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A NEW PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

J PILLAY

ABSTRACT. We introduce a method, which we use to analyse the factors of polynomial functions $P(x)$. We then establish criteria for when the zeros of $P(x)$ with real coefficients are real, then we go on to analyse $P(x)$ with complex coefficients by using the same method introduced. Upon completion of our analysis we will have established that $P(x)$ with real or complex has at least one zero.

INTRODUCTION

This proof was first reviewed in the period 2001-2002 and stored in its *draft* state at UP tech-ref India Madras. This proof was presented by myself as an acceptance of a challenge put forward for scholarship purposes and asked students to attempt to develop some ideas or to provide insights into this old theorem. I later obtained the full scholarship after providing the proof presented herein. Only syntactical changes have been made from the original version that I presented at the age of eighteen. I mention the age as the version presented lacked a professional standard in English.

We credit the first ever proof of the Fundamental Theorem of Algebra to Carl Fredrich Gauss who first proved it in 1799.(See [1][2]) This paper analyses the criteria necessary for factors to exist, after which, it tests to see whether the criteria can be mathematically met.

Gauss proof shows that Zeros of polynomials always exist, but this paper I believe offers a different proof, which gives a different explanation as to why, whilst still taking a geometric approach.

1. NOTATION.

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + c_0$ will denote a single determinate polynomial with real/complex coefficients $a_n \in \mathbb{C}$.

We will make use of $[a, b, c, d, e]$ to denote the complex coefficients of arbitrary functions. $r, q \in \mathbb{R}_+$, indexed $m_n \in \mathbb{R}$ and $p_n \in \mathbb{R}$ will be used to denote arbitrary real numbers where necessary, and $x, \lambda \in \mathbb{C}$ will denote

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arbitrary complex variables. Indexed $\alpha_n \in \mathbb{R}$ will denote radian measures where appropriate.

2. THE ANALYSIS.

We wish to show that

$$(1) \quad P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + c_0$$

can always be written in the form

$$(2) \quad (x - m)(b_{n-1} x^{n-1} + \dots + c_1)$$

The summation actually forms criteria necessary in order for [m] to exist. To analyse the criteria more clearly, we set out the terms of the summation individually; in the following tower:

$$(3) \quad \begin{array}{r} b_{n-1} x^n =_{\text{needed}} a_n x^n \\ (-m)b_{n-1} x^{n-1} + b_{n-2} x^{n-1} =_{\text{needed}} a_{n-1} x^{n-1} \\ \dots \\ (-m)c_1 =_{\text{needed}} c_0 \end{array}$$

In each of these rows we can divide by the common factors: $x^n, x^{n-1}, x^{n-2}, \dots, x$ respectively, hence x as a variable need not fathom in our equations. If we can prove that the above criteria can always be met, then any polynomial function $P(x)$ can be written as a product of its factor and a polynomial of degree one less than $P(x)$. Now to satisfy (eq.1 in 3) we choose $b_{n-1} = a_n$; in the final equation; we can choose $(-m)$ and c_1 such that it equals c_0 , in (eq.2 in 3) we let $b_{n-2} = a_{n-1} + mb_{n-1}$, and let $b_{n-3} = a_{n-2} + mb_{n-2}$, we continue this process until we reach the semi final equation where we let $b_1 = a_2 + mb_2$. Now in the semifinal such equation we arrive at a problem whereby we have run out of variables to alter, to satisfy this criterion, as m, b_1 and c_1 have already been used. To rectify this problem we refer to equation's (3), and notice that m can be made as small or as large as desired and c_1 can be chosen appropriately to keep $(-m)c_1$ constant, i.e equal to c_0 .

Proof. If $(-m)c_1 = c_0$ then $c_1 = c_0/(-m)$ □

It is evident from this that the smaller we chose m to be, the larger c_1 becomes to keep $(-m)c_1$ constant. Bearing this in mind we refer back to equations (3) and notice that variation of m and c_1 can make $(-m)c_1$ as large or as small as we desire. We wish to see if variations exist that can be made to satisfy all the criteria simultaneously. We can do this in two parts, where we perform the case for the real values of m then move onto the complex, or proceed with the complex case alone. We opt for the

latter. Before we move on; in order to observe the nature of the variations possible by means of the real valued variable $(m, \lambda) \in \mathbb{R}$; we let $m = \frac{1}{\lambda}$ and let $c_0 = \frac{c_1}{\lambda}$.

The actual analysis is conducted in the following section with $\lambda \in \mathbb{C}$. We should also take note of the following. By means of simple backward substitution; we may combine the first $n - 1$ equations of the tower of requirement as stipulated, into a reciprocal function; and the final along with the semi final into another quadratic function. By means of this we can simply say that the requirements are met if a λ exists; that allows us to equate both functions. The equations are as follows, in the real case: $(-1/\lambda)b_1 + (c_0(-\lambda)) = a_1$, which can be written as: $Q(\lambda) = b_1 = -a_1\lambda - c_0\lambda^2$. Grouping of the first $(n - 2)$ equations in (3); has the following reciprocal form: $I(\lambda) = b_1 = c + a/\lambda + b/\lambda^2 + \dots + d/\lambda^{n-2}$. As this curve is relevant due only to its general non-specific shape, we don't evaluate the full expression.

3. THE COMPLEX ANALYSIS.

: In what may seem as a disconnect between the previous and current sections, we provide this brief description for this section.

We have within the previous section derived two functions, which if/when intersect provide a root for P . We extent the two functions along with their characteristics along with some basic conditioning to demonstrate that these, for complex coefficients, always have points at which they intersect simultaneously in their real and complex parts.

We make the following substitutions: $\lambda = l = r(\cos(q) + i\sin(q))^1$, setting $a_k = m_k(\cos(\alpha_k) + i\sin(\alpha_k))$ and making use of de Moivre's theorem, we can express $I(l)$ as follows:

$$(4) \quad \sum_{k=1}^{n-2} \frac{m_k(\cos(\alpha_k) + i\sin(\alpha_k))}{r^{k-1}(\cos((k-1)q) + i\sin((k-1)q))}$$

With:

$$(5) \quad ReI(l) = m_1(\cos(\alpha_1)) + (m_2/r^1)(\cos(\alpha_2 - q)) + (m_3/r^2)(\cos(\alpha_3 - 2q)) + \dots + (m_n/r^n)(\cos(\alpha_n - nq)).$$

$$(6) \quad ImI(l) = im_1(\sin(\alpha_1)) + (im_2/r^1)(\sin(\alpha_2 - q)) + (im_3/r^2)(\sin(\alpha_3 - 2q)) + \dots$$

¹making both functions a variable of r and q

$$\dots + (im_n/r^n)(\sin(\alpha_n - nq))$$

Similarly for Q , we let $b_k = p_k(\cos(\alpha_k) + i\sin(\alpha_k))$ which when following the necessary substitutions results in: $-p_1r(\cos(\alpha_1) + i\sin(\alpha_1))(\cos(q) + i\sin(q)) - p_2r^2(\cos(\alpha_2) + i\sin(\alpha_2))(\cos(2q) + i\sin(2q))$.

In this complex case, we require that $Re(L) = Re(Q)$ and $Im(L) = Im(Q)$ simultaneously, i.e. for a single pair $r, q \in \mathbb{R}_+$.

Without loss of generality we can divide throughout by (i) in the imaginary cases of both imaginary portions of the functions derived and treat them both as real functions.

Given the above, it remains to show that there always exist r and q values, such that $ReQ(l)$ equals $ReI(l)$ and $ImQ(l)$ equals $ImI(l)$ simultaneously.

3.1. Some basic Differential Geometry. We will be using cylindrical coordinates from now on, to visualize $I(l)$ and $Q(l)$.

We can see clearly that $ReI(l)$ is a reciprocal type function in 3-dimensions. i.e this function diverges to infinity at the Z-axis and converges while moving away from the Z-axis, because in $ReI(l) = m_1(\cos(\alpha_1)) + (m_2/r)(\cos(\alpha_2 - q)) + (m_3/r^2)(\cos(\alpha_3 - 2q)) + \dots + (m_n/r^{n-1})(\cos(\alpha_n - nq))$, where $[m_1..m_n]$ are constants, for any one particular q value the $Lim_{r \rightarrow 0} ReI(l) = \infty$, and for any one particular q value the $Lim_{r \rightarrow \infty} ReI(l) = m_1(\cos(\alpha_1))$.

The same can be done for $ImI(l)$ to show that, $ImI(l)$ is also a reciprocal type function in 3-dimensions, omitting without loss of generality i from our functions.

Following this convention throughout the remainder of the article, we take note of the general shapes of $ReQ(l)$, and $ImQ(l)$. $ReQ(l)$ can be written as: $-p_1r(\cos(\alpha_1)\cos(q) - \sin(\alpha_1)\sin(q)) - p_2r^2(\cos(\alpha_1)\cos(2q) - \sin(\alpha_1)\sin(2q))$. Let $g = \cos(\alpha_1)$ and $h = \sin(\alpha_1)$. for large values of r , i.e large enough such that $[-p_1r(\cos(\alpha_1)\cos(q) - \sin(\alpha_1)\sin(q))]$ contributes almost nothing to this function, the general shape of the graph will be determined by $[-p_2r^2(g\cos(2q) - h\sin(2q))]$. It is therefore evident that as r increases, the z values of the function increase.

To analyse the shape of the function we take note of the following:

$$(7) \quad \cos(2q + A) = \cos(A)\cos(2q)\sin(A)\sin(2q),$$

$$(8) \quad \cos(2q - A) = \cos(A)\cos(2q) + \sin(A)\sin(2q),$$

$$(9) \quad \sin(2q + A) = \sin(A)\cos(2q) + \cos(A)\sin(2q),$$

$$(10) \quad \sin(2q - A) = \sin(A)\cos(2q)\cos(A)\sin(2q)$$

The zeros of $\sin(2q)$ occur at $0, \pi/2, \pi, 3\pi/2, 2\pi$, and $\sin(2q)$ is positive in the following intervals $: [0; \pi/2], [\pi; 3\pi/2]$, and negative in the remaining. The zeros of $\cos(2q)$ occur at $\pi/4, 3\pi/4, 5\pi/4, 6\pi/4$, and $\cos(2q)$ is positive in the following intervals $: [6\pi/4; \pi/4], [3\pi/4; 5\pi/4]$. It is easy to see now that $ReQ(l)$ and $ImQ(l)$ exist both above and below the (r, q) plane simultaneously, as $\sin(2q - \alpha_1), \cos(2q - \alpha_1), \sin(2q + \alpha_1)$ and $\cos(2q + \alpha_1)$ have equal amounts of positive and negative intervals over which they exist (See Attached Figures). $Dx[-p_1r(\cos(\alpha_1 + q)) - p_2r^2(\cos(\alpha_1 + 2q))] = p_1r(\sin(\alpha_1 + q)) + 2p_2r^2(\sin(\alpha_1 + 2q))$. The existence of derivatives for all four, real and imaginary implies that the functions are smooth and continuous $\forall r \in \mathbb{R}_+$ and $\theta \in [0, 2\pi]$, but for $r = 0$.² Analysis of the derivative of $ReQ(l)$, will show us the general shape of the function. For our purpose we can choose r to be large enough such that $p_1r(\sin(\alpha_1 + q))$, contributes almost nothing to $DxReQ(l)$. Since $(\sin(\alpha_1 + 2q))$ is positive and negative over certain intervals we can see that the function clearly has two rises and two falls, also, since $\cos(\alpha_1 + 2q)$ has zeros and is negative and positive over certain intervals, we can see that the function has two hills and two valleys [in the $0 < \theta < 2\pi$ interval]. The valleys clearly go below the plane since the zeros of $\cos(\alpha_1 + 2q)$ lie between the intervals over which it is positive and negative and the zeros of $(\sin(\alpha_1 + 2q))$ lie between pairs of zeros of $\cos(\alpha_1 + 2q)$. The hills are above the plane for the same reasons. The same process can be repeated to determine the general shape of $ImQ(l)$, and you will notice that here also the graph forms two hills and two valleys. There will always be regions of the plane; where the graph lies below and other regions where it lies above. This simply because the dominating terms of $Im[Q]$, and $Re[Q]$ have regions over which they are positive and other regions; negative. Since we are working in cylindrical co-ordinates the hills and valleys rotate around the z-pole. Since $ReI(l)$ converges as r increases and $ReQ(l)$ diverges as r increases, there necessarily exists an intersection of $ReQ(l)$ and $ReI(l)$, regardless of whether $ReI(l)$ exists below or above the plane.

The intersection forms two U-like shapes that extend to Infinity, where $ReI(l)$, cuts either the two hills or the two valleys of $ReQ(l)$. The same applies to the case of $ImI(l)$ and $ImQ(l)$, i.e regardless of whether $ImI(l)$

²We leave this verification to the interested reader.

exists above or below the plane, it intersects $ImQ(l)$ forming two U-like shapes. It is easy to see from the analysis that these intersections are manifest; due precisely to the nature of both graphs.

3.2. Projections of the intersections on the (r, θ) Plane. We now consider the curves formed on $Z = K$. Let $\Phi := \theta_1, \theta_2, \theta_3, \theta_4, \dots, \theta_8$ denote the angels respectively, of the curves formed by the intersection of any of the Quadratic functions with the $Z = 0$ Plane and the circle centred at 0 and radius r . Taking any θ_t close to either of the θ 's we note that as $r \rightarrow \infty$; $Q(r, \theta_t)$ intersects $Z = K$ a finite r away. In order to make $Q(r, \theta_t)$ intersect $Z = K$ as $r \rightarrow \infty$, we note that $\forall i, \theta_i \in \Phi_r, |(r, \theta_t) - (r, \theta_p)| \rightarrow \infty$.

From the above we can see that as $r \rightarrow \infty$; intersections of the projections of the curves formed by the intersections of ReQ ; ReI and ImQ ; ImI on the (r, θ) plane are eminent, as the zeros of $Sin(2\theta + p)$ lie between pairs of the zeros of $Cos(2\theta + p)$. As such; (r, θ) Pairs always exist that satisfy the tower of equations.

REFERENCES

- [1] C. F. Gauss, Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse, in Gauss Werke Vol. 3, 1-30. An English translation is available at http://www.quantresearch.info/gauss_phd_dissertation.pdf.
- [2] https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra#
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