A new Proof of the Fundamental Theorem of Algebra

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Abstract

Firstly, we introduce a method, which we use to analyse the factors of polynomial functions P(x). We then establish criteria for when the zeros of P(x) with real coefficients are Real. Then finally we go on to analyse P(x) with complex coefficients by using the same method introduced. Upon completion of our analysis we would have established that P(x) with real or complex coefficients, or a mixture of both, has at least one zero i.e at least one factor.

Keywords: Polynomials (Zeros Of); Factorization of polynomials; Algebraic Functions, Degree of a polynomial.

Introduction

We credit the first ever proof of the Fundamental Theorem of Algebra to Carl Fredrich Gauss Who first proved it in 1799. Like most of the Theorems proved today, that are proved using other existing theorems, Gauss begins his proof with a theorem, which seems to hold no other reason why it was chosen, other than that it could be used to prove the Fundamental Theorem of Algebra. But this method, although very often efficient, offers little or no insight into the actual mechanics of the Theorem that it proves. For this reason alone I chose for this paper to offer a proof, which is to be more analytic in nature, than the one presented by Gauss. This paper analyses the criteria necessary for factors to exist, after which, it tests to see whether the criteria can be mathematically met. Gauss proof shows that Zeros of polynomials always exist, but this paper I believe offers a different proof, which gives a more clear explanation as to why.

Notation

$P(\lambda)$ will denote the polynomial function with complex numbers, $I(\lambda)$ will denote the complex reciprocal type function and $Q(\lambda)$ will denote the complex quadratic function. $[a_1, a_2, a_3, .., a_n]$ will denote the complex coefficients of $I(\lambda)$ and $[b_1, b_2, b_3, .., b_n]$ will denote the complex coefficients of $Q(\lambda)$. We will also use $[a, b, c, d, e]$ to denote the complex coefficients of arbitrary functions. $[m_1, m_2, .., m_n]$ and $[p_1, p_2, .., p_n]$ will be used to denote real numbers. x and λ will denote arbitrary complex variables. $q, [f_1, f_2, .., f_n]$ and $[\alpha_1, \alpha_2, .., \alpha_n]$ will denote angles and r will denote a positive real number variable.

1. We wish to show that $P(x)$; [2] can always be written in the form $(x -$

[1] My sincerest of thanks goes to P.Pillay, M.Be.Devaraj.N, L.Naick and family, along with Ms.Sharada and Ms.Rashida.

[2] Given by : $a_n x^n + .. + a_1 x^1 + c_1$

$m)(b_{n-1}x^{n-1}... + b_1x^1 + c_2)$, if we can accomplish this then we have essentially proven the Theorem. To analyse the possibility of this, we establish the criteria that needs to be met, in order for factors of this type to exist. Firstly we note that if:

$(b_{n-1}x^n + b_{n-2}x^{n-1} + .. + b_1x^2 + c_2x^1) + ((-m)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + .. + b_1x^1 + c_2)) = P(x)$

$[m]$ Becomes a zero of $P(x)$. The summation actually forms criteria necessary in order for $[m]$ to exist.
To analyse the criteria more clearly, we set out the terms of the summation individually; in the following tower:

1)$b_{n-1}x^n$ should equal a_nx^n,
2)$(-m)b_{n-1}x^{n-1} + b_{n-2}x^{n-1}$ should equal $a_{n-1}x^{n-1}$;
3)$(-m)b_{n-2}x^{n-2} + b_{n-3}x^{n-2}$ should equal $a_{n-2}x^{n-2}$...
.....
.....
Finally $(-m)c_2$ should equal c_1.

In each of these rows we can divide by the common factors:
$x^n, x^{n-1}, x^{n-2}, .., x$
respectively, hence x as a variable need not fathom in our equations.
If we can prove that the above criteria can always be met, then any polynomial function $P(x)$ can be written as a product of its factor and a polynomial of degree one less than $P(x)$. Now to satisfy (1) we choose $b_{n-1} = a_n$; in the final equation; we can choose $(-m)$ and c_2 such that it equals c_1, in (2) we let $b_{n-2} = a_{n-1} + mb_{n-1}$, in (3) we let $b_{n-3} = a_{n-2} + mb_{n-2}$, we continue this process until we reach the semi semi final equation where we let $b_1 = a_2 + mb_2$.

Now in the semifinal such; equation we arrive at a problem whereby we have run out of variables to alter, to satisfy this criterion, because m, b_1 and c_2 have already been chosen. To rectify this problem we refer to equation (6), and notice that m can be made as small or as large as desired and c_2 can be chosen appropriately to keep $(-m)c_2$ constant, i.e equal to c_1.

Proof
If $(-m)c_2 = c_1$ then $c_2 = c_1/(-m)$

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It is evident from this that the smaller we chose m to be, the larger c_2 becomes to keep $(-m)c_2$ constant. Bearing this in mind we refer back to equation (5) and notice that the fluctuation of m and c_2 can make (5) : $(-m)b_1 + c_2$ as large or as small as we desire. We wish to see if these fluctuations, which is our last resort to help satisfy all the criteria simultaneously, will suffice. We can do this in two parts, where we perform the case for the real values of m then move onto the complex, or proceed with the complex case alone. We opt for the latter.

Before we move on; in order to observe the nature of the fluctuations possible by means of the variable (m); we let $m = \frac{1}{\lambda}$ and let $c_1 = \frac{c_2}{\lambda}$.
The actual analysis is conducted in the following section with λ as a complex variable.
We should also take note of the following. By means of simple backward substitution; we may combine the first $n-1$ equations of the tower of requirement into a reciprocal function; and the final along with the semi final into another quadratic function.
By means of this we can simply say that the requirements are met if a λ exists; that allows us to equate both functions.

The equations are as follows in the real case:

$(-1/\lambda)b_1 + (c_1(-\lambda)) = a_1$ Which can be written as : $Q(\lambda) = b_1 = -a_1\lambda - c_1\lambda^2$.
And the grouping of the first (n-2) Equations; has the following reciprocal form:
$I(\lambda) = b_1 = c + a/\lambda^1 + b/\lambda^2 + .. + d/\lambda^n$. **The Complex Analysis**

Even though we are dealing with complex variables, the criteria does not change, i.e the formation, structure and the criteria required by the tower of equations does not change.
Furthermore, since the algebra of complex numbers is the same as the algebra of real numbers, we can safely say that $I(\lambda)$ and $Q(\lambda)$ can be constructed in the same way that it was constructed before, except now $[a_1, a_2, a_3, .., a_n]$ and $[b_1, b_2, b_3, .., b_n]$ are all complex numbers.

We wish to construct the complex reciprocal and complex quadratic functions for further analysis.

$I(\lambda) = c + a/l_1 + b/l_2 + .. + d/l_n$,

where a, b, c, d are complex numbers. We let $l = r(cos(q) + isin(q))$, and the other coefficients of $I(l)$, will be written as follows:

$a_1 = m_1(cos(\alpha_1) + isin(\alpha_1))$
$a_2 = m_2(cos(\alpha_1) + isin(\alpha_1))$
$. = .$
$. = .$
$a_n = m_n(cos(\alpha_1) + isin(\alpha_1))$

Therefore we can re-write $I(l)$ as:
$m_1(cos(\alpha_1)+isin(\alpha_1))+m_2(cos(\alpha_2)+isin(\alpha_2))/[r(cos(q)+isin(q))]1+m_3(cos(\alpha_3)+$

$isin(\alpha_3))/[r(cos(q)+isin(q))]2+..+m_n(cos(\alpha_n)+isin(\alpha_n))/[r(cos(q)+isin(q))]n,$

which equals to,

$m_1(cos(\alpha_1)+isin(\alpha_1))+m_2(cos(\alpha_2)+isin(\alpha_2))/r^1(cos(q)+isin(q))+m_3(cos(\alpha_3)+ isin(\alpha_3))/r^2(cos(2q) + isin(2q)) + .. + m_n(cos(\alpha_n) + isin(\alpha_n))/r^n(cos(n_q) + isin(n_q)),$

which becomes,

$b_1 = m_1(cos(\alpha_1)+isin(\alpha_1))+(m_2/r^1)(cos(\alpha_2-q)+isin(\alpha_2-q))+(m_3/r^2)(cos(\alpha_3- 2q) + isin(\alpha_3 - 2q)) + .. + (m_n/r^n)(cos(\alpha_n - nq) + isin(\alpha_n - nq)).$

In $Q(l)$ we let :

$b_1 = p_1(cos(\alpha_1) + isin(\alpha_1))$
$b_2 = p_2(cos(\alpha_2) + isin(\alpha_2))$

then $Q(l)$ becomes,

$-p_1r(cos(\alpha_1)+isin(\alpha_1))(cos(q)+isin(q))-p_2r^2(cos(\alpha_2)+isin(\alpha_2))(cos(2q)+ isin(2q)),$

which becomes,

$b_1 = -p_1r(cos(\alpha_1 + q) + isin(\alpha_1 + q)) - p_2r^2(cos(\alpha_1 + 2q) + isin(\alpha_1 + 2q)).$

It is therefore clear that $I(l)$ and $Q(l)$ are functions of r and q. If we can show that there always exists a l, such that $I(l) = Q(l)$, then $P(x)$ has at lest one factor.

Now $ReI(l) = m_1(cos(\alpha_1)) + (m_2/r^1)(cos(\alpha_2 - q)) + (m_3/r^2)(cos(\alpha_3 - 2q)) + .. + (m_n/r^n)(cos(\alpha_n - nq)).$

$ImI(l) = im_1(sin(\alpha_1)) + (im_2/r^1)(sin(\alpha_2 - q)) + (im_3/r^2)(sin(\alpha_3 - 2q)) + .. + (im_n/r^n)(sin(\alpha_n - nq)), and, ReQ(l) = -p_1r(cos(\alpha_1 + q)) - p_2r^2(cos(\alpha_1 + 2q)).$

$ImQ(l) = -p_1r(isin(\alpha_1 + q)) - p_2r^2(isin(\alpha_1 + 2_q)).$

Therefore in order for $Q(l)$ to equal to $I(l), ReQ(l)$ should equal $ReI(l)$ and $ImQ(l)$ should equal $ImI(l)$. It is easy to see that, when checking for $ImQ(l) = ImI(l)$, without loss of generality we can divide throughout by (i) and treat them both as real functions. So the problem at hand is now to show that there always exist r and q values, such that $ReQ(l)$ equals $ReI(l)$ and $ImQ(l)$ equals$ImI(l)$ simultaneously. To do this, (i) need not fathom in our functions, because it does not alter the shape of the function, nor does it alter where its zeros occur.

We are primarily interested in seeing where $ImQ(l)$ intersects $ImI(l)$ and you will also notice that dividing both equations by (i), does not effect where they will intersect, in 3-dimensional space. We will be using cylindrical coordinates from now on, to visualize $I(l)$ and $Q(l)$. We can see clearly that $ReI(l)$ is an reciprocal type function in 3-dimensions. i.e this function diverges to infinity at the Z-axis and converges while moving away from the Z-axis, because in

$ReI(l) = m_1(cos(\alpha_1)) + (m_2/r^1)(cos(\alpha_2 - q)) + (m_3/r^2)(cos(\alpha_3 - 2q)) + .. + (m_n/r^n)(cos(\alpha_n - nq))$,

where $[m_1..m_n]$ are constants, for any one particular q value the $Lim_{r\to 0}ReI(l) = \infty$, and for any one particular q value the $Lim_{r\to\infty}ReI(l) = m_1(cos(\alpha_1))$.

The same can be done for $ImI(l)$ to show that, $ImI(l)$ is also an reciprocal type function in 3-dimensions, omitting without loss of generality i from our functions. We follow this convention throughout the remainder of the article.

Now we take note of the general shapes of $ReQ(l)$, and $ImQ(l)$.

$ReQ(l)$ can be written as:$-p_1r(cos(\alpha_1)cos(q)-sin(\alpha_1)sin(q))-p_2r^2(cos(\alpha_1)cos(2q)-sin(\alpha_1)sin(2q))$.

Let $g = cos(\alpha_1)$ and $h = sin(\alpha_1)$. for large values of r, i.e large enough such that$[-p_1r(cos(\alpha_1)cos(q) - sin(\alpha_1)sin(q))]$ contributes almost nothing to this function, the general shape of the graph will be determined by $[-p_2r^2(gcos(2q)-hsin(2q))]$.

It is therefore evident that as r increases, the z values of the function increase. To analyse the shape of the function we take note of the following:

$cos(2q + A) = cos(A)cos(2q)sin(A)sin(2q), cos(2q - A) = cos(A)cos(2q) + sin(A)sin(2q), sin(2q + A) = sin(A)cos(2q) + cos(A)sin(2q), sin(2q - A) = sin(A)cos(2q)cos(A)sin(2q)$.

And that the zeros of $sin(2q)$ occur at 0, $\pi/2$, π,$\frac{3\pi}{2}$,2π, and $sin(2q)$ is positive in the following intervals :$[0; \frac{\pi}{2}]$, $[\pi; \frac{3\pi}{2}]$, and negative in the remaining. The zeros of $cos(2q)$ occur at $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{6\pi}{4}$,and $cos(2q)$ is positive in the following intervals : $[\frac{6\pi}{4}; \frac{\pi}{4}], [\frac{3\pi}{4}; \frac{5\pi}{4}]$.

It is easy to see now that $ReQ(l)$ and $ImQ(l)$ exist both above and below the (r, q) plane simultaneously,

since $sin(2q - \alpha_1), cos(2q - \alpha_1), sin(2q + \alpha_1)$ and $cos(2q + \alpha_1)$ have equal

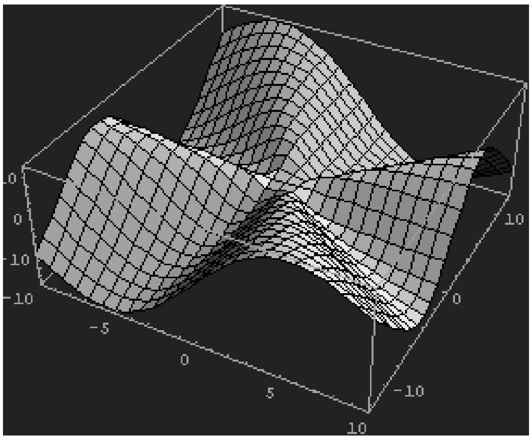

Figure 1: Graph of r Sin[2θ]

amounts of positive and negative intervals over which they exist.

$Dx[-p_1r(cos(\alpha_1+q))-p_2r^2(cos(\alpha_1+2q))] = p_1r(sin(\alpha_1+q))+2p_2r^2(sin(\alpha_1+2q))$,

from this we can see that the function is smooth and continuous.
Analysis of the derivative of $ReQ(l)$, will show us the general shape of the function. For our purpose we can choose r to be large enough such that $p_1r(sin(a1+q))$, contributes almost nothing to $DxReQ(l)$. Since $(sin(\alpha_1+2q))$ is positive and negative over certain intervals we can see that the function clearly has two rises and two falls, also, since $cos(\alpha_1+2q)$ has zeros and is negative and positive over certain intervals, we can see that the function has two hills and two valleys [in the $0 < \theta < 2\pi$ interval].

The valleys clearly go below the plane since the zeros of $cos(\alpha_1+2q)$ lie between the intervals over which it is positive and negative and the zeros of $(sin(\alpha_1+2q))$ lie between pairs of zeros of $cos(\alpha_1+2q)$.

The hills are above the plane for the same reasons. The same process can be repeated to determine the general shape of $ImQ(l)$, and you will notice that here also the graph forms two hills and two valleys. There will always be regions of the plane; where the graph lies below and other regions where it lies above. This simply because the dominating terms of $Im[Q]$, and $Re[Q]$ have regions over which they are positive and other regions; negative.
Since we are working in cylindrical co-ordinates the hills and valleys rotate around the z-pole.
Since $ReI(l)$ converges as r increases and $ReQ(l)$ diverges as r increases, there

necessarily exists an intersection of $ReQ(l)$ and $ReI(l)$, regardless of whether $ReI(l)$ exists below or above the plane.

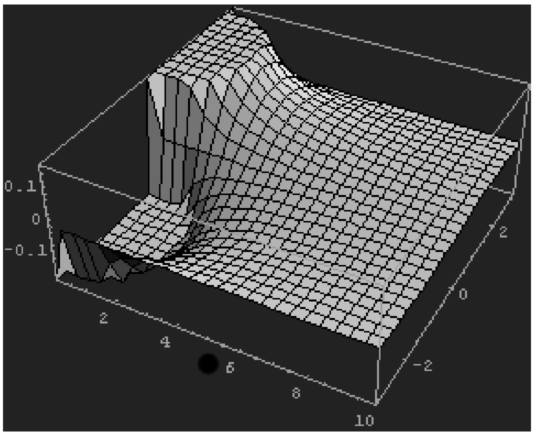

Figure 2: Figure of Reciprocal Function in Cartesian Coordinates

The intersection forms two U-like shapes that extend to Infinity, where $ReI(l)$, cuts either the two hills or the two valleys of $ReQ(l)$.

The same applies to the case of $ImI(l)$ and $ImQ(l)$, i.e regardless of whether $ImI(l)$ exists above or below the plane, it intersects $ImQ(l)$ forming two U-like shapes. It is easy to see from the analysis that these intersections are manifest; due precisely to the nature of either graphs.

We now consider the properties of the projections of these intersections on the (r, θ) Plane.

One important result is the following.
Let K denote the the constant term in either of the reciprocal type functions.
Let R_∞ denote the curve $Lim_{r\to\infty}(r, \theta)$ for the variable θ; where $\{0 \leq \theta \leq 2\pi\}$.
Let the intersection of the plane $Z = K$ with either quadratic function form the connected curves $S := \{U_1, U_2, U_3, U_4\}$.
We wish to show that the length of the distance between the points $\{x \in (R_t)\} \cap \{x \in (U_i \in S)\}$; approaches infinity as $r \to \infty$.

Proof
In order to prove this, we consider the curves U_p formed by the intersection of $Z = 0$ with either quadratic function.
Since $Lim_{r\to\infty} r\theta = \infty$; we can conclude that the distance formed between the points $\{x \in (R_\infty)\} \cap \{x \in (U_p)\}$ also approaches infinity.

We now consider the curves formed on $Z = K$.
Let $\Theta := \{\theta_1, \theta_2, \theta_3, \theta_4, .., \theta_8\}$ denote the angels respectively, of the lines formed by the intersection of any of the Quadratic functions with the $Z = 0$ Plane.
Taking any θ_t close to either of the $\theta' s$ we note that as $r \to \infty$; $Q(r, \theta_t)$ intersects $Z = K$ a finite r away. In order to make $Q(r, \theta_t)$ intersect $Z = K$ as $r \to \infty$, we note that as $\theta_t \to \theta_p \in \Theta$ $r \to \infty$.

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From the above we can see that as $r \to \infty$; intersections of the projections of the curves formed by the intersections of $ReQ; ReI$ and $ImQ; ImI$ on the (r, θ) plane are eminent, as the zeros of $Sin(2\theta + p)$ lie singly between pairs of the zeros of $Cos(2\theta + p)$.
As such; (r, θ) Pairs always exist that satisfy the tower of equations.

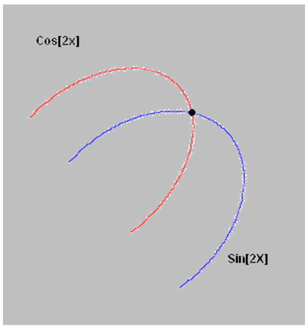

Figure 3: Figure of Reciprocal Function in Cartesian Coordinates. Blue Representing the projection onto the plane of the intersection of $Re[Q]$ and $Re[I]$; Red Representing the projection onto the plane of the intersection of $Im[Q]$ and $Im[I]$