the Borel parameter τ in terms of condensates of increasing dimension. In practice, one has a truncated double series for which at most a few lowest-order terms can be calculated. This truncated series for the correlator depends explicitly on the renormalization scheme and scale (even in those cases where the full correlator does not), and the magnitude of the unknown higher-order terms crucially depends on the relevant choice of this scheme and scale. Therefore, controlling higher-order perturbative corrections poses a serious problem.

Since the pioneering work [2], the correlator expressed in terms of the on-shell (or pole) heavy-quark mass has been employed for the extraction of the decay constant. However, after the three-loop result for the correlator has appeared [3] it became evident that the perturbative expansion in terms of the on-shell mass shows no signal of convergence: the LO, NLO, and NNLO terms all give comparable contributions to the decay constant. In contrast to this, a reshuffling of the perturbative expansion making use of the $\overline{\text{MS}}$ mass of the heavy quark leads to a clear hierarchy of the perturbative contributions to the decay constant [4].¹ Following [4], we adopt the OPE of the relevant correlator in terms of the $\overline{\text{MS}}$ heavy-quark mass, denoted hereafter by $m_Q \equiv \overline{m}_Q(\overline{m}_Q)$. Therefore, the on-shell mass does not appear in our analysis.

Second, the knowledge of the correlator for only moderate values of τ allows one to extract the characteristics of the bound state with some error which reflects the intrinsic uncertainty of the method of QCD sum-rules. Gaining control over this systematic uncertainty is a rather subtle problem, as it has been shown in [6]. Moreover, since higher multiloop perturbative calculations are becoming available and the knowledge of the fundamental QCD parameters is improving, the accuracy of the OPE of the relevant correlators is increasing. Therefore, the intrinsic systematic uncertainty of the QCD sum-rule method may become competitive with the decreased OPE uncertainties, as we shall show to be the case for the D -meson decay constant f_D .

This work presents a detailed analysis of the decay constants of the $D_{(s)}$ and $B_{(s)}$ mesons, with emphasis on acquiring control over all the uncertainties — of both OPE and intrinsic (i.e., systematic) origin — in these quantities. Recently, we formulated a novel algorithm for extracting bound-state parameters from OPEs for the correlators which opens the possibility to arrive at realistic error estimates for the extracted hadron parameters [7]. The efficiency of our algorithm has been established in potential models: there the exact ground-state decay constants may be computed by solving the Schrödinger equation. Moreover, it has been explicitly demonstrated that the *extraction procedures* of the ground-state parameters in QCD and in potential models are very close to each other quantitatively as soon as the (quark–hadron) duality is implemented in both theories in the same way [8].

This paper is organized as follows: Section 2 summarizes existing results on the OPEs for pseudoscalar heavy-light two-point functions and presents details of our algorithm for extracting the ground-state contribution to this correlator. Section 3 sketches our analysis of f_D and f_{D_s} , testing and proving the efficiency of our formalism. Section 4 provides the corresponding analysis of f_B and f_{B_s} . Section 5 summarizes our conclusions.

¹ As rather unpleasant consequence, the decay constant extracted from the three-loop correlator in terms of the on-shell mass turns out to be *considerably* smaller than the estimate obtained from the three-loop correlator in terms of the running $\overline{\text{MS}}$ mass [5]. For the translation from one scheme to the other one employs the three-loop relation between the on-shell and the running quark masses.

2. CORRELATOR AND SUM RULE

We consider the correlator

$$\Pi(p^2) = i \int d^4x e^{ipx} \langle 0 | T \left(j_5(x) j_5^\dagger(0) \right) | 0 \rangle \quad (2.1)$$

of two pseudoscalar heavy–light currents

$$j_5(x) = (m_Q + m) \bar{q}(x) i \gamma_5 Q(x). \quad (2.2)$$

Here and below, m denotes the running current masses of the light u , d , and s quarks in the $\overline{\text{MS}}$ renormalization scheme, evaluated at renormalization scale $\mu = 2$ GeV. The Borel-transformed OPE series for our correlator (2.1) is of the form

$$\Pi(\tau) = \int_{(m_Q+m)^2}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu), \quad (2.3)$$

where the perturbative spectral density $\rho_{\text{pert}}(s, \mu)$ may be obtained as expansion in powers of the strong coupling $\alpha_s(\mu)$:

$$\rho_{\text{pert}}(s, \mu) = \rho^{(0)}(s) + \frac{\alpha_s(\mu)}{\pi} \rho^{(1)}(s) + \left(\frac{\alpha_s(\mu)}{\pi} \right)^2 \rho^{(2)}(s) + \dots \quad (2.4)$$

The Borel-transformed correlator (2.3) does not depend on the renormalization scale μ ; however, both the perturbative expansion truncated to some fixed order in α_s and the truncated power corrections depend on μ . Moreover, the relative magnitudes of the lowest-order contributions strongly depend on the choice of the renormalization scheme and/or scale.

A crucial issue for the reliability of a *truncated* perturbative expansion is the magnitude of the unknown higher-order corrections. One can play around with the choice of the renormalization scale in order to obtain some properties of the known terms of the perturbative expansion. For example, one may choose the scale μ by minimizing the highest-order known correction or by requiring some hierarchy of the known perturbative contributions. Unfortunately, even a clear hierarchy of several lowest-order perturbative corrections does not mean that the subsequent corrections are also small. Very often, the variation of the renormalization scale μ in some range is used as an attempt to probe the magnitude of the unknown higher-order corrections. We shall pursue this strategy too, although there seems to be no rigorous way to estimate the size of these corrections without explicitly calculating them.

In all expressions below, the quark masses m_Q and m and the strong coupling α_s denote the respective $\overline{\text{MS}}$ running quantities at the scale μ . Note that $\rho^{(0)}(s)$ depends on μ implicitly through the quark masses, whereas all higher-order spectral densities $\rho^{(n)}(s)$, $n \geq 1$, depend on μ implicitly through the quark masses and contain, in addition, explicitly μ -dependent logarithmic terms. Both perturbative spectral density and power corrections are given below for $\mu = m_Q$.

A. Perturbative spectral density

The leading-order spectral density is well-known:

$$\rho^{(0)}(s) = \frac{N_c}{8\pi^2} (m_Q + m)^2 \frac{s - (m_Q - m)^2}{s} \sqrt{[s - (m_Q - m)^2][s - (m_Q + m)^2]}. \quad (2.5)$$

For the spectral density of order α_s we make use of the first two terms of its expansion in small mass m derived in [4]:²

$$\begin{aligned} \rho^{(1)}(s) &= \rho_{m^0}^{(1)}(s) + \rho_m^{(1)}(s) + O(m^2), \\ \rho_{m^0}^{(1)}(s) &= \frac{N_c}{16\pi^2} C_F (m_Q + m)^2 s (1-x) \{ (1-x) [4L_2(x) + 2 \ln x \ln(1-x) - (5-2x) \ln(1-x)] \\ &\quad + (1-2x)(3-x) \ln x + (17-33x)/2 \}, \\ \rho_m^{(1)}(s) &= \frac{N_c}{8\pi^2} C_F (m_Q + m)^2 m_Q m \{ (1-x) [4L_2(x) + 2 \ln x \ln(1-x) - 2(4-x) \ln(1-x)] \\ &\quad + 2(3-5x+x^2) \ln x + 2(7-9x) \}, \end{aligned} \quad (2.6)$$

where $x \equiv m_Q^2/s$, $N_c = 3$, and $C_F = (N_c^2 - 1)/2N_c$. The order- α_s^2 spectral density reads

$$\rho^{(2)}(s) = R^{(2),s} + \Delta_1 \rho^{(2)} + \Delta_2 \rho^{(2)} + O(m). \quad (2.7)$$

² Note that the $O(\alpha_s m^2)$ corrections to $\Pi(\tau)$ cannot be obtained by expanding the spectral density $\rho^{(1)}$ in powers of m : Starting from the order m^3 , the functions $\rho_{m^3}^{(1)}$ etc. contain poles of increasing orders at $s = m_Q^2$ (see Eqs. (C3) and (C4) of Ref. [4]). Therefore, after the s -integration all these terms yield contributions of the same order $O(\alpha_s m^2)$ to the spectral function. For the same reason, the expansion of $\rho^{(0)}(s)$ in powers of m does not allow one to get the terms of order m^3 in $\Pi(\tau)$. We therefore use the exact expression for the spectral density $\rho^{(0)}$ instead of expanding it in powers of m .

Here, $R^{(2),s}$ is the spectral function defined by Eq. (8) of Ref. [3], which is provided by the authors through the publicly available program `rvs.m`. The authors of [3] have calculated the spectral function $\rho_{\text{pert}}(s)$ for the case $m = 0$ in terms of the heavy-quark on-shell mass. Rewriting the $O(1)$ and $O(\alpha_s)$ spectral densities $R^{(0),s}$ and $R^{(1),s}$ of [3] in terms of the running mass generates the corrections $\Delta_1\rho^{(2)}$ and $\Delta_2\rho^{(2)}$ to the spectral density $\rho^{(2)}(s)$. The explicit expressions for these corrections are given by Eqs. (14) and (15) of Ref. [4] and will not be reproduced here.

B. Power corrections

For the power corrections we also make use of the expression from [4]:

$$\begin{aligned} \Pi_{\text{power}}(\tau, \mu = m_Q) &= (m_Q + m)^2 e^{-m_Q^2 \tau} \\ &\times \left\{ -m_Q \langle \bar{q}q \rangle \left[1 + \frac{2C_F \alpha_s}{\pi} \left(1 - \frac{m_Q^2 \tau}{2} \right) - \frac{m}{2m_Q} (1 + m_Q^2 \tau) + \frac{m^2}{2} m_Q^2 \tau^2 + \frac{m_0^2 \tau}{2} \left(1 - \frac{m_Q^2 \tau}{2} \right) \right] + \frac{1}{12} \left\langle \frac{\alpha_s}{\pi} GG \right\rangle \right\}. \end{aligned} \quad (2.8)$$

The parameter m_0^2 describes the dimension-5 mixed quark–gluon condensate [4]. It is worth noticing that the radiative corrections to the condensates increase rather fast with the Borel parameter τ .

In summary, we make use of the expressions for $\Pi(\tau)$ from [4] with minor modifications:

(i) We adopt the “natural” threshold $(m_Q + m)^2$ in the spectral representation for $\Pi(\tau)$ and therefore encounter only running $\overline{\text{MS}}$ masses in our formulæ.

(ii) For $\rho^{(0)}(s)$ we use the exact expression without performing an expansion in powers of m , and for $\rho^{(1)}(s)$ we do not include terms of order m^2 and higher.

The OPE parameters required for our analysis are

$$\begin{aligned} m(2 \text{ GeV}) &= (3.5 \pm 0.5) \text{ MeV}, & m_s(2 \text{ GeV}) &= (100 \pm 10) \text{ MeV}, \\ \langle \bar{q}q \rangle(2 \text{ GeV}) &= -((267 \pm 17) \text{ MeV})^3, & \frac{\langle \bar{s}s \rangle(2 \text{ GeV})}{\langle \bar{q}q \rangle(2 \text{ GeV})} &= 0.8 \pm 0.3, & \left\langle \frac{\alpha_s}{\pi} GG \right\rangle &= (0.024 \pm 0.012) \text{ GeV}^4, \\ m_0^2 &= (0.8 \pm 0.2) \text{ GeV}^2, & \alpha_s(M_Z) &= 0.1176 \pm 0.0020. \end{aligned} \quad (2.9)$$

As is a rather common practice in the literature, we present the numerical values of all renormalization-scale-dependent QCD parameters at their respective relevant (and consequently differing) scales μ : both light-quark masses and vacuum condensates at the scale $\mu = 2 \text{ GeV}$, the strong coupling α_s at the weak scale $\mu = M_Z$, and heavy-quark $\overline{\text{MS}}$ masses \overline{m}_Q at the scale $\mu = \overline{m}_Q$ equal to the respective heavy-quark $\overline{\text{MS}}$ mass. Needless to say, both the spectral densities (2.6) and the power corrections (2.8) involve these parameters at one and the same scale μ . Accordingly, in our actual calculations we properly evolved these QCD parameters to their common scale μ , by exploiting their running to order α_s^2 . Only these latter values of the renormalization-scale-dependent parameters enter our formulæ. At the final stage, the sensitivity of the extracted decay constants to the particular choice of μ is investigated. The outcome of this analysis constitutes part of the statistical uncertainty of our numerical findings.

We perform the calculations for two sets of c - and b -quark masses $m_Q \equiv \overline{m}_Q(\overline{m}_Q)$: the values from PDG [9]

$$m_c = (1.27_{-0.09}^{+0.07}) \text{ GeV}, \quad m_b = (4.19_{-0.06}^{+0.18}) \text{ GeV}, \quad (2.10)$$

and the very accurate values reported recently in [10]³

$$m_c = (1.279 \pm 0.013) \text{ GeV}, \quad m_b = (4.163 \pm 0.016) \text{ GeV}. \quad (2.11)$$

C. Sum rule

The correlator (2.1) may be evaluated by inserting a complete set of hadronic intermediate states:

$$\Pi(\tau) = f_Q^2 M_Q^4 e^{-M_Q^2 \tau} + \text{contributions of higher states}, \quad (2.12)$$

where f_Q is the decay constant of the P_Q meson, defined according to

$$(m_Q + m) \langle 0 | \bar{u} i \gamma_5 Q | P_Q \rangle = f_Q M_Q^2. \quad (2.13)$$

For large values of τ , the contributions of the excited states decrease faster than the ground-state contribution and the correlator $\Pi(\tau)$ is dominated by the ground state. Unfortunately, the truncated OPE does not allow us to calculate the correlator at sufficiently large τ , such that the excited states give a sizable contribution to $\Pi(\tau)$.

³ As a matter of fact, our predictions would not be affected by errors of the charmed-quark mass twice as large [11] as those given in (2.11).

According to the quark–hadron duality assumption, the contributions of excited states and continuum are described by the QCD perturbative contribution above an effective continuum threshold s_{eff} . This leads to the following relation:

$$f_Q^2 M_Q^4 e^{-M_Q^2 \tau} = \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)) \equiv \int_{(m_Q+m)^2}^{s_{\text{eff}}(\tau)} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu). \quad (2.14)$$

In the region near the physical continuum threshold at $s = (M_{V_Q} + m_\pi)^2$, V_Q being the lightest vector meson containing a quark Q , the QCD perturbative spectral density and the hadron spectral density are rather different. Consequently, the effective continuum threshold as defined by (2.14) turns out to be necessarily a function of the Borel parameter τ .

We introduce the dual invariant mass M_{dual} and the dual decay constant f_{dual} by the definitions

$$M_{\text{dual}}^2(\tau) \equiv -\frac{d}{d\tau} \log \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)), \quad (2.15)$$

$$f_{\text{dual}}^2(\tau) \equiv M_Q^{-4} e^{M_Q^2 \tau} \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)). \quad (2.16)$$

Notice that the deviation of the dual mass from the actual ground-state mass provides an indication of the excited-state contributions picked up by the dual correlator.

In order to determine the decay constant f_Q of the P_Q meson from the OPE we must execute the following two steps.

1. Borel window

First, we have to fix the working τ -window where, on the one hand, the OPE yields a sufficiently accurate description of the exact correlator (that is, the higher-order radiative and power corrections are small) and, on the other hand, the ground state gives a “sizable” contribution to the correlator. Since the radiative corrections to the condensates increase rather fast with τ , it is preferable to stay at the lowest possible values of τ . We shall therefore fix the Borel window by the following criteria: (a) In the window, the power corrections $\Pi_{\text{power}}(\tau)$ should not exceed 30% of the cut perturbative correlator $\Pi_{\text{pert}}(\tau, s_0)$; this gives the upper boundary of the τ -window. The ground-state contribution to the correlator at such τ values comprises about 50% of the correlator. (b) The lower boundary of the τ -window is defined by requiring the ground-state contribution not to drop below 10%. In quantum physics, our algorithm was shown to provide a good, even excellent extraction of the ground-state decay constant for Borel windows determined by these requirements [6, 7].

2. Effective continuum threshold

Second, we must formulate our criterion for the determination of $s_{\text{eff}}(\tau)$. We consider an algorithm for the extraction of f_Q which takes advantage of the knowledge of the P_Q -meson mass M_Q . Our algorithm, constructed in previous works and, in quantum-theoretical potential models, proven to work well for various correlators, is rather simple: We consider a set of τ -dependent Ansätze for the effective continuum threshold, for simplicity assumed to be all of polynomial form:

$$s_{\text{eff}}^{(n)}(\tau) = \sum_{j=0}^n s_j^{(n)} \tau^j. \quad (2.17)$$

We determine the parameters on the r.h.s. of (2.17) as follows: We calculate the dual mass squared according to (2.15) for the τ -dependent s_{eff} of Eq. (2.17). We then compute $M_{\text{dual}}^2(\tau)$ at several values of $\tau = \tau_i$ ($i = 1, \dots, N$, where N can be taken arbitrary large) chosen uniformly over the Borel window. Finally, we minimize the squared difference between M_{dual}^2 and the known value M_Q^2 :

$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [M_{\text{dual}}^2(\tau_i) - M_Q^2]^2. \quad (2.18)$$

This pins down the parameters of the effective continuum threshold. As soon as the latter is fixed, it is straightforward to calculate the decay constant f_Q .

According to our recent findings, allowing for a τ -dependence of the effective threshold leads to visible improvements compared with the traditional assumption of a τ -independent quantity: The former yields a much better stability of the dual mass calculated from the dual correlator and allows one to work at smaller values of τ , where the impact of power corrections is reduced.

3. Uncertainties of the extracted decay constant

The above discussion implies that the extracted result for the decay constant is sensitive both to the precise values of the OPE parameters and to the particular prescription for fixing the effective continuum threshold. The corresponding uncertainties of the predicted decay constant are labeled as its *OPE-related error* and its *systematic error*, respectively:

1. *OPE-related error.* The OPE-related uncertainty is estimated as follows: We perform a bootstrap analysis [12] by allowing the OPE parameters to vary over the ranges quoted in Eqs. (2.9), using 1000 bootstrap events. Gaussian distributions for all parameters but μ are employed. For μ we assume a uniform distribution in the corresponding range, which we choose to be $1 \leq \mu$ (GeV) ≤ 3 for charmed mesons and $2 \leq \mu$ (GeV) ≤ 8 for beauty mesons. The resulting distribution of the decay constant turns out to be close to a Gaussian shape. The quoted OPE-related error is therefore the Gaussian error.
2. *Systematic error.* The systematic uncertainty of some hadron parameter obtained by the sum-rule method (i.e., the error related to the intrinsically limited accuracy of this approach) represents the perhaps most subtle point in all applications of this method. So far no way to provide a *rigorous* — in the mathematical sense — systematic error has been devised. However, a *realistic* estimate of the corresponding error may be found: As prompted by detailed comparisons of the extraction of the decay constant in QCD and in potential models [8], the band of f_P values spanned by the linear, quadratic, and cubic Ansätze for the effective continuum threshold contains the true value of the decay constant. Trusting in these findings, the half-width of this band is interpreted as the systematic error of the decay constant. Presently, we do not envisage any other possibility to provide more reliable estimates for the systematic error.

3. DECAY CONSTANTS OF THE D AND D_s MESONS

A. Decay constant of the D meson

The Borel window for the charmed meson is chosen according to the criteria discussed above: $\tau = (0.1 - 0.5)$ GeV⁻². Figure 1 illustrates the application of our prescription of obtaining the effective continuum threshold and extracting the corresponding f_D . We would like to point out that τ -dependent effective thresholds lead to a much better reproduction of the meson mass in the window than a constant one (Fig. 1a). This signals that the dual correlators corresponding to the τ -dependent thresholds are less contaminated by the excited states.

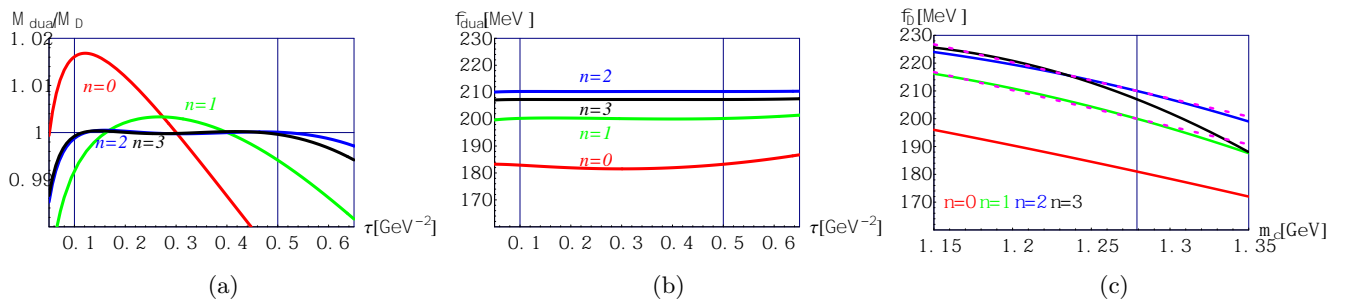


Fig. 1: Dual mass (a) and dual decay constant (b) of the D meson extracted by adopting different Ansätze (2.17) for the effective continuum threshold $s_{\text{eff}}(\tau)$ and fixing these thresholds according to (2.18). The results for $m_c \equiv \bar{m}_c(\bar{m}_c) = 1.279$ GeV, $\mu = m_c$, and central values of the other relevant parameters are presented. (c) Dual decay constant of the D meson vs. m_c for $\mu = m_c$ and central values of the other OPE parameters. The index $n = 0, 1, 2, 3$ denotes the power of our polynomial Ansatz for the effective continuum threshold in (2.17).

The dependence of the extracted value of the D -meson decay constant f_D on the c -quark mass $m_c \equiv \bar{m}_c(\bar{m}_c)$ and the quark condensate $\langle \bar{q}q \rangle \equiv \langle \bar{q}q \rangle(2 \text{ GeV})$ may be parameterized in the form

$$f_D^{\text{dual}}(m_c, \mu = m_c, \langle \bar{q}q \rangle) = \left[206.2 - 13 \left(\frac{m_c - 1.279 \text{ GeV}}{0.1 \text{ GeV}} \right) + 4 \left(\frac{|\langle \bar{q}q \rangle|^{1/3} - 0.267 \text{ GeV}}{0.01 \text{ GeV}} \right) \pm 5.1_{(\text{syst})} \right] \text{ MeV}. \quad (3.1)$$

This relation describes the band of values delimited by the two dotted lines in Fig. 1c, which include the results derived with the linear, quadratic, and cubic Ansätze for the effective continuum threshold. Figure 2a displays the results of the bootstrap analysis of the OPE uncertainties. The distribution has a Gaussian shape, and therefore the corresponding OPE uncertainty is the Gaussian error. Adding the width of the band provided by the τ -dependent $n = 1, 2, 3$ Ansätze for the effective continuum threshold as the (intrinsic) systematic error of our approach, we obtain the following result:

$$f_D = (206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{syst})}) \text{ MeV}. \quad (3.2)$$

We have considered for m_c the two ranges in (2.10) and (2.11). The OPE-related error is practically the same for both ranges (see Figs. 2a,b), so the main source of the OPE uncertainty in the extracted f_D comes from the OPE parameters other than m_c (mainly, the renormalization scale and the quark condensate).

Notice that the bootstrap procedure for a τ -independent effective threshold gives a substantially lower f_D range, viz., $f_D(n=0) = (181.3 \pm 7.4_{(\text{OPE})})$ MeV, which deviates from our τ -dependent result (3.2) by almost three times the OPE uncertainty. Moreover, as we have already shown in our previous works [6], making use of merely a constant Ansatz for the effective continuum threshold does not allow one to probe at all the intrinsic systematic error of the QCD sum rule. From (3.2) the latter turns out to be of the same order as the OPE uncertainty.

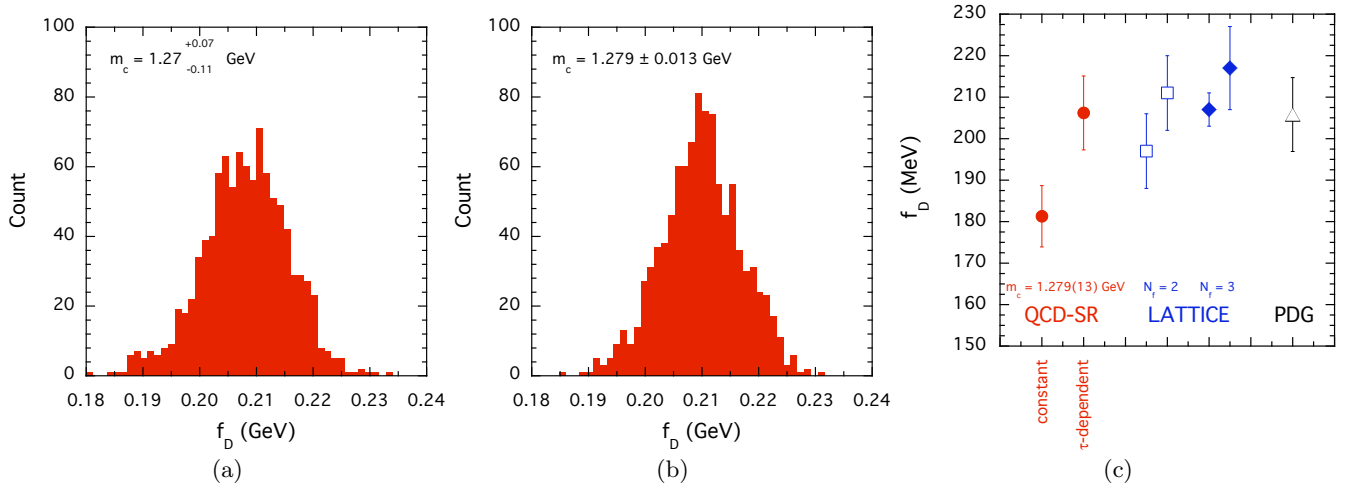


Fig. 2: Distribution of f_D as obtained by a bootstrap analysis of the OPE uncertainties. For all OPE parameters but μ Gaussian distributions with corresponding errors quoted in (2.9) are employed. A variation of m_c in the interval (2.10) (a) and (2.11) (b) is allowed. For μ we assume a uniform distribution in the range $1 \text{ GeV} < \mu < 3 \text{ GeV}$. (c) Summary of our results for f_D . For comparison, the lattice results are shown for two dynamical light flavors ($N_f = 2$) [13, 14] and three dynamical flavors ($N_f = 3$) [15, 16]. The triangle represents the experimental result from PDG [9]. For the τ -dependent QCD-SR result the depicted error is the sum of the OPE and systematic uncertainties given in (3.2), added in quadrature.

The τ -dependent threshold leads to a clearly discernible effect and brings the results from QCD sum rules into perfect agreement with recent lattice results and the data (Fig. 2b). A perfect agreement of our result with the lattice ones and with experiment provides a further confirmation of the reliability of our procedure.

B. Decay constant of the D_s meson

The corresponding τ -window is $\tau = (0.1 - 0.6) \text{ GeV}^{-2}$. Figure 3 provides the details of our extraction procedure. Our results for f_{D_s} may be represented in the form

$$f_{D_s}^{\text{dual}}(m_c, \mu = m_c, \langle \bar{s}s \rangle) = \left[245.3 - 18 \left(\frac{m_c - 1.279 \text{ GeV}}{0.1 \text{ GeV}} \right) + 3.5 \left(\frac{|\langle \bar{s}s \rangle|^{1/3} - 0.248 \text{ GeV}}{0.01 \text{ GeV}} \right) \pm 4.5_{(\text{sys})} \right] \text{ MeV}. \quad (3.3)$$

This relation describes the band of f_{D_s} values as a function of $m_c \equiv \bar{m}_c(\bar{m}_c)$ indicated by the two dotted lines in Fig. 3c, as well as the dependence on the quark condensate $\langle \bar{s}s \rangle \equiv \langle \bar{s}s \rangle(2 \text{ GeV})$. Performing the bootstrap analysis of the OPE uncertainties, we obtain the following estimate (cf. Fig. 4):

$$f_{D_s} = (245.3 \pm 15.7_{(\text{OPE})} \pm 4.5_{(\text{sys})}) \text{ MeV}. \quad (3.4)$$

As in the case of f_D , a constant threshold yields a substantially lower f_D value: $f_{D_s}(n=0) = (218.8 \pm 16.1_{(\text{OPE})}) \text{ MeV}$.

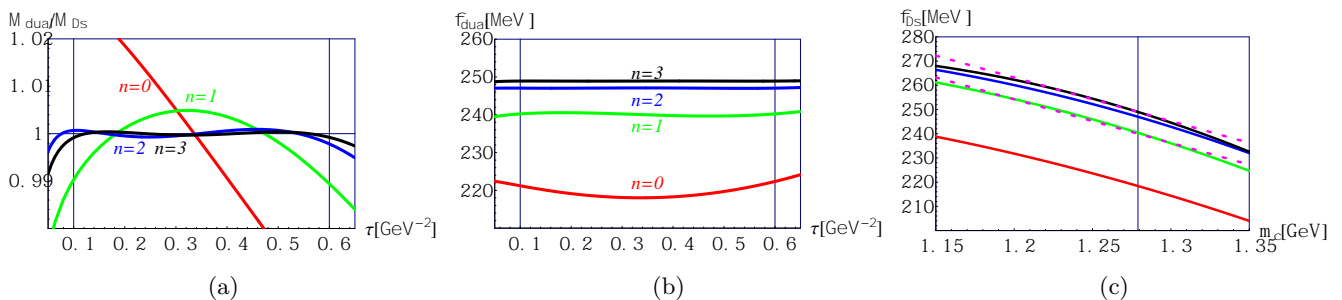


Fig. 3: Same as Fig. 1 but for the D_s meson.

C. f_{D_s}/f_D

For the ratio of the $D_{(s)}$ -meson decay constants, we find

$$f_{D_s}/f_D = 1.193 \pm 0.025_{(\text{OPE})} \pm 0.007_{(\text{sys})}, \quad (3.5)$$

to be compared with the PDG average, $f_{D_s}/f_D = 1.25 \pm 0.06$ [9], and the recent lattice results $f_{D_s}/f_D = 1.24 \pm 0.03$ [13] at $N_f = 2$, and $f_{D_s}/f_D = 1.164 \pm 0.011$ [15] and $f_{D_s}/f_D = 1.20 \pm 0.02$ [16] at $N_f = 3$. The error in (3.5) comes mainly from the uncertainties in the quark condensates: $\langle \bar{s}s \rangle / \langle \bar{q}q \rangle = 0.8 \pm 0.3$.

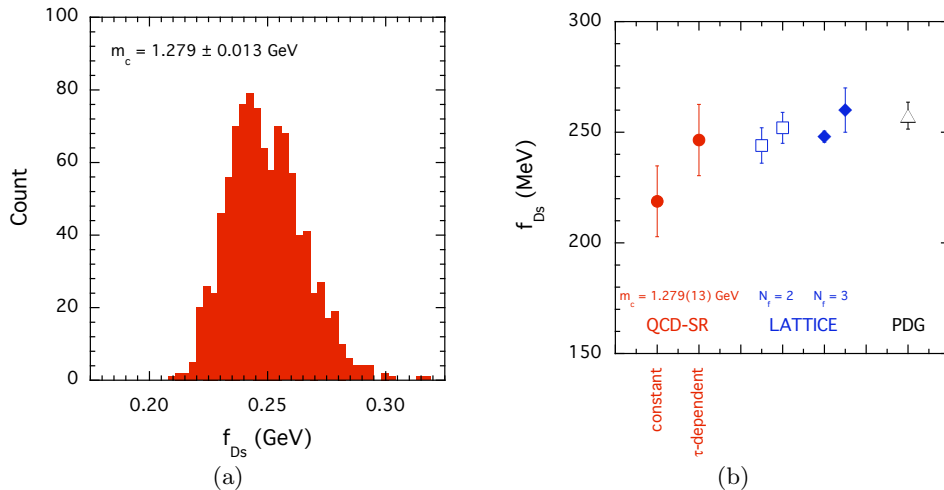


Fig. 4: (a) Distribution of f_{D_s} as obtained by the bootstrap analysis of the OPE uncertainties. For all our OPE parameters but μ Gaussian distributions with corresponding errors given in (2.9) are adopted. A variation of m_c in the interval (2.11) is allowed. For μ we assume a uniform distribution in the range $1 \text{ GeV} < \mu < 3 \text{ GeV}$. (b) Summary of our results for f_{D_s} . Lattice results are depicted for $N_f = 2$ [13, 14] and $N_f = 3$ [15, 16]. The experimental results, summarized by a triangle, are from PDG [9]. For the τ -dependent QCD-SR result the error shown is the sum of the OPE and systematic uncertainties in (3.4), added in quadrature.

4. DECAY CONSTANTS OF THE B AND B_s MESONS

We set the Borel window as $\tau = (0.05 - 0.18) \text{ GeV}^{-2}$. Note that the radiative corrections to the condensates increase rather fast with τ , so it is preferable to stay at lower values of τ .

A. Decay constant of the B meson

Figure 5 shows the application of our prescription to the extraction of f_B . The correlator $\Pi(\tau)$, which has dimension six, is extremely sensitive to the precise value of m_b . The dependence of the extracted value of the decay constant f_B on the b -quark mass $m_b \equiv \overline{m}_b(\overline{m}_b)$ and the condensate $\langle \bar{q}q \rangle \equiv \langle \bar{q}q \rangle(2 \text{ GeV})$ may be parameterized in the form

$$f_B^{\text{dual}}(m_b, \mu = m_b, \langle \bar{q}q \rangle) = \left[193.4 - 37 \left(\frac{m_b - 4.245 \text{ GeV}}{0.1 \text{ GeV}} \right) + 4 \left(\frac{|\langle \bar{q}q \rangle|^{1/3} - 0.267 \text{ GeV}}{0.01 \text{ GeV}} \right) \pm 4_{(\text{syst})} \right] \text{ MeV}. \quad (4.1)$$

The above relation describes the band of values delimited by the two dotted lines in Fig. 5c which, as before, include the results obtained with the linear, quadratic, and cubic Ansätze for the effective continuum threshold. In addition, it also encodes the dependence on the value of the quark condensate.

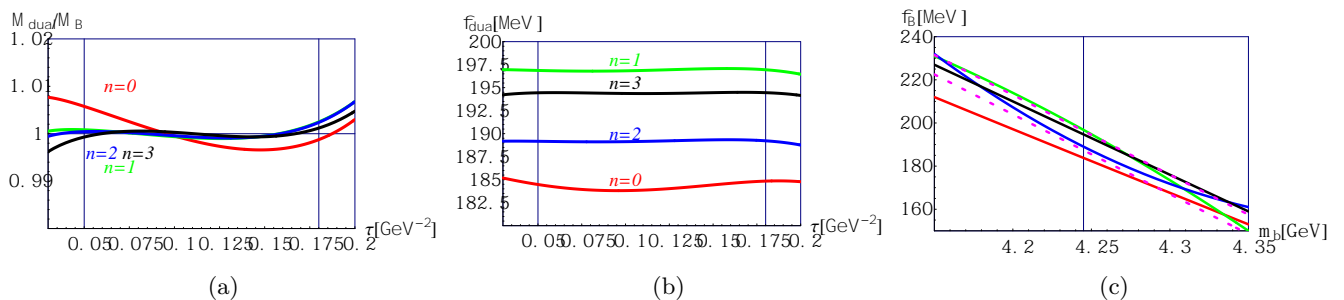


Fig. 5: Dual mass (a) and dual decay constant (b) of the B meson obtained by using different Ansätze for the effective continuum threshold $s_{\text{eff}}(\tau)$ (2.17) and fixing the coefficients according to (2.18). The results for $m_b \equiv \overline{m}_b(\overline{m}_b) = 4.245 \text{ GeV}$, $\mu = m_b$, and central values of all other relevant parameters are presented. (c) Dual decay constant of the B meson vs. m_b for $\mu = m_b$ and central values of all other OPE parameters. The index $n = 0, 1, 2, 3$ denotes the power of the polynomial Ansatz for the effective continuum threshold in (2.17).

We now perform a bootstrap analysis of f_B combining all OPE uncertainties. We assume Gaussian distributions for the OPE parameters (quark masses, condensates) with corresponding errors. The renormalization scale μ is assumed to be uniformly distributed in the interval $2 \leq \mu \text{ (GeV)} \leq 8$.

Because of the high sensitivity of the correlator to the b -quark mass, the sum-rule estimate for f_B strongly depends on the range of m_b used. For the PDG range $m_b = (4.19_{-0.06}^{+0.18}) \text{ GeV}$ [9], the OPE uncertainties are very large; therefore, no reasonable estimate of f_B may be obtained (see Fig. 6).

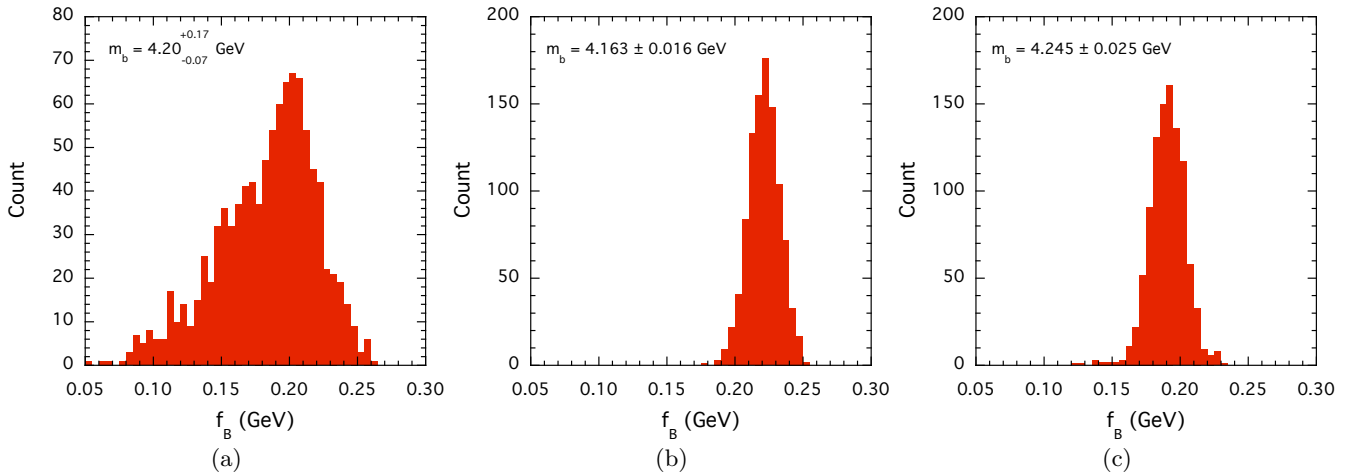


Fig. 6: Distribution of f_B for three ranges of m_b : the range (2.10) (a), the range (2.11) (b), and the range specified by (4.2) (c).

Adopting the recently reported precise range $m_b = (4.163 \pm 0.016)$ GeV from [10] leads to a rather accurate estimate: $f_B = (225.6 \pm 11.3_{(\text{OPE})} \pm 2.2_{(\text{syst})})$ MeV. However, we observe (see Fig. 7) some tension between this value and recent lattice calculations of f_B , which yield, on average, $f_B(\text{lattice}) = (193 \pm 130)$ MeV. Requiring the sum-rule estimate to match the lattice average leads to a rather accurate determination of m_b :

$$\overline{m}_b(\overline{m}_b) = (4.245 \pm 0.025) \text{ GeV}. \quad (4.2)$$

This differs from the range found in [10] as well as from the recent finding $\overline{m}_b(\overline{m}_b) = 4.164 \pm 0.023$ GeV [20], obtained from a (perturbative) QCD analysis similar to the one used in [10] but applied to the moments of heavy-quark current-current correlators calculated in lattice QCD with $N_f = 3$. However, the value (4.2) is in perfect agreement with the lattice determinations $\overline{m}_b(\overline{m}_b) = 4.26 \pm 0.03_{\text{stat}} \pm 0.09_{\text{syst}}$ GeV [21] and $\overline{m}_b(\overline{m}_b) = 4.25 \pm 0.02_{\text{stat}} \pm 0.11_{\text{syst}}$ GeV [22], as well as with the recent preliminary result of the Alpha Collaboration [23], all of these obtained using HQET on the lattice with $N_f = 2$.

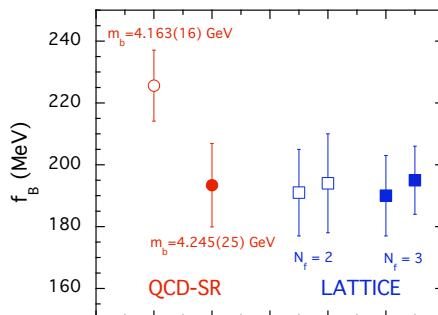


Fig. 7: Our estimates for f_B , with m_b in the ranges (2.11) and (4.2), vs. recent results from lattice QCD for $N_f = 2$ [14, 17] and $N_f = 3$ [18, 19].

The B-meson decay constant and its uncertainties corresponding to the b -quark mass (4.2) are

$$f_B = (193.4 \pm 12.3_{(\text{OPE})} \pm 4.3_{(\text{syst})}) \text{ MeV}. \quad (4.3)$$

Finally, let us mention that, for the range (4.2), a constant effective threshold gives $f_B(n=0) = (184 \pm 13_{(\text{OPE})})$ MeV.

B. Decay constant of the B_s meson

Figure 8 depicts the application of our procedure to the extraction of f_{B_s} . The dependence of the extracted value of the decay constant f_{B_s} on the b -quark mass and the strange-quark condensate is given by the relation

$$f_{B_s}^{\text{dual}}(m_b, \mu = m_b, \langle \bar{s}s \rangle) = \left[232.5 - 43 \left(\frac{m_b - 4.245 \text{ GeV}}{0.1 \text{ GeV}} \right) + 3.5 \left(\frac{|\langle \bar{s}s \rangle|^{1/3} - 0.248 \text{ GeV}}{0.01 \text{ GeV}} \right) \pm 2.4_{(\text{syst})} \right] \text{ MeV}. \quad (4.4)$$

This formula describes the band of values indicated by two dotted lines in Fig. 8c and, in addition, gives the dependence on the value of the quark condensate at renormalization scale $\mu = 2$ GeV.

We now perform a bootstrap analysis of f_{B_s} combining all OPE uncertainties. We assume Gaussian distributions of the OPE parameters (quark masses, condensates) with corresponding errors. The renormalization scale μ is assumed to

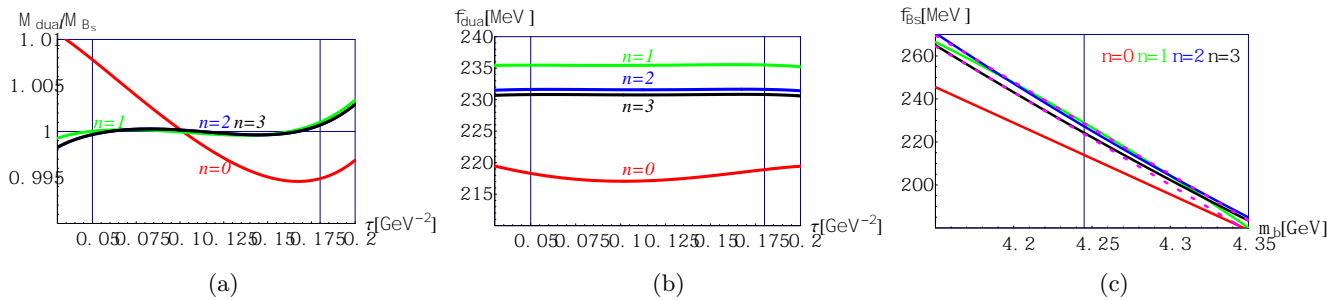


Fig. 8: Extraction of f_{B_s} . Line identification as in Fig. 5.

be uniformly distributed in the interval $2 \leq \mu$ (GeV) ≤ 8 . Making use of the range $m_b = (4.163 \pm 0.016)$ GeV [10] one has $f_{B_s} = (262.0 \pm 18.1_{(\text{OPE})} \pm 2.9_{(\text{syst})})$ MeV, while the range (4.2), $m_b = (4.245 \pm 0.025)$ GeV, leads to

$$f_{B_s} = (232.5 \pm 18.6_{(\text{OPE})} \pm 2.4_{(\text{syst})}) \text{ MeV}. \quad (4.5)$$

Figure 9b compares our results with recent lattice determinations.

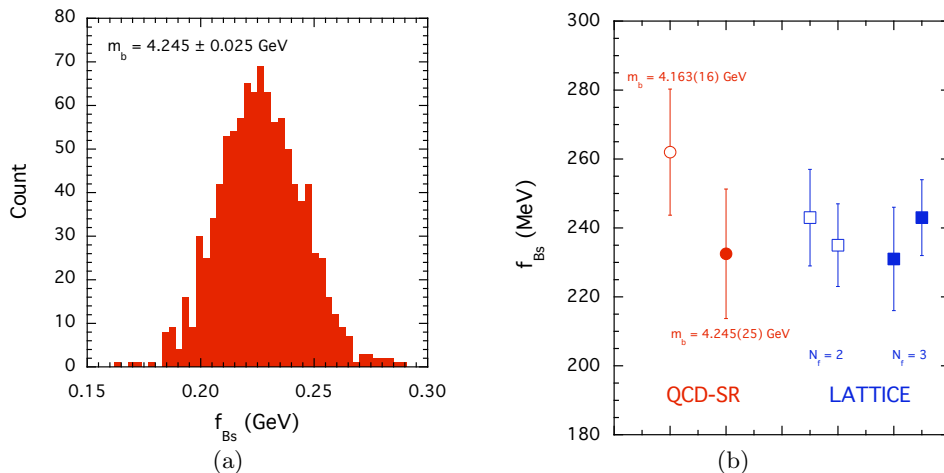


Fig. 9: (a) Distribution of f_{B_s} for $m_b = (4.245 \pm 0.025)$ GeV [see (4.2)]. (b) Our results, corresponding to the m_b -ranges (2.11) and (4.2), vs. lattice-QCD results for $N_f = 2$ [14, 17] and $N_f = 3$ [18, 19].

C. f_{B_s}/f_B

For the ratio of the decay constants of beauty mesons, we find

$$f_{B_s}/f_B = 1.203 \pm 0.020_{(\text{OPE})} \pm 0.007_{(\text{syst})}, \quad (4.6)$$

which has to be compared with the lattice results $f_{B_s}/f_B = 1.27 \pm 0.05$ at $N_f = 2$ [17] and $f_{B_s}/f_B = 1.226 \pm 0.026$ [18] and $f_{B_s}/f_B = 1.245 \pm 0.043$ [19] at $N_f = 3$. Similar to the charmed ratio (3.5), the error in (4.6) comes mainly from the uncertainty in the ratio of quark condensates: $\langle \bar{s}s \rangle / \langle \bar{q}q \rangle = 0.8 \pm 0.3$.

D. Double ratio $(f_{B_s}/f_B)/(f_{D_s}/f_D)$

The double ratio of the decay constants — the Grinstein ratio R_1 [24] — is practically free from OPE uncertainties. We obtain the particularly accurate value

$$\frac{f_{B_s}/f_B}{f_{D_s}/f_D} - 1 = 0.013 \pm 0.011_{(\text{syst})}, \quad (4.7)$$

which is consistent with the lattice determination $(f_{B_s}/f_B)/(f_{D_s}/f_D) = 0.018 \pm 0.006 \pm 0.010$ [25].

5. SUMMARY AND CONCLUSIONS

In summary, we performed a detailed analysis of the extraction of the decay constants of pseudoscalar heavy mesons from the correlator of pseudoscalar currents. Particular emphasis was laid on the investigation of the uncertainties in

the extracted values of the decay constants: namely, on the OPE uncertainty related to the not precisely known QCD parameters, and on the intrinsic uncertainty of the method related to the limited accuracy of the extraction procedure. According to our recent findings, the accuracy of the sum-rule estimates may be considerably improved and the intrinsic uncertainties in hadron parameters may be probed by studying systematically the Borel-parameter dependence of the effective continuum thresholds; the parameters of these effective thresholds may be fixed by minimizing the deviation of the dual mass from the known meson mass in the Borel window. In the present work, this strategy has been applied to the decay constants of heavy mesons. Our main results are as follows:

(i) We obtain the following estimates for the decay constants of the charmed D and D_s mesons:

$$f_D = (206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{syst})}) \text{ MeV}, \quad (5.1)$$

$$f_{D_s} = (245.3 \pm 15.7_{(\text{OPE})} \pm 4.5_{(\text{syst})}) \text{ MeV}. \quad (5.2)$$

We would like to point out that we provide both the OPE uncertainties and the intrinsic (systematic) uncertainty of the method of sum rules related to the limited accuracy of the extraction procedure. In the case of f_D , the latter turns out to be of the same order of magnitude as the OPE uncertainty. Noteworthy, assuming τ -independence of the effective continuum threshold leads to the substantially lower decay-constant range $f_D(n=0) = (181.3 \pm 7.4_{(\text{OPE})}) \text{ MeV}$, which differs by almost three times the OPE uncertainty from our result (5.1) found from a τ -dependent effective threshold. The ratio of the charmed-meson decay constants (5.2) and (5.1) is

$$f_{D_s}/f_D = 1.193 \pm 0.025_{(\text{OPE})} \pm 0.007_{(\text{syst})}. \quad (5.3)$$

Instinctively one might worry about the relative magnitude of higher-order corrections and the quality of convergence of the perturbative expansion. However, a closer inspection [5] reveals that there is no reason for any concern of this kind:

- At NNLO, the $O(\alpha_s^2)$ contribution to f_D accounts for some 7% at scale $\mu = m_c$ and some 5% at scale $\mu = 3 \text{ GeV}$.
- At NLO — where all $O(\alpha_s^2)$ contributions are absent — the effective threshold emerging from our procedure turns out to be slightly larger than the one extracted at NNLO. This increase of the threshold partially compensates the absence of the $O(\alpha_s^2)$ contribution: at $\mu = m_c$, $f_D^{[\text{NLO}]} = 198 \text{ MeV}$ at NLO, to be compared with $f_D = 206 \text{ MeV}$ at NNLO. In other words, at the scale $\mu = m_c$ our result for f_D at NLO is merely 4% smaller than the one at NNLO.

(ii) The decay constants of the B and B_s mesons are very sensitive to the precise value of m_b . Using the PDG range of m_b does not allow us to obtain a reasonable estimate. For the very narrow range $\overline{m}_b(\overline{m}_b) = (4.163 \pm 0.0016) \text{ GeV}$ [10], our analysis gives $f_B = (225.6 \pm 11.3_{(\text{OPE})} \pm 2.2_{(\text{syst})}) \text{ MeV}$ and $f_{B_s} = (262.0 \pm 18.1_{(\text{OPE})} \pm 2.9_{(\text{syst})}) \text{ MeV}$. We observe some tension between the above sum-rule result for f_B and the average of recent lattice calculations [14, 17–19], namely, $f_B^{(\text{lattice})} = 193 \pm 13 \text{ MeV}$.

We emphasize that the observed strong sensitivity of f_B to the precise value of m_b provides an interesting alternative way of obtaining m_b from the analysis of the decay constant: Using the lattice average for f_B as input yields the rather accurate estimate for the b -quark mass

$$\overline{m}_b(\overline{m}_b) = (4.245 \pm 0.025) \text{ GeV}. \quad (5.4)$$

This new range of m_b corresponds to

$$f_B = (193.4 \pm 12.3_{(\text{OPE})} \pm 4.3_{(\text{syst})}) \text{ MeV} \quad (5.5)$$

and yields

$$f_{B_s} = (232.5 \pm 18.6_{(\text{OPE})} \pm 2.4_{(\text{syst})}) \text{ MeV}. \quad (5.6)$$

For the ratio of the decay constants (5.6) and (5.5) we get

$$f_{B_s}/f_B = 1.203 \pm 0.020_{(\text{OPE})} \pm 0.007_{(\text{syst})}, \quad (5.7)$$

(iii) The double (Grinstein) ratio of the decay constants,

$$\frac{f_{B_s}/f_B}{f_{D_s}/f_D} - 1 = 0.013 \pm 0.011_{(\text{syst})}, \quad (5.8)$$

is practically free from OPE uncertainties and, consequently, may be predicted with rather high accuracy.

Acknowledgments

We are indebted to Matthias Jamin for providing us with his Mathematica code for the calculation of the two-point function. DM expresses his gratitude to the Institute of Theoretical Physics of the Heidelberg University for hospitality during his visit to Heidelberg, where this work was started. DM was supported, in part, by: the Austrian Science Fund (FWF) under project P20573, the Alexander von Humboldt-Stiftung, the Federal Agency for Science and Innovation of Russian Federation under state contract 02.740.11.0244, and EU Contract No. MRTN-CT-2006-035482 “FLAVIANet”.

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